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Local analysis for semi-bounded groups

by

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Abstract. An o-minimal expansion $\mathcal{M} = \langle M, <, +, 0, \dots \rangle$ of an ordered group is called *semi-bounded* if it does not expand a real closed field. Possibly, it defines a real closed field with bounded domain $I \subseteq M$. Let us call a definable set *short* if it is in definable bijection with a definable subset of some I^n, and *long* otherwise. Previous work by Edmundo and Peterzil provided structure theorems for definable sets with respect to the dichotomy 'bounded versus unbounded'. Peterzil (2009) conjectured a refined structure theorem with respect to the dichotomy 'short versus long'. In this paper, we prove Peterzil's conjecture. In particular, we obtain a quantifier elimination result down to suitable existential formulas in the spirit of van den Dries (1998). Furthermore, we introduce a new closure operator that defines a pregeometry and gives rise to the refined notions of 'long dimension' and 'long-generic' elements. Those are in turn used in a local analysis for a semi-bounded group G, yielding the following result: on a long direction around each long-generic element of G the group operation is locally isomorphic to $\langle M^k, + \rangle$.

1. Introduction. For an o-minimal expansion $\mathcal{M} = \langle M, <, +, 0, \dots \rangle$ of an ordered group, there are naturally three possibilities: $\mathcal{M}$ is either (a) linear, (b) semi-bounded (and non-linear), or (c) it expands a real closed field. Let us define the first two.

Definition 1.1. Let Λ be the set of all partial $\emptyset$-definable endomorphisms of $\langle M, <, +, 0 \rangle$, and $\mathcal{B}$ the collection of all bounded definable sets. Then $\mathcal{M}$ is called *linear* ([LP]) if every definable set is already definable in $\langle M, <, +, 0, \{\lambda\}_{\lambda \in \Lambda} \rangle$, and it is called *semi-bounded* ([Ed, Pet1]) if every definable set is already definable in $\langle M, <, +, 0, \{\lambda\}_{\lambda \in \Lambda}, \{B\}_{B \in \mathcal{B}} \rangle$.

Obviously, if $\mathcal{M}$ is linear then it is semi-bounded. By [PeSt], $\mathcal{M}$ is not linear if and only if there is a real closed field defined on some bounded interval. By [Ed], $\mathcal{M}$ is not semi-bounded if and only if $\mathcal{M}$ expands a real closed field if and only if for any two intervals there is a definable bijection between them.

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An important example of a semi-bounded non-linear structure is the expansion of the ordered vector space $\langle \mathbb{R}, <, +, 0, x \mapsto \lambda x\rangle_{\lambda\in\mathbb{R}}$ by all bounded semialgebraic sets.

It is largely evident from the literature that among the three cases, (a) and (c) have provided the most accommodating settings for studying general mathematics. For example, the definable sets in a real closed field are the main objects of study in semialgebraic geometry (a classical reference is [DK]). Moreover, o-minimal linear topology naturally extends the classical subject of piecewise linear topology and has the potential to tackle problems that arise in the study of algebraically closed valued fields (see, for example, [HL]). From an internal aspect, the study of definable groups in both of these two settings has been rather successful (see further comments below).

On the other hand, the middle case (b) remains as elusive as interesting from a classification point of view. Although a local field may be definable, and thus the definable structure can get quite rich, there is no global field, and hence many known techniques do not apply. In particular, little is known on the structure of definable groups in this setting. In this paper, we set forth a project of analyzing semi-bounded groups, mainly motivated by two conjectures asked by Peterzil in [Pet3]. Let us describe our project.

For the rest of the paper, we fix a semi-bounded o-minimal expansion $\mathcal{M} = \langle M, <, +, 0, \dots\rangle$ *of an ordered group which is not linear. We fix an element* $1 > 0$ *such that a real closed field whose universe is* $(0, 1)$ *and whose order agrees with* $<$ *is definable in* $\mathcal{M}$.

Let $\mathcal{L}$ denote the underlying language of $\mathcal{M}$. By ‘definable’ we mean ‘definable in $\mathcal{M}$’ possibly with parameters. A group G is said to be *definable* if both its domain and its group operation are definable. Definable sets and groups in this setting are also referred to as *semi-bounded*. If they are defined in the *linear reduct* $\mathcal{M}_{\text{lin}} = \langle M, <, +, 0, \{\lambda\}_{\lambda\in\Lambda}\rangle$ of $\mathcal{M}$, we call them *semi-linear*. The underlying language of $\mathcal{M}_{\text{lin}}$ is denoted by $\mathcal{L}_{\text{lin}}$.

Following [Pet3], an interval $I \subseteq M$ is called *short* if there is a definable bijection between I and $(0, 1)$; otherwise, it is called *long*. Equivalently, an interval $I \subseteq M$ is short if a real closed field whose domain is I is definable. An element $a \in M$ is called *short* if either $a = 0$ or $(0, |a|)$ is a short interval; otherwise, it is called *tall*. A tuple $a \in M^n$ is called *short* if $|a| := |a_1| + \cdots + |a_n|$ is short, and *tall* otherwise. A definable set $X \subseteq M^n$ (or its defining formula) is called *short* if it is in definable bijection with a subset of $(0, 1)^n$; otherwise, it is called *long*. Notice that this is compatible, for $n = 1$, with the notion of a short interval.

In [Pet1] and [Ed] the authors proved structure theorems about definable sets and functions. (See also [Bel] for an analysis of semibounded sets in a different context.) The gist of those theorems was that the definable sets can be decomposed into 'cones', which are bounded sets 'stretched' along some unbounded directions. Conjecture 1 from [Pet3] asks if we can replace 'bounded' by 'short', and 'unbounded' by 'long', in the definition of a cone and still obtain a structure theorem. We answer this affirmatively (the precise terminology to be given in Section 2 below).

THEOREM 3.8 (Refined Structure Theorem). *Every A-definable set $X \subseteq M^n$ is a finite union of A-definable long cones.* (*In particular, a short set is a* 0*-long cone.*) *Furthermore, for every A-definable function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$, there is a finite collection $\mathcal{C}$ of A-definable long cones, whose union is X and such that f is almost linear with respect to each long cone in $\mathcal{C}$.*

As noted in Remark 3.9 below, it is not always possible to achieve *disjoint* unions in our theorem.

This theorem implies, in particular, a quantifier elimination result down to suitable existential formulas in the spirit of [vdD1] (see our Corollary 3.10). The proof of the Refined Structure Theorem involves an induction on the 'long dimension' of definable sets, which is a refinement of the notion of 'linear dimension' from [Ed].

We then turn our attention to semi-bounded groups. Groups definable in o-minimal structures have been a central object of study in model theory. The climax of that study was the work around Pillay's Conjecture (PC) and Compact Domination Conjecture (CDC), stated in [Pi3] and [HPP1], respectively. In the linear case, (PC) was solved in [ElSt] and (CDC) in [El]. The proofs involved a structure theorem for semi-linear groups from [ElSt] that states that every such group is a quotient of a suitable convex subgroup of $\langle M^n, +\rangle$ by a lattice. In the field case, (PC) was solved in [HPP1], and (CDC) in [HP, HPP2] (see also [Ot] for an overview of all preceding work). In the case of semi-bounded groups, (PC) was solved in [Pet3] after developing enough theory to allow the combination of the linear and the field cases. The (CDC) for semi-bounded groups remains open. Conjecture 2 from [Pet3] asks if we can prove a structure theorem for semi-bounded groups in the spirit of [ElSt]. In the second part of this paper, we prove a local theorem for semi-bounded groups which we see as a first step towards [Pet3, Conjecture 2].

The proof of the local theorem involves a new notion of a closure operator in $\mathcal{M}$, the 'short closure operator' scl, which makes $(\mathcal{M}, \mathrm{scl})$ into a pregeometry. The arising notion of dimension coincides with the long dimension (Corollary 5.10). This allows us to make use of desirable properties of 'long-generic' elements and 'long-large' sets, by virtue of Claim 5.13 below. The local theorem is the following:

THEOREM 6.3. *Let $G = \langle G, \oplus\rangle$ be a definable group of long dimension k. Then every long-generic element a in G is contained in a k-long cone $C \subseteq G$*

such that for every $x, y \in C$,

$$x \ominus a \oplus y = x - a + y.$$

In particular, on C, the group G is locally isomorphic to $\langle M^k, +\rangle$.

We expect that Theorem 6.3 will be the starting point in subsequent work for analyzing semi-bounded groups globally.

Structure of the paper and a few words about the proofs. Section 2 contains basic definitions and preparatory lemmas about the main objects we are dealing with in this paper: the set Λ, long cones and long dimension.

Section 3 contains the proof of three main statements: Lemma on Subcones 3.1, Lemma 3.6(v) on long dimension of unions, and the Refined Structure Theorem 3.8. These statements refine the corresponding ones from [Ed], and so do their proofs. A new phenomenon, however, is that the relative position of two long cones can now range over a bigger range of possibilities. This is because long cones are not necessarily unbounded (which was the case with the cones used in [Ed]). The Lemma on Subcones, as well as Lemma 2.16 from Section 2, provide two main tools for controlling this situation.

Some difficulties involved in handling the long dimension are worked out in Section 4, and they are the following: although it is fairly easy to see that a definable set X which is the cartesian product of two definable sets with long dimensions l and m has long dimension $l + m$ (Lemma 3.6(iv)), it is not a priori clear why if a definable set X is the union of a definable family of fibers each of long dimension m over a set of long dimension n, then X has long dimension $n + m$. We establish this in Lemma 4.2.

Section 5 deals with the new pregeometry coming from the 'short closure operator'.

In Section 6 we prove the local theorem for semi-bounded groups.

2. Basic notions and lemmas. We assume familiarity with the basic notions from o-minimality, such as the inductive definition of cells either as graphs or 'cylinders' of definable continuous functions, the cell decomposition theorem, dimension, generic elements, definable closure, etc. The reader may consult [vdD2] or [Pi2] for these notions.

LEMMA 2.1. *Let $f : I \to M$ be a definable function, where I is a long interval. If $f(I)$ is short, then f is piecewise constant except for a finite collection of short subintervals of I.*

Proof. The function f is piecewise strictly monotone or constant. If it were strictly monotone on a long subinterval of I, then on that subinterval f would be a definable bijection between a long interval and a short set. ■

LEMMA 2.2. *Let $f : X \subseteq M^n \to M$ be a definable function. For every $i = 1, \dots, n$, and $\bar{x}^i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in M^{n-1}$, let*

$$X_{\bar{x}^i} = \{x_i \in M : (x_1, \dots, x_n) \in X\}$$

be the fiber of X above $\bar{x}^i$ and $f_{\bar{x}^i} : X_{\bar{x}^i} \to M$ the map $f_{\bar{x}^i}(x_i) = f(\bar{x})$. Consider the set

$$A = \{\bar{a} \in X : \forall i \in \{1, \dots, n\},\ f_{\bar{a}^i} \text{ is monotone in an interval containing } a_i\}.$$

Then $\dim(X \setminus A) < \dim(X)$.

