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Multiplicative maps from $H\mathbb{Z}$ to a ring spectrum R —a naive version

by

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Abstract. The paper is devoted to the study of the space of multiplicative maps from the Eilenberg–MacLane spectrum $H\mathbb{Z}$ to an arbitrary ring spectrum R. We try to generalize the approach of Schwede [Geom. Topol. 8 (2004)], where the case of a very special R was studied. In particular we propose a definition of a formal group law in any ring spectrum, which might be of independent interest.

0. Introduction. For a commutative ring B Stefan Schwede described in [S1] a surprising connection between stable homotopy theory of commutative B-algebras and formal group laws over B. The stable homotopy operations of commutative simplicial B-algebras are described by the algebra π_*DB, where DB is a certain "classical" spectrum studied by Bousfield, Dwyer and others (see for example [D]). Schwede was able to describe the weak homotopy type of the space of multiplicative maps from the Eilenberg–MacLane spectrum $H\mathbb{Z}$ to DB in terms of formal group laws over B and their isomorphisms. But his methods seem to be much more general and should work in other situations as well. On the other hand the formal group laws over B are present in the description of DB, so in order to generalize his results one should start from defining something like a "formal group law" even without formal power series.

In the present note we offer a definition of a formal group law in a ring spectrum R. With it we recover a weak version of the π_0-result of Schwede with any ring spectrum R in place of DB. The obvious generalization of the full Schwede result is clearly visible but we do not have any evidence to call it even a "conjecture". At present we do not see methods of attacking this problem in full generality.

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Every multiplicative map $H\mathbb{Z} \to R$ gives R the structure of a ring over $H\mathbb{Z}$. We hope that observations presented in this note can be fruitful for a better understanding of the category of such objects.

We use here the language of Lydakis [L], which we summarize in Section 1. A ring spectrum means a Γ-ring in the sense of [L], that is, a functor from the category of finite sets to simplicial sets with an extra structure. Maps between such objects are given in terms of natural transformations of functors. The word "naive" in our title refers to the fact that we work mostly with combinatorial structures only, so we do not have to use (except on the last two pages) models for our spectra which give the correct homotopy type of the mapping space (fibrant-cofibrant replacement).

1. Preliminaries on Γ-spaces and Γ-rings. For a nonnegative integer n, let $[n]$ denote the pointed set $\{0, 1, \ldots, n\}$ with 0 as the basepoint.

Three types of pointed set maps will play a crucial role in the future. Two of them map $[n] \to [n-1]$ and the third one goes the other way around. The map $p^n_i : [n] \to [n-1]$ defined by $p^n_i(j) = j$ for $j < i$, $p^n_i(i) = 0$, $p^n_i(j) = j-1$ for $j > i$ will be called the ith *restriction*. For any $i < j \leq n$ we have the "summing" map $s^n_{i,j,k} : [n] \to [n-1]$ defined by $s^n_{i,j,k}(i) = s^n_{i,j,k}(j) = k$, and mapping the other elements $a \in [n]$ bijectively onto $[n-1] \setminus \{k\}$, preserving ordering. The third map $d^n_j : [n-1] \to [n]$ is injective, misses $j \in [n]$ and preserves the order.

The category Γ^{op} is a full subcategory of the category of pointed sets, with objects all $[n]$. The category of Γ-spaces is the full subcategory of the category of functors from Γ^{op} to pointed simplicial sets with objects satisfying $F[0] = [0]$ and maps given by natural transformations of functors. The notation Γ^{op} comes from the fact that this category is dual to Segal's category Γ from [Se]. Every Γ-space can be prolonged by direct limits to a functor defined on the category of pointed sets. In our notation we will not distinguish between a Γ-space and the extension described above. We will use capital letters $K, L \ldots,$ to denote pointed sets. In the future, if we need an ordering of the pointed set $[n] \wedge [m]$ which identifies it with $[nm]$ we will always use the inverse lexicographical order.

Convention. If it causes no misunderstanding, for pointed sets K and L and a pointed map $f : K \to L$ we write f instead of $F(f)$ for the induced map $F(K) \to F(L)$.

