FUNDAMENTA
MATHEMATICAE
184 (2004)

Maximal actions of finite 2-groups on $\mathbb{Z}_2$-homology 3-spheres

by

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Abstract. It is known that a finite 2-group acting on a $\mathbb{Z}_2$-homology 3-sphere has at most ten conjugacy classes of involutions; the action of groups with the maximal number of conjugacy classes of involutions is strictly related to some questions concerning the representation of hyperbolic 3-manifolds as 2-fold branched coverings of knots. Using a low-dimensional approach we classify these maximal actions both from an algebraic and from a geometrical point of view.

1. Introduction. The 2-fold branched coverings of knots and links are a class of 3-manifolds that is diffusely studied in low-dimensional topology. In [R] the following fact is proved: there are at most nine inequivalent knots in S^3 with the same hyperbolic 3-manifold as 2-fold branched covering (for an alternative proof see [MR]). Both proofs also hold in the more general case of knots in $\mathbb{Z}_2$-homology 3-spheres (i.e. 3-manifolds with the $\mathbb{Z}_2$-homology of the 3-sphere); in this case, the upper bound nine is the best possible (see [MZ1]). For the classical case of knots in S^3 examples of six knots with the same hyperbolic 2-fold branched covering are known but we suppose that also for knots in S^3 nine is the best possible upper bound.

The 2-fold branched covering of a knot in a $\mathbb{Z}_2$-homology 3-sphere is again a $\mathbb{Z}_2$-homology 3-sphere (see [Sa, Sublemma 15.4]) and thus we have a natural relation between this setting and the classical problem to determine the finite groups which act on (homology) 3-spheres. In fact the key point of the proofs in [R] and [MR] is that a finite 2-group acting smoothly and orientation-preservingly on a $\mathbb{Z}_2$-homology 3-sphere has at most ten conjugacy classes of involutions, and at most nine classes of involutions with non-empty fixed-point set. In the following, *maximal 2-groups* will be the 2-groups acting on $\mathbb{Z}_2$-homology 3-spheres and realizing the maximum number ten of conjugacy classes of involutions.

2000 *Mathematics Subject Classification*: 57M60, 57M12.

The maximal 2-groups and their actions on $\mathbb{Z}_2$-homology 3-spheres are the main subject of this paper; we classify these maximal actions both from an algebraic and from a geometrical point of view.

First we give an algebraic classification of the maximal 2-groups; each group in this class is uniquely identified by two integers, the maximal order of an element and the maximal order of an element with non-empty fixed-point set. To determine the groups we use a direct approach that consists of a mix of 3-dimensional topological methods and elementary group theory.

The second point in our result is the analysis of the orbifolds that are quotients of $\mathbb{Z}_2$-homology 3-spheres by maximal 2-groups; in particular we prove that the algebraic structure of the group determines the combinatorial structure of the singularity graph of the quotient orbifold. The description of the singularity graphs of the quotient orbifolds seems the most natural way to describe the relation between different knots with the same hyperbolic 2-fold branched covering (see [MR] and [M]).

As a by-product it turns out that the possible actions are standard: the maximal 2-groups appear also in the list of the finite subgroups of $SO(4)$ in [DV] and, if we consider the combinatorial structure of the singularity graph of the quotient, the action of a maximal 2-group on a generic $\mathbb{Z}_2$-homology 3-sphere resembles an orthogonal action (for a discussion about standard and non-standard actions see [KwS]).

There also exists an alternative approach to determine the maximal 2-groups acting on $\mathbb{Z}_2$-homology 3-spheres. A result in [DoHa] states that any finite p-group acting on a $\mathbb{Z}_p$-homology n-sphere has a representation as a group of isometries of S^n; this representation preserves the dimension of the global fixed-point set of any subgroup. Combining this result and the list of finite subgroups of $SO(4)$ in [DV] we can determine the groups with ten conjugacy classes. The paper of Dotzel and Hamrick is based on deep results on transformation groups and in particular on the Borel Formula (see [Bo]); their result gives a powerful tool that works in a general setting. In the present paper we prefer to propose a direct and more elementary approach that gives explicit information about the geometry of the actions on 3-manifolds and that is more related to the low-dimensional problems involving actions of maximal 2-groups.

We now present the results of the paper in more detail.

Let (m, n) be a couple of positive integers such that $m \geq n$; we denote by $G_{m,n}$ the group which has the following presentation:

$$\langle s, t, f, g \mid s^2 = t^2 = f^{2^m} = g^{2^n} = 1,\ gf = fg,\ st = ts,\ tft^{-1} = f^{-1},\\ tgt^{-1} = g^{-1},\ sfs^{-1} = f,\ sgs^{-1} = f^{2^{m-n}} g^{-1} \rangle.$$

The group $G_{m,n}$ is a semidirect product $(\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes (\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m})$ of the normal subgroup generated by f and g with the subgroup generated by t and s; the involution t acts on $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m}$ by inverting each element.