Proof. We may assume that f and X are $\emptyset$-definable. The set A is then also $\emptyset$-definable and it clearly contains every generic element of X. ∎

2.1. Properties of Λ. The definition of a long cone in the next subsection requires the notion of M-independence for elements of Λ^n. We define this notion and elaborate on it sufficiently in this subsection. Let us first fix some of our standard terminology and notation.

By a *partial endomorphism* of $\langle M, <, +, 0\rangle$ we mean a map $f : (a, b) \to M$ such that for every $x, y, x + t, y + t \in (a, b)$,

$$f(x + t) - f(x) = f(y + t) - f(y).$$

As we said in the introduction, Λ denotes the set of all $\emptyset$-definable partial endomorphisms. A definable function $f : A \subseteq M^n \to M$ is called *affine on A* if it has the form

$$f(x_1, \dots, x_n) = \lambda_1 x_1 + \dots + \lambda_n x_n + a$$

for some fixed $\lambda_i \in \Lambda$ and $a \in M$. For every $i = 1, \dots, n$, we denote by

$$e_i = (0, \dots, 0, 1, 0, \dots, 0)$$

the standard ith unit vector from Λ^n, where $1 : M \to M$ is the identity map. For $v \in \Lambda$, we denote by $\mathrm{dom}(v)$ and $\mathrm{ran}(v)$ the domain and range of v, respectively. We write vt for $v(t)$. Following [Pet3], we consider the equivalence relation $\sim$ on Λ where $\lambda, \mu \in \Lambda$ are said to be *$\sim$-equivalent* if there are $\epsilon > 0$, $a \in \mathrm{dom}(\lambda)$ and $b \in \mathrm{dom}(\mu)$ such that the restrictions of the maps $\lambda(a + x) - \lambda(a)$ and $\mu(b + x) - \mu(b)$ to $(-\epsilon, \epsilon)$ are the same. (That is, those maps have the same germ at 0.) It is observed in [Pet3, Section 6] that Λ modulo $\sim$ can be given the structure of an ordered field with multiplication given by composition. This implies in particular that

(2.1) for every $\lambda, \mu \in \Lambda$ and $x \in \mathrm{dom}(\lambda\mu) \cap \mathrm{dom}(\mu\lambda)$, $\lambda\mu(x) = \mu\lambda(x)$.

We also recall from [LP, Proposition 4.1] that

(2.2) if two partial endomorphisms agree at some non-zero point of their domain then they agree at any other point of their common domain.

It is a standard practice in this paper that whenever we write an expression of the form 'vt', with $v \in \Lambda$ and $t \in M$, we mean in particular that $t \in \operatorname{dom}(v)$. Sometimes, however, we say explicitly that $t \in \operatorname{dom}(v)$. For a matrix $A = (a_{ij})$ with entries from Λ, the *rank* of A is the rank of the matrix $\bar{A} = (\bar{a}_{ij})$, where $\bar{a}_{ij}$ is the $\sim$-equivalence class of a_{ij}. It is then a routine matter to check, using notes (2.1) and (2.2) above, that various classical results from linear algebra hold for matrices with entries from Λ. For example, an $n \times n$ linear system with coefficients from Λ has a unique solution if and only if the coefficient matrix has rank n. We freely use such results in this paper.

We now proceed to the notion of M-independence.

DEFINITION 2.3. If $v = (v_1, \ldots, v_n) \in \Lambda^n$ and $t \in M$, we define $vt := (v_1 t, \ldots, v_n t)$ and $\operatorname{dom}(v) := \bigcap_{i=1}^n \operatorname{dom}(v_i)$. We say that $v_1, \ldots, v_k \in \Lambda^n$ are *M-independent* if for all $t_1, \ldots, t_k \in M$ with $t_i \in \operatorname{dom}(v_i)$,

$$v_1 t_1 + \cdots + v_k t_k = 0 \quad \text{implies} \quad t_1 = \cdots = t_k = 0.$$

If $v = (v_1, \ldots, v_n) \in \Lambda^n$ and $\mu \in \Lambda$, we denote $\mu v := (\mu v_1, \ldots, \mu v_n)$. We say that $v_1, \ldots, v_k \in \Lambda^n$ are *Λ-independent* if for all $\mu_1, \ldots, \mu_k$ in Λ, with $\operatorname{ran}(v_i) \subseteq \operatorname{dom}(\mu_i)$,

$$\mu_1 v_1 + \cdots + \mu_k v_k = 0 \quad \text{implies} \quad \mu_1 = \cdots = \mu_k = 0.$$

The proofs of the following two lemmas are straightforward computations but we include them anyway for completeness.

LEMMA 2.4. *For $v_1, \ldots, v_l \in \Lambda^n$ with common domain $(-a, a) \subseteq M$, the following are equivalent:*

(i) *$v_1, \ldots, v_l$ are M-independent.*
(ii) *$v_1, \ldots, v_l$ are Λ-independent.*
(iii) *The set*

$$H = \{v_1 t_1 + \cdots + v_l t_l : -a < t_i < a\}$$

has dimension l. (This was called an 'open l-parallelogram' in [ElSt].)

Proof. (i)$\Rightarrow$(ii). This is essentially a straightforward application of (2.1) and (2.2) above, but we include the complete proof in the interests of completeness. If $v_1, \ldots, v_l$ are Λ-dependent, then there are $\mu_1, \ldots, \mu_l \in \Lambda$ with $\operatorname{ran}(v_i) \subseteq \operatorname{dom}(\mu_i)$, not all 0, such that $\mu_1 v_1 + \cdots + \mu_l v_l = 0$. In particular, the domain of each μ_i contains some interval containing 0 (because so does the range of v_i). So we can restrict μ_i so that its range contains that interval and is contained in the domain of v_i.

We claim that for any $t \neq 0$ in the domain of all μ_i's, we have $v_1(\mu_1 t) + \cdots + v_l(\mu_l t) = 0$, which will show that the v_i's are M-dependent. To prove the claim we will need to use commutativity between elements of Λ. We explain how this is precisely done.

By restricting μ_i even more, we can assume that the domain of μ_i is also contained in the domain of v_i. Let us call that new restriction μ_i'. We want to show that for some $t \neq 0$ in the domain of μ_i', we have

$$v_i\mu_i'(t) = \mu_i v_i(t), \tag{2.3}$$

where now all arguments make sense.

If we look at the germs of μ_i and μ_i', they are the same. Hence the germs of the maps $v_i\mu_i'$ and $\mu_i v_i$ are also the same. So the maps $v_i\mu_i'$ and $\mu_i v_i$ are equal at any t that lies in both of their domains, by (2.2) above. This finishes the proof of (2.3).

We conclude that there is $t \neq 0$ so that

$$v_1(\mu_1' t) + \cdots + v_l(\mu_l' t) = (\mu_1 v_1 + \cdots + \mu_l v_l)(t) = 0.$$

(ii)⇒(iii). Since $v_i = \begin{pmatrix} v_i^1 \\ \vdots \\ v_i^n \end{pmatrix}$, $i = 1, \ldots, l$, are Λ-independent, the matrix

$$A = \begin{pmatrix} v_1^1 & \ldots & v_l^1 \\ \vdots & \ldots & \vdots \\ v_1^n & \ldots & v_l^n \end{pmatrix}$$

has rank l. Clearly, it is enough to prove that

the map $f : (-a, a)^l \to M^n$ with $x \mapsto Ax$ is injective and onto H.

This claim can be proved by induction on n. For the base step, if A is a 1×1 matrix, we observe that if λ is not the zero endomorphism, then it must be non-zero at any non-zero point, by (2.2). So the kernel is 0.

The inductive step is a straightforward argument, which we omit.

(iii)⇒(ii). This is an easy adaptation of the proof of [El, Corollary 2.5]. ∎

Lemma 2.5. *Let $v_1, \ldots, v_l \in \Lambda^n$ be Λ-independent with $\bigcap_{i=1}^l \mathrm{dom}(v_i) \neq \emptyset$ and denote by $\pi : \Lambda^n \to \Lambda^{n-1}$ the usual projection. The following are equivalent:*

- (i) *There are $\lambda_1, \ldots, \lambda_{n-1} \in \Lambda$ such that for all $t_1, \ldots, t_l \in M$ with $t_i \in \mathrm{dom}(v_i)$, $v_1t_1 + \cdots + v_lt_l$ has the form*

$$v_1t_1 + \cdots + v_lt_l = (a_1, \ldots, a_{n-1}, \lambda_1 a_1 + \cdots + \lambda_{n-1}a_{n-1}).$$

- (ii) *$\pi(v_1), \ldots, \pi(v_l)$ are Λ-independent.*

Proof. (i)⇒(ii). The assertion from (i) says that the last coordinate of $v_1t_1 + \cdots + v_lt_l$ is a function of the first $n-1$ coordinates. Therefore the projections under π of any two distinct elements from the set $\{v_1t_1 + \cdots + v_lt_l : t_i \in \mathrm{dom}(v_i)\}$ are distinct. We claim that the projections $\pi(v_1), \ldots, \pi(v_l)$ are Λ-independent. Indeed, if they are not, then one of them, say $\pi(v_l)$, can be written as a linear combination $\mu_1\pi(v_1) + \cdots + \mu_{l-1}\pi(v_{l-1})$. But then, for any $a \in \bigcap_{i=1}^{l} \mathrm{dom}(v_i)$, the elements $v_l a$ and $(\mu_1 v_1 + \cdots + \mu_{l-1} v_{l-1})a$ would have the same projection, a contradiction.

(ii)⇒(i). We need to compute the λ_i's and a_i's. Assume $v_i = (v_i^1, \ldots, v_i^n)$. Since $\pi(v_1), \ldots, \pi(v_l)$ are Λ-independent, the system

$$\begin{aligned} v_1^n &= \lambda_1 v_1^1 + \cdots + \lambda_{n-1} v_1^{n-1}, \\ &\vdots \\ v_l^n &= \lambda_1 v_l^1 + \cdots + \lambda_{n-1} v_l^{n-1} \end{aligned}$$

has a unique solution for $\lambda_1, \ldots, \lambda_{n-1}$. The above equations imply that

$$v_1^n t_1 + \cdots + v_l^n t_l = \lambda_1(v_1^1 t_1 + \cdots + v_l^1 t_l) + \cdots + \lambda_{n-1}(v_1^{n-1} t_1 + \cdots + v_l^{n-1} t_l)$$

and hence

$$v_1 t_1 + \cdots + v_l t_l = (a_1, \ldots, a_{n-1}, \lambda_1 a_1 + \cdots + \lambda_{n-1} a_{n-1}),$$

where $a_i = v_1^i t_1 + \cdots + v_l^i t_l$ for $i = 1, \ldots, n-1$. ■

Here is another lemma.

