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The rhombic dodecahedron and semisimple actions of $\mathrm{Aut}(F_n)$ on $\mathrm{CAT}(0)$ spaces

by

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For Mike Davis on his 60th birthday, with affection and great respect

Abstract. We consider actions of automorphism groups of free groups by semisimple isometries on complete CAT(0) spaces. If $n \geq 4$ then each of the Nielsen generators of $\mathrm{Aut}(F_n)$ has a fixed point. If $n = 3$ then either each of the Nielsen generators has a fixed point, or else they are hyperbolic and each Nielsen-generated $\mathbb{Z}^4 \subset \mathrm{Aut}(F_3)$ leaves invariant an isometrically embedded copy of Euclidean 3-space $\mathbb{E}^3 \hookrightarrow X$ on which it acts as a discrete group of translations with the rhombic dodecahedron as a Dirichlet domain. An abundance of actions of the second kind is described.

Constraints on maps from $\mathrm{Aut}(F_n)$ to mapping class groups and linear groups are obtained. If $n \geq 2$ then neither $\mathrm{Aut}(F_n)$ nor $\mathrm{Out}(F_n)$ is the fundamental group of a compact Kähler manifold.

1. Introduction. This article is part of a project to understand the ways in which mapping class groups and automorphism groups of free groups can act on CAT(0) spaces. Here we focus mainly (but not exclusively) on actions that are by semisimple isometries. This includes all cellular actions on polyhedral complexes with only finitely many isometry types of cells [3].

The action of $\mathrm{Aut}(F_n)$ on the abelianisation of F_n gives an epimorphism $\mathrm{Aut}(F_n) \to \mathrm{GL}(n, \mathbb{Z})$. The inverse image of $\mathrm{SL}(n, \mathbb{Z})$ is generated by *Nielsen transformations*, which are the obvious lifts of the elementary matrices: fixing a basis $\{a_1, \dots, a_n\}$ for F_n, one defines the left Nielsen transformations λ_{ij} by $[a_i \mapsto a_j a_i,\ a_k \mapsto a_k\ (k \neq i)]$ and the right Nielsen transformations ρ_{ij} by $[a_i \mapsto a_i a_j,\ a_k \mapsto a_k\ (k \neq i)]$.

Semisimple isometries of CAT(0) spaces divide into elliptics (those with fixed points) and hyperbolics (those that have a non-trivial axis of translation). Conjugate isometries are of the same type. The Nielsen transfor-

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mations are all conjugate in $\mathrm{Aut}(F_n)$, so the semisimple actions of $\mathrm{Aut}(F_n)$ on CAT(0) spaces divide into two classes: those where the Nielsen transformations act as hyperbolic isometries and those where they act as elliptic isometries. We shall prove that if $n \geq 4$ then there are no actions of the former type. The proof (which establishes something more—Proposition 3.2) is based on an idea of Gersten [12].

THEOREM 1.1. *If $n \geq 4$, then each Nielsen transformation in* $\mathrm{Aut}(F_n)$ *fixes a point whenever* $\mathrm{Aut}(F_n)$ *acts by semisimple isometries on a complete* CAT(0) *space.*

If the dimension of the CAT(0) space is sufficiently small, one can promote the existence of fixed points for the individual Nielsen transformations to a fixed point for the whole group—see [2], [5]. And by considering induced actions one can extend Theorem 1.1 to finite-index subgroups $\Gamma < \mathrm{Aut}(F_n)$: any power of a Nielsen transformation that lies in Γ fixes a point whenever Γ acts by semisimple isometries on a complete CAT(0) space (see Section 2.1). In particular, if $\lambda_{ij}^p \in \Gamma$ then λ_{ij}^p must have finite order in $H_1(\Gamma, \mathbb{Z})$; for if not then there would be a homomorphism $\Gamma \to \mathbb{Z}$ mapping λ_{ij}^p non-trivially.

Despite recent progress in the understanding of maps between automorphism groups of free groups and mapping class groups, many issues remain unresolved. For example, it is unknown whether every homomorphism from $\mathrm{Aut}(F_n)$ to a mapping class group has to have finite image if $n \geq 4$. Closed surfaces of negative Euler characteristic admit semisimple actions on complete CAT(0) spaces where the elliptic isometries are the roots of multi-twists [4, Theorem A], so Theorem 1.1 constrains putative maps $\mathrm{Aut}(F_n) \to \mathrm{Mod}(S)$.

COROLLARY 1.2. *If $n \geq 4$ and $\Gamma <$* $\mathrm{Aut}(F_n)$ *is a subgroup of finite index, then every homomorphism from Γ to a mapping class group sends powers of Nielsen transformations to roots of (possibly trivial) multi-twists.*

We shall prove in Theorem 7.1 that this corollary fails for $\Gamma = \mathrm{Aut}(F_3)$. Theorem 1.1 fails even more starkly when $n = 3$. The key difference in the case $n = 3$ lies in an observation from the work of Grunewald and Lubotzky [13] on linear representations of $\mathrm{Aut}(F_n)$. They proved that $\mathrm{Out}(F_3)$ has a subgroup of finite index that maps onto a non-abelian free group. This gives rise to an abundance of semisimple actions of $\mathrm{Aut}(F_3)$ and $\mathrm{Out}(F_3)$ in which the Nielsen transformations act hyperbolically, as we shall explain in Section 6. But across this enormous range of actions there is a striking geometric feature that remains constant. This concerns the way in which abelian subgroups of maximal rank generated by Nielsen transformations act. Each such subgroup is conjugate ([1]) to $\Lambda := \langle \lambda_{21}, \rho_{21}, \lambda_{31}, \rho_{31} \rangle \cong \mathbb{Z}^4$; we call it a *Nielsen* $\mathbb{Z}^4$.