For a Γ-space F let RF denote the $\Gamma \times \Gamma$-space defined as $RF(K, L) = F(K \wedge L)$. Having two Γ-spaces F and F' we can form their exterior smash product $\Gamma \times \Gamma$-space $F \tilde{\wedge} F'$ which is defined by $F \tilde{\wedge} F'(K, L) = F(K) \wedge F'(L)$. Then the smash product of F and F' is a universal Γ-space F'' with a map of $\Gamma \times \Gamma$-spaces $F \tilde{\wedge} F' \to RF''$ (see [L, Remark 2.4]). Moreover, if we denote by $\mathcal{GS}$ the category of Γ-spaces and by $\mathcal{GSS}$ the category of $\Gamma \times \Gamma$-spaces then for given Γ-spaces F_1, F_2 and F_3 we have (following [L, Theorem 2.2])

$$\mathcal{GS}(F_1 \wedge F_2, F_3) = \mathcal{GSS}(F_1 \tilde{\wedge} F_2, RF_3).$$

REMARK 1.1. The symmetric group Σ_n acts on $\{0, 1, \ldots, n\}$ by permuting $\{1, \ldots, n\}$ and hence acts on $F[n]$ for any Γ-space F. We will use this action restricted to various subgroups of Σ_n.

Let $\mathbf{S}$ denote the Γ-space defined by the identity functor. We say that a Γ-space F is a *Γ-ring* if there are maps $\eta : \mathbf{S} \to F$ called the *unit* and $\mu : F \wedge F \to F$ called *multiplication* satisfying the usual associativity and unit conditions (see [L, 2.13]).

REMARK 1.2. By our previous observations μ is determined by a map $\tilde{\mu} : F \tilde{\wedge} F \to RF$, which is fully determined by a collection of maps $\tilde{\mu} : F[n] \wedge F[m] \to F([n] \wedge [m])$ natural in $[n]$ and $[m]$ and satisfying the obvious associativity conditions.

Let us introduce one more notion. We will say that a Γ-ring R is *discrete* if for any pointed set K, $R(K)$ is just a set considered as a simplicial set in the trivial way. Assume that R is a discrete Γ-ring. Then $R[1]$ is a unital monoid with zero. Moreover η takes $1 \in \mathbf{S}[1]$ to the unit of $R[1]$.

REMARK 1.3. Assume that R is a discrete Γ-ring. Then the map $\tilde{\mu} : R(K) \wedge R(L) \to R(K \wedge L)$ of 1.2 is a map of sets which is associative with respect to the smash product of pointed sets. This means that if $p \in R(K)$ and $q \in R(L)$ then it makes sense to say that the product of p and q belongs to $R(K \wedge L)$, which, of course, means that $\tilde{\mu}(p, q) \in R(K \wedge L)$. We will usually write this product as $pq \in R(K \wedge L)$.

2. Multiplicative maps from $H\mathbb{N}$ to a discrete Γ-ring R. We want to study multiplicative maps from the Eilenberg–MacLane spectrum $H\mathbb{Z}$ to a Γ-ring R. This is a bit technical and postponed until Section 3. In the present section we will consider maps from the spectrum stably equivalent to $H\mathbb{Z}$ which is easier to study. Let us start by recalling a Γ-ring model of $H\mathbb{Z}$. As a functor, $H\mathbb{Z}$ takes K to a reduced free abelian group generated by K. The map

$$\eta : \mathbf{S} \to H\mathbb{Z}$$

is given by the embedding of generators. The multiplication map

$$\mu : \tilde{\mathbb{Z}}(K) \wedge \tilde{\mathbb{Z}}(L) \to \tilde{\mathbb{Z}}(K \wedge L)$$

is defined by the formula

$$\Big(\sum_{k \in K} a_k k\Big) \wedge \Big(\sum_{l \in L} b_l l\Big) \mapsto \sum_{k \wedge l \in K \wedge L} a_k b_l (k \wedge l).$$

The Γ-ring $H\mathbb{N}$ is defined by the same formulas but with the additive monoid of natural numbers (with 0) instead of the integers. The embedding $H\mathbb{N} \to H\mathbb{Z}$ induces a stable equivalence of spectra because for any $k > 0$ the map $H\mathbb{N}(S^k) \to H\mathbb{Z}(S^k)$ is a homotopy equivalence by a theorem of Spanier [Sp, Theorem 4.4]. This map is obviously multiplicative but there is no nontrivial multiplicative map going the other way.

It is easy to see that Schwede's map $H\mathbb{Z} \to DB$ associated to a formal group F is uniquely determined by saying that the image of $(1,1) \in H\mathbb{Z}[2]$ is F. We comment on this more later but now we should define the possible images of $(1,1) \in H\mathbb{N}[2]$ in the case of an arbitrary Γ-ring R. Below we give the first definition of a formal group law in a Γ-ring R.