In the list of finite subgroups of the orthogonal group $SO(4)$ in [DV] this group is presented as a central product of two quaternion groups. Note that $SO(4)$ is isomorphic to $S^3 \times_{\mathbb{Z}_2} S^3$, the central product of two copies of the unit quaternions. Consider $Q_{2^a} = \langle x, y \mid x^2 = y^{2^{a-2}}, xyx^{-1} = y^{-1} \rangle$, the quaternion group of order 2^a; each quaternion group has a unique central involution and we denote by $Q_{2^a} \times_{\mathbb{Z}_2} Q_{2^b} \subset S^3 \times_{\mathbb{Z}_2} S^3$ the central product of Q_{2^a} and Q_{2^b} with the two central involutions identified. Thus we have verified that $G_{m,n}$ is isomorphic to $Q_{2^{m+1}} \times_{\mathbb{Z}_2} Q_{2^{n+2}}$.

If $m > n$ there are ten conjugacy classes of involutions (fewer if $m = n$) and the greatest order of an element of $G_{m,n}$ is 2^m (see Section 2); this last fact also implies that these groups are non-isomorphic for different couples of integers.

Suppose now that $G_{m,n}$ acts smoothly on a $\mathbb{Z}_2$-homology 3-sphere. Then we prove in Section 2 that the central involution of $G_{m,n}$ acts freely and the other involutions of the group have non-empty fixed-point set; the greatest order of an element with non-empty fixed-point set is 2^n.

The main result of the paper is the following.

THEOREM 1. *Let M be a $\mathbb{Z}_2$-homology 3-sphere and S a finite 2-group of orientation-preserving diffeomorphisms acting on M. Suppose that S has the maximal number ten of conjugacy classes of involutions. Then the following hold.*

(i) *There is a unique couple of integers $m > n > 0$ such that S is isomorphic to $G_{m,n}$, where 2^n is the greatest order of an element with non-empty fixed-point set.*

(ii) *The underlying topological space of the quotient orbifold M/S is a $\mathbb{Z}_2$-homology 3-sphere. If S is isomorphic to $G_{n+1,n}$ the singularity graph of M/S has the combinatorial type of the graph I in Figure 1 (the Kuratowski graph); if S is isomorphic to $G_{m,n}$ with $m > n+1$ the singularity graph of M/S has the combinatorial type of the graph II in Figure 1. In any case the singularity graph of M/S has one edge with singularity index 2^n and the other edges with singularity index two.*

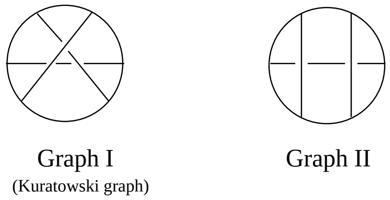

Fig. 1

We remark that the minimum order of a maximal 2-group is 32; there is only one group with this order, which is $G_{2,1}$. This is the group studied in [MZ1] where it is used to prove the existence of nine inequivalent knots in integer homology 3-spheres with the same hyperbolic 2-fold branched covering. Also for the successive order 64 there is only one group, which is $G_{3,1}$. In general for each order 2^a the number of groups is equal to the integer part of $(a-3)/2$.

In the following theorem we collect the main applications of Theorem 1 to 2-fold branched coverings of knots.

THEOREM 2. (i) *Let M be a hyperbolic 3-manifold which is the 2-fold branched covering of $q \geq 1$ inequivalent knots in (possibly different) $\mathbb{Z}_2$-homology 3-spheres; then M is a $\mathbb{Z}_2$-homology 3-sphere. If q assumes the maximal possible value nine, then the Sylow 2-subgroup of the orientation-preserving isometry group of M is one of the groups $G_{m,n}$; if all the nine knots are knots in S^3 the Sylow 2-subgroup is one of the groups $G_{m,1}$. The nine knots correspond to the nine edges of the singularity graph of the quotient orbifold $M/G_{m,n}$ which is combinatorially equivalent to one of the graphs in Figure 1.*

(ii) *Each group $G_{m,n}$ acts on a hyperbolic homology 3-sphere M such that $G_{m,n}$ coincides with the orientation-preserving isometry group of M, and consequently M is the 2-fold branched covering of nine inequivalent knots in homology 3-spheres.*

Theorem 2 is obtained by combining Theorem 1 with some other results. Part (i) follows from Theorem 1 and the Orbifold Geometrization Theorem ([BP]) which implies that, if the 2-fold branched covering of a knot is a hyperbolic 3-manifold M, then the covering involution is conjugate to an isometry (and then also to an element in the Sylow 2-subgroup of the isometry group of M). We consider in (i) also the case of knots in S^3. In general if M is the hyperbolic 2-fold branched covering of nine knots and the Sylow 2-subgroup of the isometry group of M is $G_{m,n}$, one of the nine knots admits a symmetry of order 2^{n-1} that fixes pointwise the knot; then if the nine knots are all in S^3, by the positive solution of the Smith Conjecture [MoBa], we obtain $n = 1$. An action of $G_{m,n}$ on a hyperbolic homology 3-sphere can be obtained from the standard orthogonal action of $G_{m,n}$ on S^3 by applying Kawauchi's imitation theory ([K1], [K2]); see [MZ1] for a similar application.

2. Preliminaries. In this section we present some preliminary results about finite groups acting on 3-manifolds; we also consider some elementary properties of the groups $G_{m,n}$ and of their actions on $\mathbb{Z}_2$-homology 3-spheres.

Proposition 1 follows from classical Smith theory (see [Br] for a review of this theory).