The *rhombic dodecahedron* is the Catalan solid that is dual to the Archimedean cuboctahedron; it is described in more detail in Section 4. Notice that in the following theorem no assumption is made about the discreteness of the action.

THEOREM 1.3. *Whenever* $\mathrm{Aut}(F_3)$ *acts by semisimple isometries on a complete* $\mathrm{CAT}(0)$ *space* X, *either each Nielsen transformation fixes a point, or else each Nielsen* $\mathbb{Z}^4 \subset \mathrm{Aut}(F_3)$ *leaves invariant an isometrically embedded* 3*-flat* $\mathbb{E}^3 \hookrightarrow X$ *on which it acts as a discrete group of translations with Dirichlet domain a rhombic dodecahedron.*

When $n \geq 6$ one can sharpen Theorem 1.1 by proving that in any action of $\mathrm{Aut}(F_n)$ by isometries on a complete $\mathrm{CAT}(0)$ space the Nielsen transformations have zero translation length, i.e. they are either elliptic or neutral parabolic. Actions of the second type arise from linear representations $\mathrm{Aut}(F_n) \to \mathrm{GL}(d, \mathbb{R})$ via the action of $\mathrm{GL}(d, \mathbb{R})$ on its symmetric space. The fact that Nielsen transformations must act as neutral parabolics imposes constraints on the representation theory of $\mathrm{Aut}(F_n)$ that supplement [10] and [16]. In Section 9 we explain how the rigidity of the standard linear representation [10] can be combined with Simpson's results [17] to prove:

THEOREM 1.4. *If* $n \geq 2$, *then neither* $\mathrm{Aut}(F_n)$ *nor* $\mathrm{Out}(F_n)$ *is the fundamental group of a compact Kähler manifold.*

2. Isometries of $\mathrm{CAT}(0)$ **spaces and centralizers.** Our basic reference for $\mathrm{CAT}(0)$ spaces and their isometries is [6]. Let X be a complete $\mathrm{CAT}(0)$ space. The *translation length* of an isometry $\gamma \in \mathrm{Isom}(X)$ is $\|\gamma\| := \inf\{d(x, \gamma.x) \mid x \in X\}$, and we let $\mathrm{Min}(\gamma) := \{x \in X \mid d(x, \gamma.x) = \|\gamma\|\}$. An isometry is termed *semisimple* if $\mathrm{Min}(\gamma)$ is non-empty. Semisimple isometries divide into *elliptics* (when $\|\gamma\| = 0$) and *hyperbolics* ($\|\gamma\| > 0$). Isometries for which $\mathrm{Min}(\gamma) = \emptyset$ are termed *parabolic* (*neutral* or *non-neutral* [*ballistic*] according to whether $\|\gamma\| = 0$ or not).

The following basic result is proved on page 231 of [6].

PROPOSITION 2.1. *If* γ *is hyperbolic, then* $\mathrm{Min}(\gamma)$ *splits isometrically as* $Y \times \mathbb{R}$, *where* γ *acts trivially on the first factor and as* $(t \mapsto t + \|\gamma\|)$ *on the second factor. If* $\alpha \in \mathrm{Isom}(X)$ *commutes with* γ, *then it leaves* $\mathrm{Min}(\gamma)$ *invariant, preserves its splitting, and acts by translation on the second factor.*

We also need the following form of the Flat Torus Theorem [6, p. 254]. A *k-flat* is an isometrically embedded copy of k-dimensional Euclidean space.

([1]) $\mathrm{Aut}(F_3)$ has virtual cohomological dimension 4, so there is no free abelian subgroup of greater rank, but there is one other type of $\mathbb{Z}^4$, as explained on page 1710 of [11].

FLAT TORUS THEOREM. *Whenever* $\mathbb{Z}^r$ *acts by semisimple isometries on a complete* CAT(0) *space* X, *it leaves invariant a* k-*flat* $E \hookrightarrow X$ (*for some* $k \le r$) *on which it acts by translations. If* $\gamma \in \mathbb{Z}^r$ *acts hyperbolically on* X, *then* $\gamma|_E$ *is a translation of length* $\|\gamma\|$.

NOTATION. Let γ and δ be commuting hyperbolic isometries. We write $\gamma \perp \delta$ if at some (hence any) point $x \in \mathrm{Min}(\gamma) \cap \mathrm{Min}(\delta)$ the axes of γ and δ through x are orthogonal. Equivalently, the action of δ on the second factor of $\mathrm{Min}(\gamma) = Y \times \mathbb{R}$ is trivial.

