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Link bordism skein modules

by

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Abstract. We compute link bordism skein modules of colored oriented links in oriented 3-manifolds. A Hurewicz theorem relating link bordism and link homotopy skein modules is proved.

1. Statement of results. Throughout let M be a compact connected oriented 3-manifold. Homology will be considered with integer coefficients. For general background concerning link bordism in manifolds and affine linking numbers, we refer to [6, Chapter 2]. The relation between oriented singular bordism and oriented homology is discussed in [13, pp. 318–319].

We study oriented unordered smooth embeddings of circles in M, called *links*. The empty link is also considered to be a link in M.

An r*-colored* link in M is a link with an assignment of one of the colors $1, \ldots, r$ to each of its components. (Note that a sublink corresponding to a specific color may be empty.) An *elementary link bordism* between two r-colored links is an oriented homology (or oriented singular bordism) of the sublink of one of the colors in the complement of the sublink determined by all other colors (for any choice of ordering of the components of the sublinks). Two r-colored links are *link bordant* if they differ by a finite sequence of elementary link bordisms and permutations of the colors. Let $\mathfrak{b}_r(M)$ denote the set of bordism classes of r-colored links in M, and $\mathfrak{b}(M) := \bigcup_{r\geq 0} \mathfrak{b}_r(M)$.

Let R be a commutative ring with 1, which contains the invertible element q and the non-invertible element z.

The *link bordism skein module* $\mathcal{B}(M;R)$ is the quotient of $R\mathfrak{b}(M)$ by the submodule generated by all Homflypt elements

$$q^{-1}K_+ - qK_- - zK_0$$

for all crossings of different colors.

Here the two colors of the strands of $K_\pm$ at the crossing will be joined to a single color, which is assigned to the component of K_0 resulting from the

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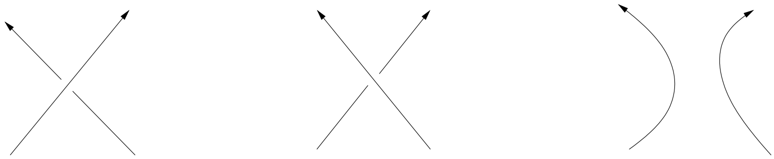

K_+ K_- K_0

smoothing. Necessarily this joined color is assigned to all the components which were colored before with one of the two colors of the two strands at the crossing. The remaining assignments of colors will be unchanged. Thus if $K_\pm$ are r-colored links then K_0 is an $(r-1)$-colored link.

It is our first goal in this paper to describe a presentation of $\mathcal{B}(M;R)$ which relates the structure of the module with the geometry of M.

Let ∂M be the boundary of M. We call M *Betti-trivial* if $2b_1(M) = b_1(\partial M)$, where $b_1(X)$ is the first Betti number of a space X. It has been shown in [6] that M is Betti-trivial if and only if each oriented surface in M is homologous into ∂M (equivalently M is a submanifold of a $\mathbb{Q}$-homology 3-sphere).

For each set T let $\mathrm{mon}(T)$ denote the set of monomials in T, which is the module basis of the symmetric algebra SRT. By definition the set $\mathrm{mon}(T)$ contains the empty monomial with no factors. For $k \geq 0$ let $\mathrm{mon}(T,k)$ denote the set of all monomials of length k.

A *standard link* for $a \in \mathrm{mon}(H_1(M))$ is a colored link in M with homology classes of colors given by a. This defines an epimorphism

$$\sigma : SRH_1(M) \to \mathcal{B}(M;R).$$

In Section 3 we will construct certain *symmetric* systems of standard links. For such a system, by using the intersection pairing (see [6, Appendix])

$$\mathfrak{j} : H_2(M) \otimes H_1(M) \to \mathbb{Z},$$

a *calibration* will be defined, which is a certain collection of labeled graphs. (The calibration is actually determined by some easy generalization of the linking pairing of M. If $H_1(M)$ is torsion-free then the calibration can be assumed trivial.) Also, in Section 3 we will define a subset

$$\mathrm{mon}_R(H_1(M)) \subset \mathrm{mon}(H_1(M))$$

from the ring R and $\mathfrak{j}$. The inclusion will be proper if and only if M is not Betti-trivial and q^2-1 is invertible in R.

THEOREM 1. *The structure of $\mathcal{B}(M)$ is completely determined by the ring R, the calibration and the intersection pairing of M. More precisely, let σ be a symmetric system of standard links.*

(a) *If M is Betti-trivial then σ induces an isomorphism*

$$SRH_1(M) \cong \mathcal{B}(M; R).$$

(b) *Suppose that M is not Betti-trivial. If $q^2 - 1$ is not invertible then $\mathcal{B}(M; R)$ has torsion. If $q^2 - 1$ is invertible then σ restricted to* $\mathrm{mon}_R(H_1(M))$ *is onto, and a further restriction is not onto.*

It is an interesting problem to find the exact conditions on M and R for which the module is free, and to give an explicit description of the torsion otherwise.

The result above follows from a general presentation of the bordism skein module by constructing an isomorphism with a certain skein module of *linking graphs*.

Two links in M are *link homotopic* if they are homotopic by a homotopy which keeps components disjoint. This equivalence relation was introduced by J. Milnor in 1954 [16]. Let $\mathfrak{h}(M)$ denote the set of link homotopy classes of links in M. The *homotopy skein module* $\mathcal{H}(M; R)$ is defined as the quotient of $R\mathfrak{h}(M)$ by the submodule generated by all Homflypt relations for crossings of different components. For $R = \mathbb{Z}[z]$ respectively $R = \mathbb{Z}[q^{\pm 1}, z]$ the module $\mathcal{H}(M; R)$ is isomorphic to the *Hoste–Przytycki skein module* respectively the *q-homotopy skein module*. These modules have been defined and studied by J. Hoste and J. Przytycki [5] and J. Przytycki [17], and by the author in [7]–[9].

The following result describes the passage from link homotopy skein modules to link bordism skein modules. This is similar to the classical Hurewicz theorem describing the passage from $\pi_1(M)$ to $H_1(M)$. Note that the classical Hurewicz homomorphism factors through the *map* $\widehat{\pi}(M) \to H_1(M)$, where $\widehat{\pi}(M)$ is the set of conjugacy classes of $\pi_1(M)$. In fact $\mathcal{H}(M; R)$ and $\mathcal{B}(M; R)$ can be considered as *quantum deformations* of $\widehat{\pi}(M)$ and $H_1(M)$ through the linking geometry contained in $\mathfrak{h}(M)$ and $\mathfrak{b}(M)$.

THEOREM 2 (Hurewicz theorem). *The surjective forgetful map*

$$\mathfrak{h}(M) \to \mathfrak{b}(M),$$

which maps an r-component link to the corresponding r-colored link, induces a Hurewicz epimorphism of skein modules

$$\mu : \mathcal{H}(M; R) \to \mathcal{B}(M; R).$$

This is an isomorphism if and only if $\pi_1(M)$ is abelian of rank not 3.

The results of this paper are based on the author's definition of affine linking numbers of links in oriented 3-manifolds, which classify the link bordism relation in M (see [6, Chapter 2]).