PROPOSITION 1. *Let f be a periodic orientation-preserving diffeomorphism of a $\mathbb{Z}_2$-homology 3-sphere whose period is a power of two. Then the fixed-point set of f is connected, that is, empty or a simple closed curve.*

Let K be a simple closed curve in a closed 3-manifold and I a group of orientation-preserving diffeomorphisms that fixes K setwise. The elements of I induce on K reflections (strong inversions) or rotations; if an element of I induces on K a reflection we call it a *K-reflection*; otherwise we call it a *K-rotation*.

PROPOSITION 2. *Let I be a finite group of orientation-preserving diffeomorphisms of a closed orientable 3-manifold which map a given simple closed curve K to itself. Then I is isomorphic to a subgroup of a semidirect product $\mathbb{Z}_2 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_m)$ for some non-negative integers n and m, where $\mathbb{Z}_2$ operates on the normal subgroup $\mathbb{Z}_n \times \mathbb{Z}_m$ by sending each element to its inverse.*

Proof. Since I is a finite group we can assume that I acts by isometries for some Riemannian metric of M. The simple closed curve K has an I-invariant tubular neighborhood in M (see [Br, p. 306, Theorem 2.2]); we can faithfully represent I as a group of isometries of the standard solid torus $S^1 \times D^2$ which fixes setwise the core $\gamma = S^1 \times 0$ of the torus. The subgroup of γ-rotations is abelian: it contains the cyclic subgroup of all elements fixing γ pointwise, with cyclic factor group acting faithfully by rotations on γ. The subgroup of γ-rotations has index 1 or 2. In the latter case I contains a γ-reflection and any reflection acts on the normal subgroup of γ-rotations by inverting each element. This finishes the proof.

PROPOSITION 3 ([MZ2, Proposition 4]). *Let I be a group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ of orientation-preserving diffeomorphisms acting on a $\mathbb{Z}_2$-homology 3-sphere. Then either I has two global fixed points or I contains exactly one involution acting freely.*

Next we present some facts about the structure of $G_{m,n}$, where m and n are integers such that $m > n > 0$.

We consider the presentation of $G_{m,n}$ given in the Introduction. We note that f and g generate an abelian and normal subgroup A that is isomorphic to $\mathbb{Z}_{2^m} \times \mathbb{Z}_{2^n}$; the involution t acts on A by inverting each element. The group $G_{m,n}$ is a semidirect product of the normal subgroup A with the subgroup generated by t and s; the latter is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Starting from the presentation we can compute the conjugacy classes of involutions contained in $G_{m,n}$; we have exactly ten classes:

$\mathcal{C}_1 = \{f^{2^{m-1}}\}$, $\mathcal{C}_2 = \{g^{2^{n-1}}, sg^{2^{n-1}}s^{-1} = g^{2^{n-1}}f^{2^{m-1}}\}$,
$\mathcal{C}_3 = \{tf^\alpha g^\beta \mid \alpha \text{ odd and } \beta \text{ even}\}$, $\mathcal{C}_4 = \{tf^\alpha g^\beta \mid \alpha \text{ odd and } \beta \text{ odd}\}$,
$\mathcal{C}_5 = \{tf^\alpha g^\beta \mid \alpha \text{ even and } \beta \text{ even}\}$, $\mathcal{C}_6 = \{tf^\alpha g^\beta \mid \alpha \text{ even and } \beta \text{ odd}\}$,
$\mathcal{C}_7 = \{sg^{-\alpha}f^{\alpha 2^{m-n-1}} \mid \alpha \text{ odd}\}$, $\mathcal{C}_8 = \{sg^{-\alpha}f^{\alpha 2^{m-n-1}} \mid \alpha \text{ even}\}$,
$\mathcal{C}_9 = \{stf^\alpha \mid \alpha \text{ even}\}$, $\mathcal{C}_{10} = \{stf^\alpha \mid \alpha \text{ odd}\}$.

With a similar calculation we also see that the elements of maximal order are in A and have order 2^m; this implies that no two groups of this family are isomorphic for different couples of integers.

We now suppose that the group $G_{m,n}$ acts smoothly and orientation-preservingly on a $\mathbb{Z}_2$-homology 3-sphere. We note that the involution $f^{2^{m-1}}$ is central in $G_{m,n}$; it acts freely, since otherwise $G_{m,n}$ fixes the non-empty fixed-point set of $f^{2^{m-1}}$ and Proposition 2 applies. All the other involutions commute with $f^{2^{m-1}}$ and by Proposition 3 they have non-empty fixed-point set. Thus we have nine conjugacy classes of involutions with non-empty fixed-point set.

3. Proof of Theorem 1(i). We start from a review of some results of [R] pointing out some properties we can deduce for the maximal case of ten conjugacy classes. Then we characterize a subgroup of index two and we analyze the possible action of S on this subgroup by conjugation; finally we deduce that the only possible case is $G_{m,n}$.

3.0. *A review of the results of* [R]. In [R] the following result is proved.