We write $Z_G(S)$ for the centralizer of a set S in a group G. We write $H_1(G)$ for the abelianisation of G and $[G, G]$ for the commutator subgroup.

LEMMA 2.2. *Suppose that* Γ *acts by isometries on a complete* CAT(0) *space* X *and that* $\gamma, \delta \in \Gamma$ *are commuting hyperbolic isometries.*

(1) *The natural map* $\langle\gamma\rangle \to H_1(Z_\Gamma(\gamma))$ *is injective.*
(2) *If* $\delta \in [Z_\Gamma(\gamma), Z_\Gamma(\gamma)]$, *then* $\gamma \perp \delta$.
(3) *If there exists* $g \in Z_\Gamma(\gamma)$ *such that* $g^{-1}\delta^{-1}g = \delta$ *then* $\gamma \perp \delta$.

Proof. The action of $Z(\gamma)$ on the second factor of $\mathrm{Min}(\gamma) = Y \times \mathbb{R} \subset X$ is by translations. The group of all such translations is torsion-free and abelian, and the image of γ is non-trivial. This proves (i) and (ii), and (iii) is a special case of (ii) since $\delta^2 = g^{-1}\delta^{-1}g\delta \in [Z_\Gamma(\gamma), Z_\Gamma(\gamma)]$ and any axis for δ is an axis for δ^2. ∎

2.1. Induced actions. Let G be a group and let $H \subset G$ be a subgroup of index d. Just as one induces an n-dimensional linear representation of H to obtain an nd-dimensional representation of G, one can induce an action of H by isometries of a metric space X to obtain an action of G by isometries on X^d (with the product metric). One way to view this induction (following [9, p. 35]) is to identify X^d with the space of H-equivariant maps $f : G \to X$ with the action $(g.f)(\gamma) = f(\gamma g^{-1})$.

The following lemma is covered in [6, pp. 231–232].

LEMMA 2.3. *Suppose* H *has index* d *in* G *and that* H *acts by isometries on a complete* CAT(0) *space* X. *If* $g^p \in H$ *acts as a hyperbolic isometry, where* p *is a non-zero integer, then* g *acts hyperbolically in the induced action of* G *on* X^d.

One can also express induction in its group-theoretic form, the *wreath product*. Recall that $A \wr B$ is the semidirect product $W = B \ltimes \bigoplus_{b\in B} A_b$, where the A_b are isomorphic copies of A permuted by left translation. $\bigoplus_{b\in B} A_b$ is called the *base* of the wreath product. If B is finite and A acts on X by isometries, then there is an obvious action of W by isometries on X^B, with B permuting the factors and A_b acting as A in the b-coordinate and trivially in the others. The induction described above is then just a manifestation of the following standard lemma. To avoid complications we assume that H is normal in G.

Lemma 2.4. *If H is normal in G and $\phi : H \to Q$ is a homomorphism, then there is a homomorphism $\Phi : G \to Q \wr (G/H)$ so that $\Phi(H) \subset \bigoplus_{x \in G/H} Q_x$ and for all $h \in H$ the coordinate of $\Phi(h)$ in Q_1 is $\phi(h)$.*

3. Automorphisms that cannot act hyperbolically

Conventions. Throughout this article, F_n denotes a free group of rank n. We work with the left action of $\mathrm{Aut}(F_n)$ on F_n and the commutator convention $[\alpha,\beta] = \alpha\beta\alpha^{-1}\beta^{-1}$. (So $[\alpha,\beta]$ is the automorphism "β^{-1}, followed by $\alpha^{-1}, \ldots$".) We write ad_γ to denote the inner automorphism $x \mapsto \gamma x \gamma^{-1}$ (so $\mathrm{ad}_{\gamma\delta} = \mathrm{ad}_\gamma \circ \mathrm{ad}_\delta$). Note that $\phi\, \mathrm{ad}_\gamma\, \phi^{-1} = \mathrm{ad}_{\phi(\gamma)}$.

The following lemma is a variation on the argument that Gersten [12] used to show that $\mathrm{Aut}(F_n)$ cannot act properly and cocompactly on a CAT(0) space if $n \geq 3$ (cf. [6, p. 253]).

Lemma 3.1. *Let $G_{p,q} = \langle \alpha, \beta, \gamma, t \mid [t,\alpha],\ t\beta t^{-1} = \beta\alpha^p,\ t\gamma t^{-1} = \gamma\alpha^q \rangle$, where p and q are non-zero integers. If $p \neq q$, then α fixes a point whenever $G_{p,q}$ acts by semisimple isometries on a complete* CAT(0) *space.*

Proof. Suppose $G_{p,q}$ acts semisimply on a complete CAT(0) space X. Note that β conjugates t to $\alpha^p t$ and γ conjugates t to $\alpha^q t$, so $\|t\| = \|\alpha^p t\| = \|\alpha^q t\|$. Thus if t is elliptic then $\alpha^p t$ is elliptic, hence $\alpha^p = (\alpha^p t)t^{-1}$ as a product of commuting elliptics is elliptic, and therefore α is elliptic.