Outlook. The ideas of this paper suggest a more involved construction of *strong linking graphs* involving multi-component versions of the affine linking numbers of Chernov and Rudyak [2]. (See also [8] for a skein-theoretic version in the case of 3-manifolds.) This will incorporate a non-abelian version of some of the constructions in Sections 2–4 of this paper. Indeterminacies will be determined by string topology operations [1], as in [2] and [8]. In general, the strong linking graph leads to a more detailed understanding of link homotopy skein theory. It can be proved that, if $q^2 - 1$ is invertible, then the image of a link in the homotopy skein module is determined by its strong linking graph [11]. Also, for M a cylinder over a surface this image can be computed by a state-sum formula on the strong linking graph, which generalizes Przytycki's formula for links in S^3 (see [12] and [17]).

Plan of the paper. In Section 2 we develop a skein theory of labeled graphs and prove a presentation result. In Section 3 this is applied to deduce a presentation of the link bordism skein module, and prove Theorem 1. In Section 4 a corresponding theory is developed in the link homotopy case with abelian fundamental group. This is applied in Section 5 to prove the Hurewicz Theorem 2.

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2. Skein modules of abstract labeled graphs. Let

$$H = \bigoplus_{a \in A} H_a$$

be an abelian group graded over an abelian group A. Suppose there is a bilinear pairing

$$\iota : \mathbb{Z}A \otimes H \to \mathbb{Z}.$$

Its restriction to $A \otimes H_a$ is denoted ι_a.

Remark 1. Note that the group ring $\mathbb{Z}A$ has two abelian group operations: a formal group ring addition, and the group ring multiplication induced from the group operation in A. Linearity in the first argument refers to the first structure. Note that addition in A defines a natural homomorphism

$$\mathbb{Z}A \to A.$$

Thus given a pairing

$$A \otimes H \to \mathbb{Z}$$

there is always an induced pairing

$$\mathbb{Z}A \otimes H \to A \otimes H \to \mathbb{Z}.$$

Note that the collection $(\iota_a)_{a \in A}$ determines ι.

EXAMPLE 1. For M a 3-manifold let $A := H_1(M)$ and $H_a := H_2(M)$ for all $a \in A$. For each $a \in H_1(M)$ let $\iota_a = \mathfrak{j}$ be the intersection pairing

$$\mathfrak{j} : H_1(M) \otimes H_2(M) \to \mathbb{Z}.$$

Let

$$h_i(M) := H_i(M)/j_*(H_i(\partial M))$$

for $i = 0, 1, 2, 3$, where $j : \partial M \hookrightarrow M$ is the inclusion. It has been proved in [6] that M is Betti-trivial if and only if $h_2(M) = 0$ (equivalently $h_1(M)$ is torsion). In fact $\mathfrak{j}$ factors through the *non-degenerate* inner homology pairing

$$h_1(M)/\mathrm{Tor}(h_1(M)) \otimes h_2(M) \to \mathbb{Z}.$$

EXAMPLE 2. Let M be a 3-manifold with abelian fundamental group. Then $A := \pi_1(M) \cong H_1(M) \cong \widehat{\pi}(M)$. Let LM be the free loop space of M and let $H := H_1(LM)$. Note that H is naturally graded over A. In this case the Chas–Sullivan [1] intersection pairing

$$H_0(LM) \otimes H_1(LM) \to H_0(LM)$$

is a bilinear pairing

$$\mathbb{Z}A \otimes H \to \mathbb{Z}A.$$

Let $\iota = \mathfrak{j}'$ be the composition of this pairing with the coefficient sum homomorphism $\mathbb{Z}A \to \mathbb{Z}$. Note that because of the definition of the Chas–Sullivan pairing it is clear that the image of some element $b \otimes c \in A \otimes H_a$ is an integer multiple of $b + a$. Thus the composition with the coefficient sum homomorphism is injective when restricted to $\{b\} \otimes H_a$.

An A-*graph* is a *complete* graph with vertices labeled by elements of A and edges labeled by integers. For $k \geq 0$ let $\mathcal{G}[k]$ be the set of A-graphs with k distinguished ordered edges, with multiple edges allowed. Let $\mathcal{G}[2]_{\mathrm{I}}$ denote the subset of those graphs where the distinguished edges are distinct, and let $\mathcal{G}[2]_{\mathrm{II}} := \mathcal{G}[2] \setminus \mathcal{G}[2]_{\mathrm{I}}$.

A *rooted A-graph* is an A-graph with a choice of vertex. Let $a \in A$ be the root label. There is an action of H_a on the set of rooted A-graphs with root label a, defined as follows: Let $y \in H_a$ act by adding *simultaneously*, for all edges with a root vertex, the numbers $\iota_a(b, y) \in \mathbb{Z}$ to the edge labels incident to the root with vertex label b.

This defines an equivalence relation on the set of A-graphs: the relation is generated by $\gamma_1 \sim \gamma_2$ if γ_1 and γ_2 differ by the action of some $y \in H_a$ for some choice of root with label a. The equivalence classes of A-graphs are simply called *graphs* (or more precisely *ι-graphs*). For $a \in \mathrm{mon}(A)$ let $\mathcal{G}(a)$ denote the set of graphs with vertex labels corresponding to a. If a is the empty monomial then $\mathcal{G}(a)$ consists of a single empty graph. Let $\mathcal{G}$ denote the set of all graphs.

For $a_0 \in H_1(M)$ and $a \in \mathrm{mon}(H_1(M))$ let $\mathcal{G}(a_0, a)$ denote the set of rooted graphs with root label a_0 and the collection of remaining labels given by a.

Similarly an *A-spider* is a rooted height 1 tree with labels as above. An A-spider is *symmetric* if any two edges whose vertices have the same labels in A have the same edge labels.

Note that A-spiders are naturally rooted. The action of H_a on the set of A-spiders with root a is defined as in the graph case. Equivalence classes of A-spiders are *spiders* (or more precisely *ι-spiders*). Note that the equivalence class of a symmetric A-spider only contains symmetric A-spiders. Thus *symmetric spiders* are defined naturally. Let $\mathcal{W}$ respectively $\mathcal{W}_{\mathrm{s}}$ denote the sets of spiders respectively symmetric spiders.

Let $\mathcal{W}(a_0, a)$ denote the set of spiders with root label a_0, and with order 1 vertices labeled by $a \in \mathrm{mon}(A)$. Similarly define $\mathcal{W}_{\mathrm{s}}(a_0, a)$. Note that for the empty monomial a, $\mathcal{W}_{\mathrm{s}}(a_0, a) = \mathcal{W}(a_0, a)$ is the graph with one vertex labeled a_0 and no edges.

Lemma 1. *Addition of corresponding edge labels defines maps*

$$\mathcal{W}_{\mathrm{s}}(a_0, a) \times \mathcal{W}(a_0, a) \to \mathcal{W}(a_0, a)$$

which restrict to define the structures of abelian groups on $\mathcal{W}_{\mathrm{s}}(a_0, a)$, *and extend to define actions*

$$\mathcal{W}_{\mathrm{s}}(a_0, a) \times \mathcal{G}(a_0, a) \to \mathcal{G}(a_0, a).$$

Similarly, *exterior multiplication* on the level of representative spiders,

$$\widetilde{\mathcal{W}}_{\mathrm{s}}(a_0, a) \times \widetilde{\mathcal{W}}_{\mathrm{s}}(a_0', a) \to \widetilde{\mathcal{W}}_{\mathrm{s}}(a_0 + a_0', a),$$

is defined by addition of edge labels. Here $\widetilde{\mathcal{W}}_{\mathrm{s}}(a_0, a)$ is the corresponding set of A-spiders. Note that if the actions of H_{a_0} respectively H'_{a_0} are trivial then the exterior multiplication is well defined on symmetric spiders. It is also well defined in the case when H_a does not depend on a.