DEFINITION 2.1. A *formal sum law* in a Γ-ring R is an element $w \in R[2]$ with the following properties:

1. $p_1^2(w) = 1$, $p_2^2(w) = 1$,
2. any power $w^k \in R[2^k]$ is fixed under the action of the symmetric group Σ_{2^k}.

THEOREM 2.2. *Let R be a discrete Γ-ring. Then every formal sum law in R determines a multiplicative map $\phi : H\mathbb{N} \to R$.*

Proof. Let w be a formal sum law in R. Let $1_n = (1, \ldots, 1) \in H\mathbb{N}[n]$. We will show that $1_{2^n} \mapsto w^n$ defines the desired map ϕ.

Observe first that any $(n_1, \ldots, n_k) \in \mathbb{N}[k]$ can be presented as the image of 1_n for a certain n. So our map is uniquely determined on the elements 1_n: if for a pointed map $f : [n] \to [k]$ we have $f(1_n) = (n_1, \ldots, n_k)$ then we must have

$$\phi(n_1, \ldots, n_k) = f(w^n).$$

Hence we only have to show that ϕ is well defined by the formula above.

First of all, for a given k-tuple $(n_1, \ldots, n_k) \in \mathbb{N}[k]$ there is a minimal n such that 1_{2^n} maps to $(n_1, \ldots, n_k)$. Obviously n is the minimal natural number such that $2^n \geq \sum_{i=1}^k n_i$. Of course there are many ways of mapping 1_{2^n} to $(n_1, \ldots, n_k)$ but all of them give the same definition of $\phi(n_1, \ldots, n_k)$ because of condition 2 of Definition 2.1.

Assume now that $g(1_{2^m}) = (n_1, \ldots, n_k)$ for a certain map g with $n < m$. Then it is easy to see that g factors through 1_{2^n}. Hence our proof that ϕ is well defined will be finished if we show:

LEMMA 2.2.1. *For any k, w^{2^k} maps to $w^{2^{k-1}}$ under any map f which satisfies $f(1_{2^k}) = 1_{2^{k-1}}$.*

Proof of 2.2.1. Assume first that f takes the last 2^{k-1} coordinates to zero. In other words, using the fact that

$$[2^k] = [2^{k-1}] \wedge [2]$$

we can write

$$f = \mathrm{Id}_{[2^{k-1}]} \wedge p_2^2.$$

Then

$$f(w^{2^k}) = f(w^{2^{k-1}} \cdot w) = (\mathrm{Id}_{[2^{k-1}]} \wedge p_2^2)(w^{2^{k-1}} \cdot w) = w^{2^{k-1}} \cdot 1 = w^{2^{k-1}}.$$

Now observe that, by condition 2, it is enough to consider maps like f above. Any other f' which takes 1_{2^k} to $1_{2^{k-1}}$ differs from f by the action of an element from Σ_{2^k}.

We will finish the proof of the theorem if we show that the map ϕ obtained above is multiplicative. To check this observe first that if $f(1_{2^n}) = (n_1, \dots, n_k)$ and $g(1_{2^m}) = (m_1, \dots, m_l)$ then $(f \wedge g)(1_{2^n} \cdot 1_{2^m}) = (f \wedge g)(1_{2^{n+m}}) = (n_1, \dots, n_k)(m_1, \dots, m_l)$ as elements of $H\mathbb{N}[kl]$. Further,

$$\begin{aligned} \phi((n_1, \dots, n_k) \cdot (m_1, \dots, m_l)) &= \phi(f \wedge g(1_{2^{n+m}})) = f \wedge g\phi(1_{2^{n+m}}) \\ &= f \wedge g(w^n \cdot w^m) = f(w^n) \cdot g(w^m) \\ &= \phi(f(1_{2^n})) \cdot \phi(g(1_{2^m})) \\ &= \phi(n_1, \dots, n_k) \cdot \phi(m_1, \dots, m_l), \end{aligned}$$

and the proof is finished.

We will come to the issue when two formal sum laws give homotopic maps later in a more general setting. From the proof of Theorem 2.2 it is easy to derive the following observation:

REMARK 2.3. Our assumption that R is discrete is not important. We could define a formal sum law in R as a 0-simplex of $R[2]$ and the rest would go through by the same arguments.