LEMMA 2.6. *Let $v_1, \ldots, v_l \in \Lambda^n$ be M-independent. Then, for every $t_1, \ldots, t_l \in M$ with $t_i \in \mathrm{dom}(v_i)$,*

$$v_1 t_1 + \cdots + v_l t_l \textit{ is short} \ \Rightarrow\ t_1, \ldots, t_l \textit{ are short.}$$

Proof. Since $v_i = \begin{pmatrix} v_i^1 \\ \vdots \\ v_i^n \end{pmatrix}$, $i = 1, \ldots, l$, are Λ-independent, the matrix

$$A = \begin{pmatrix} v_1^1 & \ldots & v_l^1 \\ \vdots & \ldots & \vdots \\ v_1^n & \ldots & v_l^n \end{pmatrix}$$

has rank l. Let B be an $l \times l$ submatrix of A of rank l. Then $B \begin{pmatrix} t_1 \\ \vdots \\ t_l \end{pmatrix} = \begin{pmatrix} s_1 \\ \vdots \\ s_l \end{pmatrix}$ for some short $s_1, \ldots, s_l \in M$. Hence $\begin{pmatrix} t_1 \\ \vdots \\ t_l \end{pmatrix} = B^{-1} \begin{pmatrix} s_1 \\ \vdots \\ s_l \end{pmatrix}$ and each row of the last matrix consists of a short element. ■

The following two lemmas will be used in the proof of the Lemma on Subcones 3.1 below.

LEMMA 2.7. *Let $w, v_1, \ldots, v_m \in \Lambda^n$, with* $\mathrm{dom}(w) = (0, a)$ *and* $\mathrm{dom}(v_i) = (-a_i, a_i)$ *for some positive $a, a_i \in M$. Assume that*

$$wt = v_1 t_1 + \cdots + v_m t_m$$

for some $t, t_1, \ldots, t_m \in M$ with $t \in \mathrm{dom}(w)$ and $t_i \in \mathrm{dom}(v_i)$. Then for every $s \in \mathrm{dom}(w)$ with $s < t$, there are $s_1, \ldots, s_m \in M$ with $|s_i| < |t_i|$ such *that*

$$ws = vs_1 + \cdots + v_m s_m.$$

Moreover, s_i has the same sign as t_i.

Proof. This follows from [ElSt, Lemma 3.4], whose proof used only the fact that $\mathcal{M}$ is an o-minimal expansion of an ordered group. Indeed, since $s < t$, from the convexity of the set $A = \{v_1x_1 + \cdots + v_m x_m : x_i \in \mathrm{dom}(v_i)\}$ and the aforementioned lemma we deduce that $ws \in A$. ∎

LEMMA 2.8. *Let $w_1, \ldots, w_n \in \Lambda^n$ be M-independent and $\lambda_1, \ldots, \lambda_n \in \Lambda^n$. Let $t_1, \ldots, t_n \in M$ be non-zero elements. Assume that*

$$\begin{aligned} w_1 t_1 &= \lambda_1 s_1^1 + \cdots + \lambda_n s_1^n, \\ &\;\;\vdots \\ w_n t_n &= \lambda_1 s_n^1 + \cdots + \lambda_n s_n^n \end{aligned}$$

for some $s_i^j \in M$. Then there are non-zero $a_1, \ldots, a_n \in M$ and $b_i^j \in M$, $i, j = 1, \ldots, n$, such that

$$\begin{aligned} \lambda_1 a_1 &= w_1 b_1^1 + \cdots + w_n b_1^n, \\ &\;\;\vdots \\ \lambda_n a_n &= w_1 b_n^1 + \cdots + w_n b_n^n. \end{aligned}$$

Proof. See the Appendix. ∎

2.2. Long cones. Here we refine the notion of a 'cone' from [Ed].

DEFINITION 2.9. Let $k \in \mathbb{N}$. A *k-long cone* $C \subseteq M^n$ is a definable set of the form

$$\Big\{b + \sum_{i=1}^{k} v_i t_i : b \in B,\ t_i \in J_i\Big\},$$

where $B \subseteq M^n$ is a short cell, $v_1, \ldots, v_k \in \Lambda^n$ are M-independent and $J_1, \ldots, J_k$ are long intervals each of the form $(0, a_i)$, $a_i \in M^{>0} \cup \{\infty\}$, with $J_i \subseteq \mathrm{dom}(v_i)$. So a 0-long cone is just a short cell. A *long cone* is a k-long cone for some $k \in \mathbb{N}$. We say that the long cone C is *normalized* if for each $x \in C$ there are unique $b \in B$ and $t_1 \in J_1, \ldots, t_k \in J_k$ such that $x = b + \sum_{i=1}^{k} v_i t_i$. In this case, we write

$$C = B + \sum_{i=1}^{k} v_i t_i | J_i.$$

In what follows, all long cones are assumed to be normalized, and we thus drop the word 'normalized'. We also often refer to $\bar{v} = (v_1, \ldots, v_k) \in \Lambda^{kn}$ as the *direction* of the long cone C. If we want to distinguish some v_j, say v_k, from the rest of the v_i's, we write

$$C = B + \sum_{i=1}^{k-1} v_i t_i | J_i + v_k | J_k.$$

By a *subcone of* C we simply mean a long cone contained in C.

REMARK 2.10. By Lemma 2.4, a (normalized) k-long cone $C = B + \sum_{i=1}^{k} v_i t_i | J_i$ has dimension k if and only if B is finite. In fact, $\dim(C) = \dim(B) + k$.

DEFINITION 2.11. Let $C = B + \sum_{i=1}^{k} v_i t_i | J_i$ be a k-long cone and $f : C \to M$ a definable continuous function. We say that f is *almost linear with respect to* C if there are $\mu_1, \dots, \mu_k \in \Lambda$ and an extension $\tilde{f}$ of f to $\{b + \sum_{i=1}^{k} v_i t_i : b \in B,\, t_i \in \{0\} \cup J_i\}$ such that

(2.4) $\quad \forall b \in B,\, t_1 \in \{0\} \cup J_1, \dots, t_k \in \{0\} \cup J_k,$

$$\tilde{f}\Big(b + \sum_{i=1}^{k} v_i t_i\Big) = \tilde{f}(b) + \sum_{i=1}^{k} \mu_i t_i.$$

REMARK 2.12. Let $C = B + \sum_{i=1}^{k} v_i t_i | J_i$ be a k-long cone.

(i) If $f : C \to M$ is almost linear with respect to C, then, since C is normalized, the $\mu_1, \dots, \mu_k$ and $\tilde{f}$ as above are unique. In particular, $\tilde{f}$ is continuous. For this reason, we often abuse notation and write f for $\tilde{f}$. Indeed, we simply replace (2.4) by

$$f\Big(b + \sum_{i=1}^{k} v_i t_i\Big) = f(b) + \sum_{i=1}^{k} \mu_i t_i.$$

(ii) If $B = \{b\}$ and $f : C \to M$ is a definable function, then f is almost linear with respect to C if and only if f is affine on C. More generally, f is almost linear with respect to $B + \sum_{i=1}^{k} v_i t_i | J_i$ if and only if there are $\mu_1, \dots, \mu_k \in \Lambda$ such that for every $b \in B$ and $s_i, s_i + t_i \in J_i$, we have

$$f\Big(b + \sum_{i=1}^{k} v_i (s_i - t_i)\Big) - f\Big(b + \sum_{i=1}^{k} v_i s_i\Big) = \sum_{i=1}^{k} \mu_i t_i.$$

(iii) If $f : C \to M$ is almost linear with respect to C, then the graph of f is also a k-long cone, with the short cell being $\{(b, f(b)) : b \in B\}$:

$$\mathrm{Graph}(f) = \Big\{(b, f(b)) + \sum_{i=1}^{k} (v_i, \mu_i) t_i : b \in B,\, t \in J_i\Big\}.$$

(iv) Let $j \in \{1, \dots, k\}$ and assume $J_j = (0, a_j)$ with $a_j \in M$. Then

$$C = B + v_j a_j + \sum_{i=1}^{k} v_i' t_i | J_i,$$

where $v'_j = -v_j$ and for $i \neq j$, $v'_i = v_i$. Indeed, if $x = b + \sum_{i=1}^k v_i t_i$ is in C, then for $s_j = a_j - t_j \in J_j$ we have $x = b + v_j a_j - v_j s_j + \sum_{i \neq j} v_i t_i$. If, moreover, $f : C \to M$ is almost linear with respect to C and has the form

$$f\Big(b + \sum_{i=1}^k v_i t_i\Big) = f(b) + \sum_{i=1}^k \mu_i t_i,$$

then

$$f\Big(b + v_j a_j + \sum_{i=1}^k v'_i t_i\Big) = f(b + v_j a_j) + \sum_{i=1}^k \mu'_i t_i,$$

where $\mu'_j = -\mu_j$ and for $i \neq j$, $\mu'_i = \mu_i$.

COROLLARY 2.13. *If $D = b + \sum_{i=1}^l v_i t_i | J_i \subseteq M^n$ is an l-long cone, then some projection $\pi : M^n \to M^l$, restricted to D, is a bijection onto an l-long cone.*

Proof. By Lemmas 2.4 and 2.5. ∎

NOTATION. If $J = (0, a)$, we denote $\pm J := (-a, a)$. Let $C = B + \sum_{i=1}^m v_i t_i | J_i$ be an m-long cone. We set

$$\langle C \rangle := \Big\{ \sum_{i=1}^m v_i t_i : t_i \in \pm J_i \Big\}.$$

COROLLARY 2.14. *Let $C = b + \sum_{i=1}^k v_i t_i | J_i$ be a k-long cone. Let $\lambda \in \Lambda^k$ be such that for some positive $t \in M$, $\lambda t \in \langle C \rangle$. Then there is a tall $b \in M$ such that $\lambda b \in \langle C \rangle$.*

Proof. Fix i. Let $a = \sup\{x \in M : \lambda x \in \langle C \rangle\}$. It is easy to see that $a = v_1 t_1 + \cdots + v_k t_k$ with at least one of $t_1, \dots, t_k$, say t_i, equal to $\pm |J_i|$. Hence, by Lemma 2.6, a is tall. Take $b = \frac{1}{2} a$ (since a is not in $\langle C \rangle$). ∎

2.3. Long dimension. Here we refine the notion of 'linear dimension' from [Ed].

DEFINITION 2.15. Let $Z \subseteq M^n$ be a definable set. Then the *long dimension* of Z is defined to be

$$\mathrm{lgdim}(Z) := \max\{k : Z \text{ contains a } k\text{-long cone}\}.$$

Equivalently, the long dimension of Z is the maximum k such that Z contains a definable homeomorphic image of J^k for some long interval J. Indeed, this follows from the proof of Lemma 2.4, (ii)⇒(iii).

Some main properties of long dimension will be proved in Section 3.2 below, after proving the Lemma on Subcones in Section 3.1. For the moment, we state a lemma which says that given a cone we can always find subcones of suitable direction. An analogous statement fails in the context of [Ed], where all cones were unbounded.

LEMMA 2.16. *Let $C = b + \sum_{i=1}^{k} v_i t_i | J_i$ be a k-long cone. Let $w_1, \ldots, w_k \in \Lambda^n$ be M-independent such that for every i, there is a positive $s_i \in M$ with $w_i s_i \in \langle C \rangle$. Then there is a k-long subcone $C' \subseteq C$ of the form $C' = c + \sum_{i=1}^{k} w_i t_i | (0, \kappa_i)$ for some tall $\kappa_i \in M$.*

Proof. By Corollary 2.14, we may assume that each s_i is tall. Assume $J_i = (0, a_i)$. Let $c = b + \sum_{i=1}^{k} \frac{1}{2} v_i a_i$ and for each i, let $\kappa_i = \frac{1}{2k} |s_i|$. Using Lemma 2.7, one can easily check that $C' = c + \sum_{i=1}^{k} w_i t_i | (0, \kappa_i) \subseteq C$. ∎

The following lemma will be used in the proof of the Refined Structure Theorem.