PROPOSITION 4. *Let M be a $\mathbb{Z}_2$-homology 3-sphere and S a finite 2-group of orientation-preserving diffeomorphisms of M. Then one of the following cases occurs.*

- *The group S is cyclic, dihedral of order at least 16, quasidihedral or a quaternion group, and the unique central involution acts freely.*
- *The group S contains, with index at most two, the centralizer $C_S h$ of an involution h with non-empty fixed-point set. The group $C_S h$ is a subgroup of a semidirect product $\mathbb{Z}_2 \ltimes (\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m})$ for some non-negative integers n and m, where $\mathbb{Z}_2$ acts on $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m}$ by inverting each element.*

Now we suppose that S contains exactly ten conjugacy classes of involutions.

There are fewer than three conjugacy classes of involutions in the groups appearing in the first case of Proposition 4, so we consider only groups of the second type. We consider $\mathrm{Fix}(h)$, the fixed-point set of h, and we denote by A the subgroup of Fix(h)-rotations in $C_S h$. By Proposition 2 we know that A is an abelian group of rank at most two.

From the proof of Theorem 1 in [R] we deduce that, if S has ten conjugacy classes of involutions, then it has the following properties.

1. The subgroup A has rank two.
2. The subgroup A has index four in S and $S = stA \cup sA \cup tA \cup A$, where t is a Fix(h)-reflection and s is an involution not in $C_S h$.
3. The involution shs is the only Fix(h)-rotation of order two with non-empty fixed-point set that is different from h.
4. Each conjugacy class of involutions is contained in one of the four cosets of A; in particular two classes must be contained in A, four classes in tA and two classes in each of stA and sA.

3.1. *The structure of A.* Since A is abelian of rank two, we suppose that $A = \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m}$, with n and m integers such that $m \geq n > 0$.

A is normal. By the previous considerations, elements conjugate to involutions of A are in A; moreover all elements in $C_S h$ with order strictly greater than two are in A. These facts imply that A is normal in S.

Fixed-point sets of elements of A and generators of A. By Proposition 3 the subgroup A contains exactly two distinct involutions with non-empty fixed-point set, h and $h' = shs$. We denote by K (resp. K') the fixed-point set of h (resp. h'); the simple closed curves K and K' are disjoint. Since $h' = shs$ we have $K = sK'$. Any element of A with non-empty fixed-point set fixes pointwise either K or K', otherwise we can obtain an involution with non-empty fixed-point set distinct from K and K'. We let B and B' be the cyclic subgroups of A fixing pointwise K and K' respectively; B and B' are disjoint because they have distinct fixed-point sets. It is easy to see that $sB's = B$ and so the orders of B and B' are equal. We denote by g a generator of B, by g' the element sgs that is a generator of B', and by F the subgroup of A generated by all elements with non-empty fixed-point set; in particular, F is generated by g and g'.

We prove that A/F is a cyclic group. Consider the quotient orbifold M/F. It can be seen as the double quotient $(M/B)/(F/B)$ and both B and F/B act with non-empty global fixed-point set; so the underlying topological space of M/F is a $\mathbb{Z}_2$-homology sphere. The set $A-F$ contains only elements acting freely on M, thus the group A projects to a group isomorphic to A/F composed of diffeomorphisms that act freely on M/F. By Proposition 3 the group A/F is cyclic. This implies that the order of g is 2^n.

We consider an element f of maximal order 2^m. The diffeomorphism f fixes both K and K' and we suppose that f acts with order 2^ϕ and 2^ψ respectively on K and K'. By Proposition 1 either ϕ or ψ is equal to m, where 2^m is the order of f. Since $sK = K'$, s inverts the roles of K and K' and sfs acts with order 2^ϕ on K' and 2^ψ on K. Replacing f with $fsfs$ if necessary, we can suppose that f acts with order 2^m both on K and K'; in particular all the non-trivial elements that are powers of f act freely on M. Thus the cyclic group generated by f is disjoint from B and the subgroup A is generated by f and g.

The integer m is strictly greater than n. We note that if $F = A$ then the action by conjugation of s on A is determined; by a computation similar to that which will be presented in 3.3, we can prove that in this case we cannot obtain ten conjugacy classes of involutions. Thus we can suppose that $m > n$.

The action of generators of A on K and K'. Any element of A fixes setwise K (the fixed-point set of g) and K' (the fixed-point set of $g' = sgs$). We fix a point P_0 of K and denote by Q_0 the point of K' that coincides with $s(P_0)$. We consider the orbit of P_0 under the action of f. We fix an orientation on K; this gives an order to the points of the orbit; denote them by $\{P_0, P_1, \ldots, P_{2^m-1}\}$ according to this order. We fix on K' the orientation induced from K by s and we denote the points of the orbit of Q_0 under f by $\{Q_0, Q_1, \ldots, Q_{2^m-1}\}$, respecting the order induced by this orientation. In particular, since s is a diffeomorphism we have $s(P_i) = Q_i$. Replacing f with a power of f we can suppose that $f(P_0) = P_1$. We denote by a the odd integer such that $f(Q_0) = Q_a$.

We note that by the proof of Proposition 2 the action on K and K' can locally be considered standard; in particular the orbit of Q_0 under g is a subset of the orbit of Q_0 under f, and the orbit of P_0 under g' $(= sgs)$ is a subset of the orbit of P_0 under f. We can also suppose, replacing g with a power of g, that $g(Q_0) = Q_{2^{m-n}}$; then $g'(P_0) = P_{2^{m-n}}$. Finally, replacing f with fg^ϕ for some ϕ we can suppose that $0 < a < 2^{m-n}$.