3. Multiplicative maps from $H\mathbb{Z}$ to a discrete Γ-ring R. Now we turn to multiplicative maps $H\mathbb{Z} \to R$. We would like to define formal group laws in this situation in such a way that we get the same statement as in 2.2. But first of all let us identify the complications which occur when we allow negative coordinates in our Γ-ring. The problem is that for an arbitrary Γ-ring R there is no natural way of defining maps coming from multiplying one "variable" by -1. That was not a problem in the case of DB. More generally this is not a problem in the case of any Γ-ring coming from the composition of functors

$$T \circ L : \Gamma^{\mathrm{op}} \to \mathrm{Sets}_*$$

where L is the linearization functor from sets to the category B_{free} of free modules over some ring B, and $T : B_{\mathrm{free}} \to \mathrm{Sets}_*$. We plan to study such situations in another paper; now we would like to define the formal group law in full generality overcoming the difficulty described above.

Before the definition we have to describe a particular type of action of $\Sigma_{2^{k-1}} \times \Sigma_{2^{k-1}}$ on $F[2^k]$ for any Γ-space F. This action will be called *special* later on. For any k, let $\pm 1_k := (1,-1)^k \in H\mathbb{Z}[2^k]$. Our convention on ordering smash products of pointed sets yields a splitting $[2^k] = A_+ \vee A_-$ according to the rule that $(1,-1)^k$ has 1 at the coordinates from A_+ and -1 at A_-. In a more formal way we can say that an element $i \in \{1,\ldots,2^k\}$ belongs to A_+ if and only if the binary expansion of $i-1$ has an even number of digits "1". There is also a "coordinate-free" way of describing the splitting. If we identify the set $[2^k]$ with the set of subsets of $\{1,\ldots,k\}$ then A_+ (resp. A_-) consists of sets of even (resp. odd) order.

The special action of $\Sigma_{2^{k-1}} \times \Sigma_{2^{k-1}}$ on $F[2^k]$ is defined as follows: if $a \times b \in \Sigma_{2^{k-1}} \times \Sigma_{2^{k-1}}$ then a permutes the coordinates from A_+ and b permutes the remaining coordinates. Let σ be the nontrivial element of Σ_2.

Definition 3.1. A *formal difference law* in a Γ-ring R is an element $r \in R[2]$ with the following properties:

1. $p_2^2(r) = 1$, $s_{1,2,1}^2(r) = 0$,
2. $p_1^2(r)r = rp_1^2(r) = \sigma(r)$ in $R[2]$,
3. any power $r^k \in R[2^k]$ is fixed under the special action of $\Sigma_{2^{k-1}} \times \Sigma_{2^{k-1}}$,
4. for any k , $i < j$ and l such that $i \in A_+$ and $j \in A_-$ or $j \in A_+$ and $i \in A_-$ we have
$$s_{i,j,l}^{2^k}(r^k) = d_l^{2^k-1} p_i^{2^k-1} p_j^{2^k}(r^k).$$

Observe first that $p_1^2(r)$ plays the role of -1 in $R[1]$ because

$$(p_1^2(r))^2 = (p_1^2 \wedge p_1^2)(r^2) = (p_1^2 \wedge p_1^2) \circ \tau)(r^2) = (p_2^2 \wedge p_2^2)(r^2) = (p_2^2(r))^2 = 1$$

where τ is the special permutation in $\Sigma_2 \times \Sigma_2$ given by the transposition $(1,4)$. We can now compare our new definition with the results and definitions from Section 2. We check that if r is a formal difference law in a Γ-ring R then $w = p_2^3 \circ p_2^4(r^2) \in R[2]$ is a formal sum law in the sense of Definition 2.1. Indeed,

$$p_2^2(w) = p_2^2 \wedge p_2^2(r^2) = (p_2^2(r))^2 = 1$$

and similarly

$$p_1^2(w) = p_1^2 \wedge p_1^2(r^2) = (p_1^2(r))^2 = 1.$$

Moreover

$$\sigma(w) = p_2^3 \circ p_2^4(\tau(r^2)) = p_2^3 \circ p_2^4(r^2)$$

and hence w is fixed under the action of Σ_2. Let p denote $p_2^3 \circ p_2^4$. By naturality of the smash product and multiplication maps we have the following commutative diagram:

$$\begin{array}{ccc} R[4] \wedge R[4] & \longrightarrow & R[16] \\ \big\downarrow & & \big\downarrow \\ R[2] \wedge R[2] & \longrightarrow & R[4] \end{array}$$

where the left vertical arrow is given by $p \wedge p$ and the horizontal arrows are multiplication maps. Then the right vertical map is defined by the set map which takes four elements of A_+ bijectively to nonzero elements of $[4]$ and the remaining elements of $[16]$ to 0. Hence the action of any permutation from Σ_4 on $w \wedge w$ lifts to the special permutation acting on $R[16]$. This argument generalizes easily to higher degrees because $p^{\wedge k}$ maps bijectively 2^k elements of $A_+ \subset [4^k]$ to nonzero elements of $[2^k]$ and has value 0 elsewhere.