Definition 1. A *calibration* is a choice of

$$w(a_1, a_2, a) \in \mathcal{W}_{\mathrm{s}}(a_1 + a_2, a),$$

for all $a_1, a_2 \in A$ and $a \in \mathrm{mon}(A)$, such that

$$w(a_1, a_2, a) = w(a_2, a_1, a).$$

A *zero-calibration* is a calibration of the form $w(a_1, -a_1, a)$.

Let $\gamma \in \mathcal{G}[1]$ with a_1, a_2 the vertex labels of the distinguished edge e. Let $a \in \mathrm{mon}(A)$ be the monomial of the other vertex labels. Let $\gamma_- \in \mathcal{G}$ be the graph defined by the obvious projection $\mathcal{G}[1] \to \mathcal{G}$, which forgets the choice of distinguished edge, and only keeps the equivalence class of the graph. Similarly let $\gamma_+ \in \mathcal{G}$ be defined by increasing the edge label of e by 1 and then applying the forgetful map into graphs. Finally, let $\gamma_* \in \mathcal{G}_*$ be defined by (i) collapsing e to a single vertex and adding the vertex labels in A, and (ii) identifying the corresponding edges in pairs in the obvious way and adding the edge labels in $\mathbb{Z}$ for each pair. ($\mathcal{G}_*$ is the set of rooted A-trees.) Then the *Conway smoothing* γ_0 is defined by applying the action of Lemma 1 to $w(a_1, a_2, a)$ and γ_*. Similarly γ defines a skein relation, where γ_+ is given by the projection to $\mathcal{G}$ and γ_- is defined by subtracting 1 from the distinguished edge label, while γ_0 is defined in the same way.

DEFINITION 2. For given ι and calibration w define the skein module $\mathcal{D}(w, \iota)$ to be the quotient of the free R-module with basis $\mathcal{G}$ by the submodule generated by all elements

$$q^{-1}\gamma_+ - q\gamma_- - z\gamma_0$$

for all $\gamma \in \mathcal{G}[1]$.

THEOREM 3. *Let w be a calibration. Then the injective map*

$$\sigma : \mathrm{mon}(A) \to \mathcal{G}$$

which assigns to each monomial in A the graph with all edge labels 0 *induces an epimorphism*

$$\sigma : SRA \to \mathcal{D}(w, \iota).$$

Proof. The proof is by induction over the following complexity function on $\mathcal{G}$:

$$c(\gamma) := (c_1(\gamma), c_2(\gamma)) \in \mathbb{N} \times \mathbb{N},$$

where $\mathbb{N}$ is the set of non-negative integers and $\mathbb{N} \times \mathbb{N}$ has the lexicographic order. Here $c_1(\gamma)$ is the number of vertices. For $\gamma' \in \mathcal{G}[0]$ let $c_2(\gamma')$ be the sum of the absolute values of the labels over all edges in γ'. Then let $c_2(\gamma)$ denote the minimum of the $c_2(\gamma')$ over all $\gamma' \in \mathcal{G}[0]$ mapping to γ. We only have to show that, up to skein relations, each $\gamma \in \mathcal{G}$ is in the image of σ. Now let $\gamma \in \mathcal{G}$ be any graph with $c(\gamma) = (n, 0)$; then obviously γ is in the image of σ. Note that for $n = 0$ the graph is the empty graph and is the image of the empty monomial. Assume that $c(\gamma) = (n, n')$ with $n' > 0$, and all graphs with smaller complexity are already contained in the image of σ. Consider a representative γ' with $n > 0$ and $n' > 0$ minimal. Then there is an edge with non-zero label and by application of a skein relation we get a new graph $\gamma_\pm$ with complexity (n, n'') and $n'' < n'$ and the smoothing has complexity $(n - 1, n''')$. ∎

Let $\gamma \in \mathcal{G}[2]_{\mathrm{I}}$ with distinguished ordered edges e, e'. Let $\gamma_{(0),0}$ respectively $\gamma_{0,(0)}$ be the results of applying Conway smoothings first to e and then to e' respectively with order changed.

Definition 3. A calibration is *local* if

$$\gamma_{(0),0} = \gamma_{0,(0)}$$

for all $\gamma \in \mathcal{G}[2]_{\mathrm{I}}$.

It is easy to show that the trivial calibration is local.

In the following we will identify $a \in \mathrm{mon}(A)$ with $\sigma(a) \in \mathcal{G}$, which is the graph with only zero edge labels. We want to describe the kernel of the homomorphism σ. For each $a \in \mathrm{mon}(A)$ choose a lifting a' to $\mathcal{G}[1]$. Then let $a'' =: a(y, *)$ be another lifting resulting from the action of some element $y \in h_2(M)$ and a choice of vertex $*$. If y acts non-trivially then $c_2(a'') > 0$. We apply skein relations and expand a'' respectively its projection into $\mathcal{G}$ (which is equal to a') as a sum of $q^{2\varepsilon}a'$ and graphs with a smaller number of vertices. Expand these graphs as R-linear combinations of elements in the image of σ. This will define an element $(q^{2\varepsilon} - 1)a +$ lower order terms. Here the lower order refers to the number of colors. (Each such term is divisible by z.) In this way, by induction over the number of vertices, we define a submodule U of SRA. It suffices to consider the elements $a(y, *)$ for a generating set of $h_2(M)$.

Theorem 4. *Suppose that the calibration w is local. Then the kernel of σ is the submodule U.*

Proof. Obviously U is contained in the kernel of σ. Let σ also denote the induced epimorphism $SRA/U \to \mathcal{D}(w, \iota)$. We define inductively a map $\varrho : \mathcal{G} \to SRA/U$, and so a homomorphism $\varrho : R\mathcal{G} \to SRA/U$, which maps the skein relations trivially and satisfies $\varrho \circ \sigma = \mathrm{id}$. (Here ϱ also denotes the induced $\mathcal{D}(w, \iota) \to SRA/U$.) This will prove that σ as above is also injective, which yields the result.

Suppose that γ has complexity $(n, 0)$, so $\gamma = \sigma(a)$ for some $a \in \mathrm{mon}(A, n)$. Then define $\varrho(\gamma) = [a] \in SRA/U$, where $[a]$ is the equivalence class of $a \in \mathrm{mon}(A) \subset SRA$ in the quotient module. Define ϱ of the empty graph to be the empty monomial. Assume that we have defined ϱ on all graphs with complexity smaller than (n, n') with $n > 0$ and $n' > 0$. Consider γ with complexity (n, n'). Choose $\gamma_e \in \mathcal{G}[1]$ mapping to γ with distinguished edge e, and apply the skein relation to e. We can choose e and the type of application such that the other two graphs in the Conway relation, which are γ_0 and either γ_+ or γ_-, have smaller complexity than γ. It remains to prove that this construction does not depend on the choice of a lifting γ_e. Note that each lifting involves both a choice of lifting to $\mathcal{G}[0]$ and a lifting from there to $\mathcal{G}[1]$. A different choice of lifting to $\mathcal{G}[0]$ will be resolved in U as long as we can prove independence of the lifting to $\mathcal{G}[2]$.

