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L^2-homology and reciprocity for right-angled Coxeter groups

by

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Dedicated to Mike Davis on the occasion of his 60th birthday

Abstract. Let W be a Coxeter group and let μ be an inner product on the group algebra $\mathbb{R}W$. We say that μ is *admissible* if it satisfies the axioms for a Hilbert algebra structure. Any such inner product gives rise to a von Neumann algebra $\mathcal{N}_\mu$ containing $\mathbb{R}W$. Using these algebras and the corresponding von Neumann dimensions we define L^2_μ-Betti numbers and an L^2_μ-Euler charactersitic for W. We show that if the Davis complex for W is a generalized homology manifold, then these Betti numbers satisfy a version of Poincaré duality. For arbitrary Coxeter groups, finding interesting admissible products is difficult; however, if W is right-angled, there are many. We exploit this fact by showing that when W is right-angled, there exists an admissible inner product μ such that the L^2_μ-Euler characteristic is $1/W(\mathbf{t})$ where $W(\mathbf{t})$ is the growth series corresponding to a certain normal form for W. We then show that a reciprocity formula for this growth series that was recently discovered by the second author is a consequence of Poincaré duality.

1. Introduction. A systematic study of the L^2-(co)homology of Coxeter groups was started by Davis and Okun [6] in an effort to prove the Singer Conjecture for right-angled Coxeter groups. Subsequent work of Dymara and Januszkiewicz computing the L^2-cohomology of buildings [9] then led to a "weighted" L^2-theory for Coxeter groups obtained by deforming multiplication in the group algebra according to a (multi)parameter $\mathbf{q}$ which encoded the thickness of the corresponding building [8]. This work was further developed by all four authors in [5] where various vanishing and decomposition theorems were established for sufficiently small (or large) values of the parameter $\mathbf{q}$.

What it means for the parameter $\mathbf{q}$ to be sufficiently small or large is given in terms of the region of convergence of a certain power series in $\mathbf{q}$, called the *growth series*, associated to the Coxeter group. Given a Coxeter

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group W, we let $W(\mathbf{q})$ denote the corresponding growth series. It turns out that for values of $\mathbf{q}$ for which the growth series converges, the weighted L^2-cohomology of W is concentrated in dimension 0 and the corresponding weighted L^2-Betti number is $b^0_{\mathbf{q}} = 1/W(\mathbf{q})$. In fact, the growth series is known to be a rational function of $\mathbf{q}$, and for arbitrary $\mathbf{q}$, one obtains a formula for the weighted L^2-Euler characteristic

$$\chi^{(2)}_{\mathbf{q}}(W) = \frac{1}{W(\mathbf{q})}$$

(see [8]). Thus, the growth series of a Coxeter group can be expressed geometrically in terms of weighted L^2-cohomology. It was further observed by Dymara [8] that the reciprocity formula

$$W(\mathbf{q}^{-1}) = \pm W(\mathbf{q}),$$

first proved by Serre [16] for affine Coxeter groups and generalized to Coxeter groups with Eulerian nerve (see Section 7) by Charney and Davis [3], actually follows from a version of Poincaré duality for weighted L^2-cohomology.

More recently Scott [15] introduced a more general growth series, called the *greedy growth series*. This power series depends on a different multiparameter $\mathbf{t}$ and is defined in terms of elements of greatest length in finite parabolic subgroups of W (see Section 11). This power series, which we denote by $W(\mathbf{t})$, specializes to the usual growth series $W(\mathbf{q})$ with an appropriate substitution of variables. It was shown by Scott [15] that $W(\mathbf{t})$ is also a rational function, and in the case that W is right-angled and has Eulerian nerve, one has the reciprocity formula

$$W(\mathbf{t}^{-1}) = \pm W(\mathbf{t}).$$

The goal of the present paper is to describe an L^2-(co)homology theory for right-angled Coxeter groups that provides a geometric context for this greedy growth series and the corresponding reciprocity formula.

The basic technical tool that allows one to obtain nonintegral Betti numbers and Euler characteristics is the theory of von Neumann algebras and von Neumann dimensions. Roughly speaking, in the nonweighted L^2-theory, one uses the standard inner product on the group algebra $\mathbb{R}W$ to complete the action of $\mathbb{R}W$ on itself. The resulting operator algebra $\mathcal{N}$ is called the *von Neumann algebra* of W. As in the case of equivariant homology, one then considers a cell complex X on which W acts cocompactly with finite cell stabilizers (the standard choice is the Davis complex Σ described in Section 2, below). The space of ordinary chains (or compactly supported cochains) on X can then be completed with respect to the standard inner product, resulting in a (co)chain complex consisting of $\mathcal{N}$-modules. Such modules are called *Hilbert modules* and have a well-defined dimension, called the *von Neumann* *dimension*, obtained by taking the trace of a suitable projection. One defines L^2-Betti numbers as the dimensions of the corresponding (co)homology groups and the L^2-Euler characteristic as the alternating sum of these Betti numbers.

In the weighted version, one deforms both the algebra $\mathbb{R}W$ and the inner product with respect to the parameter $\mathbf{q}$. The deformed algebra $\mathbb{R}_{\mathbf{q}}W$ coincides with the classical Hecke algebra, and the inner product is orthogonal (but not orthonormal) with respect to the standard basis. One then obtains a family of von Neumann algebras $\mathcal{N}_{\mathbf{q}}$ indexed by $\mathbf{q}$, and a resulting weighted $L^2_{\mathbf{q}}$-(co)homology theory.

In the event that the Coxeter group W is right-angled, all of the Hecke algebras $\mathbb{R}_{\mathbf{q}}W$ (for different values of $\mathbf{q}$) are canonically isomorphic, hence can all be identified with the group algebra $\mathbb{R}W$. This allows one to interpret the $\mathbf{q}$-deformation described above not as a simultaneous deformation of the group algebra and the inner product, but rather as a deformation *only of the inner product* on the fixed group algebra $\mathbb{R}W$. Moreover, in the right-angled setting, the group algebra $\mathbb{R}W$ has many natural automorphisms, allowing one to move any given inner-product to many others. By taking convex combinations, one then obtains a large family of inner products on $\mathbb{R}W$ with respect to which one can form a von Neumann algebra.

We adopt this perspective of fixing the algebra $\mathbb{R}W$ and varying the inner product throughout the paper. We call any such inner product μ *admissible*, and we let $\mathcal{N}_\mu$ denote the resulting von Neumann algebra. We refer to the resulting homology theory as L^2_μ-homology. We let b^i_μ denote the corresponding L^2_μ-Betti numbers and χ_μ the L^2_μ-Euler characteristic of W.

For any Coxeter group W, mapping each generator $s \in S$ to its negative $-s$ induces an automorphism of the group algebra $\mathbb{R}W$. Given an admissible inner product μ, we let μ^* denote the (admissible) inner product obtained by pulling back μ via this automorphism. With the assumption that the Davis complex Σ is a generalized homology n-manifold, we prove (in Theorem 8.1) a version of Poincaré duality:

$$b^i_\mu = b^{n-i}_{\mu^*}.$$

As a consequence, we obtain the formula

$$\chi_\mu = (-1)^n \chi_{\mu^*}.$$

In fact, we prove that this last formula holds more generally whenever the Coxeter group W has Eulerian nerve.

Although the calculations above hold for any Coxeter group, finding interesting admissible inner products in this generality is challenging. However, when one restricts to right-angled Coxeter groups, the isomorphism with Hecke algebras and the abundance of automorphisms gives rise to a large family of admissible inner products. The main result of the paper is the following theorem, which establishes the desired connection between the L^2_μ-Euler characteristic and the greedy growth series $W(\mathbf{t})$ when W is right-angled.

THEOREM. *Let W be a right-angled Coxeter group with greedy growth series $W(\mathbf{t})$. Then if $\mathbf{t}$ is in the region of convergence of $W(\mathbf{t})$, there exists an admissible inner product μ such that $\chi_\mu = 1/W(\mathbf{t})$. If, in addition, the nerve of W is Eulerian, then $\chi_{\mu^*} = 1/W(\mathbf{t}^{-1})$.*

(The first claim of this theorem is Theorem 11.3 below, and the second is the displayed equation (12.3) at the end of the paper.) Applying this last result to the aforementioned Euler characterstic equation, we obtain the desired reciprocity formula

$$W(\mathbf{t}^{-1}) = (-1)^n W(\mathbf{t})$$

for the greedy growth series.

2. Coxeter groups. In this section, we recall some standard facts about Coxeter groups and refer the reader to any standard reference for proofs and further details (see, e.g., [1, 2, 4, 12]).

Let W be a Coxeter group with standard generating set S. In particular, W admits a presentation of the form

$$W = \langle s \in S \mid (st)^{m(s,t)} = 1 \text{ for all } s, t \in S \rangle$$

where $m(s,t)$ are integers (or ∞) satisfying $m(s,t) = m(t,s) \geq 2$ for $s \neq t$ and $m(s,s) = 1$. The group W is called *right-angled* if $m(s,t) \in \{2, \infty\}$ for all $s \neq t$.

Let $l : W \to \mathbb{Z}_{\geq 0}$ denote the length function, defined with respect to S. Any subset $T \subset S$ generates a *parabolic subgroup* of W, which we denote by W_T. By convention, we set $W_\emptyset = \{1\}$. Parabolic subgroups are themselves Coxeter groups whose length function is the restriction of l to W_T. A subset $T \subset S$ is called *spherical* if W_T is finite, and we let N denote the collection of all spherical subsets of S. The set N, called the *nerve* of W, forms an abstract simplicial complex on the vertex set S. We shall often denote spherical subsets by $\tau, \sigma, \ldots$ as a reminder that they represent simplices in N.

Let Δ be a simplex whose codimension-1 faces are indexed by S. Then each $\sigma \in N$ corresponds to a codimension-$|\sigma|$ face of Δ, which we denote by Δ_σ, and we let v_σ denote the barycenter of Δ_σ. The *Davis chamber* is the subcomplex of the barycentric subdivision of Δ that is spanned by the vertices v_σ, $\sigma \in N$. We let D_σ denote the subcomplex $D_\sigma = D \cap \Delta_\sigma$. The *Davis complex* Σ for (W,S) is then defined to be $\Sigma = W \times D/{\sim}$ where $(w,p) \sim (u,q)$ whenever there exists a $\sigma \in N$ such that $p = q \in D_\sigma$ and

$w^{-1}u \in W_\sigma$. The simplicial decomposition of D induces a triangulation of Σ, and the group W acts on Σ (on the left) preserving this triangulation.

The Davis complex also admits a cell decomposition into "Coxeter cells". For each $\sigma \in N$, we let c_σ denote the union of all simplices $c \subset \Sigma$ such that $c \cap D_\sigma = v_\sigma$. The boundary of c_σ is cellulated by wc_τ where $w \in W_\sigma$ and $\sigma \subset \tau \in N$. As a simplicial complex, this boundary is the Coxeter complex for the finite Coxeter group (W_σ, σ) which is a sphere. Hence c_σ and its W-translates are disks, which we call *Coxeter cells*. We denote this decomposition into Coxeter cells by Σ_{cc} to distinguish it from the triangulated Davis complex. Σ_{cc} is a regular cell complex whose poset of cells can be identified with the set of cosets $WN = \{wW_\sigma \mid w \in W,\ \sigma \in N\}$ partially ordered by inclusion. The simplicial structure on Σ coincides with the geometric realization of this poset, hence Σ is the barycentric subdivision of Σ_{cc}.