Thinking about our definition as if it were the definition of a formal group law in the ordinary sense we can give an interpretation of most of the structure described in 3.1. The element $p_1^2(r)$ plays the role of -1 in the "commutative group structure" defined by r. Hence it commutes "with other elements". Condition 1 is always included in the general definition of a formal group law. The same can be said about condition 3—in the classical case of formal power series this kind of invariance property is indirectly in the definition of a formal group law.

Condition 4 is new and makes the situation technically more complicated. It is hard to attach an abstract meaning to it. This condition is strongly related to the fact that $1 + (-1) = 0$ in $\mathbb{Z}$, which is a very additive condition, having no meaning in the structure of an arbitrary Γ-ring. The simplest explanation which one can imagine for the need of condition 4 is the following: this condition is an extension of the second formula from condition 1 to higher degrees, which is needed when we face the lack of additivity. The good news is that condition 4 is often satisfied in interesting cases, namely in Γ-rings coming from algebraic theories. This is the case of the Γ-ring DB. We are not going to define here what an algebraic theory is and what is the definition of a Γ-ring associated to it. Instead we refer the interested reader to [S3, Section 2].

REMARK 3.2. Let T^s be a Γ-ring associated to the algebraic theory T. Then condition 4 of 3.1 is satisfied for T^s as a consequence of condition 1.

Proof (sketch). We will follow [S2, Section 2] without further explanations. Observe first that condition 4 for $k = 1$ is equivalent to the second equality of condition 1 and hence is satisfied. By definition $T^s[n] = \hom_T([n], [1])$ and the multiplication

$$T^s[n] \wedge T^s[m] \to T^s[nm]$$

is obtained from composition. This means that we can write it as

$$\alpha \wedge \beta \mapsto \beta \circ (\alpha, \dots, \alpha)$$

with our convention of identifying $[n] \wedge [m]$ with $[nm]$. So in the notation above $r^k = r^{k-1} \circ (r, \dots, r)$ and the value of the map $s^{2^k}_{i,j,l}$ on r^k is the same as if we apply $s^2_{1,2,1}$ to one of the r's in the bracket, by naturality and condition 3. So

$$s^{2^k}_{i,j,l}(r^k) = r^{k-1} \circ (r, \dots, r, 0, r, \dots, r) = d_l^{2^k-1} p_i^{2^k-1} p_j^{2^k}(r^k).$$

DEFINITION 3.3. A *homomorphism* $a : r_1 \to r_2$ of formal difference laws in R is an element $a \in R[1]$ satisfying $ar_1 = r_2 a$. An invertible homomorphism is called an *isomorphism*. An isomorphism is called *strict* if it maps to the unit component of R under the map $R[1] \to \pi_0 R$.

Perhaps for completeness it is worth recalling here the definition of the map $R[1] \to \pi_0 R$. According to [S3, Lemma 1.2], $\pi_0 R$ can be presented as the cokernel of the map

$$\tilde{Z}p_2^2 + \tilde{Z}p_1^2 - \tilde{Z}s^2_{1,2,1} : \tilde{Z}[R[2]] \to \tilde{Z}[R[1]].$$

Then our map can be described as an embedding of generators composed with the quotient map described above.

THEOREM 3.4. *Let R be a discrete Γ-ring. Then every formal difference law in R determines a multiplicative map $\phi : H\mathbb{Z} \to R$.*

Proof. Let r be a formal difference law in R. We show that $\pm 1_n \mapsto r^n$ defines the desired map.

Observe first that any $(n_1, \dots, n_k) \in \mathbb{Z}[k]$ can be presented as the image of $\pm 1_n$ for a certain n. Hence our map is uniquely determined on elements $\pm 1_n$ and we only have to show that ϕ is well defined.

We would like to follow the proof of 2.2 but the situation is different now. The proof of 2.2 was based on the fact that $(n_1, \dots, n_k) \in \mathbb{N}[k]$ was the image of 1_n in a unique way up to a permutation which acted trivially on the corresponding power of r. This is not the case now: 1 and -1 from different coordinates in $\mathbb{Z}[k]$ can cancel either by mapping coordinates to the basepoint or by the summing map. In the case of the proof of 2.2 we had only to consider the first possibility.