If t is hyperbolic, then by the Flat Torus Theorem, $A = \langle \alpha, t \rangle$ would act by translations on a geodesic line or Euclidean plane E in $\mathrm{Min}(t) \cap \mathrm{Min}(\alpha)$ with t, $t\alpha^p$, $t\alpha^q$ all acting as translations of length $\|t\|$. But the images of any point $e \in E$ under these three translations are collinear. This is incompatible with the fact that they are equidistant from e unless we are in the degenerate situation where the points coincide, i.e. α is acting trivially on E and hence is elliptic. ∎

For the next lemma it is convenient to work with a basis $\{a_0, \ldots, a_n\}$ for F_{n+1}, setting $F_n = \langle a_1, \ldots, a_n \rangle$, and for each $w \in F_n$ defining $R_w \in \mathrm{Aut}(F_{n+1})$ by $R_w(a_0) = a_0 w$ and $R_w(a_i) = a_i$ for $i > 0$.

As R_{a_i} is a Nielsen transformation, the following proposition implies Theorem 1.1.

Proposition 3.2. *If $w \in F_n$ lies in a free factor of rank $n-2$, then R_w fixes a point whenever* $\mathrm{Aut}(F_{n+1})$ *acts by semisimple isometries on a complete* CAT(0) *space.*

Proof. Without loss of generality we may assume that $w \in \langle a_1, \ldots, a_{n-2} \rangle$. Let $p \neq q$ be non-zero integers and let $T \in \mathrm{Aut}(F_{n+1})$ be the automorphism defined by $T(a_{n-1}) = a_{n-1}w^p$, $T(a_n) = a_n w^q$ and $T(a_i) = a_i$ for $i = 0, \dots, n-2$. An elementary calculation shows that the assignment $[\alpha \mapsto R_w,\ \beta \mapsto R_{a_{n-1}},\ \gamma \mapsto R_{a_n},\ t \mapsto T]$ respects the defining relations of the group $G_{p,q}$ of Lemma 3.1 and hence defines a homomorphism $G_{p,q} \to \mathrm{Aut}(F_{n+1})$. Thus any action of $\mathrm{Aut}(F_{n+1})$ by semisimple isometries on a complete CAT(0) space gives rise to an action of $G_{p,q}$, and Lemma 3.1 tells us that R_w, the image of $\alpha \in G_{p,q}$, must act elliptically. ∎

Our interest in the following lemma lies with the case $n = 3$.

LEMMA 3.3. *Let $n \geq 3$. Whenever* $\mathrm{Aut}(F_n)$ *acts by semisimple isometries on a complete* CAT(0) *space, the inner automorphisms corresponding to primitive elements are elliptic.*

Proof. If $n \geq 4$, then the preceding proposition implies that the right Nielsen transformations ρ_{ij} are elliptic, hence so are their conjugates λ_{ij}. The inner automorphism corresponding to a primitive element, say a_1 from the basis $\{a_1, \dots, a_n\}$, is a composition of commuting Nielsen transformations: $\mathrm{ad}_{a_1} = (\lambda_{21}^{-1}\rho_{21}) \dots (\lambda_{n1}^{-1}\rho_{n1})$, and a product of commuting elliptic isometries is elliptic.

When $n = 3$ the Nielsen transformations need not be elliptic, so we require a different argument. Let $\{a, b, c\}$ be a basis for F_3 and let $\tau \in \mathrm{Aut}(F_3)$ be the automorphism $[a \mapsto a,\ b \mapsto ba^p,\ c \mapsto ca^q]$. The assignment $[\alpha \mapsto \mathrm{ad}_a,\ \beta \mapsto \mathrm{ad}_b,\ \gamma \mapsto \mathrm{ad}_c,\ t \mapsto \tau]$ defines a homomorphism $G_{p,q} \to \mathrm{Aut}(F_3)$ sending α to ad_a, so Lemma 3.1 implies that ad_a must be elliptic. ∎

REMARK 3.4. The maps $G_{p,q} \to \mathrm{Aut}(F_{n+1})$ and $G_{p,q} \to \mathrm{Aut}(F_3)$ used in the preceding proofs are injective, but since we do not need this fact we omit the proof.

4. The rhombic dodecahedron. The *rhombic dodecahedron* is one of the Catalan solids; it is dual to the Archimedean solid known as the cuboctahedron. It has fourteen vertices and twelve faces. The faces are rhombi whose diagonals have lengths in the ratio $1 : \sqrt{2}$.

Consider the standard tiling of $\mathbb{R}^3$ by unit cubes and fix a vertex u. The rhombic dodecahedron can be constructed as the convex hull of the endpoints of the six edges incident at u together with the centres of the eight cubes incident at u. In other words, it is the Voronoĭ cell for the face-centred cubic lattice (and therefore occurs naturally in many crystal formations).

It follows from this description that the rhombic dodecahedron is a Dirichlet domain for the free abelian group of rank 3 that translates $\mathbb{R}^3$ by integer vectors (a, b, c) with $a + b + c$ even. This is the group generated by the edge-vectors of an octahedron dual to a cube with edge-length 2. For our purposes, it is convenient to rephrase this as follows.