In the event that the Davis chamber is a generalized homology disk (so Σ is a generalized homology manifold), we obtain another decomposition of Σ into homology cells. Recall that $(D, \partial D)$ is a generalized homology n-disk if it is a homology manifold with boundary and its relative homology groups are the same as the homology of an n-disk relative to its boundary. In this case, each D_σ is also a homology $(n-|\sigma|)$-disk, and the W-translates of the D_σ's form a homology cell structure on Σ, which we denote by Σ_{ghd} (as in [8]). It follows that Σ_{ghd} and Σ_{cc} have the simplicial complex Σ as their common barycentric subdivision.

3. Hilbert algebras and Hilbert modules. In this section we work over the group algebra $\mathbb{R}W$ of an arbitrary Coxeter group W. We let $x \mapsto x^*$ denote the linear involution on $\mathbb{R}W$ induced by $w \mapsto w^{-1}$.

DEFINITION 3.1. Let $\mu = \langle\,,\rangle$ be a positive definite inner product on $\mathbb{R}W$. We say that μ is *admissible* if, together with the involution $*$, it satisfies the axioms for a Hilbert algebra structure in the sense of [7, A.54]. These axioms are

(i) $(xy)^* = y^*x^*$,
(ii) $\langle x, y\rangle = \langle y^*, x^*\rangle$,
(iii) $\langle xy, z\rangle = \langle y, x^*z\rangle$,
(iv) the map $y \mapsto xy : \mathbb{R}W \to \mathbb{R}W$ is continuous for every x, and
(v) the set $\{xy \mid x, y \in \mathbb{R}W\}$ is dense in $\mathbb{R}W$.

The first axiom is immediate from our definition of $*$, the last axiom follows since the algebra $\mathbb{R}W$ has a unit element 1. Axioms (ii) and (iii) ensure that $*$ corresponds to taking adjoints for left and right multiplication. Axiom (iv) ensures that left and right multiplication by any element of the algebra are bounded operators.

The standard inner product on $\mathbb{R}W$ is clearly admissible. In Section 10 we will show that if W is right-angled then there are many admissible inner products on $\mathbb{R}W$ (arising from natural Hilbert algebra structures on Hecke algebras). Any inner product $\mu = \langle\, , \rangle$ on $\mathbb{R}W$ has a *dual* inner product defined as follows. Let $\overline{j} : \mathbb{R}W \to \mathbb{R}W$ denote the algebra involution that maps each $s \in S$ to $-s$. This involution commutes with the involution $*$. Given any inner product $\mu = \langle\, , \rangle$ on $\mathbb{R}W$, we define its *dual* $\mu^* = \langle\, , \rangle^*$ by

$$\langle x, y\rangle^* = \langle \overline{j}(x), \overline{j}(y)\rangle.$$

Note that with respect to the standard basis for $\mathbb{R}W$, we have

$$\langle v, w\rangle = (-1)^{l(w)+l(v)} \langle v, w\rangle^*$$

for all $v, w \in W$. A routine computation shows that μ^* is admissible if and only if μ is admissible.

Let $\mu = \langle\, , \rangle$ be an admissible inner product on $\mathbb{R}W$, and let L^2_μ denote the Hilbert space completion of $\mathbb{R}W$ with respect to μ. As described in [7], one obtains a von Neumann algebra $\mathcal{N}_\mu$ which acts on the left of L^2_μ by taking all bounded linear endomorphisms that commute with the right $\mathbb{R}W$-action on L^2_μ. Similarly, one obtains another von Neumann algebra acting on the right of L^2_μ. As in [5], we use the same notation for both algebras and rely on the context to distinguish between them.

We define an $\mathbb{R}$-linear trace $\mathrm{tr}_\mu : \mathcal{N}_\mu \to \mathbb{R}$ by

$$\mathrm{tr}_\mu(\phi) = \langle 1 \cdot \phi, 1\rangle$$

where ϕ is any element of the von Neumann algebra acting from the right on L^2_μ. In general, given any bounded linear map $\Phi : \bigoplus_{i=1}^n L^2_\mu \to \bigoplus_{i=1}^n L^2_\mu$ of left $\mathbb{R}W$-modules, we can represent it as right multiplication by an $n \times n$ matrix (ϕ_{ij}) with entries in $\mathcal{N}_\mu$. We then define the trace of Φ to be

$$\mathrm{tr}_\mu(\Phi) = \sum_{i=1}^n \mathrm{tr}_\mu(\phi_{ii}).$$

By definition, a *Hilbert $\mathcal{N}_\mu$-module* is any closed subspace V of $(L^2_\mu)^n$ (an orthogonal direct sum) that is stable under the diagonal left action of $\mathbb{R}W$. Given a Hilbert module $V \subseteq (L^2_\mu)^n$, we let $p_V : (L^2_\mu)^n \to (L^2_\mu)^n$ denote the orthogonal projection onto V, and we define the corresponding von Neumann dimension of V to be

$$\dim_\mu V = \mathrm{tr}_\mu(p_V).$$

A standard argument shows that $\dim_\mu V$ is independent of the choice of embedding of V.

A *map* of Hilbert $\mathcal{N}_\mu$-modules U and V is any bounded linear map $f : U \to V$ that commutes with the left $\mathbb{R}W$-actions. The following lemma provides our main source of $\mathcal{N}_\mu$-module maps.

LEMMA 3.2. *Let $\Phi : \mathbb{R}W^m \to \mathbb{R}W^n$ be a morphism of left $\mathbb{R}W$-modules. Then the induced map $\overline{\Phi} : (L^2_\mu)^m \to (L^2_\mu)^n$ is a map of $\mathcal{N}_\mu$-modules. More generally, suppose $U \subseteq \mathbb{R}W^m$ and $V \subseteq \mathbb{R}W^n$ are $\mathbb{R}W$-invariant subspaces with closures $L^2_\mu U \subseteq (L^2_\mu)^m$ and $L^2_\mu V \subseteq (L^2_\mu)^n$, respectively. If U is projective (i.e., a direct summand of $\mathbb{R}W^m$), then any $\mathbb{R}W$-morphism $\Phi : U \to V$ induces a map of $\mathcal{N}_\mu$-modules $\overline{\Phi} : L^2_\mu U \to L^2_\mu V$.*

Proof. The first statement follows from the fact that Φ can be written as right multiplication by an $m \times n$ matrix with entries in $\mathbb{R}W$ and that right multiplication by elements of $\mathbb{R}W$ is bounded (this follows from the axioms for an admissible inner product). See [10, Lemma 2.2.1] for details. For the second statement, write $\mathbb{R}W^m = U \oplus U'$ and extend the map Φ by zero to get a map $\widetilde{\Phi} : \mathbb{R}W^m \to \mathbb{R}W^n$. Then apply the first statement. ∎

4. L^2_μ-homology of W. Let W be a Coxeter group, and let μ be an admissible inner product on $\mathbb{R}W$. Let X be a cell complex on which W acts with finite stabilizers. By fixing an orientation for each cell in X, we identify ordinary chains (over $\mathbb{R}$) with compactly supported cochains,

$$C_*(\Sigma_{cc}) = C^*_c(\Sigma_{cc}).$$

These inherit the structure of (left) $\mathbb{R}W$-modules from the W-action on X.

We let $X^{(i)}$ denote the set of i-cells in X, and using our choice of orientations, we regard $X^{(i)}$ as a subset of (in fact, a basis for) $C_i(X)$. For any $c \in X^{(i)}$, we let $W_c \subseteq W$ denote the stabilizer, and we let $\deg : W_c \to \{-1, +1\}$ denote the degree homomorphism induced by the action of W_c on the orientations of c. For each cell c, define $p_c \in \mathbb{R}W$ by

$$p_c = \frac{1}{|W_c|} \sum_{w \in W_c} \deg(w) w.$$

LEMMA 4.1. *The element p_c is a self-adjoint (with respect to μ) idempotent.*

Proof. Since deg is a homomorphism to $\{-1, +1\}$,

$$p_c^2 = \frac{1}{|W_c|^2} \sum_{v,w \in W_c} \deg(v)\deg(w) vw = \frac{|W_c|}{|W_c|^2} \sum_{w \in W_c} \deg(w) w = p_c.$$

Since $w^* = w^{-1}$ and $\deg(w^{-1}) = \deg(w)$, we get $p_c^* = p_c$. ∎

From each W-orbit of i-cells, choose a basis cell c. This gives an $\mathbb{R}W$-module direct sum decomposition

$$C_i(X) = \bigoplus_c C_i(Wc). \tag{4.1}$$

For each summand, we define an embedding $\phi_c : C_i(Wc) \to L^2_\mu$ by mapping c to $\sqrt{|W_c|}\, p_c$ and extending left-equivariantly. We pull back the inner product μ to $C_i(Wc)$, and let $L^2_\mu C_i(Wc)$ denote the Hilbert space completion of $C_i(Wc)$. Thus, ϕ_c extends to an isomorphism of $\mathcal{N}_\mu$-modules

$$\phi_c : L^2_\mu C_i(Wc) \to L^2_\mu p_c.$$

Doing this for each of the summands in (4.1) and declaring the summands to be orthogonal, we let $L^2_\mu C_i(X)$ denote the Hilbert space completion of $C_i(X)$. We then obtain an isomorphism of $\mathcal{N}_\mu$-modules

$$\varPhi := \bigoplus_c \phi_c : L^2_\mu C_i(X) \to \bigoplus_c L^2_\mu p_c \subseteq \bigoplus_c L^2_\mu.$$

Let $\partial : C_i(X) \to C_{i-1}(X)$ and $\delta : C_i(X) \to C_{i+1}(X)$ denote the ordinary boundary and coboundary maps. More precisely, with respect to the basis $X^{(i)}$ for $C_i(X)$, we have

$$\partial(c) = \sum_{d \in X^{(i-1)}} [c:d]d \quad \text{and} \quad \delta(c) = \sum_{d \in X^{(i+1)}} [d:c]d$$

where $[c:d]$ denotes the incidence number for the two cells. Both are maps of (projective) $\mathbb{R}W$-modules, hence by Lemma 3.2 induce maps of $\mathcal{N}_\mu$-modules

$$\partial : L^2_\mu C_i(X) \to L^2_\mu C_{i-1}(X), \qquad \delta : L^2_\mu C_i(X) \to L^2_\mu C_{i+1}(X).$$

We define the (reduced) L^2_μ-(co)homology of X by

$$L^2_\mu H_i(X) := \frac{\ker(\partial : L^2_\mu C_i(X) \to L^2_\mu C_{i-1}(X))}{\overline{\mathrm{Im}}(\partial : L^2_\mu C_{i+1}(X) \to L^2_\mu C_i(X))},$$
$$L^2_\mu H^i(X) := \frac{\ker(\delta : L^2_\mu C_i(X) \to L^2_\mu C_{i+1}(X))}{\overline{\mathrm{Im}}(\delta : L^2_\mu C_{i-1}(X) \to L^2_\mu C_i(X))}$$

where $\overline{\mathrm{Im}}$ denotes the closure of the image.