LEMMA 2.17. *Let $X = (f, g)_{\pi(X)}$ be a cylinder in M^{n+1} such that $\pi(X)$ is a k-long cone and f and g are almost linear with respect to $\pi(X)$. If there is an $x \in \pi(X)$ such that $\pi^{-1}(x)$ is long, then* $\operatorname{lgdim}(X) = k + 1$.

Proof. If $k = 0$, then there is a 1-long cone $\pi^{-1}(x) \subseteq X$. Now assume that $k > 0$ and for some $x \in \pi(X)$, $\pi^{-1}(x) = (f(x), g(x))$ is long. Since f, g are almost linear on $\pi(X)$, there is clearly a k-long cone $C_x = x + \sum_{i=1}^{k} v_i t_i | (0, a_i) \subseteq \pi(X)$ such that for each $y \in C_x$, $g(y) - f(y)$ must be tall. Let $\alpha = \inf\{g(y) - f(y) : y \in C_x\}$. Since f is affine,

$$\forall t_1 \in J_1, \ldots, t_k \in J_k, \qquad f\Big(x + \sum_{i=1}^{k} v_i t_i\Big) = f(x) + \sum_{i=1}^{k} \mu_i t_i$$

for some $\mu_1, \ldots, \mu_k \in \Lambda^n$. Then clearly the $(k+1)$-long cone

$$(x, f(x)) + \sum_{i=1}^{k} (v_i, \mu_i) t_i | J_i + e_{n+1} t_{k+1} | (0, \alpha)$$

is contained in X. ∎

3. Structure theorem for semi-bounded sets. In this section we prove the main results for semi-bounded sets.

3.1. Generalizing the Lemma on Subcones [Ed, Lemma 3.4]. The Lemma on Subcones can be viewed as a kind of converse to Lemma 2.16. Recall from Section 2 that if $C = B + \sum_{i=1}^{m} v_i t_i | J_i$ is an m-long cone, we denote $\langle C \rangle = \{\sum_{i=1}^{m} v_i t_i : t_i \in \pm J_i\}$.

LEMMA 3.1 (Lemma on Subcones). *If $C' = B' + \sum_{i=1}^{m'} w_i t_i | J'_i$ and $C = B + \sum_{i=1}^{m} v_i t_i | J_i$ are two long cones such that $C' \subseteq C \subseteq M^n$, then $\langle C' \rangle \subseteq \langle C \rangle$ (and hence $m' \le m$).*

Proof. Clearly, we may assume that B' is a singleton. Moreover, we can translate both C' and C, so that C' gets the form $C' = \sum_{i=1}^{m'} w_i t_i | J'_i$. Let $j \in \{1, \ldots, m'\}$, and denote for convenience $J := J'_j$. Then $w_j u \in C' \subseteq C$ for all $u \in J$, so there exist a unique $b \in B$ and, for each $i \in \{1, \ldots, m\}$, a unique $t_i \in J_i$ such that $w_j u = b + \sum_{i=1}^m v_i t_i$. This yields the following definable functions:

- $\beta : J \to B$, with $u \mapsto \beta(u)$,
- for each $i \in \{1, \dots, m\}$, $\tau_i : J \to J_i$, with $u \mapsto \tau_i(u)$,

where

$$w_j u = \beta(u) + \sum_{i=1}^m v_i(\tau_i(u)).$$

By Lemma 2.1 and o-minimality, there are long subintervals $I_1, \dots, I_l \subseteq J$ such that $J \setminus (I_1 \cup \dots \cup I_l)$ is short and on each of them $\beta(u)$ is constant. Let $I = (p, q)$ be an interval with maximum length among the I_i's, and assume that on I the map $\beta(u)$ is equal to b. Now let $u_1 < u_2$ in I, with u_1 close enough to p and u_2 close enough to q, so that, if $u := u_2 - u_1$, then $J \subseteq (0, ku)$ for some $k \in \mathbb{N}$ (this is possible by the choice of I). We have

$$w_j u = w_j(u_2 - u_1) = \sum_{i=1}^m v_i(\tau_i(u_2) - \tau_i(u_1)).$$

If we define $t_i := \tau_i(u_2) - \tau_i(u_1)$, then

$$w_j u = \sum_{i=1}^m v_i t_i. \tag{3.1}$$

Hence the condition of Lemma 2.7 is satisfied for $w = w_j$.

Now pick any $t \in J$. We have to show that $w_j t \in \langle C \rangle$. We distinguish two cases.

Case I: $t \le u$. By Lemma 2.7, we have $w_j t = \sum_{i=1}^m v_i s_i$ for some $0 < |s_i| \le |t_i|$, and we are done.

Case II: $t > u$. By the choice of u, there is $k \in \mathbb{N}$ such that $t - u < ku$. Hence, by Lemma 2.7 again, we have $k^{-1} w_j(t - u) = \sum_{i=1}^m v_i s_i$ for some $0 < |s_i| \le |t_i|$ and s_i having the same sign as t_i. Equivalently,

$$w_j(t - u) = \sum_{i=1}^m v_i k s_i. \tag{3.2}$$

By (3.1) and (3.2), we obtain

$$w_j t = \sum_{i=1}^m v_i(t_i + k s_i), \tag{3.3}$$

so it remains to show that $-a_i < t_i + k s_i < a_i$, where $J_i = (0, a_i)$. Again we distinguish two subcases.

Subcase II(a): $t_i > 0$. We observe that, since $C' \subseteq C$, we have

$$w_j t = b' + \sum_{i=1}^m v_i r_i$$

for some $r_i \in J_i$ and $b' \in B$. Together with (3.3), this gives

$$\sum_{i=1}^{m} v_i(t_i + ks_i) = b' + \sum_{i=1}^{m} v_i r_i.$$

If $t_i + ks_i > r_i$, then we would have $b' = \sum_{i=1}^{m} v_i z_i$ for some positive $z_i < t_i + ks_i$. By (3.3), this would imply that $b' = w_j s$ for some $0 < s < t$. In particular, $b' \in C'$, a contradiction. So $0 < t_i + ks_i \le r_i < a_i$, as required.

SUBCASE II(b): $t_i < 0$. Then also $s_i < 0$. Since $0 \in C'$, we have

$$0 = b' + \sum_{i=1}^{m} v_i r_i$$

for some $r_i \in J_i$ and $b' \in B$. Together with (3.3), this shows that

$$w_j t = b' + \sum_{i=1}^{m} v_i(r_i + t_i + ks_i).$$

Hence, $0 < r_i + t_i + ks_i$, and therefore $-a_i < -r_i < t_i + ks_i < 0 < a_i$, as required.

Finally, the fact that $m' \le m$ is now a consequence of Lemma 2.4. ∎

REMARK 3.2. Observe that it is not always possible to get $w_j t \in \langle C\rangle^{>0} := \{\sum_{i=1}^{m} v_i t_i : t_i \in J_i\}$, as in the corresponding conclusion of [Ed, Lemma 3.4].

We can now characterize exactly the subcones of a given long cone C.

COROLLARY 3.3. *The subcones of a long cone C are exactly those cones whose direction $\bar v = (v_1, \dots, v_k)$ satisfies the following condition: for every $i = 1, \dots, k$, there is a positive $s \in M$ such that $v_i s \in \langle C\rangle$.*

Proof. By Corollary 2.16 and the Lemma on Subcones. ∎

LEMMA 3.4. *Let $C' = B' + \sum_{i=1}^{k'} v'_i t_i | J'_i \subseteq C = B + \sum_{i=1}^{k} v_i t_i | J_i$ be two long cones and $f : C \to M$ be a definable function which is almost linear with respect to C. Then f is almost linear with respect to C'.*

Proof. By the Lemma on Subcones, for each $i = 1, \dots, k'$ and $t \in J'_i$, we have $v'_i t \in \langle C\rangle$. It is then an easy exercise to check that f is affine in each v'_i, uniformly on $b' \in B'$; that is, there are $\mu_1, \dots, \mu_{k'} \in \Lambda$ such that for every $b' \in B'$ and $s_i, s_i + t_i \in J'_i$, we have

$$f\Big(b' + \sum_{i=1}^{k} v_i(s_i - t_i)\Big) - f\Big(b + \sum_{i=1}^{k} v_i s_i\Big) = \sum_{i=1}^{k} \mu_i t_i.$$

This exactly means (Remark 2.12(ii)) that f is almost linear with respect to C'. ∎

COROLLARY 3.5. *Let $C \subseteq C'$ be two k-long cones and let $\bar v$ be the direction of C'. Then there is a k-long cone of direction $\bar v$ contained in C.*

Proof. By the Lemma on Subcones, Lemma 2.8 and Corollary 2.14. ∎

3.2. Properties of long dimension

Lemma 3.6. *Let $X, Y, X_1, \ldots, X_k$ be definable sets. Then:*

(i) $\operatorname{lgdim}(X) \leq \dim(X)$.
(ii) $X \subseteq Y \subseteq M^n \Rightarrow \operatorname{lgdim}(X) \leq \operatorname{lgdim}(Y) \leq n$.
(iii) *If C is a n-long cone, then* $\operatorname{lgdim}(C) = n$.
(iv) $\operatorname{lgdim}(X \times Y) = \operatorname{lgdim}(X) + \operatorname{lgdim}(Y)$.
(v) $\operatorname{lgdim}(X_1 \cup \cdots \cup X_k) = \max\{\operatorname{lgdim}(X_1), \ldots, \operatorname{lgdim}(X_k)\}$.

Proof. (i) is by Lemma 2.4, and (ii) is clear. Item (iii) follows from the Lemma on Subcones 3.1. The proof of (iv) is word-for-word the same as the proof of [EdEl, Fact 2.2(3)] after replacing 'ldim' by 'lgdim' and the notion of a cone by that of a long cone we have here.

For (v), we prove by parallel induction on $n \geq 1$ the following two statements:

$(1)_n$ *For all definable X_1, X_2 such that* $\operatorname{lgdim}(X_1 \cup X_2) = n$, *either* $\operatorname{lgdim}(X_1) = n$ *or* $\operatorname{lgdim}(X_2) = n$.
$(2)_n$ *Let $C \subseteq M^n$ be an n-long cone. For any definable set $X \subseteq C$ with* $\dim(X) \leq n-1$ *we have* $\operatorname{lgdim}(C \setminus X) = n$.

Statement (v) then clearly follows from $(1)_n$ by induction on k.

Step I: $(2)_1$ follows from [Pet3, Lemma 3.4(2)].