LEMMA 4.1. *Let $A \subset \mathbb{R}^3$ be the $\mathbb{Z}$-module generated by unit vectors u_1, u_2, v_1, v_2. If*

(1) $u_1 + u_2 = v_1 + v_2$,
(2) $u_1 \perp u_2$ *and* $v_1 \perp v_2$, *and*
(3) $(u_1 - u_2) \perp (v_1 - v_2)$,

then A is discrete and $\{x \mid \|x\| \le \|x - a\|$ for all $a \in A\}$ is a rhombic dodecahedron.

REMARK 4.2. Note that the conclusion of the lemma includes the observation that A is not contained in a plane. Thus one cannot find such vectors u_1, u_2, v_1, v_2 in $\mathbb{R}^d$ for $d < 3$.

PROPOSITION 4.3. *Let Γ be a group. Suppose that $\alpha_1, \alpha_2, \beta_1, \beta_2$ commute and are conjugate in Γ. Let $A = \langle \alpha_1, \alpha_2, \beta_1, \beta_2 \rangle$ and suppose that*

(i) $\alpha_1\alpha_2 = \beta_1\beta_2$,
(ii) $\alpha_1 \in [Z_\Gamma(\alpha_2), Z_\Gamma(\alpha_2)]$ *and* $\beta_1 \in [Z_\Gamma(\beta_2), Z_\Gamma(\beta_2)]$, *and*
(iii) *there exists* $\gamma \in Z_\Gamma(\alpha_1\alpha_2^{-1})$ *with* $\gamma^{-1}(\beta_1\beta_2^{-1})\gamma = \beta_2\beta_1^{-1}$.

Then, whenever Γ acts by semisimple isometries on a complete CAT(0) *space X, either A fixes a point, or A leaves invariant an isometrically embedded 3-flat $\mathbb{E}^3 \hookrightarrow X$, acting properly and cocompactly on it by isometries, with Dirichlet domain a rhombic dodecahedron.*

Proof. Since the given generators of A are all conjugate, either all are elliptic or all are hyperbolic. If they are elliptic, then they have a common fixed point, since they commute. If they are hyperbolic, then by the Flat Torus Theorem, A leaves invariant a k-flat $E \subset X$, with $k \le 3$, and acts on it by translations; since the a_i and b_i are hyperbolic, they act non-trivially on E, and since they are conjugate in Γ, their translation lengths are equal. Let u_i (resp. v_i) be the translation vector of $\alpha_i|_E$ (resp. $\beta_i|_E$). According to Lemma 2.2, condition (ii) implies that $u_1 \perp u_2$ and $v_1 \perp v_2$, and condition (iii) implies that $(u_1 - u_2) \perp (v_1 - v_2)$. Lemma 4.1 (with the remark that follows it) completes the proof. ∎

REMARK 4.4. The above proposition admits obvious variations in which conditions (ii) and (iii) are altered to allow parts (2) and (3) of Lemma 2.2 to be applied in different combinations.

5. The shape of Nielsen-hyperbolic actions of Aut(F_3)**.** The free abelian subgroups of Aut(F_3) have rank at most 4 (the virtual cohomological dimension of Aut(F_3)). Each Nielsen $\mathbb{Z}^4$ is conjugate to $\Lambda = \langle \lambda_{21}, \rho_{21}, \lambda_{31}, \rho_{31} \rangle$. Let ε_3 denote the involution $a_3 \leftrightarrow a_3^{-1}$.

Proposition 4.3 applies directly to the image of Λ in $\mathrm{Out}(F_3)$.

LEMMA 5.1. *In $\Gamma = \mathrm{Out}(F_3)$, the elements $\alpha_1 := \lambda_{21}^{-1}$, $\alpha_2 := \rho_{21}$, $\beta_1 := \rho_{31}^{-1}$, $\beta_2 := \lambda_{31}$ and $\gamma := \varepsilon_3$ satisfy the conditions of Proposition* 4.3.

Proof. In $\mathrm{Out}(F_3)$ we have $\mathrm{ad}_{a_1} = \lambda_{21}^{-1}\rho_{21}\lambda_{31}^{-1}\rho_{31} = 1$, so $\alpha_1\alpha_2 = \beta_1\beta_2$. A direct calculation yields the well-known relation $[\lambda_{23}^{-1}, \lambda_{31}^{-1}] = \lambda_{21}^{-1}$, and both λ_{23}, λ_{31} commute with ρ_{21}. Thus λ_{21} lies in the commutator subgroup of $Z(\rho_{21})$. Likewise, $[\rho_{23}^{-1}, \rho_{31}^{-1}] = \rho_{21}^{-1}$ implies that ρ_{21} is in the commutator subgroup of $Z(\lambda_{21})$. Conjugation by ε_3 leaves λ_{21} and ρ_{21} fixed while interchanging ρ_{31}^{-1} and λ_{31}. ∎

Proposition 4.3 does not apply directly to $\Lambda \subset \mathrm{Aut}(F_3)$ but the difficulty is a minor one.