3.2. *The action of s on A by conjugation.* We first consider the element sfs. We know that it maps Q_0 to Q_1 and P_0 to P_a.

By Proposition 1 the elements of A are completely determined by their action on K and K', and we obtain

$$sfs = f^a g^\mu,$$

where μ is an integer such that

$$a^2 + \mu 2^{m-n} = 1 \bmod 2^m;$$

so there exists an integer α such that

$$(*) \qquad a^2 - 1 = (a-1)(a+1) = \alpha 2^m - \mu 2^{m-n}.$$

We also consider the element $f^{2^{m-n}}$ that maps P_0 to $P_{2^{m-n}}$ and Q_0 to $Q_{a2^{m-n}}$ mod 2^m and we obtain the following relations:

$$f^{2^{m-n}} = g'g^a, \qquad sgs = g' = f^{2^{m-n}} g^{-a}.$$

If $a = 1$ we find that $\mu = \alpha 2^n$ and consequently

$$sfs = f, \qquad sgs = f^{2^{m-n}} g^{-1}.$$

In this case the action of s on A by conjugation is completely determined.

We now suppose that $a > 1$; since we can suppose $0 < a < 2^{m-n}$ we have $m - n > 1$. In this case there exists an odd integer β (with $2^{m-n-1} - 1 \geq \beta \geq 1$) and an integer δ (with $m - n - 1 \geq \delta \geq 1$) such that

$$a - 1 = \beta 2^{\delta}.$$

First we suppose that $\delta > 1$; in this case from $(*)$ we obtain

$$\beta(\beta 2^{\delta-1} + 1)2^{\delta+1} = \alpha 2^m - \mu 2^{m-n}.$$

Since $\beta 2^{\delta-1} + 1$ is odd we see that 2^{m-n} divides $2^{\delta+1}$ and in particular from the inequality $m - n - 1 \geq \delta$ we find that $\delta = m - n - 1$ and $\beta = 1$. Finally, we conclude that $\mu = -1 - 2^{m-n-2} + \alpha 2^n$ and s acts by conjugation in the following way:

$$sfs = f^{2^{m-n-1}+1} g^{-1-2^{m-n-2}}, \qquad sgs = f^{2^{m-n}} g^{-2^{m-n-1}-1}.$$

Now, we suppose that $\delta = 1$; in this case $(*)$ yields

$$4\beta(\beta + 1) = \alpha 2^m - \mu 2^{m-n}.$$

Thus 2^{m-n-2} divides $\beta + 1$ and since $2^{m-n-1} - 1 \geq \beta$ we have two possibilities: either $\beta = 2^{m-n-1} - 1$ or $\beta = 2^{m-n-2} - 1$ and respectively $a = 2^{m-n} - 1$ or $a = 2^{m-n-1} - 1$. In the first case we can replace f with fg^{-1} and g with g^{-1} to obtain

$$sfs = f^{-1}, \qquad sgs = f^{-2^{m-n}} g.$$

If we suppose that $a = 2^{m-n-1} - 1$ we obtain $\mu = 1 - 2^{m-n-2} + \alpha 2^n$ and finally

$$sfs = f^{2^{m-n-1}-1} g^{1-2^{m-n-2}}, \qquad sgs = f^{2^{m-n}} g^{-2^{m-n-1}+1}.$$

Altogether, the four possibilities are:

(I) $$sfs = f, \qquad sgs = f^{2^{m-n}} g^{-1};$$

(II) $$sfs = f^{-1}, \qquad sgs = f^{-2^{m-n}} g,$$

(III) $$sfs = f^{2^{m-n-1}+1} g^{-1-2^{m-n-2}}, \qquad sgs = f^{2^{m-n}} g^{-2^{m-n-1}-1};$$

(IV) $$sfs = f^{2^{m-n-1}-1} g^{1-2^{m-n-2}}, \qquad sgs = f^{2^{m-n}} g^{-2^{m-n-1}+1}.$$

3.3. *The action of s by conjugation on K-reflections.* With the notation introduced for A we see that the four conjugacy classes of involutions in tA are the following:

$$\{tf^{\psi}g^{\phi} \mid \psi \text{ even}, \phi \text{ even}\}; \quad \{tf^{\psi}g^{\phi} \mid \psi \text{ odd}, \phi \text{ odd}\};$$
$$\{tf^{\psi}g^{\phi} \mid \psi \text{ even}, \phi \text{ odd}\}; \quad \{tf^{\psi}g^{\phi} \mid \psi \text{ odd}, \phi \text{ even}\}.$$

The involution s acts by conjugation on the set of conjugacy classes in tA; if we want to get the maximal number of conjugacy classes, the action of s must be trivial. In particular $sts = tf^{\lambda}g^{\omega}$ with λ and ω even. Moreover if we consider $stfs$ we see that cases III and IV cannot occur with ten conjugacy classes when $m-n>2$. If $m-n=2$ case IV coincides with case I and, for case III, we can replace f with fg^{-1} obtaining the same presentation as in case II; thus we can consider only cases I and II.