Lemma 4.2. *The maps ∂ and δ are adjoints.*

Proof. Let c be a basic i-cell and d a basic $(i-1)$-cell. With respect to the sum decomposition (4.1), the matrix entry of ∂ corresponding to the orbits cW and dW is

$$\partial_{c,d} = \sum_{e \in W_d} [c:e]e = \frac{1}{|W_d|} \sum_{w \in W} [c:wd]wd.$$

With respect to the isomorphisms ϕ_c and ϕ_d, we have

$$\partial_{c,d}(p_c) = \partial_{c,d}(c/\sqrt{|W_c|}) = \frac{1}{\sqrt{|W_c|\,|W_d|}} \sum_{w \in W} [c:wd]wp_d.$$

Since p_c is idempotent, we also have

$$\partial_{c,d}(p_c) = \partial_{c,d}(p_c^2) = p_c\partial_{c,d}(p_c) = \frac{1}{\sqrt{|W_c|\,|W_d|}} \sum_{w \in W} [c:wd]p_c wp_d.$$

Thus, as a map $L^2_\mu \to L^2_\mu$, $\partial_{c,d}$ is given by right multiplication by

$$\frac{1}{\sqrt{|W_c|\,|W_d|}} \sum_{w\in W} [c : wd] p_c w p_d.$$

Similarly, the matrix element $\delta_{d,c}$ is given by

$$\frac{1}{\sqrt{|W_c|\,|W_d|}} \sum_{w\in W} [wc : d] p_d w p_c.$$

Since $[gc : d] = [c : g^{-1}d]$, $g^* = g^{-1}$, and p_c and p_d are self-adjoint, we see that $\partial_{c,d}$ and $\delta_{d,c}$ are adjoints. Since different orbits are orthogonal, it follows that δ and ∂ are also adjoints. ∎

It follows from Lemma 4.2 that we have a Hodge decomposition

$$L^2_\mu C_i(X) = (\ker \partial_i \cap \ker \delta_{i+1}) \oplus \overline{\operatorname{Im}}\, \partial_{i+1} \oplus \overline{\operatorname{Im}}\, \delta_i.$$

In particular, $L^2_\mu H_i(X)$ and $L^2_\mu H^i(X)$ can both be identified with the space of "harmonic chains", $\ker \partial_i \cap \ker \delta_{i+1}$.

Since ∂ and δ are both $\mathcal{N}_\mu$-module maps, $L^2_\mu H_i(X)$ is an $\mathcal{N}_\mu$-module. We define the ith *L^2_μ-Betti number* of X to be

$$b^i_\mu(X) := \dim_\mu L^2_\mu H_i(X) = \dim_\mu L^2_\mu H^i(X).$$

Since the von Neumann dimension is additive with respect to direct sums, we can then define the L^2_μ-Euler characteristic by either of the formulas

$$\chi_\mu(X) := \sum_i (-1)^i b^i_\mu(X), \qquad \chi_\mu(X) := \sum_i (-1)^i \dim_\mu L^2_\mu C_i(X).$$

The following theorem establishes the topological invariance of L^2_μ-Betti numbers.

THEOREM 4.3. *If X is $\mathbb{R}$-acyclic, then the Betti numbers $b^\mu_i(X)$ and Euler characteristic $\chi_\mu(X)$ depend only on W and μ (not on X).*

Proof. The pair $(\mathbb{R}W, \tau)$, where $\tau : \mathbb{R}W \to \mathbb{R}$ is the functional given by $\tau(x) = \langle 1 \cdot x, 1\rangle$, is a traced $*$-algebra in the sense of Paschke [14]. Since X is acyclic, the augmented chain complex

$$\cdots \to C_2(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \to \mathbb{R} \to 0$$

gives a finitely generated projective resolution of the trivial $\mathbb{R}W$-module $\mathbb{R}$. The invariance of $b^\mu_i(X)$ and, hence, $\chi_\mu(X)$ then follow from the standard chain homotopy argument and Lemma 3.2 (see, e.g., Theorem 2.2 in [14]). ∎

REMARK 4.4. Our perspective differs from that of Dymara and coauthors in [4, 5, 8]. Because they work with von Neumann completions of the Hecke algebra (rather than the group algebra), the ordinary boundary map is not a morphism of Hilbert modules. Thus, they use the ordinary coboundary map δ (which is a module map) and its adjoint $\partial^{\mathfrak{q}}$ to define weighted L^2-cohomology and homology, respectively.

5. Standard Hilbert modules. In this section we describe the basic Hilbert modules that will play a role in the rest of the paper. Let W be a Coxeter group with nerve N, and let $\mu = \langle\, ,\rangle$ be an admissible inner product on $\mathbb{R}W$. For each $\sigma \in N$, we let a_σ and h_σ denote the usual "averaging" and "alternating" idempotents in $\mathbb{R}W$:

$$a_\sigma = \frac{1}{|W_\sigma|}\sum_{w\in W_\sigma} w \quad \text{and} \quad h_\sigma = \frac{1}{|W_\sigma|}\sum_{w\in W_\sigma} (-1)^{l(w)} w.$$

For any $\sigma \in N$, we let W^σ denote the set of minimal length coset representatives for W/W_σ. Equivalently, W^σ is the set of σ-*reduced* elements:

$$W^\sigma = \{w \in W \mid l(ws) > l(w) \text{ for all } s \in \sigma\}.$$

To simplify notation, for $\tau \subseteq \sigma$, we let W_σ^τ denote the set $(W_\sigma)^\tau$ consisting of minimal coset representatives for W_σ/W_τ. The following properties of the idempotents are routine computations.

LEMMA 5.1. *For any $\sigma \in N$ and $\tau \subseteq \sigma$, we have*

(1) $a_\sigma a_\tau = a_\sigma = a_\tau a_\sigma$,
(2) $h_\sigma h_\tau = h_\sigma = h_\tau h_\sigma$,
(3) $a_\sigma = p_\sigma^\tau a_\tau$ *where* $p_\sigma^\tau = \dfrac{|W_\tau|}{|W_\sigma|}\displaystyle\sum_{v\in W_\sigma^\tau} v$,
(4) $h_\sigma = q_\sigma^\tau h_\tau$ *where* $q_\sigma^\tau = \dfrac{|W_\tau|}{|W_\sigma|}\displaystyle\sum_{v\in W_\sigma^\tau} (-1)^{l(v)} v$.

Multiplying $\mathbb{R}W$ on the right by (the idempotents) a_σ and h_σ, we obtain (projective) left $\mathbb{R}W$-submodules:

$$\mathbb{R}Wa_\sigma \subseteq \mathbb{R}W, \qquad \mathbb{R}Wh_\sigma \subseteq \mathbb{R}W.$$

Taking the closures in L^2_μ, we obtain the $\mathcal{N}_\mu$-modules

$$A_\sigma := L^2_\mu a_\sigma \quad \text{and} \quad H_\sigma := L^2_\mu h_\sigma.$$

Since the subgroups W_σ are closed under inversion and $l(w) = l(w^{-1})$, the elements a_σ and h_σ satisfy $a_\sigma^* = a_\sigma$ and $h_\sigma^* = h_\sigma$. In other words, they are self-adjoint idempotents, hence right multiplication by them defines an orthogonal projection $L^2_\mu \to L^2_\mu$ onto their respective images. It follows that

$$\dim_\mu H_\sigma = \langle h_\sigma, 1\rangle \quad \text{and} \quad \dim_\mu A_\sigma = \langle a_\sigma, 1\rangle.$$

When σ is the singleton set $\{s\}$, we denote a_σ by a_s (likewise for h_σ, A_σ, and H_σ). For a subspace $E \subseteq L^2_\mu$, we let $E^\perp$ denote its orthogonal complement.

LEMMA 5.2. *For all $\sigma \in N$, we have*
$$A_\sigma = \bigcap_{s\in\sigma} A_s = \Big(\sum_{s\in\sigma} H_s\Big)^{\perp} \quad \text{and} \quad H_\sigma = \bigcap_{s\in\sigma} H_s = \Big(\sum_{s\in\sigma} A_s\Big)^{\perp}$$

Proof. The arguments are the same as those for Lemmas 19.2.11–19.2.13 in [4]. ∎

As a corollary, using inclusion-exclusion and the additivity of the von Neumann dimension with respect to direct sums, we obtain the following.

LEMMA 5.3. *For all $\sigma \in N$, we have*
$$\dim_\mu H_\sigma = \sum_{\tau\subseteq\sigma} (-1)^{|\tau|} \dim_\mu A_\tau \quad \text{and} \quad \dim_\mu A_\sigma = \sum_{\tau\subseteq\sigma} (-1)^{|\tau|} \dim_\mu H_\tau.$$

REMARK 5.4. If W is right-angled, then the idempotents a_σ and h_σ factor as
$$a_\sigma = \frac{1}{2^{|\sigma|}} \prod_{s\in\sigma}(1+s) = \prod_{s\in\sigma} a_s \quad \text{and} \quad h_\sigma = \frac{1}{2^{|\sigma|}} \prod_{s\in\sigma}(1-s) = \prod_{s\in\sigma} h_s.$$
In this case, the formulas in Lemma 5.3 actually hold in the group ring $\mathbb{R}W$, i.e.,
$$h_\sigma = \sum_{\tau\subseteq\sigma} (-1)^{|\tau|} a_\tau \quad \text{and} \quad a_\sigma = \sum_{\tau\subseteq\sigma} (-1)^{|\tau|} h_\tau.$$

The $\overline{\jmath}$-involution. Let $\overline{\jmath} : \mathbb{R}W \to \mathbb{R}W$ denote the algebra involution that maps each $s \in S$ to $-s$. It follows from the definition of the dual inner product μ^* that $\overline{\jmath}$ defines an isometric isomorphism from $(\mathbb{R}W, \mu)$ to $(\mathbb{R}W, \mu^*)$, hence induces an isomorphism of Hilbert spaces
$$\overline{\jmath} : L^2_\mu \to L^2_{\mu^*}.$$
Since $\overline{\jmath}$ takes bounded elements to bounded elements, it also extends to an isomorphism of von Neumann algebras
$$\overline{\jmath} : \mathcal{N}_\mu \to \mathcal{N}_{\mu^*}.$$
In other words, $\overline{\jmath}$ defines a $\overline{\jmath}$-equivariant isomorphism between the Hilbert $\mathcal{N}_\mu$-module L^2_μ and the Hilbert $\mathcal{N}_{\mu^*}$-module $L^2_{\mu^*}$.

Applying $\overline{\jmath}$ to the definitions of a_σ and h_σ, we obtain $\overline{\jmath}(a_\sigma) = h_\sigma$ and $\overline{\jmath}(h_\sigma) = a_\sigma$. This implies the following.

LEMMA 5.5. *For all $\sigma \in N$, the isometry $\overline{\jmath} : L^2_\mu \to L^2_{\mu^*}$ restricts to $\overline{\jmath}$-equivariant isomorphisms*
$$\overline{\jmath} : A_\sigma \to H_\sigma \quad \text{and} \quad \overline{\jmath} : H_\sigma \to A_\sigma$$
from $\mathcal{N}_\mu$-modules to $\mathcal{N}_{\mu^*}$-modules. In particular,
$$\dim_\mu A_\sigma = \dim_{\mu^*} H_\sigma \quad \text{and} \quad \dim_\mu H_\sigma = \dim_{\mu^*} A_\sigma.$$

6. The cell complex Σ_{cc}. Let W be a Coxeter group with nerve N. In this section, we consider the L^2_μ-homology of W using the cell decomposition Σ_{cc}. Recall from Section 2 that Σ_{cc} consists of all W-translates of the special cells c_σ, $\sigma \in N$. Moreover, each such cell wc_σ can be identified with the coset wW_σ in a face-preserving fashion. Fix an orientation for each c_σ arbitrarily. For any other cell wc_σ, let u be the minimal coset representative for wW_σ, and assign the orientation to wc_σ so that left multiplication by u maps the oriented cell c_σ to wc_σ in an orientation-preserving manner ([1]).

Let $N^{(i)}$ denote the set of $(i-1)$-simplices in the nerve N. Since each $\sigma \in N^{(i)}$ corresponds to a unique W-orbit Wc_σ, we have the $\mathbb{R}W$-module direct sum decomposition

$$C_i(\Sigma_{cc}) = \bigoplus_{\sigma \in N^{(i)}} C_i(Wc_\sigma). \tag{6.1}$$

As a vector space, the summand $C_i(Wc_\sigma)$ has a basis consisting of the oriented cells $\{uc_\sigma \mid u \in W^\sigma\}$. For each summand, we let ψ_σ denote the embedding

$$\psi_\sigma := \phi_{c_\sigma} : C_i(Wc_\sigma) \to L^2_\mu.$$

It is given by $wc_\sigma \mapsto \sqrt{|W_\sigma|}\, uh_\sigma$ where u is the minimal coset representative for wW_σ. Thus, $L^2_\mu C_i(Wc_\sigma)$ gets identified with the standard module $H_\sigma = L^2_\mu h_\sigma$, giving us the $\mathcal{N}_\mu$-module isomorphism

$$L^2_\mu C_i(\Sigma_{cc}) = \bigoplus_{\sigma \in N^{(i)}} H_\sigma. \tag{6.2}$$

In light of this decomposition, we have the following formula for χ_μ.