Proof of Theorem 1.3. Suppose that $\mathrm{Aut}(F_3)$ is acting by isometries on a complete CAT(0) space X with the Nielsen moves λ_{ij} acting as hyperbolic isometries. According to the Flat Torus Theorem, $\Lambda \cong \mathbb{Z}^4$ leaves invariant a k-flat $E \subset X$ on which it acts by translations, with the generators λ_{ij} and ρ_{ij} (which are conjugate in $\mathrm{Out}(F_3)$) acting non-trivially by translations of the same length. Lemma 3.3 tells us that $\mathrm{ad}_{a_1} = \lambda_{21}^{-1}\rho_{21}\lambda_{31}^{-1}\rho_{31}$ acts elliptically on X and hence trivially on E. Replacing E by the convex hull of a Λ-orbit, we may assume that it has dimension at most 3.

The calculations in the proof of Lemma 5.1 allow us to apply Lemma 2.2: as in the proof of Proposition 4.3, the translation vectors of $\lambda_{21}|_E, \rho_{21}|_E$, $\lambda_{31}|_E, \rho_{31}|_E$ satisfy the conditions of Lemma 4.1. Hence E has dimension 3 (Remark 4.2), the action of $\Lambda/\langle \mathrm{ad}_{a_1}\rangle$ is proper and cocompact, and a Dirichlet domain for the action is a rhombic dodecahedron. ∎

6. An abundance of actions when $n = 3$. Let Γ be an arbitrary finitely generated group. In this section we shall assign to each action of Γ on a CAT(0) space X actions of $\mathrm{Aut}(F_3)$ and $\mathrm{Out}(F_3)$ on the Cartesian product of finitely many copies of X. The assignment is such that if the generators of Γ act as hyperbolic isometries, then the induced actions of $\mathrm{Aut}(F_3)$ and $\mathrm{Out}(F_3)$ will be Nielsen-hyperbolic.

The heart of the construction is the following proposition. Here, as usual, F_k denotes the free group of rank k.

PROPOSITION 6.1. *For each positive integer k, there exists a subgroup of finite index $H_k \subset \mathrm{Out}(F_3)$ and a surjective homomorphism $\pi_k : H_k \to F_k$ such that F_k is generated by the images of powers of conjugates of the Nielsen transformations λ_{ij}.*

Proof. Fix a basis $\{a_1, a_2, a_3\}$ for F_3. We first construct a map from a subgroup of finite index in $\mathrm{Out}(F_3)$ to a free group of rank 2 so that the image is generated by powers of λ_{12} and λ_{21}. This map is based on a construction of Grunewald and Lubotzky [13] (cf. [7, Qu. 9]).

Regard F_3 as the fundamental group of a graph R with one vertex. The loops of length one are labelled $\{a_1, a_2, a_3\}$. Consider the 2-sheeted covering $\hat{R} \to R$ with fundamental group $\langle a_1, a_2, a_3^2, a_3a_1a_3^{-1}, a_3a_2a_3^{-1}\rangle$ and let $G \subset \mathrm{Aut}(F_3)$ be the stabilizer of this subgroup.

G acts on $H_1(\hat{R}, \mathbb{Q})$ leaving invariant the eigenspaces of the involution that generates the Galois group of the covering. The eigenspace corresponding to the eigenvalue -1 is two-dimensional with basis $\{a_1 - a_3a_1a_3^{-1}, a_2 - a_3a_2a_3^{-1}\}$. The action of G with respect to this basis gives an epimorphism $G \to \mathrm{GL}(2, \mathbb{Z})$. By replacing G with a subgroup of finite index one can ensure that the inner automorphisms in G act trivially on $H_1(\hat{R}, \mathbb{Q})$ (because $\mathrm{Inn}(F_n) \cap G$ has finite image in $\mathrm{GL}(2, \mathbb{Z})$, which is virtually torsion-free). If Q is the image of G in $\mathrm{Out}(F_3)$, then we have a map $\mu : Q \to \mathrm{GL}(2, \mathbb{Z})$ whose image is of finite index.

Since $\mathrm{GL}(2, \mathbb{Z})$ has a free subgroup of finite index, by passing to a further subgroup if necessary we may assume that $\mu(Q)$ is free. We choose p so that $\lambda_{12}^p, \lambda_{21}^p \in Q$. The images of $\mu(\lambda_{12})$ and $\mu(\lambda_{21})$ generate a subgroup of finite index in $\mathrm{GL}(2, \mathbb{Z})$, so the subgroup L_2 generated by $a := \mu(\lambda_{12}^p)$ and $b := \mu(\lambda_{21}^p)$ is free of rank 2. Applying Marshall Hall's theorem, we pass to a subgroup of finite index in $\mu(Q)$ that retracts onto L_2. Let H_2 be the inverse image of this subgroup in Q and let $\pi : H_2 \to L_2$ be its surjection to L_2.

In $L_2 = \langle a, b\rangle$ one has the subgroup L_k of index $k - 1$ with basis $\{a^iba^{-i}, a^{k-1} \mid i = 0, \ldots, k-2\}$; this is free of rank k and is generated by conjugates of powers of a and b. Thus it suffices to take $H_k = \pi^{-1}(L_k)$. ∎

6.1. Induced actions. Let Γ be a group and let $\psi : F_r \to \Gamma$ be an epimorphism. Suppose that Γ acts by isometries on a complete CAT(0) space X. By composing ψ with the map $H_r \to F_r$ constructed in the preceding proposition, we obtain a surjection $H_r \to \Gamma$ sending powers of certain Nielsen transformations to generators of Γ. And by inducing the action we obtain an action of $\mathrm{Aut}(F_3)$ on a product of finitely many copies of X. If the action of Γ on X is by hyperbolic isometries, then the Nielsen transformations act as hyperbolic isometries in this induced action.