Let us consider case I and analyze the exponents λ and ω. From the equality

$$t = sstss = stf^{\lambda}g^{\omega}s = tf^{2\lambda+\omega 2^{m-n}}$$

we deduce that there exists an integer γ such that

$$2\lambda + \omega 2^{m-n} = \gamma 2^{m}.$$

If γ is even we can replace t with $tg^{\omega/2}$ and if γ is odd we can replace t with $tg^{(\omega/2)+2^{n-1}}$; in any case we obtain $sts = t$ and so S is isomorphic to $G_{m,n}$.

Finally, consider case II and in particular the following equality:

$$t = sstss = stf^{\lambda}g^{\omega}s = tf^{-\omega 2^{m-n}}g^{2\omega}.$$

It implies that $-\omega 2^{m-n}$ is divisible by 2^m; then ω is a multiple of 2^n and $sts = tf^{\lambda}$. Replacing t with $tf^{\lambda/2}$ (λ is even) we obtain $sts = t$. This relation implies that $(st)^2 = 1$, $(st)t(st) = t$, $(st)f(st) = f$ and $(st)g(st) = f^{2^{m-n}}g^{-1}$. Interchanging the roles of s and st we conclude that also in this case S is isomorphic to $G_{m,n}$.

Thus we have proved that S is always isomorphic to $G_{m,n}$; at this point it is easy to see that all elements with non-empty fixed-point set and of order strictly greater than two are in A, and we conclude that 2^n is the greatest order of an element that does not act freely. This finishes the proof of Theorem 1(i).

4. Proof of Theorem 1(ii). In this section we consider the action of groups with ten conjugacy classes of involutions on $\mathbb{Z}_2$-homology 3-spheres; let M be a $\mathbb{Z}_2$-homology 3-sphere and S a group of orientation-preserving diffeomorphisms of M isomorphic to $G_{m,n}$. We are interested in the combinatorial type of the singularity graph of the quotient M/S.

We recall some basic facts about the singularity graphs of orientable 3-orbifolds. In this case we consider a generalized definition of graphs because in general the singular set of an orbifold is a disjoint union of trivalent graphs and knots. We consider the knots as edges with no endpoint; moreover in these graphs we can find edges with two coinciding endpoints and multiple edges (these are different edges that join the same two vertices).

We also remark that a diffeomorphism of finite order that fixes setwise one edge with two vertices of a singularity graph either exchanges the two endpoints and fixes a single internal point of the edge, or fixes pointwise the whole edge.

In the proof of Theorem 1(ii) we refer to two particular graphs: theta curves and tetrahedral graphs (see Figure 2). A *theta curve* is a graph with two vertices and three edges such that all three edges join the two vertices. A *tetrahedral graph* is a complete graph with four vertices (that is, a graph with four vertices and six edges such that each edge joins a different couple of vertices).

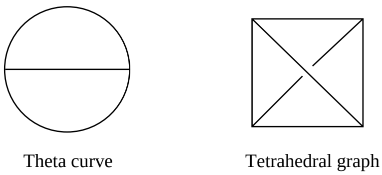

Fig. 2

PROPOSITION 5. *Let M be a $\mathbb{Z}_2$-homology 3-sphere and S a finite 2-group of orientation-preserving diffeomorphisms acting on M. Then the number of conjugacy classes of involutions with non-empty fixed-point set in S is equal to the number of edges in the singularity graph of the quotient orbifold M/S.*

Proof. It is clear that the projections of the fixed-point sets of two involutions that are conjugate in S coincide in M/S. Thus to each conjugacy class of involutions we can associate a subset of the singularity graph of M/S, namely the projection of the fixed-point set of any of the involutions contained in the conjugacy class.

By Proposition 1 the fixed-point set of an involution that does not act freely is a simple closed curve; thus its projection to M/S is homeomorphic to S^1 or to an interval $[0,1]$, and the projection is a connected union of edges.

Now we suppose that the projections of the fixed-point sets of two distinct involutions h and h' contain a common edge e. Consider the preimages of e in $\mathrm{Fix}(h)$ and $\mathrm{Fix}(h')$. They have the same projection in M/S, so there exists $g \in S$ that maps an arc of $\mathrm{Fix}(h)$ to an arc of $\mathrm{Fix}(h')$; by Propositions 1 and 2 this implies that $ghg^{-1} = h'$. Thus, if the projections of the fixed-point sets of two distinct involutions contain a common edge, the two involutions are conjugate.

Finally, we prove that the image of an involution with non-empty fixed-point set contains exactly one edge. Let h be such an involution. By the properties of 2-groups we can consider a subnormal series

$$H_0 = \{\mathrm{Id}\} \subset H_1 = \{h, \mathrm{Id}\} \subset H_2 \subset \cdots \subset H_n = S$$

such that the factor groups are cyclic of order two. We can factorize the quotient M/S by successive quotients M/H_i using the fact that H_{i+1} projects to an involution of M/H_i. In the first quotient the image of $\mathrm{Fix}(h)$ is a knot that is a single edge with no endpoint; the quotient of a single edge by an involution remains a single edge (whatever the number of endpoints) and thus in the final quotient M/S the projection of $\mathrm{Fix}(h)$ remains a single edge.

This finishes the proof of the proposition.

Proof of Theorem 1(ii). For elements of S, we use the same notations as in Section 3.