PROPOSITION 6.1. *The Euler characteristic χ_μ is given by the formula*

$$\chi_\mu = \sum_{\sigma \in N} (-1)^{|\sigma|} \dim_\mu H_\sigma.$$

7. Eulerian nerves. The existence of the homology cell decomposition Σ_{ghd} relies on the Davis chamber D being a generalized homology disk. A weaker version of this condition is to require that the nerve N resemble a sphere "up to Euler characteristics". Although imposing this condition on W is not enough to ensure Poincaré duality for L^2_μ-homology, it does suffice to prove a duality formula for the L^2_μ-Euler characteristic.

DEFINITION 7.1. Let N be an abstract simplicial complex on the vertex set S. Then N is an *Eulerian $(n-1)$-sphere* if for every simplex $\sigma \in N$, we

([1]) As a word of caution, the oriented cell wc_σ need not coincide with the oriented cell obtained by translating c_σ by w. For example, if $s \in \sigma$, then $sc_\sigma = c_\sigma$ as an oriented cell, but $s(c_\sigma) = -c_\sigma$ since s is a reflection that reverses the orientation of c_σ.

have

$$\sum_{\tau \supseteq \sigma} (-1)^{|\tau|} = (-1)^n.$$

Note that for $\sigma = \emptyset$, the condition says that the Euler characteristic of the geometric realization of N is the same as that of an $(n-1)$-sphere; and more generally, the link of any k-simplex σ in N has the same Euler charactersitic as the link of a k-simplex in an $(n-1)$-sphere. In particular, if the Davis chamber D is a generalized homology disk of dimension n, then the nerve N is an Eulerian $(n-1)$-sphere. Under the assumption that N be Eulerian, we obtain the following alternative formula for the L^2_μ-Euler characteristic of W.

Proposition 7.2. *Let W be any Coxeter group whose nerve N is an Eulerian $(n-1)$-sphere. Then for any admissible inner product μ, we have*

$$\chi_\mu = (-1)^n \sum_{\sigma \in N} (-1)^{|\sigma|} \dim_\mu A_\sigma.$$

Proof. We have

$$\begin{aligned}
\chi_\mu &= \sum_{\sigma \in N} (-1)^{|\sigma|} \dim_\mu H_\sigma \\
&= \sum_{\sigma \in N} (-1)^{|\sigma|} \sum_{\tau \subseteq \sigma} (-1)^{|\tau|} \dim_\mu A_\tau && \text{by Lemma 5.3} \\
&= \sum_{\tau \in N} (-1)^{|\tau|} \dim_\mu A_\tau \sum_{\sigma \supseteq \tau} (-1)^{|\sigma|} && \text{changing the order of summation} \\
&= \sum_{\tau \in N} (-1)^{|\tau|} \dim_\mu A_\tau (-1)^n && \text{since } N \text{ is Eulerian.} \ \blacksquare
\end{aligned}$$

Combining Lemma 5.5 and Propositions 6.1 and 7.2, we obtain the Euler characterstic equation.

Theorem 7.3. *Let W be a Coxeter group and assume that the nerve of W is an Eulerian $(n-1)$-sphere. Then for any admissible inner product on $\mathbb{R}W$, we have*

$$\chi_\mu = (-1)^n \chi_{\mu^*}.$$

8. The dual cell complex and Poincaré duality. Assume now that the Davis chamber D is a generalized homology n-disk and Σ_{ghd} is the corresponding homology cell structure. In this case, Σ is the common barycentric subdivision of both Σ_{cc} and Σ_{ghd} and is a contractible (hence, orientable) homology n-manifold. Fix an orientation for Σ. For each $\sigma \in N^{(i)}$, the face $D_\sigma \subseteq D$ is an $(n-i)$-dimensional homology disk. Each such D_σ intersects the i-cell c_σ transversely, and we choose the orientation for D_σ so that the (already) chosen orientation for c_σ followed by the orientation for D_σ yields the orientation for Σ. We then extend these orientations equivariantly to all homology cells wD_σ.

Although we work with a homology cell structure in this case, everything in Section 4 still applies if we let $C_*(\Sigma_{ghd})$ denote the homology chain complex

$$C_i(\Sigma_{ghd}) := H_*(\Sigma_{ghd}^{(i)}, \Sigma_{ghd}^{(i-1)}).$$

In this case, the sum decomposition of $C_i(\Sigma_{ghd})$ is

$$C_i(\Sigma_{ghd}) = \bigoplus_{\sigma \in N^{(n-i)}} C_i(WD_\sigma). \tag{8.1}$$

Each summand $C_i(WD_\sigma)$ has a basis consisting of oriented cells $\{uD_\sigma \mid u \in W^\sigma\}$. For each summand, we let ϕ_σ denote the embedding

$$\phi_\sigma = \phi_{D_\sigma} : C_i(WD_\sigma) \to L^2_\mu.$$

This time, it is given by $\phi_\sigma(wD_\sigma) = \sqrt{|W_\sigma|}\, wa_\sigma$, and hence ϕ_σ identifies $C_i(WD_\sigma)$ with $A_\sigma = L^2_\mu a_\sigma$. The map $\Phi = \bigoplus \phi_\sigma$ then gives the isomorphism of $\mathcal{N}_\mu$-modules

$$\Phi_i : L^2_\mu C_i(\Sigma_{ghd}) = \bigoplus_{\sigma \in N^{(n-i)}} A_\sigma. \tag{8.2}$$

If the Davis complex Σ is a generalized homology manifold, the isomorphisms (8.2) and Theorem 4.3 provide an alternative argument for Proposition 7.2, and hence for the Euler-charactersitic duality

$$\chi_\mu = (-1)^n \chi_{\mu^*}$$

(Theorem 7.3). However, since requiring that Σ be a generalized homology manifold is much stronger than just requiring Eulerian links, one should expect a stronger duality. This is indeed the case.

THEOREM 8.1. *Assume Σ is a generalized homology manifold of dimension n, and μ is any admissible inner product on $\mathbb{R}W$. Then there exists an isometry $\mathcal{D} : L^2_\mu H_i(\Sigma_{cc}) \to L^2_{\mu^*} H^{n-i}(\Sigma_{ghd})$ which is a $\overline{j}$-equivariant isomorphism from an $\mathcal{N}_\mu$-module to an $\mathcal{N}_{\mu^*}$-module. In particular,*

$$b^i_\mu = b^{n-i}_{\mu^*}.$$

Proof. The argument is the same as the one given in Theorem 20.4.2 in [4] (but with the multiparameter $\mathbf{q}$ set equal to $\mathbf{1}$). We first define a map

$$\mathcal{D} : L^2_\mu C_i(\Sigma_{cc}) \to C_{n-i}(\Sigma_{ghd})$$

with respect to the basis $\{uc_\sigma \mid \sigma \in N^{(i)},\ u \in W^\sigma\}$ by

$$\mathcal{D}(uc_\sigma) = (-1)^{l(u)} D_\sigma.$$

It then suffices to show that (1) $\mathcal{D}$ intertwines (up to sign) the boundary operator ∂ on Σ_{cc} with the coboundary operator δ on Σ_{ghd}, and that (2) $\mathcal{D}$ is an isometric $\bar{j}$-equivariant isomorphism of modules.

To establish (1), first note that with our choice of orientations, we have $[c_\sigma : c_\tau] = \pm[D_\tau : D_\sigma]$ where the sign depends only on i and n. Next we calculate

$$\partial(c_\sigma) = \sum_{\tau\subseteq\sigma}[c_\sigma : c_\tau]q_\sigma^\tau c_\tau \quad \text{and} \quad \delta(D_\sigma) = \sum_{\tau\subseteq\sigma}[D_\tau : D_\sigma]p_\sigma^\tau D_\tau$$

where q_σ^τ and p_σ^τ are as in Lemma 5.1. And finally, for $u \in W^\sigma$, we compute

$$\begin{aligned}\mathcal{D}(\partial(uc_\sigma)) &= \mathcal{D}(u\partial(c_\sigma)) = \mathcal{D}\Big(u\sum_{\tau\subseteq\sigma}[c_\sigma : c_\tau]q_\sigma^\tau c_\tau\Big)\\ &= \sum_{\tau\subseteq\sigma}[c_\sigma : c_\tau]\mathcal{D}(uq_\sigma^\tau c_\tau) = \sum_{\tau\subseteq\sigma}[c_\sigma : c_\tau](-1)^{l(u)}up_\sigma^\tau D_\tau,\end{aligned}$$

and

$$\begin{aligned}\delta(\mathcal{D}(uc_\sigma)) &= (-1)^{l(u)}\delta(uD_\sigma) = (-1)^{l(u)}u\delta(D_\sigma)\\ &= (-1)^{l(u)}u\Big(\sum_{\tau\subseteq\sigma}[D_\tau : D_\sigma]p_\sigma^\tau D_\tau\Big) = \pm\mathcal{D}(\partial(uc_\sigma)).\end{aligned}$$

To establish (2), it suffices to show that the diagram

$$\begin{tikzcd} L^2_\mu C_i(Wc_\sigma) \arrow[r, "\psi_\sigma"] \arrow[d, "\mathcal{D}"'] & H_\sigma \arrow[d, "\bar{j}"] \\ L^2_{\mu^*} C_i(WD_\sigma) \arrow[r, "\phi_\sigma"] & A_\sigma \end{tikzcd}$$

commutes (this is because the horizontal maps are isometric isomorphisms of Hilbert modules and the vertical map $\bar{j}$ is a $\bar{j}$-equivariant isomorphism of Hilbert modules). Computing with the basis elements, we have

$$\bar{j}(\psi_\sigma(uc_\sigma)) = \bar{j}(\sqrt{|W_\sigma|}\,uh_\sigma) = \sqrt{|W_\sigma|}\,\bar{j}(u)a_\sigma = \sqrt{|W_\sigma|}\,(-1)^{l(u)}ua_\sigma,$$

and

$$\phi_\sigma(\mathcal{D}(uc_\sigma)) = \phi_\sigma((-1)^{l(u)}uD_\sigma) = (-1)^{l(u)}\sqrt{|W_\sigma|}\,ua_\sigma.$$

It follows that $\bar{j}\circ\psi_\sigma = \phi_\sigma\circ\mathcal{D}$. ∎

9. Hecke algebras. In this section we recall the basic properties of Hecke algebras that we will need in order to define admissible products on $\mathbb{R}W$ when W is right-angled. At the same time, we try to clarify relationships between various isomorphisms and involutions appearing in the literature.

Let (W, S) be a Coxeter group. Let $\mathbf{q} = (q_s)_{s\in S} \in \mathbb{R}^S$ be a fixed S-tuple satisfying $q_s = q_{s'}$ whenever $m(s, s')$ is odd. For any element $w \in W$, we let $\mathbf{q}^w$ denote the product $q_{s_1} \cdots q_{s_n}$ where $s_1 \cdots s_n$ is any reduced expression for w. It follows from Tits' solution to the word problem that $\mathbf{q}^w$ does not depend on the choice of reduced expression.