7. Homomorphisms from $\mathrm{Out}(F_3)$ to mapping class groups. We saw in the introduction that when $n \geq 4$, any homomorphism from $\mathrm{Aut}(F_n)$ to a mapping class group must send Nielsen transformations to roots of multi-twists. We also noted that no homomorphisms with infinite image are known to exist. The situation for $n = 3$ is completely different. Let S_g be the closed surface of genus g and let $\mathrm{Mod}(S_g)$ be its mapping class group. Let $\mathrm{Mod}(S_{g,1})$ be the mapping class group of the genus g surface with one boundary component.

Theorem 7.1. *For certain positive integers g, there exist homomorphisms* $\mathrm{Out}(F_3) \to \mathrm{Mod}(S_g)$ *sending Nielsen transformations to elements of infinite order that are not roots of multi-twists.*

I do not know the value of the least integer g for which there is such a homomorphism (and likewise for $\mathrm{Aut}(F_3)$).

Proposition 7.2. *For all positive integers h and all non-trivial finite groups G, there exist integers g for which there is an injective homomorphism* $\mathrm{Mod}(S_{h,1}) \wr G \to \mathrm{Mod}(S_g)$.

Proof. Every finite group has a faithful realisation as a group of symmetries of a closed hyperbolic surface (and therefore embeds in the mapping class group of that surface). We realise G on the surface Y and equivariantly delete an open disc about each point in a free orbit. We then glue a copy of $S_{h,1}$ to each of the resulting boundary components and extend the action of G in the obvious manner. Let S_g be the resulting surface.

Corresponding to each of the attached copies of $S_{h,1}$ there is an injective homomorphism $\mathrm{Mod}(S_{h,1}) \to \mathrm{Mod}(S_g)$ obtained by extending homeomorphisms to be the identity on the complement of the attached copy of $S_{h,1}$. (The extension respects isotopy classes.) Thus we obtain $|G|$ commuting copies $\{M_\gamma \mid \gamma \in G\}$ of $\mathrm{Mod}(S_{h,1})$ in $\mathrm{Mod}(S_g)$. The action of $G \subset \mathrm{Mod}(S_g)$ by conjugation permutes the M_γ, and the canonical map $G \ltimes \bigoplus_{\gamma\in G} M_\gamma \to \mathrm{Mod}(S_g)$ is injective. ∎

Proof of Theorem 7.1. Let F_r be a free group of rank r that maps onto $\Gamma = \mathrm{Mod}(S_{h,1})$ sending at least one basis element, say a_1, to an element of infinite order $\psi \in \Gamma$ that is not a root of a multi-twist. Proposition 6.1 provides a normal subgroup of finite index $H_r \subset \mathrm{Out}(F_3)$ that maps onto F_r sending a power of a Nielsen transformation, say λ^p, to a_1. Thus we obtain a homomorphism $H_r \to \Gamma$ sending λ^p to ψ. We replace H_r with a subgroup of finite index that is normal in $\mathrm{Out}(F_3)$ (with quotient Ω, say) and consider the induced homomorphism $\mathrm{Out}(F_3) \to \Gamma \wr \Omega$ as in Lemma 2.4. As in Proposition 7.2, this wreath product acts on a closed surface S_g. By construction, a non-zero power of λ^p acts on one of the subsurfaces $S_{h,1}$ as a non-zero power of ψ and hence the image of λ in $\mathrm{Mod}(S_g)$ is not a root of a multi-twist. ∎

8. Nielsen generators are not ballistic if $n \geq 6$

Lemma 8.1. *Let $\lambda \in \mathrm{Aut}(F_n)$ be a Nielsen transformation and let Z be its centralizer. If $n \geq 6$, then $[\lambda] = 0$ in $H_1(Z, \mathbb{Z})$.*

Proof. For any $n \geq 2$ one can realise λ as a Dehn twist δ about a non-separating curve on a compact orientable surface S that has genus $\lfloor n/2 \rfloor$ and Euler characterisic $1-n$. More precisely, taking a basepoint on the boundary of S, there is an isomorphism ([2]) from $\pi_1 S$ to F_n that induces a homomorphism from the mapping class group Mod(S) to Aut(F_n) sending δ to λ.