If S is isomorphic to $G_{n+1,n}$, then the singularity graph of M/S is of type I (Kuratowski graph) with one edge of singularity index 2^n and the others of singularity index two; the underlying topological space of M/S is a $\mathbb{Z}_2$-homology sphere.

We obtain the quotient M/S as the final output of a series of successive quotients that corresponds to a subnormal series of S.

We remark that, even if the action of a diffeomorphism on the singularity graph of an orbifold is determined by a combinatorial point of view, the combinatorial type of the singularity graph of the quotient orbifold in general is not determined and it depends on the topological situation. The idea is to avoid ambiguous cases and to choose quotients such that the "combinatorial" action determines the combinatorial type of the singularity graph in the quotient orbifold.

We recall that the quotient of a $\mathbb{Z}_2$-homology sphere by an involution with non-empty fixed-point set is again a $\mathbb{Z}_2$-homology sphere; we use this fact at each step to prove that all the quotient orbifolds of the series have as underlying topological space a $\mathbb{Z}_2$-homology sphere. During the construction we shall also frequently use Proposition 1 without further mention.

The series of quotients we construct is represented in Figure 3.

First we consider the cyclic subgroup H_0 of order 2^n generated by g (in Section 3 we call this group B and it is one of the maximal subgroups with non-empty fixed-point set contained in A, the subgroup of K-rotations). The singularity graph of $\mathcal{O}_0 = M/H_0$ is a knot with singularity index 2^n at each point.

We recall that t is a $\mathrm{Fix}(g)$-reflection and t normalizes H_0; the projection $\bar{t}$ of t to $\mathcal{O}_0$ is a strong reflection of the singularity graph of $\mathcal{O}_0$. We define H_1

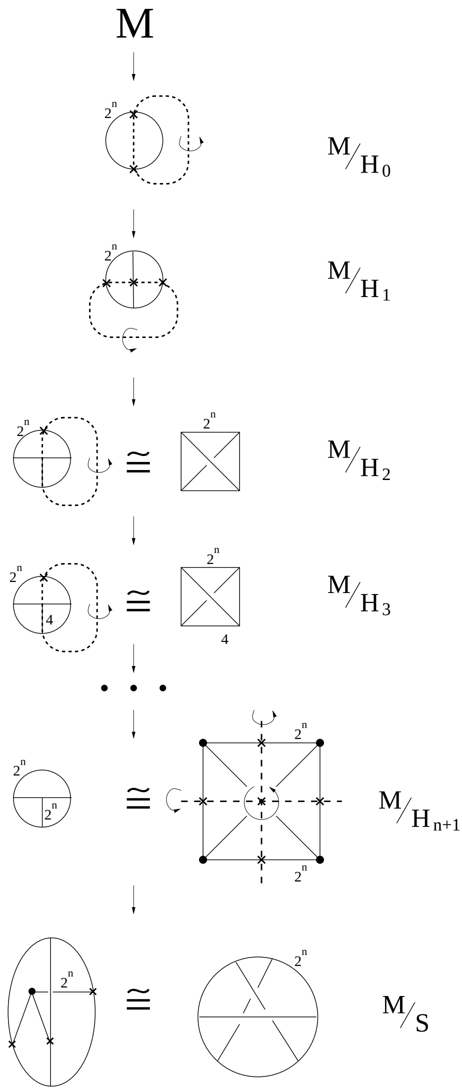

Fig. 3

to be the subgroup of S generated by H_0 and t, and consider the quotient orbifold $\mathcal{O}_1 = M/H_1$; it corresponds to the quotient of $\mathcal{O}_0$ by $\bar{t}$, and its singularity graph is a theta curve with one edge of singularity index 2^n and the other two edges of singularity index two.

The next group H_2 is generated by H_1 and $(sgs)^{2^{n-1}}$; the latter involution has non-empty fixed-point set and commutes with each element of H_1. These facts imply that the projection of $(sgs)^{2^{n-1}}$ to O_1 has non-empty fixed-point set and it fixes setwise all edges of the singularity graph. The only possibility is that the projection of $(sgs)^{2^{n-1}}$ acts as a strong reflection of the theta curve (i.e. it fixes setwise each edge and exchanges the two vertices), and the singularity graph of M/H_2 is a tetrahedral graph with one edge of singularity index 2^n and the others of singularity index two.

At this point if $n = 1$ we go directly to the final step, otherwise we have to insert in the series $n-1$ steps to arrive at the final one. We define iteratively for $i = 3, \ldots, n+1$ the subgroup H_i generated by H_{i-1} and $(sgs)^{2^{n-i+1}}$; we note that each H_{i-1} has index two in H_i and for each i the element $(sgs)^{2^{n-i+1}}$ fixes pointwise the same simple closed curve. We consider the quotient M/H_3; the projection of $(sgs)^{2^{n-2}}$ to $\mathcal{O}_2$ fixes pointwise the edge of the singularity graph of $\mathcal{O}_2$ that corresponds to the fixed-point set of $(sgs)^{2^{n-1}}$. The only possibility is that the projection of $(sgs)^{2^{n-2}}$ to $\mathcal{O}_2$ fixes two vertices of the graph and the corresponding edge, and inverts the other two vertices. The singularity graph of M/H_3 is a tetrahedral graph with one edge of singularity index 2^n, one edge of singularity index 4, and the others of singularity index two. We repeat the same argument for each H_i until we get the quotient M/H_{n+1}; the singularity graph of $\mathcal{O}_{n+1} = M/H_{n+1}$ is a tetrahedral graph with two (non-adjacent) edges of singularity index 2^n and the others of singularity index two.