Step II: $(1)_{n-1}$ and $(2)_l$ for $l \leq n-1$ imply $(2)_n$, for $n \geq 2$. Assume $(1)_{n-1}$ and $(2)_l$ for all $l \leq n-1$. We perform a subinduction on $\dim(X)$. Observe that after some suitable linear transformation we may assume that C has the form

$$C = \sum_{i=1}^{n} e_i t_i | J_i,$$

where the e_i's are the standard basis vectors.

If $\dim(X) = 0$, then X is finite and, without loss of generality, we may assume that X contains only one point a. Then it is easy to see that $C \setminus \{a\}$ contains 2^n disjoint long cones of the form $a + \sum_{i=1}^{n} e_i t_i | J'_i$ such that, for at least one of them, all J'_i's are long.

Suppose the result holds for all X with $\dim(X) \leq l < n-1$, and assume now that $\dim(X) = l+1$. If $l+1 < n-1$, then $\dim(\pi(X)) \leq n-2$ and by $(2)_{n-1}$, $\operatorname{lgdim}(\pi(C) \setminus \pi(X)) = n-1$, which implies that $\operatorname{lgdim}(C \setminus X) = n$, by (iv).

So now assume that $\dim(X) = n-1$. By cell decomposition and by the subinductive hypothesis, we may assume that X is a finite union of cells $X_1, \ldots, X_k$, each of dimension $n-1$. We perform a second subinduction on k.

Base Step. Suppose $k = 1$. If X_1 is not the graph of a function or $\operatorname{lgdim}(X_1) < n-1$, then by $(2)_{n-1}$ or $(1)_{n-1}$, respectively, we have $\operatorname{lgdim}(\pi(C) \setminus \pi(X_1)) = n-1$, which implies $\operatorname{lgdim}(C \setminus X_1) = n$, by (iv). Thus it remains to examine the case where X_1 is the graph of a function $f : \pi(X_1) \to M$ and $\operatorname{lgdim}(X_1) = n-1$. In this case, $\operatorname{lgdim}(\pi(X_1)) = \operatorname{lgdim}(X_1) = n-1$, where the first equality is by Lemma 2.5. Let $D \subseteq \pi(X_1)$ be an $(n-1)$-long cone. Let

$$A = \{\bar{a} \in D : \forall i \in \{1, \dots, n-1\},\ f_{\bar{a}^i} \text{ is monotone around } a_i\},$$

according to the notation of Lemma 2.2. By that lemma,

$$\dim(D \setminus A) < \dim(D) = n-1.$$

Hence, by $(2)_{n-1}$, A contains an $(n-1)$-long cone E, and by Lemma 2.16, we may assume that $E = b + \sum_{i=1}^{n-1} e_i t_i|(0,\kappa)$ for some tall κ. Let $\bar{a} = b + \sum_{i=1}^{n-1} e_i \frac{1}{2}\kappa$. Since f is continuous on E, each $f_{\bar{x}^i}$ is monotone on its domain $(0,\kappa)$. Without loss of generality, we may assume that for all $i \in \{1, \dots, n-1\}$, $f_{\bar{x}^i}$ is increasing on $(0,\kappa)$. We distinguish two cases.

Case 1: $f(\bar{a})$ is short. Then the n-long cone

$$E_1 = (b, f(\bar{a})) + \sum_{i=1}^{n-1} e_i t_i|(0,\kappa/2) + e_n t_n|J_n/2$$

is contained in X_1.

Case 2: $f(\bar{a})$ is tall. Then the n-long cone

$$E_2 = (\bar{a}, 0) + \sum_{i=1}^{n-1} e_i t_i|(0,\kappa/2) + e_n t_n|J_n/2$$

is contained in X_1.

This completes the case $k = 1$.

Inductive Step. Suppose the result holds for any X which is a union of less than k cells of dimension $n-1$, and assume now that X is the union of the cells $X_1, \dots, X_k$, each of dimension $n-1$. By the second subinductive hypothesis, there is an n-long cone F contained in $C \setminus (X_1 \cup \dots \cup X_{k-1})$. Now, we reduce to the Base Step for C equal to F and X_1 equal to X_k. This completes the proof of the second subinduction, as well as that of Step II of the original induction.

Step III: $(2)_n \Rightarrow (1)_n$. Without loss of generality, we may assume that X_1 and X_2 are disjoint. Since $\operatorname{lgdim}(X_1 \cup X_2) = n$, we may also assume that $X_1 \cup X_2$ is an n-long cone C of dimension n. If $X = \operatorname{bd}(X_1) \cup \operatorname{bd}(X_2)$, then $\dim(X) \le n-1$. By $(2)_n$, we conclude that either X_1 or X_2 contains an n-long cone. ∎

The following corollary will not be used until Section 6.

COROLLARY 3.7. *Let $X \subseteq M^n$ be a definable set of long dimension k. If $C \subseteq X \times X$ is a $2k$-long cone, then there are k-long cones $C_1, C_2 \subseteq X$ such that $C_1 \times C_2 \subseteq C$.*

Proof. We may assume that $C = b + \sum_{i=1}^{2k} v_i t_i | J_i$. Let $\pi : M^{2n} \to M^{2k}$ be the projection given by Corollary 2.13, whose restriction $\pi_{\restriction C}$ is a bijection onto the $2k$-long cone $\pi(C)$. Moreover, as can easily be checked, its inverse $(\pi_{\restriction C})^{-1}$ can be written as $\pi_{\restriction C} = (f_1, \dots, f_{2n})$ for some affine maps $f_j : M^{2k} \to M$. By Remark 2.12(ii)&(iii), the graph of $\pi_{\restriction C}^{-1}$ on a k-long cone contained in $\pi(C)$ is a k-long cone contained in C.

Now let $p_1 : M^{2k} \to M^k$ and $p_2 : M^{2k} \to M^k$ be the suitable projections, so that $\pi(C) \subseteq p_1\pi(C) \times p_2\pi(C)$. Since $\pi(C)$ has long dimension k, by the Lemma on Subcones and 3.6(iv), each of $p_1\pi(C)$ and $p_2\pi(C)$ must have long dimension k. In particular, for each $i = 1, \dots, 2k$, there is $t > 0$ with $e_i t \in \langle \pi(C) \rangle$. By Lemma 2.16, $\pi(C)$ contains a $2k$-long cone

$$C' = (b_1, b_2) + \sum_{i=1}^{2k} e_i t_i | (0, a).$$

The k-long cones

$$C_1' = b_1 + \sum_{i=1}^{k} e_i t_i | (0, a) \quad \text{and} \quad C_2' = b_2 + \sum_{i=k}^{2k} e_i t_i | (0, a)$$

are clearly contained in $p_1\pi(C)$ and $p_2\pi(C)$, respectively. By the first paragraph of this proof, the set

$$D = \pi_{\restriction C}^{-1}(C')$$

is a $2k$-long cone contained in C, and each of

$$D_1 = \pi_{\restriction C}^{-1}(C_1' \times \{b_2\}) \quad \text{and} \quad D_2 = \pi_{\restriction C}^{-1}(\{b_1\} \times C_2')$$

is a k-long subcone of D. If we take the projection C_1 of D_1 onto the first n coordinates, and the projection C_2 of D_2 onto the last n coordinates, then both C_1 and C_2 are k-long cones, contained in X, such that

$$C_1 \times C_2 = D \subseteq C,$$

as desired. ∎

3.3. The Refined Structure Theorem. We are now in a position to prove the first main result of this paper. For a given definable function $f : A \times M \to M$, with $A \subseteq M^n$, let us denote

$$\Delta_t f(a, x) := f(a, x + t) - f(a, x)$$

for all $x, t \in M$ and $a \in A$.

Theorem 3.8 (Refined Structure Theorem). *Let $X \subseteq M^n$ be an A-definable set. Then*

(i) *X is a finite union of A-definable long cones.*
(ii) *If X is the graph of an A-definable function $f : Y \to M$ for some $Y \subseteq M^{n-1}$, then there is a finite collection $\mathcal{C}$ of A-definable long cones whose union is Y and such that f is almost linear with respect to each long cone in $\mathcal{C}$.*

Proof. By cell decomposition we may assume that X is an A-definable cell. We will prove (i) and (ii), together with (iii) below, by induction on $\langle n, \mathrm{lgdim}(X)\rangle$.

(iii) *In the notation from* (ii), *Y contains an A-definable* $\mathrm{lgdim}(Y)$*-long cone such that f is almost linear with respect to it.*

If $n = 1$, then (i)–(iii) are clear. Assume the inductive hypothesis (IH): (i)–(iii) hold for $\{\langle n, k\rangle\}_{k\le n}$, and let $X \subseteq M^{n+1}$ with $\mathrm{lgdim}(X) = k \le n+1$.

Case (I): $\dim(X) < n+1$. So, after perhaps permuting the coordinates, we may assume that X is the graph of a continuous A-definable function $f : Y \to M$.

(i) This is clear, by (IH)(ii) and Remark 2.12(iii).

(ii) By (IH)(i), we may further assume that $Y = B' + \sum_{i=1}^{k} v_i t_i|J_i$ is an A-definable k-long cone, where $k \le n$.

Claim. *We may assume that $Y = B + \sum_{i=1}^{k} e_{n-k+i} t_i|J_i$.*

Proof. To see this, we will define a suitable affine transformation from Y into M^n. The idea is to map elements of the form $v_i t$ to $e_{n-k+i} t$. Since the v_i's are not necessarily global endomorphisms, we need to explain how this transformation works.

First extend each v_i, $1 \le i \le k$, to a vector u_i in Λ^n with domain $2J_i$. More precisely, if $J_i = (0, a_i)$, let $u_i : (0, 2a_i) \to M^n$ be equal to $v_i(t)$ for $t \in (0, a_i)$, and equal to $(\lim_{s\to a_i} v_i s) + v_i(t - a_i)$ for $t \in (a_i, 2a_i)$. Also, choose $u_{k+1}, \dots, u_n \in \Lambda^n$ with long domains $J_{k+1}, \dots, J_n$ so that all $u_1, \dots, u_n$ are M-independent (in fact, $u_{k+1}, \dots, u_n$ can be chosen among the unit vectors in Λ^n).

Now, fix any $b \in B'$ and let $C = \sum_{i=1}^{n} v_i t_i|J_i$. By Lemma 2.4, $b + \langle C\rangle$ is open. We claim that $b + \langle C\rangle$ contains Y. First we observe that B' is contained in $b + \langle C\rangle$. Since B' is connected and contains b, if B' were not contained in $b + \langle C\rangle$, we would have a definable path that starts from b and ends outside $b + \langle C\rangle$. This path has short domain but long range, a contradiction.

Now we want to see that every element x in Y is contained in $b + \langle C\rangle$. Let $x = b' + \sum_{i=1}^{k} v_i t_i$. Since b' is in $b + \langle C\rangle$, we have $b' = b + \sum_{i=1}^{k} v_i s_i +$

$\sum_{i=k+1}^{n} u_i s_i$. Therefore, $x = b + \sum_{i=1}^{k} u_i(s_i + t_i) + \sum_{i=k+1}^{n} u_i s_i$, that is, $x \in b + \langle C\rangle$.