Let $\mathbb{R}^{(W)}$ denote the free $\mathbb{R}$-module with basis $\{e_w \mid w \in W\}$. According to [2, Exercise 23], there is a unique ring structure on $\mathbb{R}^{(W)}$ such that

$$e_s e_w = \begin{cases} e_{sw} & \text{if } l(sw) > l(w), \\ q_s e_{sw} + (q_s - 1)e_w & \text{if } l(sw) < l(w). \end{cases} \tag{9.1}$$

Let $\mathbb{R}_{\mathbf{q}}W$ denote this ring. If $\mathbf{q}$ is the constant S-tuple $\mathbf{1} = (1, \ldots, 1)$, then $\mathbb{R}_{\mathbf{q}}W$ is the ordinary group algebra $\mathbb{R}W$. Thus, $\mathbb{R}_{\mathbf{q}}W$ is a deformation of $\mathbb{R}W$, called the *Hecke algebra* of W associated to $\mathbf{q}$.

It follows from (9.1) that $\mathbb{R}_{\mathbf{q}}W$ is generated as an algebra by $\{e_s \mid s \in S\}$. For any pair $s, s' \in S$ with $s \neq s'$, let

$$w = \underbrace{ss'ss'\cdots}_{m(s,s')} = \underbrace{s'ss's\cdots}_{m(s,s')}.$$

Then e_w can be written two different ways as a product of e_s and $e_{s'}$, giving the relation

$$\underbrace{e_s e_{s'} e_s e_{s'} \cdots}_{m(s,s')} = \underbrace{e_{s'} e_s e_{s'} e_s \cdots}_{m(s,s')}. \tag{9.2}$$

Applying (9.1) to the case $w = s$, we also obtain the relation

$$e_s^2 = q_s + (q_s - 1)e_s. \tag{9.3}$$

Proposition 9.1. *The Hecke algebra* $\mathbb{R}_{\mathbf{q}}W$ *is isomorphic to the associative algebra generated by* e_s, $s \in S$, *with relations* (9.2) *and* (9.3).

Proof. Let A be the abstract associative $\mathbb{R}$-algebra (with unit 1) that is generated by $\overline{e}_s$, $s \in S$, with relations of the form (9.2) and (9.3). Since these relations hold for $\mathbb{R}_{\mathbf{q}}W$ there exists a surjective homomorphism $\phi : A \to \mathbb{R}_{\mathbf{q}}W$ such that $\phi(\overline{e}_s) = e_s$ for all $s \in S$. Given any $w \in W$, let $s_1 \cdots s_n$ be a reduced expression for w, and define $\overline{e}_w = \overline{e}_{s_1} \cdots \overline{e}_{s_n}$. It follows from Tits [17] and the relations (9.2) that $\overline{e}_w$ is independent of the reduced expression. We now prove by induction that any product $x = \overline{e}_{s_1} \cdots \overline{e}_{s_n} \in A$ can be written as a linear combination of these $\overline{e}_w$'s. If the word $s_1 \cdots s_n$ is reduced, we are done. Otherwise, it follows from [17] that, after applying the relation (9.2) if necessary, the product x has consecutive repetitions. We can then apply (9.3) to write x as a linear combination of shorter products, which by induction can each be written as linear combinations of the $\overline{e}_w$'s. Thus, x is a linear combination of the $\overline{e}_w$'s. Since $\phi(\overline{e}_w) = e_w$ and the e_w's form an $\mathbb{R}$-basis for $\mathbb{R}_{\mathbf{q}}W$, it follows that ϕ is injective. ∎

Another useful set of generators for $\mathbb{R}_{\mathbf{q}}W$ is the set of idempotents a_s, $s \in S$, defined by

$$a_s = \frac{1+e_s}{1+q_s}.$$

Substituting $(1+q_s)a_s - 1$ for e_s in the relations (9.2) and (9.3), we obtain

$$(9.4) \qquad \underbrace{a_s a_{s'} a_s a_{s'} \cdots}_{m(s,s')} = \underbrace{a_{s'} a_s a_{s'} a_s \cdots}_{m(s,s')} + \sum_{k=1}^{m(s,s')-2} \alpha_k(\mathbf{q}) \Big(\underbrace{a_s a_{s'} a_s a_{s'} \cdots}_{k} - \underbrace{a_{s'} a_s a_{s'} a_s \cdots}_{k} \Big)$$

and

$$(9.5) \qquad a_s^2 = a_s$$

where the coefficients $\alpha_k(\mathbf{q})$ appearing in (9.4) are all rational functions in $\mathbf{q}$ (depending on the pair (s, s')). Note that if W is right-angled, all of these coefficients are zero, in which case none of the relations involve the parameter $\mathbf{q}$. Hence, we have the following.

Corollary 9.2. *The Hecke algebra $\mathbb{R}_{\mathbf{q}}W$ is isomorphic to the associative algebra generated by a_s, $s \in S$, with relations* (9.4) *and* (9.5). *In particular, if W is right-angled, then for any $\mathbf{q}$, there exists a canonical isomorphism of algebras $\mathbb{R}W \to \mathbb{R}_{\mathbf{q}}W$ taking a_s to a_s.*

A similar algebra presentation can be described using the idempotents h_s, $s \in S$, defined by

$$h_s = 1 - a_s = \frac{q_s - e_s}{1+q_s}.$$

The only difference in the resulting relations is that the rational functions in (9.4) are different. If W is right-angled, however, these rational functions all vanish, so one obtains the same algebra presentation. It follows that in this case, mapping $a_s \mapsto h_s$ for all $s \in S$ induces an involution of the algebra $\mathbb{R}_{\mathbf{q}}W$. In fact, for any subset $T \subset S$, exchanging a_s and h_s if and only if $s \in T$ also defines an involution in the right-angled case. We discuss these involutions further below, but in the context of some more well-known isomorphisms that hold for general Coxeter groups.

The j-isomorphism. For the multiparameter $\mathbf{q} = (q_s)_{s\in S}$, we let $1/\mathbf{q}$ denote the I-tuple $(1/q_s)_{s\in S}$. For each $e_s \in \mathbb{R}_{\mathbf{q}}W$, let $j(e_s)$ denote the element $j(e_s) = -q_s e_s \in \mathbb{R}_{1/\mathbf{q}}W$. An easy calculation also shows that (9.2) and (9.3) continue to hold after replacing e_s with $j(e_s)$ and q_s with $1/q_s$ (for all $s \in S$). It follows that $e_s \mapsto j(e_s)$ also induces an isomorphism $j : \mathbb{R}_{\mathbf{q}}W \to \mathbb{R}_{1/\mathbf{q}}W$. Note that with respect to the generators a_s, we have $j(a_s) = h_s$.

The Kazhdan–Lusztig isomorphism. It follows from (9.3) that in $\mathbb{R}_{\mathbf{q}}W$, the element e_s is invertible with inverse given by

$$e_s^{-1} = \left(\frac{1}{q_s} - 1\right) + \frac{1}{q_s} e_s.$$

Although the relations (9.2) hold when the e_s's are replaced by their inverses, the relations (9.3) do not. However, if e_s *and* q_s are replaced by their inverses, then both (9.2) and (9.3) do hold. It follows that $e_s \mapsto e_s^{-1}$ extends (by linearity) to an isomorphism $\mathbb{R}_{\mathbf{q}}W \to \mathbb{R}_{1/\mathbf{q}}W$. Following [13], we denote this isomorphism by $x \mapsto \overline{x}$. Note, in particular, that with respect to the idempotents a_s we have

$$\overline{a}_s = \frac{1+\overline{e}_s}{1+q_s} = \frac{1}{1+q_s} + \frac{1}{1+q_s}((q_s-1)+q_s e_s) = \frac{1+e_s}{1+\frac{1}{q_s}} = a_s.$$

Involutions on $\mathbb{R}_{\mathbf{q}}W$. Both the j-isomorphism and the Kazhdan–Lusztig isomorphism described above are "involutions" in the sense that applying them twice is the identity map. But they are not actually automorphisms of the same Hecke algebra. Composing the j-isomorphism with the Kazhdan–Lusztig isomorphism, however, does result in an involution on $\mathbb{R}_{\mathbf{q}}W$, which we denote by $\overline{j}$. In terms of e_s's we have

$$\overline{j}(e_s) = (q_s - 1) - e_s,$$

and in terms of a_s's we have

$$\overline{j}(a_s) = 1 - a_s = h_s.$$

(Note that in the special case $\mathbf{q} = \mathbf{1}$, this involution $\overline{j}$ is precisely the involution $\overline{j} : \mathbb{R}W \to \mathbb{R}W$ introduced in Section 3.)

In the right-angled case, $\overline{j}$ is the involution mentioned above in the paragraph after Corollary 9.2 that exchanges a_s with h_s for each $s \in S$. More generally, in the right-angled case, for any subset $T \subset S$, we obtain an involution $\overline{j}_T = \mathbb{R}_{\mathbf{q}}W \to \mathbb{R}_{\mathbf{q}}W$ defined on generators by

$$\overline{j}_T(e_s) = \begin{cases} (q_s - 1) - e_s & \text{for } s \in T, \\ e_s & \text{for } s \notin T \end{cases}$$

or

$$\overline{j}_T(a_s) = \begin{cases} h_s & \text{for } s \in T, \\ a_s & \text{for } s \notin T. \end{cases}$$

Remark 9.3. For general (not just right-angled) Coxeter groups, there do exist "partial j-isomorphisms". These are defined as follows. Let $T \subset S$ be any subset satisfying $wTw^{-1} \cap S \subseteq T$ for all $w \in W$ (i.e., T is closed with respect to the conjugacy relation on the generating set S). Let $\mathbf{q}^T = (r_s)$

denote the S-tuple defined by

$$r_s = \begin{cases} 1/q_s & \text{for } s \in T, \\ q_s & \text{for } s \notin T. \end{cases}$$

For any generator $e_s \in \mathbb{R}_{\mathbf{q}}W$, let $j_T(e_s) \in \mathbb{R}_{\mathbf{q}^T}W$ be the element

$$j_T(e_s) = \begin{cases} -q_s e_s & \text{for } s \in T, \\ e_s & \text{for } s \notin T. \end{cases}$$

As in the case of the (full) j-isomorphism, the relations (9.2) and (9.3) continue to hold after replacing e_s with $-q_s e_s$ and q_s with $1/q_s$ *for only the generators* $s \in T$. Thus, j_T extends to an isomorphism $j_T : \mathbb{R}_{\mathbf{q}}W \to \mathbb{R}_{\mathbf{q}^T}W$. One can also verify that $j_T(a_s) = h_s$ for $s \in T$ and $j_T(a_s) = a_s$ for $s \notin T$.

In general, however, there does not exist a "partial Kazhdan–Lusztig isomorphism" that one might compose j_T with to get an involution on $\mathbb{R}_{\mathbf{q}}W$. This is because replacing only some of the generators e_s with their inverses in the relations (9.2) need not result in the same relations. Thus, the involutions $\overline{j}_T$ are specific to the right-angled setting.

10. Admissible inner products for right-angled Coxeter groups. The goal for this section is to show that for right-angled Coxeter groups, the collection of admissible inner products is large enough that the set of dimensions $\{\dim_\mu A_\sigma \in \mathbb{R} \mid \sigma \in N,\ \sigma \neq \emptyset\}$ can be varied independently by changing μ. This will allow us (in Section 11) to relate the Euler characteristic χ_μ to the growth series $W(\mathbf{t})$ for a suitable choice of μ.