It is easy to prove that two applications of the skein relation to the same edge (with appropriate signs) lead to the following skein expansions:

$$\gamma_{++} \rightsquigarrow q^2\gamma_{-+} + qz\gamma_0 \rightsquigarrow q^4\gamma_{--} + (q^3z + qz)\gamma_0,$$
$$\gamma_{++} \rightsquigarrow q^2\gamma_{+-} + qz\gamma_0 \rightsquigarrow q^4\gamma_{--} + (q^3z + qz)\gamma_0,$$

where γ_0 is the result of Conway smoothing for the edge. Other signs will lead to similar calculations. So it suffices to show that changing the order of two applications to two *different* edges gives rise to the same result in SRA/U. (Any change in the order of applications can be reduced to transpositions.) We assume by induction that the ϱ-expansion is well defined for all graphs of smaller complexity. After application of the relation to one of the edges we have already reduced the complexity in the way we want, and we are free to expand in any possible way.

Thus let $\gamma_{e,e'} \in \mathcal{G}[2]_I$ be given with distinguished edges e, e'. For simplicity assume that we apply the skein relation whenever we replace a γ_+ by γ_- and γ_0. The argument in the other cases is similar and will be omitted. As before, the bracketed index will denote the edge where the skein relation has been first applied. In cases where it is obvious that the corresponding graph (with a choice of vertex) does not depend on the choice of order we omit the brackets. Thus we have to compare

$$\gamma_{--} \rightsquigarrow q^2\gamma_{-+} + qz\gamma_{0+} \rightsquigarrow q^4\gamma_{--} + q^3z\gamma_{-0} + q^3z\gamma_{0-} + q^2z^2\gamma_{(0),0}$$

with

$$\gamma_{--} \rightsquigarrow q^2\gamma_{+-} + qz\gamma_{+0} \rightsquigarrow q^4\gamma_{--} + q^3z\gamma_{0-} + q^3z\gamma_{-0} + q^2z^2\gamma_{0,(0)}.$$

The two expansions agree by induction because w is local. So the terms with coefficients q^2z^2 agree. ∎

COROLLARY 1. *Suppose that w is local and ι is trivial. Then σ induces an isomorphism*

$$SRA \cong \mathcal{D}(w, \iota).$$

REMARK 2. The algebraic constructions above are natural in the following sense. Suppose two quadruples (A, H, ι, w) and (A', H', ι', w') as above are given. Then a morphism is a pair of group homomorphisms

$$(\phi : A \to A',\ \psi : H \to H')$$

compatible with the grading, i.e. for all $a \in A$,

$$\psi(H_a) \subset H'_{\phi(a)},$$

and a pairing ι' such that the diagram

$$\begin{array}{ccc} \mathbb{Z}A \otimes H & \xrightarrow{\iota} & \mathbb{Z} \\ {\scriptstyle \phi\otimes\psi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbb{Z}A' \otimes H' & \xrightarrow{\iota'} & \mathbb{Z} \end{array}$$

commutes. We have induced maps of the sets of graphs and spiders preserving symmetry. Suppose that for all $a_1, a_2 \in A$ and $a \in \mathrm{mon}(A)$ the image of the symmetric spider $w(a_1, a_2, a)$ under the induced map is the symmetric spider $w(\phi(a_1), \phi(a_2), \phi(a))$. Then there is an induced homomorphism of skein modules:

$$\mathcal{D}(w, \iota) \to \mathcal{D}(w', \iota').$$

REMARK 3. In [11] we will construct a skein theory of *labeled* graphs for A an arbitrary, not necessarily abelian group π. Then graphs have vertex labels in the set of conjugacy classes of π, possibly parallel edges labeled ± 1, but no loops at a vertex. There will be suitable assignments of lifts of edge labels to π for each tree in the graph.

3. Linking graphs. In this section we apply the results of Section 2 to the situation of Example 1. Thus vertex labels of graphs are in $H_1(M)$ and the actions are defined by $H_2(M)$, respectively $h_2(M)$. Recall that $\mathcal{G} = \mathcal{G}(M) := \mathcal{G}(\mathfrak{j})$ is the set of equivalence classes of $H_1(M)$-graphs by the actions of $h_2(M)$ using $\mathfrak{j}$.

Let $\mathfrak{b}(a)$ denote the set of bordism classes of colored links with homology classes of colors given by a.

PROPOSITION 1. *There is a bijective mapping*

$$\mathfrak{l} = \mathfrak{l}_M : \mathfrak{b}(M) \to \mathcal{G}(M)$$

which maps $\mathfrak{b}(a)$ *to* $\mathcal{G}(a)$ *for all* $a \in \mathrm{mon}(H_1(M))$.

We call $\mathfrak{l}(K)$ the *linking graph* of the colored link K in M.

Proof. This is essentially proved in [6, 2.3], which considers colored links with orderings of the colors. The definition of $\mathfrak{l}$ involves a choice of standard link L_a for each $a \in \mathrm{mon}(H_1(M))$. (We will choose standard links more carefully later on in order to see the correspondence with the classical linking pairing in 3-manifolds.) It is proved in [6, 2.2 and 2.3] that each colored link in M is link bordant to the disjoint union of a standard link and a union of Hopf links in a 3-ball. Corresponding to the number of Hopf links, and by using the orientation, this associates an integer number to each pair of colors. These numbers are well defined up to the action of $h_2(M)$ determined by $\mathfrak{j}$. This is proved in [6, 2.3.4]. ∎

Next we define the calibration $w = w(M)$ such that $\mathfrak{l}$ induces a homomorphism from $\mathcal{B}(M;R)$ into $\mathcal{D}(w(M),\iota;R) =: \mathcal{D}(M;R)$. Recall that ι is determined by ι_a and $\iota_a = \jmath$ for all $a \in H_1(M)$.

Decompose $H_1(M)$ as a direct sum of cyclic subgroups. Choose a generator of each cyclic summand and represent it by a knot in M. Represent products of generators by links formed from parallel components (we say that two components are *parallel* if one is isotopic to the other within a tubular neighborhood of the former). Finally, represent each monomial by a union of colored links with each color in the previous form. A system of standard links constructed in this way is called *symmetric*. Note that each sublink of a standard link is isotopic to a standard link.

A link which is the disjoint union of a standard link and Hopf links contained in a 3-ball separated from the remaining components is said to be *in standard form.*

Suppose $a_1, a_2 \in H_1(M)$ and $a \in \mathrm{mon}(H_1(M)$. Consider the link $L_{a_1,a_2,a}$, which is the standard link $L_{a_1a_2a}$ equipped with a choice of two distinguished colors. Then merge the two distinguished colors to define a new link denoted $L_{[a_1+a_2],a}$. This link is generally not a standard link. But the sublink of the colors carrying a is standard (at least after some isotopy). Consider some oriented singular bordism of the sublink corresponding to the merged color into standard form (i.e. such that after the bordism the whole link will be standard $L_{(a_1+a_2)a}$). The transversal intersection of the oriented singular bordism, which is a mapping of an oriented surface:

$$f_{a_1,a_2} : V = V_{a_1,a_2} \to M,$$

with the sublink of the other colors is a collection of oriented points in V. After cutting small disks out of V with centers the intersection points, and after a further obvious bordism, we have defined an elementary bordism from $L_{[a_1+a_2],a}$ to a link in standard form. Note that the resulting collection of Hopf links in the 3-ball is uniquely determined by the transversal intersection numbers of f_{a_1,a_2} with the standard link L_a. This defines the calibration $w(a_1,a_2,a) \in \mathcal{W}_\mathrm{s}(a_1+a_2,a)$. (The symmetry follows from the symmetry of standard links.) If we choose another bordism then edge labels defined by the intersection numbers will change by actions of $h_2(M)$ with respect to the central vertex in the usual way.