Now that we know that $b + \langle C\rangle$ contains Y, we define the following transformation:

$$T : b + \langle C\rangle \to M^n, \quad T\Big(b + \sum_{i=1}^{n} u_i t_i\Big) = b + \sum_{i=1}^{k} e_{n-k+i} t_i + \sum_{i=k+1}^{n} e_{n-i+1} t_i.$$

This is a bijection onto its image. Clearly, $T(Y) = T(B') + \sum_{i=1}^{k} e_{n-k+i} t_i|J_i$, as the reader can verify that $T(b' + \sum_{i=1}^{k} v_i t_i) = T(b') + \sum_{i=1}^{k} e_{n-k+i} t_i$. Hence, we can let $B = T(B')$ and replace Y by $T(Y)$. This proves the Claim. ∎

Let $\pi : M^n \to M^{n-1}$ be the usual projection. By [Pet3, Lemma 4.10] and its proof, there are A-definable linear functions $\lambda_1, \dots, \lambda_l$, A-definable functions $a_0, \dots, a_m : \pi(Y) \to M$ and a short positive element $b \in \mathrm{dcl}(A)$ of M such that, for every $x \in \pi(Y)$,

- $0 = a_0(x) \le a_1(x) \le \cdots \le a_{m-1}(x) \le a_m(x) = e_n|J_k|$,
- for every i, either $|a_{i+1}(x) - a_i(x)| < b$ or the map $t \mapsto \Delta_t f(x, a_i(x))$ on $(0, a_{i+1}(x) - a_i(x))$ is the restriction of some λ_j, that is,

$$f(x, a_i(x) + t) - f(x, a_i(x)) = \lambda_j(t). \tag{3.4}$$

For every $z = (x, y) \in Y$, let $b_z := a_{i+1}(x) - a_i(x)$, where $y \in (a_i(x), a_{i+1}(x))$. Observe that $b_z \in \mathrm{dcl}(\emptyset)$. Set

$$Y_0 = \{z \in Y : b_z \ge b\},$$

and consider (by cell decomposition) a partition $\mathcal{C}$ of Y_0 into cells so that for every $Z \in \mathcal{C}$,

- there is some λ_j such that the restriction of f to Z satisfies (3.4) above,
- Z is contained in $\{(x, y) : a_i(x) \le y \le a_{i+1}(x)\}$.

By (IH)(ii), there is a finite collection $\mathcal{C}'$ of A-definable long cones whose union is $\pi(Z)$ and such that each a_i is almost linear with respect to each $C \in \mathcal{C}'$. By (IH)(i), there is a finite collection $\mathcal{C}''$ of A-definable long cones whose union is $Z \cap \pi^{-1}(C)$. Observe now that $Z \cap \pi^{-1}(C)$ is contained in some long cone W on which f is almost linear; namely, if $C = D + \sum_{i=1}^{l} w_i t_i|K_i$, then W is of the form

$$W = D \times \{d\} + \sum_{i=1}^{l} w_i t_i|K_i + e_n t_n|K_n,$$

where K_n is a long interval of length $\max\{a_{i+1}(x) - a_i(x) : x \in C\}$. By Lemma 3.4, f is almost linear with respect to each long cone in $\mathcal{C}''$.

It remains to prove (i) for $Y \setminus Y_0$. But this is given by (IH)(ii), since, in fact, $\mathrm{lgdim}(Y \setminus Y_0) < k$: assuming not, apply (IH)(iii) to get a k-long cone $C \subseteq Y \setminus Y_0 \subseteq Y$. By Corollary 3.5, there is a tall $a \in M$ such that $e_n a \in C$. But then f is linear in x_n on some long interval contained in $Y \setminus Y_0$, a contradiction. Hence $\operatorname{lgdim}(Y \setminus Y_0) < k$.

(iii) In the above notation, for every $i \in \{0, \dots, m-1\}$, the set

$$P_i := \{\bar{x} \in \pi(Y) : a_{i+1}(\bar{x}) - a_i(\bar{x}) \geq b\}$$

is A-definable and, since J_n is long, $\pi(Y) = \bigcup_{i=0}^{m-1} P_i$. By Lemma 3.6(v), one of the P_i's, say P_j, must have long dimension $k-1$. By (IH)(iii), there is a finite collection $\mathcal{C}'$ of A-definable long cones whose union is W_j and such that each a_j and a_{j+1} are almost linear with respect to each $C \in \mathcal{C}'$. By Lemma 2.17, there is an A-definable k-long cone $E \subseteq Y$ and, as before, f is almost linear with respect to E.

Case (II): $\dim(X) = n+1$. The argument in this case is a combination of the proofs of [ElSt, Lemma 3.6] and of [Pet1, Theorem 3.1]. So $X = (g,h)_Y$ is a cylinder. By (IH)(ii) and Lemma 3.4, we may assume that $Y = B + \sum_{i=1}^k v_i t_i | J_i$ is a long cone and that g, h are almost linear with respect to it. Assume they are of the form

$$g\Big(b + \sum_{i=1}^k v_i t_i\Big) = g(b) + \sum_{i=1}^k n_i t_i \quad \text{and} \quad h\Big(b + \sum_{i=1}^k v_i t_i\Big) = h(b) + \sum_{i=1}^k m_i t_i.$$

Since $g < h$ on Y, it follows that $g(b) \leq h(b)$ for every $b \in B$. One of the following two cases must occur:

Case ($\mathrm{II_a}$): For all $i = 1, \dots, k$, we have $n_i = m_i$.

Case ($\mathrm{II_b}$): For all $i = 1, \dots, k$, we have $n_i \leq m_i$, and for at least one i we have $n_i < m_i$. (We may assume so by Remark 2.12(iv): indeed, if for some i, $n_i > m_i$, then we can change B and replace n_i by $n_i' = -n_i$, and m_i by $m_i' = -m_i$, as indicated in Remark 2.12(iv). Then $n_i' < m_i'$.)

Proof of Case ($\mathrm{II_a}$). We have

$$X = \Big\{(b,y) + \sum_{i=1}^k (v_i, n_i) t_i : g(b) < y < h(b),\, b \in B,\, t_i \in J_i\Big\}.$$

It is easy to check that, if $(g(b), h(b))$ is a long interval, then

$$X = \{(b, g(b)) : b \in B\} + \sum_{i=1}^k (v_i, n_i) t_i | J_i + e_{n+1} t_{n+1} | (0, h(b) - g(b))$$

is a $(k+1)$-long cone, and if $(g(b), h(b))$ is short, then

$$X = \big\{\{b\} \times (g(b), h(b)) : b \in B\big\} + \sum_{i=1}^k (v_i, n_i) t_i | J_i$$

is a k-long cone.

Proof of Case ($\mathrm{II_b}$). We have

$$X=\Big\{\Big(b+\sum_{i=1}^{k} v_it_i, y\Big) : g(b)+\sum_{i=1}^{k} n_it_i < y < h(b)+\sum_{i=1}^{k} m_it_i,\ b\in B,\ t_i\in J_i\Big\}.$$

Notice that if $h=+\infty$ on X (similarly, if $g=-\infty$), then we are done because

$$X=\{(b,g(b)) : b\in B\}+\sum_{i=1}^{k} v_it_i|J_i+e_nt_n|(0,+\infty).$$

We partition X in the following way, going from 'top' to 'bottom':

$$X_1=\Big\{\Big(b+\sum_{i=1}^{k} v_it_i, y\Big) : h(b)+\sum_{i=1}^{k} n_it_i < y < h(b)+\sum_{i=1}^{k} m_it_i,\ b\in B,\ t_i\in J_i\Big\},$$

$$X_2=\Big\{\Big(b+\sum_{i=1}^{k} v_it_i, y\Big) : y=h(b)+\sum_{i=1}^{k} n_it_i,\ b\in B,\ t_i\in J_i\Big\},$$

$$X_3=\Big\{\Big(b+\sum_{i=1}^{k} v_it_i, y\Big) : g(b)+\sum_{i=1}^{k} n_it_i < y < h(b)+\sum_{i=1}^{k} n_it_i,\ b\in B,\ t_i\in J_i\Big\}.$$

By Remark 2.12(iii), X_2 is a k-long cone, whereas X_3 clearly satisfies the condition of Case ($\mathrm{II_a}$). Hence we only need to account for X_1.

Let $S_{X_1}=\{i=1,\dots,k : n_i<m_i\}$. By induction on $|S_{X_1}|$ we may assume that $|S_{X_1}|=1$. Indeed, if, say, $n_1<m_1$ and $n_2<m_2$, then we can partition X_1 in the following way, going again from 'top' to 'bottom':

$$\begin{aligned}X_1'=\Big\{\Big(b+\sum_{i=1}^{k} v_it_i, y\Big) &:\\ &h(b)+n_1t_1+\sum_{i=2}^{k} m_it_i < y < h(b)+\sum_{i=1}^{k} m_it_i,\ b\in B,\ t_i\in J_i\Big\},\end{aligned}$$

$$X_1''=\Big\{\Big(b+\sum_{i=1}^{k} v_it_i, y\Big) : y=h(b)+n_1t_1+\sum_{i=2}^{k} m_it_i,\ b\in B,\ t_i\in J_i\Big\},$$

$$\begin{aligned}X_1'''=\Big\{\Big(b+\sum_{i=1}^{k} v_it_i, y\Big) &:\\ &h(b)+\sum_{i=1}^{k} n_it_i < y < h(b)+n_1t_1+\sum_{i=2}^{k} m_it_i,\ b\in B,\ t_i\in J_i\Big\}.\end{aligned}$$

Observe then that X_1'' is a k-long cone, and for X_1' and X_1''', each of the corresponding $S_{X_1'}$ and $S_{X_1'''}$ has size less than $|S_{X_1}|$.

So assume now that $|S_{X_1}| = 1$ with, say, $n_1 < m_1$ and $n_i = m_i$ for $i > 1$. Let

$$A = \Big\{ \Big(\sum_{i=1}^{k} v_i t_i, y \Big) : \sum_{i=1}^{k} n_i t_i < y < \sum_{i=1}^{k} m_i t_i,\, b \in B,\, t_i \in J_i \Big\}.$$

We show that A is a union of long cones, which clearly implies that so is X_1. If $J_1 = (0, \infty)$, then

$$A = (v_1, n_1) t_1 | J_1 + \sum_{i=1}^{k} (v_i, m_i) t_i | J_i$$

is already a $(k+1)$-long cone. If $J_1 = (0, a_1)$ with $a_1 \in M$, then A is the union of the following $(k+1)$-long cones:

$$\begin{aligned} Y_1 &= (v_1, n_1) t_1 | (0, a_1/2) + (v_1, m_1) t_1 | (0, a_1/2) + \sum_{i=2}^{k} (v_i, m_i) t_i | J_i, \\ Y_2 &= (v_1, n_1) a_1/2 + (v_1, n_1) t_1 | (0, a_1/2) \\ &\quad + \sum_{i=2}^{k} (v_i, m_i) t_i | J_i + e_n t_n | (0, (m_1 - n_1) a_1/2) \\ Y_3 &= (v_1, n_1) a_1/2 + (v_1, m_1) t_1 | (0, a_1/2) \\ &\quad + \sum_{i=2}^{k} (v_i, m_i) t_i | J_i + e_n t_n | (0, (m_1 - n_1) a_1/2). \ ■ \end{aligned}$$

Remark 3.9. As opposed to the corresponding results from [Ed] and [Pet1], it is not always possible to achieve a *disjoint* union in (i) or (ii). We leave it to the reader to verify that the following set cannot be written as a disjoint union of long cones: let X be the ‘triangle’ with corners the origin, the point (a, a) and the point $(0, 2a)$, for some long element a.