First of all, as previously, for a given k-tuple $(n_1, \dots, n_k) \in \mathbb{Z}[k]$ there is a minimal n such that $\pm 1_n$ maps to $(n_1, \dots, n_k)$ by a map f'. We can assume that no n_i is zero. There is a special permutation σ such that $f = f' \circ \sigma$ takes the first $|n_1|$ coordinates in $\sigma((\pm 1)^k)$ with the same sign as n_1 to n_1, the next $|n_2|$ coordinates with correct signs to n_2 and so on. Let $N_k = \sum_{i=1}^k |n_i|$. Then all ones and minus ones on the other $n - N_k$ coordinates have to cancel to zero. Assume that $a < b$ and we have 1 on the ath coordinate and -1 on the bth and they add to 0. Then obviously $f = f \circ d_b^{2^n} \circ s_{a,b,a}^{2^n}$ as maps of pointed sets and we can iterate this process composing with more pairs of maps $d_*^{2^n} \circ s_{*,*,*}^{2^n}$. But by condition 4 we have

$$f(r^n) = f \circ d_b^{2^n} \circ s_{a,b,a}^{2^n}(r^n) = f \circ d_b^{2^n} \circ d_a^{2^n-1} \circ p_a^{2^n-1} \circ p_b^{2^n}(r^n).$$

Hence the value of f on r^n is the same as the value of a map which takes a and b to the basepoint. Iterating this process we obtain

LEMMA 3.4.1. *Let $g : [2^n] \to [k]$ be a map which agrees with f on the N_k elements chosen as described above and takes the rest to the base point. Then*

$$f(r^n) = g(r^n).$$

Hence our map ϕ is well defined. Whichever map f taking $\pm 1_n$ to $(n_1, \ldots, n_k)$ we use, it will have the same value on r^n as the map g from 3.4.1. Checking that if $f(\pm 1_n) = (n_1, \ldots, n_k) = h(\pm 1_l)$ then the two definitions of $\phi(n_1, \ldots, n_k)$ agree goes essentially as in the proof of 2.2 and is left to the reader. Similarly one can show the multiplicativity of ϕ.

REMARK 3.5. Any multiplicative map $\phi : H\mathbb{Z} \to R$ determines a formal difference law in R, given by $r = \phi(\pm 1_2)$. Hence the set of formal difference laws in R is in natural bijection with the set of multiplicative maps $H\mathbb{Z} \to R$.

We now show how our definition works in known cases, for example in the case of the spectrum DB. In Section 2 we mentioned that every formal sum law in DB determines a formal group law in the ordinary sense. Observe now that a formal difference law $r \in R[2]$ in the sense of Definition 3.1 determines its sum version $w \in R[2]$ by the formula

$$w = p_2^3 \circ p_2^4(r^2).$$

Moreover the map $H\mathbb{N} \to R$ defined by w factors through the map $H\mathbb{Z} \to R$ defined by r.

As another example we consider the case of the endomorphism Γ-ring. This notion is probably less known so we sketch the definition following [S1, 13.3].

EXAMPLE. Let $\mathcal{C}$ be a category with a 0-object and finite coproducts. The natural enrichment of $\mathcal{C}$ over Γ^{op} is given by

$$X \wedge [k] = X \sqcup \cdots \sqcup X \quad (k\text{-fold coproduct}).$$

For every object X in $\mathcal{C}$ the endomorphism Γ-ring denoted by $\mathrm{End}_{\mathcal{C}}(X)$ is defined by

$$\mathrm{End}_{\mathcal{C}}(X)([k]) = \mathrm{Hom}_{\mathcal{C}}(X, X \wedge [k]).$$

The unit map $\mathbf{S} \to \mathrm{End}_{\mathcal{C}}(X)$ comes from the identity map in $\mathrm{End}_{\mathcal{C}}(X)([1])$.

Multiplication is induced by the composition product

$$\mathrm{End}_{\mathcal{C}}([k])(X) \wedge \mathrm{End}_{\mathcal{C}}(X)([l]) \to \mathrm{End}_{\mathcal{C}}(X)([k] \wedge [l]),$$
$$f \wedge g \mapsto (f \wedge [l]) \circ g.$$

As Schwede points out, every abelian cogroup object structure on X determines a map $H\mathbb{Z} \to \mathrm{End}_{\mathcal{C}}(X)$ defined as follows. The map

$$H\mathbb{Z}([k]) = \tilde{\mathbb{Z}}[k] \to \mathrm{Hom}_{\mathcal{C}}(X, X \wedge [k])$$

is an additive extension of the map which sends $i \in [k]$ to the ith coproduct inclusion $X \to X \sqcup \cdots \sqcup X$.