Now we consider the final step. The subgroup H_{n+1} is normal in S with index four and factor group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, thus S projects to a group $\overline{S}$ of diffeomorphisms of $\mathcal{O}_{n+1}$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

By a combinatorial point of view we have four different possibilities. For three of them we have two involutions that fix pointwise one edge; for these cases it is easy to see that the final singularity graph we obtain is again a tetrahedral graph and these possibilities are excluded by Proposition 5. So the only possibility is that each involution of $\bar{S}$ fixes setwise two non-adjacent edges, exchanging their endpoints.

In Figure 3 the axes of two involutions of $\overline{S}$ appear explicitly; these axes meet the “horizontal” and the “vertical” edges of the tetrahedral graph. The axis of the third involution in $\overline{S}$ is perpendicular to the plane of the page; this axis meets the “diagonal” edges of the tetrahedral graph. We infer that the projection to M/S of the fixed-point sets of the elements in $\overline{S}$ is a theta curve; the fixed-point set of each element projects to one edge. The remaining part of the singularity graph of M/S is given by the projection of the singularity graph of $\mathcal{O}_{n+1}$. The four vertices of the tetrahedral graph project to a single one (marked by a small black disk in Figure 3). The six edges of the tetrahedral graph project pairwise to three edges; each of them meets one different edge of the theta curve in an internal point (the intersection points are marked by crosses in Figure 3). We can conclude that the singularity graph in the quotient M/S has to be of type I; one edge has singularity index 2^n and the others have index two.

If S is isomorphic to $G_{n+k,n}$ with $k > 1$, then the singularity graph of M/S is of type II with one edge of singularity index 2^n and the others of singularity index two; the underlying topological space of M/S is a $\mathbb{Z}_2$-homology sphere.

We prove this by induction on k. If $k = 2$ we consider the subgroup H of S generated by t, s, g, and f^2; H has index two and is isomorphic to $G_{n+1,n}$. The orbifold M/H has a singularity graph of combinatorial type I and S projects to an involution $\bar{f}$ that has non-empty fixed-point set because elements with non-empty fixed-point set are contained in the coset fH (e.g. tf). This implies that the underlying topological space of M/S is a $\mathbb{Z}_2$-homology sphere. Moreover by Proposition 5 the singularity graph of M/S has nine edges and a combinatorial analysis of all possibilities shows that the singularity graph of M/S is of type II with one edge of singularity index 2^n and the others of singularity index 2 (a graphical representation of the quotient is given in Figure 4).

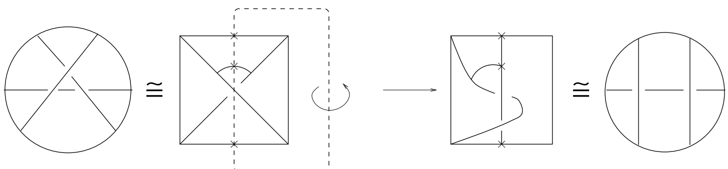

Fig. 4

Now we suppose that the assertion holds for k and we prove it for $k+1$. We consider the subgroup H of S generated by t, s, g, and f^2; H has index two and is isomorphic to $G_{n+k,n}$. The situation is the same as in the case of $k = 2$ with the only difference that the singularity graph of M/H is of type II. We denote by $\bar{f}$ the involution that is the projection of S to M/H. Since $\bar{f}$ has non-empty fixed-point set the underlying topological space of M/S is a $\mathbb{Z}_2$-homology sphere. A combinatorial analysis yields two possibilities for the action of $\bar{f}$ such that the singularity graph of M/S has nine edges (see Figure 5).

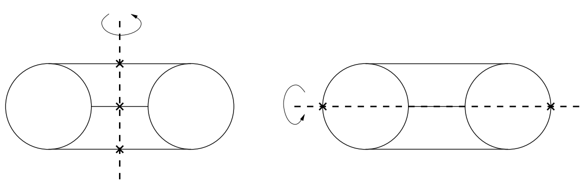

Fig. 5

We want to prove that the second possibility cannot occur (in the second case the combinatorial action does not determine the combinatorial type of the singularity graph of the quotient orbifold). It is easy to see that the only elements with non-empty fixed-point set and with order strictly greater than two are in A (in $S-A$ there are either involutions or elements acting freely); moreover, in A, elements with non-empty fixed-point set have order at most 2^n and lie either in the group generated by g or in the group generated by sgs. Since g and sgs are conjugate their fixed-point sets project to the same edge of singularity index 2^n (see Proposition 5). In the second case in Figure 5 the involution $\overline{f}$ fixes setwise one edge, and so the singularity graph of M/S contains either two edges with singularity index strictly greater than two, or one edge with singularity index 2^{n+1}; in any case we get a contradiction. So we are in the first situation presented in Figure 5 and the combinatorial type of the singularity graph of M/S is determined: the graph is of type II with one edge of singularity index 2^n and the others of singularity index two.

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Received 5 April 2004;
in revised form 6 August 2004