As a first corollary, we obtain a quantifier elimination result down to suitable existential formulas in the spirit of [vdD1].

Corollary 3.10. *Every definable subset $X \subseteq M^m$ is a boolean combination of subsets of M^m defined by*

$$\exists y_1 \ldots \exists y_m,\, B(y_1, \ldots, y_m) \wedge \varphi(x_1, \ldots, x_m, y_1, \ldots, y_m),$$

where $B(y)$ is a short formula and $\varphi(x, y)$ is a quantifier-free $\mathcal{L}_{\mathrm{lin}}$-formula. In fact, X is a finite union of such sets.

Another corollary is the following.

Corollary 3.11. *If $f : X \to M^n$ is a definable injective function, then* $\mathrm{lgdim}(X) = \mathrm{lgdim}(f(X))$.

Proof. Assume that $X \subseteq M^k$ and $f = (f^1, \ldots, f^n)$, where $f^j : X \to M$. By the Refined Structure Theorem and Lemma 3.6(v), we may assume that

X is a long cell of the form $X = b + \sum_{i=1}^{k} v_i t_i | J_i$ and each f_j is almost linear on X. Hence, for every j, there are $\mu_1^j, \ldots, \mu_k^j$ so that $f^j(b + \sum_{i=1}^{k} v_i t_i) = f^j(b) + \sum_{i=1}^{k} \mu_i^j t_i$. Thus, $f(X)$ is the long cell

$$(f^1(b), \ldots, f^n(b)) + \sum_{i=1}^{k} \mu_i t_i | J_i,$$

where each $\mu_i = (\mu_i^1, \ldots, \mu_i^n)$ is in Λ^n. ∎

4. On definability of long dimension. The following example shows that we lack 'definability of long dimension'.

EXAMPLE 4.1. Let $a > 0$ be a tall element and let

$$X = \{(x, y) : 0 \leq x \leq a,\, 0 \leq y \leq x\}.$$

Denote by $\pi : M^2 \to M$ the usual projection. Then, by [Pet3, Proposition 3.6], the set

$$X_1 = \{x \in [0, a] : \pi^{-1}(x) \text{ has long dimension } 1\}$$

is not definable.

However, X_1 clearly contains a 'suitable' definable set; namely, a definable set of long dimension 1. It follows from the lemmas of this section that the set of fibers of long dimension l of a given definable set X always lies between two definable sets each of long dimension $\mathrm{lgdim}(X) - l$ (Corollary 4.4 below).

LEMMA 4.2. *Let $X \subseteq M^{n+m}$ be a definable set such that the projection $\pi(X)$ onto the first n coordinates has long dimension k. Let $0 \leq l \leq m$. Then*

(i) $\mathrm{lgdim}(X) \leq k + m$.
(ii) $\mathrm{lgdim}(X) \geq k + l$ *if and only if* $\pi(X)$ *contains a k-long cone C such that every fiber X_c, $c \in C$, has long dimension $\geq l$.*

Proof. (i) By Lemma 3.6(ii)&(iv), since $X \subseteq \pi(X) \times M^m$.

(ii) ($\Leftarrow$) Assume that every fiber X_c, $c \in C$, has long dimension l. We prove that $\mathrm{lgdim}(X) \geq k + l$ by induction on k. For $k = 0$, this is clear, since any fiber above C contains an l-long cone. Now assume that it is proved for $\mathrm{lgdim}(C) < k$, and let $\mathrm{lgdim}(C) = k$. Clearly, we may assume that $\pi(X) = C$. For the sake of contradiction, assume $\mathrm{lgdim}(X) < k + l$. By the Refined Structure Theorem, X can be covered by finitely many long cones $X_1, \ldots, X_s$, each with $\mathrm{lgdim}(X_i) < k + l$. By the inductive hypothesis, each $\pi(X_i)$ has long dimension $< k$. But then $C = \pi(X_1) \cup \cdots \cup \pi(X_s)$ must have long dimension $< k$, a contradiction.

(⇒) This is clearly equivalent to the following:

Claim. *Let*

$$X_l = \{x \in \pi(X) : \pi^{-1}(x) \textit{ has long dimension } \geq l\}.$$

Then there is a definable set $Y_l \subseteq X_l$ *such that*

$$\operatorname{lgdim}(Y_l) = \operatorname{lgdim}(X) - l.$$

The proof of the Claim is by induction on m.

Base Step: $m = l = 1$. By cell decomposition, X is a finite union of cells, and by the Refined Structure Theorem the domain of each cell is a finite union of long cones such that the corresponding restrictions of the defining functions of the cell are almost linear with respect to each of the long cones. If a cell is the graph of a function, or if its domain has long dimension $< k$, then clearly its long dimension is at most k. Hence X contains a cylinder $X_1 = (f, g)_{\pi(X_1)}$, where $\pi(X_1)$ is a k-long cone, such that X_1 contains a $(k+1)$-long cone $C = b + \sum_{i=1}^{k+1} v_i t_i|(0, \alpha_i)$. We will first show this for some $x, y \in \overline{C}$ in the closure of C, with $x_i = y_i$ for all $i = 1, \ldots, n$, and $(y - x)_{n+1}$ tall. The argument is straightforward and we only sketch it.

The projection $\pi(C)$ is a union of long cones whose directions are tuples with elements from the set $\{v_1, \ldots, v_{k+1}\}$. By (i) and Lemma 3.6(v), there must be a subset of $\{v_1, \ldots, v_{k+1}\}$ of k elements, say $\{v_1, \ldots, v_k\}$, whose projections onto the first n coordinates is an M-independent set. Without loss of generality, assume $A = \{v_1, \ldots, v_k\}$. It is then an easy exercise to see that there is an element $y = v_1 t_1 + \cdots + v_{k+1} t_{k+1} \in \overline{C}$ such that the element

$$x = \min\{z \in \overline{C} : \forall i \leq n, z_i = y_i\}$$

has the form $x = v_1 s_1 + \cdots + v_{k+1} s_{k+1} \in \overline{C}$ and is such that $t_{k+1} - s_{k+1}$ is long. But then $y - x$ must be tall, by Lemma 2.6. It follows that $(y - x)_{n+1}$ must be tall.

Now, we conclude that there is $x \in \pi(X_1)$ such that $\pi^{-1}(x) = (f(x), g(x))$ is long. Since f, g are almost linear on $\pi(X_1)$, it is easy to see that there is a k-long cone $C_x = x + \sum_{i=1}^{k} v_i t_i|(0, a_i) \subseteq \pi(X_1)$ such that for each element $y \in C_x$, $g(y) - f(y)$ is tall. We let $Y_l = C_x$. Since, by (i), $k \geq \operatorname{lgdim}(X) - 1$, we are done.

Inductive Step. Assume we know the Claim holds for every n and $X \subseteq M^n \times M^m$, and let $X \subseteq M^n \times M^{m+1}$. Let $q : M^{n+m+1} \to M^{n+m}$ and $r : M^n \times M^m \to M^n$ be the usual projections. Of course, $\pi = r \circ q$.

Case (I): $\operatorname{lgdim}(q(X)) = \operatorname{lgdim}(X)$. In this case, by the inductive hypothesis, the set

$$q(X)_l = \{x \in \pi(X) : \operatorname{lgdim}(r^{-1}(x)) \geq l\}$$

contains a definable set A such that

$$\operatorname{lgdim}(A) = \operatorname{lgdim}(q(X)) - l = \operatorname{lgdim}(X) - l.$$

Since, clearly, $q(X)_l \subseteq X_l$, we are done.

CASE (II): $\operatorname{lgdim}(q(X)) = \operatorname{lgdim}(X) - 1$. Let

$$Y_1 = \{x \in q(X) : \operatorname{lgdim}(q^{-1}(x)) = 1\}.$$

By the Base Step, Y_1 contains some definable set Y with $\operatorname{lgdim}(Y) = \operatorname{lgdim}(X) - 1$. By the inductive hypothesis, the set

$$Y_l = \{x \in r(Y) : \operatorname{lgdim}(r^{-1}(x)) \geq l - 1\}$$

contains a definable set A with

$$\operatorname{lgdim}(A) = \operatorname{lgdim}(Y) - (l - 1) = \operatorname{lgdim}(X) - l.$$

But clearly X_l contains A, and hence we are done. ∎

On the other hand, we have the following lemma. It will not be essential until the proof of Proposition 6.4.

LEMMA 4.3. *Let $X \subseteq M^{n+m}$ be a definable set and denote by $\pi : M^{n+m} \to M^n$ the usual projection. For $0 \leq l \leq m$, let*

$$X_l = \{x \in \pi(X) : \pi^{-1}(x) \text{ has long dimension} \geq l\}.$$

Then there is a definable subset $Z_l \subseteq \pi(X)$ with $X_l \subseteq Z_l$ such that

$$\operatorname{lgdim}(Z_l) = \operatorname{lgdim}(X) - l.$$

Proof. The proof is by induction on m. For any m, if $l = 0$, then take $Z_l = \pi(X)$, since, by Lemma 4.2(ii), $\operatorname{lgdim}(\pi(X)) \leq \operatorname{lgdim}(X)$.

Base Step: $m = 1$. Let $X \subseteq M^n \times M$ and $l = 1$. By cell decomposition and Lemma 3.6(v), we may assume that X is a cell. If X is the graph of a function, then let Z_l be any subset of $\pi(X)$ of long dimension $\operatorname{lgdim}(X) - 1$. So assume X is the cylinder $(f, g)_{\pi(X)}$ between two continuous functions f and g. By the Refined Structure Theorem, we may further assume that $\pi(X)$ is a long cone such that f and g are both almost linear with respect to it. If $\operatorname{lgdim}(\pi(X)) = \operatorname{lgdim}(X) - 1$, then take $Z_l = \pi(X)$. If $\operatorname{lgdim}(\pi(X)) = \operatorname{lgdim}(X)$, then by Lemma 2.17, for every $x \in \pi(X)$, $\pi^{-1}(x)$ is short, in which case we again let Z_l be any subset of $\pi(X)$ of long dimension $\operatorname{lgdim}(X) - 1$.