Remark 4. Suppose that $H_1(M)$ is torsion free and M is Betti-trivial. (This is true for submanifolds of integral homology 3-spheres; see also [6, A.2] for a discussion of the converse.) In this case linking numbers in $\mathbb{Z}$ are well defined, and have the usual properties. It follows that $\jmath$ is trivial. In this case our *affine linking graphs* can be replaced by usual linking graphs using the integral linking number map (which is not uniquely defined but depends e.g. on an embedding in some integral homology sphere, or in the general case on the choice of a link map basis; see [6, 3.5.10]). Note that for $H_1(M)$ torsion free the bordisms f_{a_1,a_2} can be assumed to be within tubular neighborhoods of the corresponding representatives such that there will be no intersection points with L_a. So in this case the calibration is trivial.

Proposition 2. *The linear extension of the linking graph map*

$$\mathfrak{l} : R\mathfrak{b}(M) \to R\mathcal{G}(M)$$

induces an isomorphism of skein modules

$$\mathcal{B}(M;R) \to \mathcal{D}(M;R).$$

Proof. This is immediate from $\iota = \mathfrak{j}$ and the calibration. The inverse map to $\mathfrak{l}$ is defined from a symmetric system of standard links by adding Hopf links corresponding to edge labels. This induces the inverse homomorphism. ∎

In order to apply Theorem 4 we need the following:

Lemma 2. *The calibration $w(M)$ defined from a symmetric system of standard links is local.*

Proof. Let $\gamma \in \mathcal{G}[2]_{\mathrm{I}}$ and let e, e' be the distinguished edges.

Case 1. Assume that e, e' have no common vertex. Let a_1, a_2 respectively a_3, a_4 be the vertex labels of the vertices of e respectively e'. Let $a \in \mathrm{mon}(H_1(M))$ be the monomial of the remaining vertex labels. We have to show that the edge labels of $\gamma_{(0),0}$ and $\gamma_{0,(0)}$ coincide (up to the actions of $h_2(M)$). This is easy to see for all edges except for the edge e'' joining the vertices resulting from collapsing e and e'. The corresponding vertex labels are a_1+a_2 and a_3+a_4. Consider the intersection of bordisms f_{a_1,a_2} and f_{a_3,a_4} as used in the definition of the calibration. More precisely, consider the map $f_{a_1,a_2} \times f_{a_3,a_4} : V_{a_1,a_2} \times V_{a_3,a_4} \to M \times M$, which can be assumed transversal to the diagonal in $M \times M$. Consider the preimage in $V_{a_1,a_2} \times V_{a_3,a_4}$. This is an oriented 1-manifold C with boundary in $\partial(V_{a_1,a_2} \times V_{a_3,a_4})$. Now let $\partial V_{a_1,a_2} = W_{a_1,a_2} \amalg W_{a_1+a_2}$, where $W_{a_1,a_2} \to M$ is the map of circles into M defined by the merged color and $W_{a_1+a_2} \to M$ is the standard map corresponding to the symmetric choice. Similarly we have a decomposition of $\partial V_{a_3,a_4} = W_{a_3,a_4} \amalg W_{a_3+a_4}$. Then

$$\partial C \cap (V_{a_1,a_2} \times W_{a_3,a_4} \cup W_{a_1+a_2} \times V_{a_3,a_4})$$

represents the edge label of e'' in $\gamma_{(0),0}$. Similarly

$$\partial C \cap (W_{a_1,a_2} \times V_{a_3,a_4} \cup V_{a_1,a_2} \times W_{a_3+a_4})$$

represents the edge label of e'' in $\gamma_{0,(0)}$. There are no intersections in the corner manifolds, which are products of 1-manifolds, by the very construction. Thus the oriented 1-manifold C shows that the edge labels are equal.

Case 2. Assume that e and e' have a common vertex. Assume the vertex labels are a_1, a_2 and a_2, a_3. In this case the collapse of e and e' leads to a single vertex with label $a_1 + a_2 + a_3$. It follows easily that the actions of the calibrations defined above add up independently, and give rise to the same edge labels of all edges. ∎

For the statement and proof of Theorem 1(b) we introduce the following definitions. Suppose that $q^2 - 1$ is invertible in R. Note that

$$\{l \in \mathbb{Z} \mid q^{2l} - 1 \text{ is not invertible}\}$$

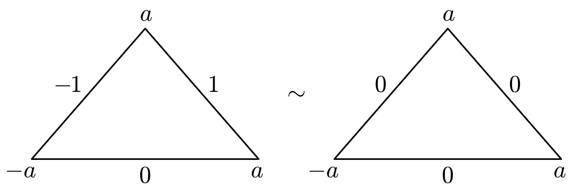
is a proper ideal (p) generated by a prime number p, or is the trivial ideal (0). Consider the multiplicative set $A_R := \mathbb{Z} \setminus (p)$. Then $q^{2l} - 1$ is invertible if and only if $l \in A_R$. For $a = a_1 \cdots a_r \in \text{mon}(H_1(M))$ let $m(a)$ be the minimum of all $|\mathfrak{j}(a/a_i, y)|$ over all $1 \leq i \leq r$ and all $y \in h_2(M)$. Define the map $\mathfrak{m} : \text{mon}(H_1(M)) \to \mathbb{Z}/(p)$ by mapping a to $m(a)$ mod p. Then let $\text{mon}_R(M) := \mathfrak{m}^{-1}(0)$. It follows that in each relation of the form $(q^{2l} - 1)a+$ (lower order terms) in U the ring element $q^{2l} - 1$ is not invertible *if and only if* $\mathfrak{m}(a) = 0$. Obviously, M is Betti-trivial if and only if $\text{mon}_R(H_1(M)) = \text{mon}(H_1(M))$. Let $\text{mon}_R(H_1(M)) := \text{mon}(H_1(M))$ for $q^2 - 1$ not invertible. We have $\text{mon}(H_1(M), 1) = H_1(M) \subset \text{mon}_R(M)$.

Proof of Theorem 1. Apply Proposition 2, Theorem 4 and Lemma 2. Thus $\mathcal{B}(M; R) \cong \mathcal{D}(M; R) \cong SRH_1(M)/U$ with the description of the submodule U given before the proof of Theorem 4. The module U is determined by the calibration and the actions on $H_1(M)$-graphs. Because of the symmetric choice of standard links and Remark 4 above, and because of the remark following Lemma 1 above, the calibration is completely determined by zero-calibrations $w(a_1, -a_1, a)$ with a_1 torsion in $h_1(M)$. But in this case the calibration can be described as follows: Consider a colored standard link in M with homology classes of components a_1' torsion in $h_1(M)$ and arbitrary $a \in H_1(M)$. Then a multiple na_1' is homologous into ∂M. The intersection numbers of a zero-bordism of this color with the other colors determine the calibrations. Note that the exterior multiplication is actually well defined in this case because $H_a = h_2(M)$ for all $a \in \text{mon}(H_1(M))$ in this case.

If M is Betti-trivial then $\mathfrak{j}$ is trivial (see [6, A.1]). Thus $U = 0$ and (a) follows.

Next assume that M is not Betti-trivial. We construct an example of torsion in $\mathcal{B}(M; R)$ following the idea of Example 6.8 in [7]. We can find $a \in H_1(M)$ and $y \in h_2(M)$ with $\mathfrak{j}(a, y) = 1$ (see [6, A.1.4]). Necessarily a is not torsion in $h_1(M)$. So we can assume without restriction that a is an infinite cyclic generator of $H_1(M)$. It follows that $w(a, -a, b) = 0$ for all $b \in H_1(M)$.