Let $C(\delta)$ be the centraliser of δ in Mod(S). The map from $\langle\delta\rangle = \langle\lambda\rangle$ to $H_1(Z,\mathbb{Z})$ factors through $H_1(C(\delta),\mathbb{Z})$, so it is enough to prove that δ has trivial image in $H_1(C(\delta),\mathbb{Z})$. If δ is the twist in a loop c, then $C(\delta)$ is the image of the natural map to Mod(S) of the mapping class group of the surface S' obtained by cutting S open along c. Thus it is enough to prove that in a surface S' of genus at least 2, the Dehn twist in any loop c parallel to a boundary curve is trivial in Mod(S'). This can be seen using the lantern relation [15]. In more detail, the lantern relation involves seven loops on a 4-holed sphere; if we embed the 4-holed sphere in S' so that one of the loops is sent to c and the remaining six loops are non-separating, then the relation takes the form $\delta_0\delta_1\delta_2\delta_3 = \delta_4\delta_5\delta_6$, where δ_0 is the twist in c and the remaining δ_i are positive twists in non-separating loops. The twists in any two non-separating loops are conjugate in Mod(S'), hence equal (to τ say) in $H_1(\mathrm{Mod}(S'),\mathbb{Z})$. So in H_1 the above relation becomes $\delta_0\tau^3 = \tau^3$. ∎

PROPOSITION 8.2. *Suppose $n \geq 6$. Whenever* Aut(F_n) *acts by isometries on a complete* CAT(0) *space, the Nielsen generators λ_{ij} and ρ_{ij} have zero translation length (i.e. they are either elliptic or neutral parabolics).*

Proof. A lemma of Karlsson and Margulis [14] can be used to show that if an isometry γ of a complete CAT(0) space has positive translation length then γ must have infinite order in the abelianisation of its centraliser (cf. the proof of [4, Theorem 1]). Lemma 8.1 completes the proof. ∎

This result places constraints on the representation theory of Aut(F_n). I shall return to this point in another article but note one example here for illustrative purposes.

COROLLARY 8.3. *Let $\lambda \in$* Aut(F_n) *be a Nielsen transformation and let Φ :* Aut$(F_n) \to$ SL$(d,\mathbb{R})$ *be an arbitrary representation. If $n \geq 6$, then the eigenvalues of $\Phi(\lambda)$ all have modulus* 1.

Proof. The symmetric space $\mathrm{SL}(d,\mathbb{R})/\mathrm{SO}(d,\mathbb{R})$ is non-positively curved and the action of $M \in \mathrm{SL}(d,\mathbb{R})$ has positive translation length if M has an eigenvalue of modulus greater than 1 (equivalently, the hyperbolic component in its Jordan decomposition is non-trivial). ∎

([2]) A 1-holed torus T is the regular neighbourhood of the union Y of two simple loops that intersect once; orient them, label them a_1, a_2 and join their intersection point to the basepoint by an arc that does not intersect the loops; the automorphism of $\pi_1 T \cong F_2$ induced by the Dehn twist in a_1 is easily seen to be one of $\lambda_{21}^{\pm 1}$ or $\rho_{21}^{\pm 1}$ (the four possibilities corresponding to the choices of orientation for the loops). For $n > 2$, one attaches a suitable surface to T along an arc in its boundary that includes the basepoint and one extends $\{a_1, a_2\}$ to a basis for π_1 where the remaining basis loops lie in the attached surface.

9. $\mathrm{Aut}(F_n)$ **and** $\mathrm{Out}(F_n)$ **are not Kähler groups.** One can deduce from Corollary 8.3 that the standard representation $\mathrm{Aut}(F_n) \to \mathrm{GL}(n, \mathbb{Z})$ (given by the action of $\mathrm{Aut}(F_n)$ on $H_1(F_n, \mathbb{Z})$) cannot be deformed locally to anything but a conjugate representation. A stronger result was proved by Dyer and Formanek [10] (see also [16, Theorem 1.2]): every representation $\mathrm{Aut}(F_n) \to \mathrm{GL}(n, \mathbb{C})$ factors through the standard representation. Using Proposition 3.1 of [8] one can extend the proof in [16] to cover the unique index-2 subgroup $\mathrm{SAut}(F_n) \subset \mathrm{Aut}(F_n)$.

Every finitely presented group is the fundamental group of a closed symplectic manifold and of a closed complex manifold, but not every finitely presented group is a Kähler group, i.e. the fundamental group of a closed Kähler manifold (see [1] for context and references). We want to prove that $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ are not Kähler groups.

A Kähler group cannot split over a finite group as an amalgamated free product or HNN extension, it cannot have a subgroup of finite index that has odd first Betti number, and it cannot be an extension of a group with infinitely many ends by a finitely generated group. This last condition covers $\mathrm{Aut}(F_2)$ and $\mathrm{Out}(F_2)$. But if $n \geq 3$ then $\mathrm{Aut}(F_n)$ has property FA and it seems likely (but is unproved) that all of its subgroups of finite index have finite abelianisation. There is, however, a more subtle obstruction coming from Simpson's work on non-abelian Hodge theory that one can use to show that $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ are not Kähler.

Simpson [17] proves that if a group Γ admits a representation $\rho : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ with image $\mathrm{SL}(n, \mathbb{Z})$, where $n \geq 3$, and if this representation cannot be deformed locally into a non-conjugate representation, then Γ is not Kähler. In more detail, if Γ were Kähler then the real Zariski closure of $\rho(\Gamma)$ would be a group of Hodge type [17, Lemma 4.4], but $\mathrm{SL}(n, \mathbb{R})$ is not of Hodge type ($n \geq 3$)—see [17, pp. 50–51]. We noted above that the standard representation of $\mathrm{SAut}(F_n)$ is rigid. And a finite index subgroup of a Kähler group is (obviously) a Kähler group.

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