Observe now that every formal difference law $r \in \mathrm{End}_{\mathcal{C}}(X)[2] = \mathrm{Hom}_{\mathcal{C}}(X, X \sqcup X)$ defines an abelian cogroup structure on X. The co-addition is given by a sum version of r,

$$p_2^3 \circ p_2^4(r^2) \in \mathrm{Hom}_{\mathcal{C}}(X, X \sqcup X).$$

It is abelian because of the invariance of formal sum laws under permutations. For the same reason the associativity condition is fulfilled. The co-inverse is given by $p_1^2(r) \in \mathrm{End}_{\mathcal{C}}(X)([1]) = \mathrm{Hom}_{\mathcal{C}}(X, X)$. The co-unit equals $s_{1,2,1}^2(r) \in \mathrm{End}_{\mathcal{C}}(X)([1]) = \mathrm{Hom}_{\mathcal{C}}(X, X)$. The map $H\mathbb{Z} \to \mathrm{End}_{\mathcal{C}}(X)$ described by Schwede (and recalled above) agrees with the one obtained by Theorem 3.4 from r.

We suggest that the reader works out by himself how our theory works in the case of matrix Γ-rings (see [S1, 13.5]). Below we come back to the question when two formal difference laws define homotopic maps $H\mathbb{Z} \to R$.

THEOREM 3.6. *Two strictly isomorphic formal difference laws determine homotopic maps in the space of maps $H\mathbb{Z} \to R$.*

Recall that if $a \in R[1]$ then multiplication by a on the left or on the right determines the map $m_a : R \to R$. Because left and right multiplications are formally the same we will assume that m_a comes from multiplication on the left. Theorem 3.6 follows easily from the following lemma.

LEMMA 3.7. *Assume that $a, b \in R[1]$ determine the same element in $\pi_0(R)$ under the obvious map $R[1] \to R$. Then the multiplication maps m_a and m_b are homotopic.*

Proof. The conclusion follows directly from the definitions if one carefully examines what it means that $a \in R[1]$ determines an element in $\pi_0(R)$. Observe that choosing $a \in R[1]$ we uniquely choose a map $f_a : \mathbf{S} \to R$: it is fully described on the set $[1]$ where we put $\mathbf{S}[1] \ni 1 \mapsto a \in R[1]$. In higher degrees our map is determined by this data because every $i \in \mathbf{S}[n]$ can be viewed as the image of the map $[1] \to [n]$ taking 1 to i.

Moreover observe that on the set $[1]$ our map can be described as the unit map η multiplied on the left by a. By naturality of the multiplication map, f_a is equal to η composed with the left multiplication by a. Observe now that we can decompose m_a as

$$R \to \mathbf{S} \wedge R \to R \wedge R \to R$$

where the first map is an isomorphism, the second is $f_a \wedge \mathrm{id}$ and the third is given by the multiplicative structure μ of the Γ-ring R.

Now we can come back to the proof of 3.7. By assumption, f_a and f_b determine homotopic maps of spectra. Therefore $f_a \wedge \mathrm{id}$ is homotopic to $f_b \wedge \mathrm{id}$ and hence m_a and m_b give homotopic maps of spectra.

Let us come back to the proof of 3.6. We know that r_1 and r_2 are strictly isomorphic, and the isomorphism is given by an invertible element $a \in R[1]$. Let ϕ_1, ϕ_2 denote the maps $H\mathbb{Z} \to R$ determined by r_1, r_2 respectively. Then

$$\phi_2 = m_{a^{-1}} \circ \phi_1 \circ m_a.$$

By assumption, m_a and $m_{a^{-1}}$ are homotopic to m_1, hence to the identity map. This finishes the proof of 3.6.

The referee suggested the following interesting generalization of the considerations above to the case when R is not discrete. For every natural n, the nth simplicial degree of $R[K]$ assemble to a discrete Γ-ring R_n. Hence, in the case of R not discrete, we can talk about the simplicial set $\mathrm{FDL}(R)$ of formal difference laws in R which in degree n has the set of formal difference laws in R_n. Similarly we can consider the simplicial set $\Gamma(H\mathbb{Z}, R)$ of multiplicative maps from $H\mathbb{Z}$ to R which in degree n has the set of such maps to R_n. Then Theorem 3.6 can be stated as

THEOREM 3.8. *The simplicial sets* $\mathrm{FDL}(R)_*$ *and* $\Gamma(H\mathbb{Z}, R)_*$ *are naturally isomorphic.*

Moreover one can take into account the action of the invertible elements of $R[1]$. Let G_n be the group of invertible elements in $R_n[1]$. They assemble to a simplicial group G_* and this simplicial group acts on both simplicial sets from 3.8 by conjugation. With this structure in mind we can generalize 3.6 to.