Consider the following two linking graphs related by the $h_2(M)$-action on the top vertex:



Apply skein relations to the left hand graph and expand in terms of standard graphs. Then take the difference of the result and the right hand diagram. This has the form

$$z(q-q^{-1})(a\cdot 0)+z^2a,$$

where we identify elements of $\mathrm{mon}(H_1(M))$ and standard graphs.

If $za=0$ then a is torsion in $\mathcal{B}(M;R)$ (because 1-colored links are never trivial in $\mathcal{B}(M;R)$; this can be proved by mapping z to 0).

Suppose that $za\neq 0$ and q^2-1 is not invertible. Then the projection of $(q^{-1}-q)(a\cdot 0)+za$ is non-zero in $\mathcal{B}(M;R/(q^2-1))$. Thus

$$(q-q^{-1})(a\cdot 0)+za\neq 0\in\mathcal{B}(M;R)$$

is z-torsion.

Suppose that q^2-1 is invertible and M is not Betti-trivial. Then the relations in U can be applied inductively to eliminate all generators from the set $\mathrm{mon}(H_1(M))\setminus\mathrm{mon}_R(H_1(M))$. It is not possible to eliminate the remaining monomials in $\mathrm{mon}_R(H_1(M))$ using the relations in U, because all the resulting relations after elimination are multiples of z. This proves the claim. ∎

Example 3. Assume that M is not Betti-trivial. Let $a\in H_1(M)$ and $y\in h_2(M)$ be such that $\jmath(a,y)=1$. Consider the action of y on the linking graph γ with one edge, vertex labels a and 0, and edge label 0. This action changes the edge label to 1. Application of the skein relation gives

$$(q^2-1)\gamma-zqa=0,$$

thus $U\neq 0$. This shows that the natural homomorphism

$$SRH_1(M)\to\mathcal{B}(M;R)$$

does not induce an isomorphism in this case.

Remark 5. Let $e:M\hookrightarrow N$ be an embedding of oriented 3-manifolds. Then there is always an induced map $\mathcal{G}(M)\to\mathcal{G}(N)$ compatible with the linking graph maps. If the induced homomorphism $e_*:H_1(M)\to H_1(N)$ is injective then it can easily be seen that there is a commuting diagram

$$\begin{array}{ccc} \mathcal{B}(M) & \xrightarrow[\cong]{\mathfrak{l}_M} & \mathcal{D}(M) \\ {\scriptstyle e_*}\big\downarrow & & {\scriptstyle e_*}\big\downarrow \\ \mathcal{B}(N) & \xrightarrow[\cong]{\mathfrak{l}_N} & \mathcal{D}(N) \end{array}$$

The induced homomorphism $\mathfrak{l}_N \circ e_* \circ \mathfrak{l}_M^{-1}$ is generally not natural with respect to inclusions because of the affine nature of the definition of the linking graph maps. In fact, the calibrations are defined using decompositions of $H_1(M)$, which might not be natural with respect to e_*.

4. Strong linking graphs for $\pi_1(M)$ abelian. Throughout this section we assume that $\pi_1(M)$ is *abelian*.

The theory of Section 2 will now be applied to the pairing defined in Example 2.

REMARK 6. For $\pi_1(M)$ abelian, it can be deduced from Sections 6 and 7 in [7] that for each $a \in H_1(M)$ the relations of $\mathcal{H}(M;R)$ can be parametrized by elements of $H_a = H_1(LM_a)$, where LM_a is the component of the free loop space of M corresponding to $a \in H_1(M) \cong \widehat{\pi}(M)$.

Let $\mathcal{G}' = \mathcal{G}'(M)$ be the resulting set of *strong linking graphs*. In general the sets $\mathcal{G}$ and $\mathcal{G}'$ are different because of the different actions.

Next we will define a *surjective* map

$$\mathfrak{l}' = \mathfrak{l}'_M : \mathfrak{h}(M) \to \mathcal{G}'(M),$$

which assigns to each link its *strong linking graph*.

For the following arguments compare also [7, Section 3]. Let $\mathcal{M}(r)$ be the space of maps $\bigcup_r S^1 \to M$ quotiented by the actions of permuting the domain circles. Let K be an r-component link. Consider the complete graph with r vertices labeled by the homotopy classes a of the components of K. For a system of standard links K_a choose a generic path in $\mathcal{M}(r)$ from K to K_a. There will be finitely many singular parameters corresponding to crossings of different components. Then the signed number of crossings between two components defines the edge label of the edge joining the corresponding components.

It follows easily from the definition of $\mathfrak{j}'$ in Example 2 and the results in [7] that the map $\mathfrak{l}'$ is well defined. For a given link K let $\mathfrak{l}'(K)$ be the *strong linking graph* of K.

Choose the standard links symmetric in the sense that multiple occurrences of elements in $\mathrm{mon}(\pi_1(M)) = \mathrm{mon}(H_1(M))$ are realized by parallel components as before. Here parallel means that there is an isotopy of M

which interchanges the two components within a tubular neighborhood of each of them. Moreover we can always choose the system of standard links so that, as before, each sublink of a standard link is isotopic to a standard link. (This can easily be constructed inductively over the number of components.)

It is easy to prove that $\mathfrak{l}'$ is surjective by choosing paths in the mapping space corresponding to a given set of edge labels.

Next we define a calibration $w'(a_1, a_2, a)$ as follows. Let $K_{a_1 a_2 a}$ be the standard link and let $K_{[a_1+a_2],a;b}$ be the result of attaching an oriented band to $K_{a_1 a_2 a}$ which connects the two distinguished components. The resulting link is standard (up to isotopy) except possibly in the banded component. As before, we find a homotopy of this component into standard form, so that the resulting final link is standard. The corresponding path in $\mathcal{M}(r)$ can be assumed generic with only finitely many points where the moving component intersects one of the other components transversely in a single point. This defines the edge labels of $w'(a_1, a_2, a) \in \mathcal{W}'_{\mathrm{s}}(a_1 + a_2, a)$. Note that the result *a priori* depends on the choice of band. Just choose a band for the definition so that the calibration $w' = w'(M)$ is well defined.

REMARK 7. Any two links $K_{[a_1+a_2],a;b}$ and $K_{[a_1+a_2],a;b'}$ for different bands b, b' have the same linking graph. In fact the difference corresponds to contributions of band-moves, which cancel.

A *band-move* between two links is a passage of two strings of one component, which are opposite sides of a band, through another component. This passage corresponds to a sequence of two crossing-changes with opposite signs.

LEMMA 3. *Suppose that $\pi_1(M)$ is abelian. Then any two links which differ by band-moves have the the same image in $\mathcal{H}(M; R)$.*

Proof. Let r be the number of components. The result is trivial for $r = 1$. Consider two 2-component links which differ by precisely one band-move. We study the difference of the two links in $\mathcal{H}(M; R)$. By applying skein relations to the two crossings of one of the links we see that their difference is the product of $q^{\pm 1}z$ with a difference of the images of two knots with the same freee homotopy class (thus are equal in $\mathfrak{h}(M)$). For $r > 2$ by application of the skein relations we see that the difference has the form $q^{\pm 1}z(K_0 - K_0')$, where K_0, K_0' differ by a sequence of band-moves between links with $r - 1$ components. So the result follows by induction on r. ∎

The following result is essential for exceptional cases, which appear in the proof of Theorem 2.