First we describe the basic inner products arising from the work of Dymara et al. In [8] (for one parameter) and [5] (in general) it is shown that the inner product $\langle\, ,\rangle_{\mathbf{q}}$ on the Hecke algebra $\mathbb{R}_{\mathbf{q}}W$ defined (with respect to the basis $\{e_w\}$) by

$$\langle e_w, e_v\rangle_{\mathbf{q}} = \begin{cases} \mathbf{q}^w & \text{if } w = v, \\ 0 & \text{if } w \neq v, \end{cases}$$

satisfies the axioms for a Hilbert algebra structure (the involution $*$ in this case is given with respect to the basis e_w by $e_w \mapsto e_{w^{-1}}$). For general W, this inner product will determine an admissible inner product on the group algebra $\mathbb{R}W$ only if $\mathbf{q} = \mathbf{1}$, in which case one simply recovers the standard inner product on $\mathbb{R}W$.

However, *if W is right-angled* then by Corollary 9.2, $\mathbb{R}_q W$ is canonically isomorphic to $\mathbb{R}W$, and this isomorphism respects the $*$-involutions. It follows that if W is right-angled, then *for any* $\mathbf{q}$, the inner product $\langle\, ,\rangle_{\mathbf{q}}$ determines an admissible inner product on $\mathbb{R}W$. To avoid cumbersome notation, we shall also denote this inner product by $\langle\, ,\rangle_{\mathbf{q}}$.

Now let $M(W)$ be the set of all admissible inner products on $\mathbb{R}W$. Since finite convex combinations of admissible inner products are admissible, $M(W)$ is convex. In general, $M(W)$ is very complicated, and we do not understand it well. However, for our purposes, it will be enough to show that a certain affine projection of $M(W)$ to $\mathbb{R}^N$ is sufficiently large.

DEFINITION 10.1. Let $\phi : M(W) \to \mathbb{R}^N$ be the map defined by $\langle\, ,\rangle \mapsto (\langle a_\sigma, 1\rangle)_\sigma$, and let $P(W)$ denote the image in $\mathbb{R}^N$.

To describe the set $P(W)$ we consider the single-parameter Hecke algebra $\mathbb{R}_q W$ (i.e., $q_s = q$ for all $s \in S$), with the standard inner product $\langle\, ,\rangle_q$ described above. More generally, for any $T \subset S$, we consider the inner product $\langle\, ,\rangle_q^T$ defined by

$$\langle x, y\rangle_q^T = \langle \bar{j}_T(x), \bar{j}_T(y)\rangle_q.$$

All of these products are admissible, and by means of the canonical isomorphism $\mathbb{R}W \to \mathbb{R}_q W$ (which maps a_s's to a_s's and h_s's to h_s's) these inner products all pull back to admissible products on the group algebra $\mathbb{R}W$. We denote the pulled back products to $\mathbb{R}W$ by the same symbols $\langle\, ,\rangle_q$ and $\langle\, ,\rangle_q^T$, but to distinguish elements $a_s, h_s \in \mathbb{R}W$ from those with the same symbols in $\mathbb{R}_q W$, we shall decorate the latter with superscripts: $a_s^{(q)}, h_s^{(q)}$.

Recall from Remark 5.4 that for a spherical subset $\sigma = \{s_1, \ldots, s_k\}$, the idempotent $a_\sigma \in \mathbb{R}W$ has the product representation

$$a_\sigma = a_{s_1} \cdots a_{s_k}.$$

Computing the inner product $\langle a_\sigma, 1\rangle_q$, we then have

$$\begin{aligned}\langle a_\sigma, 1\rangle_q &= \langle a_{s_1} \cdots a_{s_k}, 1\rangle_q = \langle a_{s_1}^{(q)} \cdots a_{s_k}^{(q)}, 1\rangle_q \\ &= \left\langle \frac{1+e_{s_1}}{1+q} \cdots \frac{1+e_{s_k}}{1+q}, 1 \right\rangle_q = \left(\frac{1}{1+q}\right)^k\end{aligned}$$

where the last equality follows from the fact that all e_w's are orthogonal to 1. More generally, for any $T \subset S$, we have

$$\begin{aligned}\langle a_\sigma, 1\rangle_q^T &= \langle \bar{j}_T(a_{s_1}) \cdots \bar{j}_T(a_{s_k}), 1\rangle_q = \langle \bar{j}_T(a_{s_1})^{(q)} \cdots \bar{j}_T(a_{s_k})^{(q)}, 1\rangle_q \\ &= \left\langle \prod_{s_i \in T} h_s^{(q)} \prod_{s_i \notin T} a_s^{(q)}, 1 \right\rangle = \prod_{s_i \in T} \frac{q}{1+q} \prod_{s_i \notin T} \frac{1}{1+q}.\end{aligned}$$

Now if we let $q \to 0$, then

$$\langle a_\sigma, 1\rangle_q^T \to \begin{cases} 1 & \text{if } \sigma \cap T = \emptyset, \\ 0 & \text{if } \sigma \cap T \neq \emptyset. \end{cases}$$

On the other hand, if we let $q \to \infty$, then

$$\langle a_\sigma, 1\rangle_q^T \to \begin{cases} 1 & \text{if } \sigma \subseteq T, \\ 0 & \text{if } \sigma \not\subseteq T. \end{cases}$$

For each $\sigma \in N$, we let δ_σ denote the corresponding standard basis vector in $\mathbb{R}^N$. For each subset $T \subseteq S$, let $v_T \in \mathbb{R}^N$ denote the vector

$$v_T = \sum_{\sigma \subseteq T} \delta_\sigma.$$

Then it follows from the formulas in the preceding paragraph that the (degenerate) inner product $\langle\, ,\rangle_0^T = \lim_{q\to 0}\langle\, ,\rangle_q^T$ projects (under $\phi : M(W) \to \mathbb{R}^N$) to the vector v_T, and the inner product $\langle\, ,\rangle_\infty^T = \lim_{q\to\infty}\langle\, ,\rangle_q^T$ projects to v_{S-T}. In particular, this implies that for each subset $T \subseteq S$, the vector v_T is contained in the closure $\overline{P(W)}$. In other words, if Δ denotes the convex hull of the vectors $\{v_T \mid T \subseteq S\}$, we have

$$\Delta \subseteq \overline{P(W)} \subseteq \mathbb{R}^N.$$

Since $\langle a_\emptyset, 1\rangle = 1$ for any admissible inner product, $\overline{P(W)}$ lies in an $(|N|-1)$-dimensional affine subspace of $\mathbb{R}^N$.

Proposition 10.2. *Let $\alpha = \sum_\sigma \alpha_\sigma \delta_\sigma$ be any point in the relative interior of Δ. Then there exists an admissible (nondegenerate) inner product $\mu = \langle\, ,\rangle$ such that $\phi(\mu) = \alpha$.*

Proof. Let $n = |N|$. Then any such point α will be contained in the relative interior of some $(n-1)$-dimensional simplex spanned by vectors of the form $v_{T_0}, \ldots, v_{T_n}$. For each such T_i, we can choose positive real numbers q_i such that the inner product $\langle\, ,\rangle_{q_i}^{T_i}$ projects via ϕ to a point arbitrarily close to v_{T_i}. Let u_{T_i} denote this point. In particular, we can choose all of the q_i's so that the point α is still in the relative interior of the simplex spanned by the vectors u_{T_i}. Let y_i denote the barycentric coordinates of the point α with respect to the vertices u_{T_i}. Then the inner product

$$\mu = \langle\, ,\rangle = \sum_{i=0}^{n} y_i \langle\, ,\rangle_{q_i}^{T_i}$$

is admissible and projects to the point α. ∎

The following corollary will be used in Section 11.

Corollary 10.3. *Let $\{\gamma_\sigma \in \mathbb{R} \mid \sigma \in N\}$ be any collection of real numbers satisfying $\gamma_\sigma > 0$ and $\sum_{\sigma\in N}\gamma_\sigma = 1$. For each $\sigma \in N$, let α_σ and β_σ be given by*

$$\alpha_\sigma = \sum_{\tau\cap\sigma=\emptyset} \gamma_\tau \quad \textit{and} \quad \beta_\sigma = \sum_{\tau\supseteq\sigma} \gamma_\tau.$$

Then

(1) *α_σ and β_σ satisfy the formulas*

$$\beta_\sigma = \sum_{\tau\subseteq\sigma} (-1)^{|\tau|}\alpha_\tau \quad \textit{and} \quad \alpha_\sigma = \sum_{\tau\subseteq\sigma} (-1)^{|\tau|}\beta_\tau,$$

(2) *there exists an admissible inner product* $\mu = \langle\, , \rangle$ *such that*

$$\dim_\mu A_\sigma = \alpha_\sigma \quad \textit{and} \quad \dim_\mu H_\sigma = \beta_\sigma.$$

Proof. For (1), we compute

$$\begin{aligned}\sum_{\tau\subseteq\sigma}(-1)^{|\tau|}\alpha_\tau &= \sum_{\tau\subseteq\sigma}(-1)^{|\tau|}\sum_{\rho\cap\tau=\emptyset}\gamma_\rho = \sum_\rho \gamma_\rho \sum_{\tau\subseteq\sigma-\rho}(-1)^{|\tau|}\\ &= \sum_{\sigma-\rho=\emptyset}\gamma_\rho = \sum_{\rho\supseteq\sigma}\gamma_\rho = \beta_\sigma,\end{aligned}$$

and the second formula follows from a similar computation.

For (2), consider the collection of "cospherical" subsets $T_\sigma = S - \sigma$, $\sigma \in N$. The corresponding vectors v_{T_σ} span an $(n-1)$-dimensional simplex in Δ_a, hence the point

$$\alpha = \sum_\sigma \gamma_\sigma v_{T_\sigma}$$

lies in the relative interior of Δ_a. With respect to the standard basis δ_σ, we have

$$\alpha = \sum_\sigma \gamma_\sigma \sum_{\tau\subseteq S-\sigma}\delta_\tau = \sum_\tau\Big(\sum_{\tau\cap\sigma=\emptyset}\gamma_\sigma\Big)\delta_\tau = \sum_\tau \alpha_\tau\delta_\tau.$$

Hence, by Proposition 10.2, there exists an admissible inner product $\mu = \langle\, , \rangle$ such that $\langle a_\sigma, 1\rangle = \alpha_\sigma$ for all $\sigma \in N$. The change of variables in Remark 5.4 and part (1) then give

$$\langle h_\sigma, 1\rangle = \Big\langle \sum_{\tau\subseteq\sigma}(-1)^{|\tau|}a_\tau, 1\Big\rangle = \sum_{\tau\subseteq\sigma}(-1)^{|\tau|}\alpha_\tau = \beta_\sigma. \ \blacksquare$$

11. Growth series. In this section, we recall the *greedy growth series* introduced in [15], and use it to measure the relative sizes of the quotients W/W_σ for $\sigma \in N$. We then show that we can choose an admissible inner product in such a way that these relative sizes become the dimensions of the submodules A_σ.

Let W be a right-angled Coxeter group with nerve N. For any $w \in W$, recall that $l(w)$ denotes the word length of w with respect to the generating set S. For each spherical subset $\sigma \in N$, we let w_σ denote the element of greatest length in W_σ. Since W is right-angled, we have

$$w_\sigma = \prod_{s\in\sigma} s,$$

thus, $l(w_\sigma) = |\sigma|$. For any spherical subset $\sigma \in N$ we define $\mathrm{St}(\sigma) \subseteq S$ by $\mathrm{St}(\sigma) = \{s \in S \mid \{s\} \cup \sigma \in N\}$. Note that $\mathrm{St}(\emptyset) = S$. The following proposition is a reformulation of Propositions 5.1 and 5.3 in [15] in the case that W is right-angled.