THEOREM 3.9. *The homotopy orbit sets* $\mathrm{FDL}(R)_{hG_*}$ *and* $\Gamma(H\mathbb{Z}, R)_{hG_*}$ *are isomorphic.*

Now we would like to comment a little on the homotopical meaning of our constructions. As one can see, the proof of 3.6 was derived directly from the definitions. But of course we would like to know whether two strictly isomorphic formal difference laws define homotopic maps in the space of multiplicative maps from $H\mathbb{Z} \to R$ or, equivalently, the same element in the 0th homotopy group of the space of multiplicative maps $H\mathbb{Z} \to R$ as in [S1]. The answer here is not easy to achieve or even to conjecture. Our constructions depended heavily on the small model of $H\mathbb{Z}$ which is not cofibrant.

Schwede's homotopical calculations were possible also because of the definition of DB-spectrum and its closed relations to symmetric algebra. With the lack of these structures we can only propose the following weak homotopical statement:

PROPOSITION 3.10. *Let r_1 and r_2 be two strictly isomorphic formal difference laws in a Γ-ring R. There exists a weak equivalence of Γ-rings $h : R \to R_3$ such that the maps defined by r_1 and r_2 composed with h are homotopic in the space of multiplicative maps $H\mathbb{Z} \to R_3$.*

Proof. We will be sketchy here because the proof is taken directly from [S1]. For any Γ-ring R, the invertible elements in $R[1]$ which map to the unit component of R form a group G which acts by conjugation on $R[2]$ and in general on R. Two formal group laws $r_1, r_2 \in R[2]$ are strictly isomorphic if they are in the same orbit of this action. Our problem would be solved if we could extend this conjugation action to the action of the whole unit component of R.

Following [S1, Section 3] we first choose R^f to be a stably fibrant replacement of R in a correct model category structure. Then we define the homotopy units R^* as the union of the invertible components of the simplicial monoid $R^f[1]$. We have $\pi_0 R^* = \mathrm{units}(\pi_0 R)$ and $\pi_i R^* = \pi_i R$ for $i \geq 1$. The stable equivalence $R \to R^f$ gives a homomorphism $\phi : G \to R^f[1]$ of simplicial monoids with image in R^*. We want to extend the conjugation action of G to R^*. The problem is that the conjugation action uses strict inverses while R^* is only a group-like simplicial monoid. Getting around this difficulty goes in several steps (see [S1, Section 4] for the details).

STEP 1. We factor the map $G \to R^*$ into $G \to cR^* \to R^*$ in the correct model category structure of simplicial monoids where the first map is a cofibration and the second an acyclic fibration. Let UR^* denote the group completion of cR^*. Then UR^* is a simplicial group and Lemma 4.3 of [S1] tells us that $cR^* \to UR^*$ is a weak equivalence.

STEP 2. Let $\mathbf{S}[cR^*]$ be the monoid Γ-ring with coefficients in the sphere spectrum. We take the obvious map $\mathbf{S}[cR^*] \to R^f$ and factor it in the model category of Γ-rings as a cofibration followed by an acyclic fibration

$$\mathbf{S}[cR^*] \to R_1 \to R^f.$$

Then we define another Γ-ring R_2 as a pushout, in the category of Γ-rings, of

$$\begin{array}{ccc} \mathbf{S}[cR^*] & \longrightarrow & R_1 \\ \downarrow & & \downarrow \\ \mathbf{S}[UR^*] & \longrightarrow & R_2 \end{array}$$

Lemma 4.4 of [S1] tells us that the map $R_1 \to R_2$ is a stable equivalence.

STEP 3. Now we define R_3 to be a stably fibrant replacement of R_2. The induced map $\mathbf{S}[UR^*] \to R_3$ induces a weak equivalence between UR^* and the invertible components of $R_3[1]$. The simplicial group UR^* acts by conjugation on R_3 via homomorphisms of Γ-rings and this action extends the action of G.

Final remark. Of course it is tempting to speculate that the weak homotopy type of the full space of multiplicative maps $H\mathbb{Z} \to R$ should be described via the classifying space of the groupoid of formal difference laws and strict isomorphism, as is proved in [S1] in the case of the spectrum DB. So far we do not see how to attack this problem in full generality.

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