THEOREM 5. *Suppose $\pi_1(M)$ is abelian. Then the strong linking graph map induces an isomorphism*

$$\mathcal{H}(M;R) \to \mathcal{D}(w'(M), \mathfrak{j}'; R) =: \mathcal{D}'(M;R).$$

Moreover, the image of a link in the homotopy skein module is determined by its strong linking graph.

Proof. Throughout the proof, linking graph will mean strong linking graph.

First we prove that the homomorphism induced by the linking graph map

$$R\mathfrak{h}(M) \to \mathcal{D}'(M;R)$$

maps skein relations of $\mathcal{H}(M;R)$ trivially. Consider the image of a skein relation $K_+ - q^2 - qzK_0$ in $\mathcal{D}'(M;R)$. Let $\gamma_\pm$ and γ_0 be the linking graphs associated to $K_\pm$ and K_0 by using paths in $\mathcal{M}(r)$ respectively $\mathcal{M}(r-1)$ as follows. There are paths for K_+ and K_- which differ precisely by a crossing change. So there is a corresponding $\gamma \in \mathcal{G}'[1]$ defining $\gamma_\pm$ as usual. We have to prove that γ_0 is equal to a linking graph of K_0, at least up to skein relations in $\mathcal{D}'(M;R)$. Add a band to K_- joining the over/undercrossing points and follow the remaining homotopy to K_a, the standard link for $K_\pm$. The band can be homotoped along, avoiding intersections with the other components. Now attach the band along the homotopy to construct a homotopy from K_0 to a link which is standard except in the banded component. Recall that we have chosen our standard links such that sublinks of standard links are isotopic to standard links. Thus the homotopy from $K_\pm$ to K_a defines a homotopy of K_0 to some link $K_{[a_1,a_2],a;b}$ for *some* band b. The homotopy between this link and the corresponding standard link now precisely contributes the action of the necessary calibration. By the remark above the linking graphs do not change under band-moves.

For $r \geq 0$ let $\mathcal{H}_r(M;R)$ respectively $\mathcal{D}'_r(M;R)$ denote the quotients of the free modules generated by link homotopy classes of links with $\leq r$ components respectively $\mathfrak{j}'$-graphs with $\leq r$ vertices, by the submodules generated by the skein relations involving at most r components respectively vertices. There are natural inclusion homomorphisms $\mathcal{H}_r(M;R) \to \mathcal{H}_{r+1}(M;R)$ respectively $\mathcal{D}'_r(M;R) \to \mathcal{D}'_{r+1}(M;R)$. Then $\mathcal{H}(M;R)$ respectively $\mathcal{D}'(M;R)$ is the direct limit of the modules with $\leq r$ components, respectively $\leq r$ vertices. It is immediate from the constructions and the proof above that there are linking graph epimorphisms for $r \geq 0$:

$$\mathcal{H}_r(M;R) \to \mathcal{D}'_r(M;R).$$

We will prove by induction on r simultaneously that:

(1) there exist homomorphisms $\mathfrak{s}_r : \mathcal{D}'_r(M;R) \to \mathcal{H}_r(M;R)$ which are inverse to $\mathfrak{l}_r$,

(2) the image of each link with $\leq r$ components in $\mathcal{H}_r(M;R)$ is determined by its linking graph.

The assertion is certainly true for $r=0$ and $r=1$. Suppose it holds for all $0 \leq i < r$. We will construct

$$\mathfrak{s}_r : \mathcal{D}'_r(M;R) \to \mathcal{H}_r(M;R)$$

by assigning to each linking graph a corresponding link, and show that this assignment maps skein relations of graphs to skein relations of links. For a $\mathfrak{j}'$-graph with r vertices choose a representative $H_1(M)$-graph. Choose an order of the vertices of the graph. Now define a path in $\mathcal{M}(r)$, beginning in the standard link, and running through crossing changes according to edge labels with appropriate signs. This can be done in such a way that for the resulting path $\mathfrak{p}$ from a *link* K to K_a the crossings between components $i<j$ precede crossings between $k<l$ if (i,j) precedes (k,l) in the lexicographic order. Moreover the crossing changes can be determined by a collection of arcs joining each pair of components. Let $\mathfrak{s}_r(\gamma)$ be the class of $\mathfrak{p}(0)$ in $\mathcal{H}_r(M;R)$. It is not hard to see that the image of the link in the homotopy skein module does not depend on the order chosen. It follows that changing the representative graph by the action on a *vertex group* also will not change the image of $\mathfrak{s}_r(\gamma)$ in $\mathcal{H}_r(M;R)$. (Just choose the order according to the vertex and realize the vertex action by a loop in the mapping space.)

We show that a skein relation $\gamma_+ - q^2\gamma_- - qz\gamma_0$, with the number of vertices of $\gamma_\pm$ less than or equal to r, maps trivially in the homotopy skein module. This is certainly true for graphs with $< r$ vertices. For a graph in $\mathcal{G}[1]$ with γ_+ having r components, define the path by a choice of order of components such that the crossing change $\gamma_+ \rightsquigarrow \gamma_-$ is between the first two components. Then the path corresponding to γ_- will be the segment of the path following the first crossing change. Let K_0 be the $(r-1)$-component link resulting from Conway smoothing at the first crossing. Let $K_0' = \mathfrak{s}_{r-1}(\gamma_0)$ be the link determined by the graph γ_0. Then K_0 and K_0' have the same linking graph. In order to see this consider the path used to define $K_\pm$. This defines a path which joins K_0 to a link which is standard except in one component. But the path in $\mathcal{M}(r-1)$ which is defined by a homotopy of this component into a form such that the complete link is in standard form, defines precisely the calibration, at least up to band-moves. Thus by Lemma 3 and the induction hypotheses for (2), the links K_0 and $K_0' = \mathfrak{s}_r(\gamma_0)$ have the same image in $\mathcal{H}(M;R)$.

Note that the linking graph of K_0' is determined by the representative of the linking graph of $K_\pm$ and the edge defining the skein relation. This proves (1) for $i=r$. By construction, $\mathfrak{s}_r$ is the inverse of $\mathfrak{l}_r$. Thus two links K_1, K_2 with the same image under $\mathfrak{l}_r$ in $\mathcal{D}'_r(M;R)$ have the same image in $\mathcal{H}(M;R)$. This shows (2) for $i=r$ and completes the proof. ∎

The next result allows us to apply Theorem 4 to the module $\mathcal{D}'(M;R)$.

LEMMA 4. *The calibration $w(M)$ is local.*

Proof. This follows similarly to the argument in the proof of Lemma 2. Consider the process of attaching two bands b_1, b_2 to at least 3 components of a standard link K_a of r components. Let $\mathfrak{p}_i$ be the two paths in LM describing corresponding homotopies of the knots resulting from the attaching of b_i for $i = 1, 2$. The product $\mathfrak{p}_1 \times \mathfrak{p}_2$, *joined* to the constant path for all the remaining components, can be assumed to be a transversal map

$$I \times I \to \mathcal{M}(r-2),$$

in the sense of [7, Section 3] and [8]. Thus the contributions for edge labels from the restrictions to the two boundary parts $(\{0\} \times I) \cup (I \times \{1\})$ and $(I \times \{0\}) \cup (\{1\} \times I)$ are the same. But these two restrictions precisely describe the spider-actions of two corresponding calibrations with different order. ∎

5. Proof of Theorem 2. Suppose that $\pi_1(M)$ is not abelian. Then the map $\widehat{\pi}(M) \to H_1(M)$ induced by the Hurewicz homomorphism is not a bijection. (A non-trivial commutator in $\pi_1(M)$ is not in the conjugacy class of $1 \in \pi_1(M)$ but maps trivially into $H_1(M)$.) Represent this element by some oriented knot in M. Since $\mathfrak{h}(M) \subset R\mathfrak{h}(M)$, this defines an element in the kernel of the homomorphism

$$R\mathfrak{h}(M) \to R\mathfrak{b}(M) \to \mathcal{B}(M; R).$$

This element represents a non-trivial element in the kernel of the Hurewicz homomorphism, because knots are non-trivial in the homotopy skein module (see [7]).