Proposition 11.1. *Any element $w \in W$ can be written uniquely as a product $w = w_{\sigma_1} \cdots w_{\sigma_n}$ where $\mathrm{St}(\sigma_i) \cap \sigma_{i-1} = \emptyset$ for $i = 2, \ldots, n$. Moreover, in this case*

$$l(w) = \sum_i l(w_{\sigma_i}) = \sum_i |\sigma_i|.$$

We call such a product representation for w its (*right*) *greedy normal form* (2). We associate a "monomial weight" to each element of W using this greedy normal form. Let $\mathbf{t}$ denote the N-tuple $\mathbf{t} = (t_\sigma)_{\sigma\in N}$ (by convention, we set $t_\emptyset = 1$), and for any $w \in W$, let $\mathbf{t}_w$ denote the monomial

$$\mathbf{t}_w = \prod_{i=1}^{n} t_{\sigma_i}$$

where $w_{\sigma_n} \cdots w_{\sigma_1}$ is the greedy normal form for w. We then define the *greedy growth series* for W to be the (multivariate) power series

$$W(\mathbf{t}) = \sum_{w\in W} \mathbf{t}_w.$$

More generally, for any subset $X \subset W$, we define the corresponding series

$$X(\mathbf{t}) = \sum_{w\in X} \mathbf{t}_w.$$

It was shown in [15] that the greedy normal form is a regular language, hence the series $W(\mathbf{t})$ can be expressed as a rational function in $\mathbf{t}$. We sketch the argument here, but refer the reader to [15] and [11] for further details. For each $\sigma \in N$, let $X_\sigma \subset W$ be the subset

$$X_\sigma = \{w \in W \mid l(ws) < l(w) \Leftrightarrow s \in \sigma\}.$$

The set X_σ consists of those elements of W whose greedy normal form ends in w_σ. It follows from Proposition 11.1 that an element of X_σ can be obtained from an element of X_τ by multiplying on the right by w_σ if and only if $\mathrm{St}(\sigma) \cap \tau = \emptyset$. This implies that the growth series for X_σ satisfies the equation

$$X_\sigma(\mathbf{t}) = t_\sigma \sum_{\mathrm{St}(\sigma)\cap\tau=\emptyset} X_\tau(\mathbf{t}).$$

In other words, the system of linear equations given by

$$\left\{ z_\sigma = t_\sigma \sum_{\mathrm{St}(\sigma)\cap\tau=\emptyset} z_\tau \;\middle|\; \text{for all } \sigma \in N \right\} \tag{11.1}$$

(2) There is also a *left* greedy normal form for w given by $w = w_{\sigma_n} \cdots w_{\sigma_1}$ where $\mathrm{St}(\sigma_i) \cap \sigma_{i-1} = \emptyset$ for $i = 2, \ldots, n$. In fact this is the one treated in [15], but since the right greedy normal form for w is the same as the left greedy normal form for w^{-1} written in reverse order, all of the relevant properties still hold.

has solution $\{z_\sigma = X_\sigma(\mathbf{t})\}_{\sigma\in N}$. On the other hand, this system can be solved explicitly by row reducing the coefficient matrix over the fraction field $\mathbb{Q}(\mathbf{t})$. It can be shown that this system has a 1-dimensional space of solutions, and that the solution becomes unique once one sets $z_\emptyset = X_\emptyset(\mathbf{t}) = 1$. It follows that each of the power series $X_\sigma(\mathbf{t})$ is a rational function in $\mathbf{t}$, and since W is the disjoint union of the X_σ's, we deduce that

$$W(\mathbf{t}) = \sum_{\sigma\in N} X_\sigma(\mathbf{t})$$

is also a rational function.

As long as $\mathbf{t}$ is in the region of convergence for the power series $W(\mathbf{t})$ and $X \subseteq W$ is nonempty, the power series $X(\mathbf{t})$ converges to a positive real number, and the ratio $X(\mathbf{t})/W(\mathbf{t})$ can be regarded as a measure of the size of the subset X relative to W. We exploit this idea by finding an inner product μ on $\mathbb{R}W$ so that the dimensions of the submodules $A_\sigma \subset L^2_\mu W$ correspond to the relative sizes of the quotients W/W_σ. To make this precise, we use the bijection $W/W_\sigma \leftrightarrow W^\sigma$ and the following decomposition.

Proposition 11.2. *The set W^σ is the disjoint union $W^\sigma = \bigcup_{\tau\cap\sigma=\emptyset} X_\tau$.*

Proof. Rewriting the definition of X_τ, we have

$$X_\tau = \{w \in W \mid l(ws) > l(w) \Leftrightarrow s \in S - \tau\}.$$

Since $W^\sigma = \{w \in W \mid l(ws) > l(w) \text{ for all } s \in \sigma\}$, it follows that $w \in W_\sigma \cap X_\tau$ if and only if $\sigma \subseteq S - \tau$ or, equivalently, if and only if $\sigma \cap \tau = \emptyset$. Since W is the disjoint union of the X_τ's the result follows. ∎

In light of this partition of W^σ, we define the rational functions $\gamma_\sigma(\mathbf{t})$, $\alpha_\sigma(\mathbf{t})$, and $\beta_\sigma(\mathbf{t})$ by

$$\gamma_\sigma(\mathbf{t}) = \frac{X_\sigma(\mathbf{t})}{W(\mathbf{t})}, \tag{11.2}$$

$$\alpha_\sigma(\mathbf{t}) = \sum_{\tau\cap\sigma=\emptyset} \gamma_\tau(\mathbf{t}), \tag{11.3}$$

$$\beta_\sigma(\mathbf{t}) = \sum_{\tau\supseteq\sigma} \gamma_\tau(\mathbf{t}). \tag{11.4}$$

Then for $\mathbf{t}$ in the region of convergence of $W(\mathbf{t})$, all of these numbers are positive and $\alpha_\sigma(\mathbf{t})$ represents the size of the set W^σ relative to W; that is,

$$\alpha_\sigma(\mathbf{t}) = \frac{W^\sigma(\mathbf{t})}{W(\mathbf{t})}.$$

Note that by (1) of Corollary 10.3, we have

$$\beta_\sigma(\mathbf{t}) = \sum_{\tau\subseteq\sigma} (-1)^{|\tau|}\alpha_\tau(\mathbf{t}).$$

In particular, the change of variables taking the α_σ's to the β_σ's is the same change of variables described in Remark 5.4 taking the idempotents a_σ's to the h_σ's.

We can now relate the Euler characteristic to the greedy growth series. Let $\alpha_\sigma(\mathbf{t})$ and $\beta_\sigma(\mathbf{t})$ be as in (11.3) and (11.4).

THEOREM 11.3. *Assume* $\mathbf{t}$ *is in the region of convergence for the greedy growth series* $W(\mathbf{t})$*. Then there exists an admissible inner product* μ *such that*

(1) $\dim_\mu A_\sigma = \alpha_\sigma(\mathbf{t})$,
(2) $\dim_\mu H_\sigma = \beta_\sigma(\mathbf{t})$, *and*
(3) $\chi_\mu = 1/W(\mathbf{t})$.

Proof. The first two statements follow from Corollary 10.3. For the third, we have

$$\begin{aligned}\chi_\mu &= \sum_\sigma (-1)^{|\sigma|}\beta_\sigma(\mathbf{t}) = \sum_\sigma (-1)^{|\sigma|}\sum_{\tau\supseteq\sigma}\gamma_\tau(\mathbf{t}) = \sum_\tau \gamma_\tau(\mathbf{t})\sum_{\sigma\subseteq\tau}(-1)^{|\sigma|}\\ &= \gamma_\emptyset(\mathbf{t}) = \frac{1}{W(\mathbf{t})}. \ \blacksquare\end{aligned}$$

REMARK 11.4. In fact, the theorem holds for any choice of $\mathbf{t}$ for which the numbers $\gamma_\sigma(\mathbf{t})$ are positive for all $\sigma \in N$.

Comparison with the L^2_q-theory. We conclude this section by showing that the dimensions of the submodules A_σ and H_σ obtained using our choice of μ in Theorem 11.3 are consistent with the dimensions obtained by Dymara in the Hecke-algebra setting. We let $W(q)$ denote the standard growth series $W(q) = \sum_{w\in W} q^{l(w)}$ for W. Let $\mathcal{N}_q$ denote the von Neumann algebra obtained from the Hecke algebra $\mathbb{R}_q W$ using the inner product $\langle e_w, e_v\rangle_q = \delta_{wv} q^{l(w)}$. We let $A_\sigma^{(q)}$ denote the $\mathcal{N}_q$-module corresponding to the W_σ-invariant functions, and let $H_\sigma^{(q)}$ denote the Hecke-algebra version of the module corresponding to the W_σ-alternating functions. (The corresponding projection operators $a_\sigma^{(q)}$ and $h_\sigma^{(q)}$ involve the weight q in the Hecke setting to ensure a Hilbert-module structure. See [8] or [4] for details.) In [8] it is shown that

$$\begin{aligned}\dim_{\mathcal{N}_q} A_\sigma^{(q)} &= \frac{1}{W_\sigma(q)} = \left(\frac{1}{1+q}\right)^{|\sigma|},\\ \dim_{\mathcal{N}_q} H_\sigma^{(q)} &= \frac{1}{W_\sigma(q^{-1})} = \left(\frac{q}{1+q}\right)^{|\sigma|}.\end{aligned}$$

The corresponding L^2_q-Euler characteristic is then computed to be

$$\chi_q = \frac{1}{W(q)}$$

(see, e.g., Theorem 17.1.9 in [4]).

The standard growth series $W(q)$ is a specialization of the greedy growth series $W(\mathbf{t})$ obtained by substituting $t_\sigma = q^{|\sigma|}$ for each $\sigma \in N$. Writing $X_\sigma(q)$ for the corresponding specialization of $X_\sigma(\mathbf{t})$, we let $\alpha_\sigma(q)$ and $\beta_\sigma(q)$ be as in (11.3) and (11.4). With μ as in Theorem 11.3, we then have

$$\begin{aligned}\dim_\mu A_\sigma &= \alpha_\sigma(q) = \frac{W^\sigma(q)}{W(q)} = \frac{1}{W_\sigma(q)} = \left(\frac{1}{1+q}\right)^{|\sigma|},\\ \dim_\mu H_\sigma &= \beta_\sigma(q) = \sum_{\tau\subseteq\sigma}(-1)^{|\tau|}\alpha_\tau(q) = \sum_{\tau\subseteq\sigma}\left(\frac{-1}{1+q}\right)^{|\tau|}\\ &= \left(1-\frac{1}{1+q}\right)^{|\sigma|} = \left(\frac{q}{1+q}\right)^{|\sigma|}.\end{aligned}$$

In particular, the dimensions we obtain (by varying the inner product on the group algebra) coincide with those obtained in the Hecke-algebra setting, and we obtain the same formula for the Euler characteristic $\chi_\mu = 1/W(q)$.

Remark 11.5. If we are only interested in finding an admissible inner product on $\mathbb{R}W$ that gives us the Euler characteristic $\chi_\mu = 1/W(q)$, then the obvious choice would be $\mu = \langle\,,\rangle_q$ (the pull back of Dymara's inner product on $\mathbb{R}_qW$ with respect to the canonical isomorphism $\mathbb{R}W \to \mathbb{R}_qW$). With this choice of μ, our cochain complex $(L^2_\mu C^*(\Sigma_{cc}),\delta)$ is isomorphic to Dymara's $(L^2_qC^*(\Sigma_{cc}),\delta)$, hence, not only are the dimensions of the cochains (and, hence, the Euler characteristics) the same, but so are the Betti numbers. There is a similar isomorphism if q is replaced with any multiparameter $\mathbf{q} \in \mathbb{R}^S$.