Suppose that $\pi_1(M)$ is abelian. Recall from [4] that in this case $\pi_1(M)$ is one of

$$\mathbb{Z}_p, \quad \mathbb{Z}, \quad \mathbb{Z} \oplus \mathbb{Z} \quad \text{or} \quad \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

and $p \geq 2$ is a prime. If M is Betti-trivial the result follows from Theorem 1.1 in [7] and Theorem 1 above. By [4, 5.1 and 11.11] and [7, lemma in Section 8] we can assume that M is one of $S^1 \times S^2$ or $S^1 \times S^1 \times S^1$.

In both cases we will apply the results of Sections 3 and 4. Note that because $H_1(M)$ is free abelian we can assume that the calibrations w are trivial. Recall that the calibrations w' are not necessarily trivial but do not depend on the choices of bands used in their definition (see Remark 7).

CASE 1. Let $M = S^1 \times S^2$. In this case $h_2(M) \cong H_2(M) \cong \pi_2(M) \cong \mathbb{Z}$ and $H_1(M) \cong \mathbb{Z}$. Thus the sets of graphs $\mathcal{G}(M)$ and $\mathcal{G}'(M)$ discussed in Sections 3 and 4 are the same. We can represent the elements of $H_1(M)$ naturally by knots wrapping around the S^1-component r times in a monotonic way. Using these representatives we can choose symmetric standard links for the definition of w' such that the calibrations are trivial. Thus by

Proposition 2, Theorem 5 and Theorem 4 there is a commuting diagram of homomorphisms

$$\begin{array}{ccc} \mathcal{H}(M;R) & \xrightarrow[\cong]{\iota'} & \mathcal{D}'(M;R) \\ {\scriptstyle\mu}\big\downarrow & & {\scriptstyle\cong}\big\downarrow \\ \mathcal{B}(M;R) & \xrightarrow[\cong]{\iota} & \mathcal{D}(M;R) \end{array}$$

and the result follows.

Case 2. Let $M = S^1 \times S^1 \times S^1$. We want to show that the homomorphism

$$\mathcal{H}(M;R) \to \mathcal{B}(M;R)$$

is not injective.

We identify $H_1(M) \cong h_1(M) \cong \mathbb{Z}^3$ and $H_2(M) \cong h_2(M) \cong \mathbb{Z}^3$. Recall that the cross product induces an isomorphism

$$\bigoplus_{i=1}^{3} (H_1(S^1) \otimes H_1(S^1)) \to H_2(M).$$

In other words, if e_i, $i = 1, 2, 3$, are the homology generators of $H_1(M)$ then the generators of $H_2(M)$ can be written in the form $t_1 = e_2 \times e_3$, $t_2 = e_3 \times e_1$, $t_3 = e_1 \times e_2$. If a, b, c are any three vectors in $\mathbb{Z}^3$ then the intersection of the torus determined by a, b with a curve respresenting c is given by the vector triple product $(a \times b) \cdot c$. Note that we can represent the three generators of $H_1(M)$ and the corresponding standard link components in the form $S^1 \times \{1\} \times \{1\}$, $\{1\} \times S^1 \times \{1\}$ and $\{1\} \times \{1\} \times S^1$. We identify standard links with elements in $\mathrm{mon}(H_1(M))$ as usual.

Assume that $q^2 \neq 1$. Let K be the two-component link in M with first component given by the boundary of a small disk on $S^1 \times S^1 \times \{1\}$ with center $\{1\} \times \{1\} \times \{1\}$, and with second component $\{1\} \times \{1\} \times S^1$. Let the standard link K_2 for $0e_3 \in \mathrm{mon}(H_1(M), 2)$ in $\mathcal{H}(M;R)$ be given by the link with the same first component but second component $\{-1\} \times \{1\} \times S^1$. In $\mathcal{B}(M;R)$ the two links are singular link bordant (they differ precisely by the action of the torus $S^1 \times S^1 \times \{1\}$ representing $e_1 \times e_2$). By application of the skein relation to K_1 we see that the element

$$(q^2 - 1)(0e_3) - qze_3,$$

respectively the corresponding linear combination of standard links, is trivial in $\mathcal{B}(M;R)$.

Recall from [7] that the relations for $\mathcal{H}(M;R)$ are determined by actions from the fundamental group of M and by actions originating from the fundamental groups of components of its free loop space. The relations from the fundamental group are divisible by z^2 because $\pi_1(M)$ is abelian. The relations from the free loop space which involve at most two link components are generated by elements of the form

$$(q^{2n}-1)(ac)-nqz(a+c),$$

with $n=(a\times b)\cdot c$ for arbitrary $b\in H_1(M)$, or the coefficients of 2-component links respectively knots will be divisible by z respectively z^2. It follows that the above element is not trivial in $\mathcal{H}(M;R)$ (because a triple product involving 0 is always 0).

Next assume that $q^2=1$. We can identify $\mathcal{B}(M;R)\cong\mathcal{D}(M;R)$ using the linking graph isomorphism. Consider the linking graph with a single edge with both vertex labels e_1 and edge label 1. An application of $t_1\in h_2(M)$ gives the following relation for $\mathcal{B}(M;R)\cong\mathcal{D}(M;R)$:

$$(*)\qquad z(1\cdot[(2e_1)])=0,$$

with $[2e_1]\in\operatorname{mon}(\mathbb{Z}^3)\cong\operatorname{mon}(H_1(M))$. We will show that such a relation between corresponding standard links cannot hold in $\mathcal{H}(M;R)$. By Theorem 5 it suffices to show that the corresponding element in $\mathcal{D}'(M;R)$ is non-trivial.

But such a relation for $\mathcal{D}'(M;R)$ necessarily has to come from actions on a strong linking graph with a single edge with vertex labels x and $2e_1-x$ and edge label

$$(x\times d)\cdot(2e_1-x)=(x\times d)\cdot 2e_1$$

for some $d\in\mathbb{Z}^3\cong H_1(M)$. The induced relations have the form

$$(2(x\times d)\cdot e_1))[2e_1].$$

If the relation $(*)$ above holds in $\mathcal{D}'(M;R)$, then there exists an integral solution vector d satisfying

$$(2(x\times d)\cdot e_1))=1,$$

which is absurd. This completes the proof of the Hurewicz theorem in the case $q^2=1$. ∎

PROBLEM. Give an explicit description of the kernel of the Hurewicz homomorphism μ. Obviously the kernel of μ contains the image in $\mathcal{H}(M;R)$ of all those link homotopy classes of links in M which map to link bordism classes represented by links with only empty components. This last set is usually large, even in cases when μ is an isomorphism (see [3], [14]).

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