On the other hand, for general $\mathbf{t} \in \mathbb{R}^N$, there is no Hecke algebra parametrized by $\mathbf{t}$. Thus, in order to have enough degrees of freedom for our choice of μ to give us $\chi_\mu = 1/W(\mathbf{t})$ for $\mathbf{t} \in \mathbb{R}^N$, we must instead use convex combinations of the inner products $\langle\,,\rangle_q^T$. This is why we need Corollary 10.3 for the proof of Theorem 11.3. Although we can control the dimensions of the cochains for such convex combinations, the coboundary maps themselves are much more subtle. In fact there are many choices of inner products having a given set of cochain dimensions, and these will typically have different coboundary maps, presumably resulting in different L^2_μ-Betti numbers.

Another important difference between the inner products $\mu = \langle\,,\rangle_{\mathbf{q}}$ and general $\mu \in M(W)$ is that for the former, the isomorphism $\mathbb{R}W \to \mathbb{R}_{\mathbf{q}}W$ gives a canonical basis for $\mathbb{R}W$ with respect to which μ is orthogonal. This means that the completion L^2_μ can be regarded as the set of functions on W that are square-summable with respect to a suitable measure. Similarly, L^2_μ-cochains can be interpreted as square-summable cochains. For general μ, however, there is no such interpretation of L^2_μ or $L^2_\mu C^*$. They are simply formal Hilbert space completions with respect to μ.

12. Reciprocity. In this section we show that under the additional assumption that the nerve of W is Eulerian, if μ is chosen as in Theorem 11.3, then switching from the Euler characteristic χ_μ to its dual χ_μ^* corresponds to replacing $\mathbf{t}$ with its reciprocal $\mathbf{t}^{-1}$. This will follow from the following purely combinatorial formula.

THEOREM 12.1. *Let W be a right-angled Coxeter group whose nerve N is an Eulerian $(n-1)$-sphere. Then the rational functions $\alpha_\sigma(\mathbf{t})$ and $\beta_\sigma(\mathbf{t})$ satisfy the reciprocity formula*

$$\alpha_\sigma(\mathbf{t}) = \beta_\sigma(\mathbf{t}^{-1}).$$

Before proving Theorem 12.1, we first describe some systems of equations that characterize the rational functions $\alpha_\sigma(\mathbf{t})$ and $\beta_\sigma(\mathbf{t})$. Recall from Section 11 that the system of equations (11.1) has solution $z_\sigma = X_\sigma(\mathbf{t})$ and that up to a scalar multiple this solution is unique. Thus by dividing each $X_\sigma(\mathbf{t})$ by $W(\mathbf{t})$, we see that $\{z_\sigma = \gamma_\sigma(\mathbf{t})\}_{\sigma\in N}$ is the unique solution to the system

$$\Big\{z_\sigma = t_\sigma \sum_{\tau\cap \mathrm{St}(\sigma)=\emptyset} z_\tau \ (\forall\sigma\in N), \text{ and } \sum_{\sigma\in N} z_\sigma = 1\Big\}. \tag{12.1}$$

Applying Möbius inversion to (11.4), we obtain the formula

$$\gamma_\sigma(\mathbf{t}) = \sum_{\tau\supseteq\sigma}(-1)^{|\tau-\sigma|}\beta_\tau(\mathbf{t}).$$

It follows that $\{y_\sigma = \beta_\sigma(\mathbf{t})\}_{\sigma\in N}$ will be the unique solution to (12.1) after the change of variables $z_\sigma = \sum_{\tau\supseteq\sigma}(-1)^{|\tau-\sigma|}y_\tau$. With this substitution, we obtain

$$\Big\{\sum_{\tau\supseteq\sigma}(-1)^{|\tau-\sigma|}y_\tau = t_\sigma \sum_{\tau\cap\mathrm{St}(\sigma)=\emptyset}\ \sum_{\rho\supseteq\tau}(-1)^{|\rho-\tau|}y_\rho \ (\forall\sigma\in N), \text{ and}$$
$$\sum_{\sigma\in N}\sum_{\tau\supseteq\sigma}(-1)^{|\tau-\sigma|}y_\tau = 1\Big\}.$$

The last equation simplifies to

$$1 = \sum_{\sigma\in N}\sum_{\tau\supseteq\sigma}(-1)^{|\tau-\sigma|}y_\tau = \sum_{\tau\in N}(-1)^{|\tau|}y_\tau\sum_{\sigma\subseteq\tau}(-1)^{|\sigma|} = y_\emptyset$$

(since the sum $\sum_{\sigma\subseteq\tau}(-1)^{|\sigma|}$ is equal to zero unless $\tau=\emptyset$). We rewrite the system more concisely as

$$\{F_\sigma(\mathbf{y}) = t_\sigma G_\sigma(\mathbf{y}) \ (\forall\sigma\in N), \text{ and } y_\emptyset = 1\} \tag{12.2}$$

where $F_\sigma(\mathbf{y})$ and $G_\sigma(\mathbf{y})$ are given by

$$F_\sigma(\mathbf{y}) = \sum_{\tau\supseteq\sigma}(-1)^{|\tau-\sigma|}y_\tau \quad\text{and}\quad G_\sigma(\mathbf{y}) = \sum_{\tau\cap\mathrm{St}(\sigma)=\emptyset}\ \sum_{\rho\supseteq\tau}(-1)^{|\rho-\tau|}y_\rho.$$

It follows that $\{y_\sigma = \beta_\sigma(\mathbf{t})\}_{\sigma\in N}$ is the unique solution to (12.2), and $\{y_\sigma = \beta_\sigma(\mathbf{t}^{-1})\}_{\sigma\in N}$ is the unique solution to

$$\{G_\sigma(\mathbf{y}) = t_\sigma F_\sigma(\mathbf{y})\ (\forall\sigma\in N),\ \text{and } y_\emptyset = 1\}.$$

It will suffice, therefore, to show that $\{y_\sigma = \alpha_\sigma(\mathbf{t})\}_{\sigma\in N}$ is also a solution to this last system. For this, we use the further change of variables $y_\sigma = \sum_{\tau\subseteq\sigma}(-1)^{|\tau|}u_\tau$ (recall that this is the same change of variables that takes a_σ to h_σ and vice versa). Let $G_\sigma(\mathbf{y}(\mathbf{u}))$ (resp., $F_\sigma(\mathbf{y}(\mathbf{u}))$) denote the expression $G_\sigma(\mathbf{y})$ (resp., $F_\sigma(\mathbf{y})$) after the substitution $y_\tau = \sum_{\rho\subseteq\tau}(-1)^{|\rho|}u_\rho$ (for all $\tau\in N$).

Lemma 12.2. *If N is an Eulerian $(n-1)$-sphere, then*

$$G_\sigma(\mathbf{y}(\mathbf{u})) = (-1)^{n+|\sigma|}F_\sigma(\mathbf{u}) \quad \textit{and} \quad F_\sigma(\mathbf{y}(\mathbf{u})) = (-1)^{n+|\sigma|}G_\sigma(\mathbf{u})$$

for all $\sigma\in N$.

Proof. Substituting variables and switching the order of summation gives

$$\begin{aligned} F_\sigma(\mathbf{y}(\mathbf{u})) &= \sum_{\tau\supseteq\sigma}(-1)^{|\tau-\sigma|}\sum_{\rho\supseteq\tau}(-1)^{|\rho|}u_\rho \\ &= (-1)^{|\sigma|}\sum_{\rho\in N}(-1)^{|\rho|}u_\rho\Big(\sum_{\tau\supseteq\rho,\,\tau\supseteq\sigma}(-1)^{|\tau|}\Big). \end{aligned}$$

The parenthetical sum evaluates to 0 if $\rho\cup\sigma\notin N$, and since N is Eulerian, it evaluates to $\sum_{\tau\supseteq\rho\cup\sigma} = (-1)^n$ if $\rho\cup\sigma\in N$. Thus, we have

$$F_\sigma(\mathbf{y}(\mathbf{u})) = (-1)^{n+|\sigma|}\sum_{\rho\cup\sigma\in N}(-1)^{|\rho|}u_\rho.$$

On the other hand, switching the order of the sums in the definition of $G_\sigma(\mathbf{u})$, we obtain

$$G_\sigma(\mathbf{u}) = \sum_{\rho\in N}(-1)^{|\rho|}u_\rho\Big(\sum_{\tau\cap\mathrm{St}(\sigma)=\emptyset,\,\tau\subseteq\rho}(-1)^{|\tau|}\Big).$$

This time the parenthetical sum evaluates to $\sum_{\tau\subseteq\rho-\nu}(-1)^{|\tau|} = 0$ if $\rho\cap\mathrm{St}(\sigma) = \nu\neq\rho$, and evaluates to $\sum_{\tau=\emptyset}(-1)^\tau = 1$ if $\rho\cap\mathrm{St}(\sigma) = \rho$ (i.e., if $\rho\cup\sigma\in N$). This gives

$$G_\sigma(\mathbf{u}) = \sum_{\rho\cup\sigma\in N}(-1)^{|\rho|}u_\rho.$$

This proves that $F_\sigma(\mathbf{y}(\mathbf{u})) = (-1)^{n+|\sigma|}G_\sigma(\mathbf{u})$. The other equation follows from this one and the observation that the change of variables formulas from y_σ's to u_σ's and from u_σ's to y_σ's are identical. ∎

Proof of Theorem 12.1. Since $\{y_\sigma = \beta_\sigma(\mathbf{t})\}_{\sigma\in N}$ is the unique solution to the system (12.2), $\{u_\sigma = \alpha_\sigma(\mathbf{t})\}_{\sigma\in N}$ is the unique solution to

$$\{F_\sigma(\mathbf{y}(\mathbf{u})) = t_\sigma G_\sigma(\mathbf{y}(\mathbf{u}))\ (\forall\sigma\in N),\ \text{and } u_\emptyset = 1\}.$$

By Lemma 12.2, $\{u_\sigma = \alpha_\sigma(\mathbf{t})\}_{\sigma\in N}$ is then the unique solution to

$$\{G_\sigma(\mathbf{u}) = t_\sigma F_\sigma(\mathbf{u})\ (\forall\sigma\in N),\ \text{and } u_\emptyset = 1\}.$$

But as already noted, $\{u_\sigma = \beta_\sigma(\mathbf{t}^{-1})\}_{\sigma\in N}$ is also a solution to this system; hence $\alpha_\sigma(\mathbf{t}) = \beta_\sigma(\mathbf{t}^{-1})$ for all $\sigma\in N$. ∎

Now suppose that μ is chosen as in Theorem 11.3 so that

$$\dim_\mu A_\sigma = \alpha_\sigma(\mathbf{t}) \quad\text{and}\quad \dim_\mu H_\sigma = \beta_\sigma(\mathbf{t}).$$

Then if W has Eulerian nerve, Theorem 12.1 and Lemma 5.5 imply that

$$\dim_{\mu^*} A_\sigma = \alpha_\sigma(\mathbf{t}^{-1}) \quad\text{and}\quad \dim_{\mu^*} H_\sigma = \beta_\sigma(\mathbf{t}^{-1}).$$

We can now give a geometric interpretation of the main result of [15]. Since

$$\chi_\mu = \sum_{\sigma\in N}(-1)^{|\sigma|}\dim_\mu H_\sigma = \sum_{\sigma\in N}(-1)^{|\sigma|}\beta_\sigma(\mathbf{t}) = \frac{1}{W(\mathbf{t})}$$

we have

$$\chi_{\mu^*} = \sum_{\sigma\in N}(-1)^{|\sigma|}\dim_{\mu^*} H_\sigma = \sum_{\sigma\in N}(-1)^{|\sigma|}\beta_\sigma(\mathbf{t}^{-1}) = \frac{1}{W(\mathbf{t}^{-1})}. \tag{12.3}$$

Thus Theorem 7.3 implies the following.

COROLLARY 12.3. *Suppose W is right-angled and the nerve N is an Eulerian $(n-1)$-sphere. Then*

$$W(\mathbf{t}^{-1}) = (-1)^n W(\mathbf{t}).$$

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