Inductive Step. Assume we know the lemma for all n and $X \subseteq M^n \times M^m$, and let $X \subseteq M^n \times M^{m+1}$. Let $q : M^{n+m+1} \to M^{n+m}$ and $r : M^n \times M^m \to M^n$ be the usual projections. Let

$$Y_1 = \{x \in q(X) : \operatorname{lgdim}(q^{-1}(x)) = 1\}.$$

By Lemma 4.2(ii), Y_1 is contains some definable set Y with $\mathrm{lgdim}(Y) = \mathrm{lgdim}(X) - 1$. Now, X_l is contained in the union of the following two sets:

$$A_1 = \{x \in r(Y) : \mathrm{lgdim}(r^{-1}(x)) \geq l - 1\},$$
$$B_1 = \{x \in r(q(X) \setminus Y) : \mathrm{lgdim}(r^{-1}(x)) = l\}.$$

By the inductive hypothesis, A_1 is contained in a definable set A with

$$\mathrm{lgdim}(A) = \mathrm{lgdim}(Y) - (l - 1) = \mathrm{lgdim}(X) - l$$

and B_1 is contained in a definable set B with

$$\mathrm{lgdim}(B) = \mathrm{lgdim}(q(X) \setminus Y) - l \leq \mathrm{lgdim}(X) - l.$$

Hence X_l is contained in the definable set $Z_l = A \cup B$, satisfying $\mathrm{lgdim}(Z_l) \leq \mathrm{lgdim}(X) - l$. By Lemma 4.2(ii), $\mathrm{lgdim}(Z_l) = \mathrm{lgdim}(X) - l$. ∎

We are finally in a position to prove the promised corollary on the definability of long dimension. Note that this corollary is not needed in the rest of the paper, but it is recorded here for completeness.

Corollary 4.4. *Let $X \subseteq M^{n+m}$ be a definable set and denote by $\pi : M^{n+m} \to M^n$ the usual projection. For $0 \leq l \leq m$, let*

$$l(X) = \{x \in \pi(X) : \pi^{-1}(x) \text{ has long dimension } l\}.$$

Then there are definable subsets $Y, Z \subseteq \pi(X)$ with $Y \subseteq l(X) \subseteq Z$ such that

$$\mathrm{lgdim}(Y) = \mathrm{lgdim}(Z) = \mathrm{lgdim}(X) - l.$$

Proof. With the notation of the previous two lemmas, let $Y = Y_l \setminus Z_{l+1}$ and $Z = Z_{l+1}$. Since $\mathrm{lgdim}(Y_l) = \max\{\mathrm{lgdim}(Y), \mathrm{lgdim}(Z_{l+1})\}$, it follows that $\mathrm{lgdim}(Y)$ is as desired (and, clearly, so is $\mathrm{lgdim}(Z)$). ∎

5. Pregeometries. In this section we develop the combinatorial counterpart of the long dimension and define the corresponding notion of 'long-genericity'. This notion is used in the application to definable groups in the next section.

Definition 5.1. A (finitary) *pregeometry* is a pair (S, cl), where S is a set and $\mathrm{cl} : P(S) \to P(S)$ is a *closure operator* satisfying, for all $A, B \subseteq S$ and $a, b \in S$:

(i) $A \subseteq \mathrm{cl}(A)$.
(ii) $A \subseteq B \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.
(iii) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.
(iv) $\mathrm{cl}(A) = \bigcup\{\mathrm{cl}(B) : B \subseteq A \text{ finite}\}$.
(v) (Exchange) $a \in \mathrm{cl}(bA) \setminus \mathrm{cl}(A) \Rightarrow b \in \mathrm{cl}(aA)$.

Definition 5.2. We define the *short closure* operator $\mathrm{scl} : P(M) \to P(M)$ as follows:

$$\mathrm{scl}(A) = \{a \in M : \text{there are } \bar{b} \subseteq A \text{ and } \phi(x, \bar{y}) \text{ from } \mathcal{L} \text{ such that } \phi(\mathcal{M}, \bar{b}) \text{ is a short interval and } \mathcal{M} \vDash \phi(a, \bar{b})\}.$$

We say that the formula $\phi(x, \bar{y}) \in \mathcal{L}$ *witnesses* $a \in \mathrm{scl}(\bar{b})$ if $\phi(\mathcal{M}, \bar{b})$ is a short interval and $\mathcal{M} \vDash \phi(a, \bar{b})$.

We will omit, as usual, the bar from tuples.

Remark 5.3. Given a formula $\phi(x, y) \in \mathcal{L}$ witnessing $a \in \mathrm{scl}(b)$, we can form a formula $S^{\phi}_{a,b}(x, y) \in \mathcal{L}$ which is satisfied by the pair (a, b) and such that for every $b' \in M^n$, $S^{\phi}_{a,b}(\mathcal{M}, b')$ is short. Indeed, let $\kappa \in M$ be short such that

$$\forall z_1, z_2\ [\phi(z_1, b) \wedge \phi(z_2, b) \to |z_1 - z_2| < \kappa].$$

By [Pet3, Corollary 3.7(3)], κ may be taken in $\mathrm{dcl}(\emptyset)$. We then let

$$S^{\phi}_{a,b}(x, y) : \phi(x, y) \wedge \forall z_1, z_2\ [\phi(z_1, y) \wedge \phi(z_2, y) \to |z_1 - z_2| < \kappa].$$

Lemma 5.4. *$a \in \mathrm{scl}(b) \Leftrightarrow \exists a' \in \mathrm{dcl}(b), a - a'$ is short.*

Proof. ($\Rightarrow$) Let f be a $\emptyset$-definable Skolem function for $S^{\phi}_{a,b}(x, y)$, where ϕ witnesses $a \in \mathrm{scl}(b)$; that is, for every $c \in M$, $\models \exists z\ [S^{\phi}_{a,b}(z, c) \to S^{\phi}_{a,b}(f(c), c)]$. Let $a' = f(b)$.

($\Leftarrow$) Assume $\phi(x, y)$ witnesses $a' \in \mathrm{dcl}(b)$. Let $\kappa \in \mathrm{dcl}(\emptyset)$ be such that $|a - a'| < \kappa$. Then a satisfies the following short formula:

$$\exists x'\ [\phi(x', b) \wedge (|x - x'| < \kappa)]. \ \blacksquare$$

Lemma 5.5. *(M, scl) is a pregeometry.*

Proof. Properties (i), (ii) and (iv) are straightforward.

(iii) This boils down to the fact that (Lemma 4.2(ii)) a short union of short sets is short. We provide the details. Let $a \in \mathrm{scl}(\bar{b})$, where $\bar{b} = (b_1, \dots, b_n) \in M^n$ is such that each b_i is in $\mathrm{scl}(\bar{c})$ for some $\bar{c} \subseteq A$. Assume that $\psi(x, \bar{b})$ witnesses $a \in \mathrm{scl}(\bar{b})$, and for each $i = 1, \dots, n$, $\phi_i(y_i, \bar{c})$ witnesses $b_i \in \mathrm{scl}(\bar{c})$, where $\psi, \phi_i \in \mathcal{L}$. Denote

$$S(\bar{y}, \bar{z}) := S^{\phi_1}_{b_1,\bar{c}}(y_1, \bar{z}) \wedge \dots \wedge S^{\phi_n}_{b_n,\bar{c}}(y_n, \bar{z}).$$

Then the set X defined by the formula

$$\exists \bar{y}\ [S(\bar{y}, \bar{c}) \wedge S^{\psi}_{a,\bar{b}}(x, \bar{y})]$$

is $\bar{c}$-definable and contains a. We show that X is short. Clearly, the set

$$X' = \bigcup_{\bar{y} \in S(\mathcal{M}, \bar{c})} \{\bar{y}\} \times S^{\psi}_{a,\bar{b}}(\mathcal{M}, \bar{y})$$

has long dimension at least the long dimension of X, since the function $f : (\bar{y}, x) \mapsto x$ maps X' onto X. But X' is a short union of short sets and, by Lemma 4.2(ii), it must have long dimension 0.

(v) Without loss of generality, assume $A = \emptyset$. Let $\phi(x, y)$ be a formula witnessing $a \in \mathrm{scl}(b)$. We assume that $b \notin \mathrm{scl}(a)$ and show $a \in \mathrm{scl}(\emptyset)$. Let $f(x)$ be a $\emptyset$-definable Skolem function for $S^{\phi}_{a,b}(x, y)$. Let $\kappa \in M$ be short and in $\mathrm{dcl}(\emptyset)$ such that

$$\forall z_1, z_2\ [\phi(z_1, b) \wedge \phi(z_2, b) \to |z_1 - z_2| < \kappa]$$

(see Remark 5.3). Let

$$Y = \{b' \in M : |f(b') - a| < \kappa\}.$$

Then since Y is a-definable and contains b, it must be long. By Lemma 2.1, there is some interval contained in Y on which f is constant, say equal to a'. But then $a' \in \mathrm{dcl}(\emptyset)$ and, by Lemma 5.4, $a \in \mathrm{scl}(\emptyset)$. ∎

Definition 5.6. Let $A, B \subseteq M$. We say that B is *scl-independent over* A if for all $b \in B$, $b \notin \mathrm{scl}(A \cup (B \setminus \{b\}))$. A maximal scl-independent subset of B over A is called a *basis for B over A*.

By the Exchange property in a pregeometric theory, any two bases for B over A have the same cardinality. This allows us to define the *rank of B over A*:

$$\mathrm{rank}(B/A) = \text{the cardinality of any basis of } B \text{ over } A.$$

Lemma 5.7. *If p is a partial type over $A \subseteq M$ and $a \models p$ with* $\mathrm{rank}(a/A) = m$, *then for any set $B \supseteq A$ there is $a' \models p$ (possibly in an elementary extension) such that* $\mathrm{rank}(a'/B) \geq m$.

Proof. The proof of the analogous result for the usual rank (coming from acl) is given, for example, in [G, p. 315]. The proof of the present lemma is word-for-word the same, after replacing 'algebraic formula' by 'short formula' in the definition of Φ^m_B ([G, Definition 2.2]) and the notion of 'algebraic independence' by that of 'scl-independence' we have here. ∎

Definition 5.8. Assume $\mathcal{M}$ is sufficiently saturated. Let p be a partial type over $A \subset M$. The *short closure dimension of p* is defined as follows:

$$\text{scl-dim}(p) = \max\{\mathrm{rank}(\bar{a}/A) : \bar{a} \subset M \text{ and } \bar{a} \models p\}.$$

Let X be a definable set. Then the *short closure dimension of X*, denoted by scl-dim(X), is the dimension of its defining formula.

In Corollary 5.10 below we establish that the scl-dimension of a definable set coincides with its long dimension we defined earlier. We note that the equivalence between the usual geometric dimension and the topological dimension was proved in [Pi1].

Lemma 5.9. *Let $\bar{a} \subseteq M$ be an n-tuple and $A \subseteq M$ a set. Then* $\mathrm{rank}(\bar{a}/A) = n$ *if and only if $\bar{a}$ does not belong to any A-definable set with long dimension $< n$.*

Proof. ($\Leftarrow$) Assume $\bar{a} = (a_1, \ldots, a_n)$ and $\mathrm{rank}(\bar{a}/A) < n$. Then for some i, say $i = 1$, $a_1 \in \mathrm{scl}(A \cup \{a_2, \ldots, a_n\})$. Let $\phi(x, \bar{y})$ be an $\mathcal{L}(A)$-formula witnessing this fact. Recall from Remark 5.3 that the $\mathcal{L}(A)$-formula $S^{\phi}_{\bar{a}}(x, \bar{y})$ is satisfied by $\bar{a}$, and $S^{\phi}_{\bar{a}}(\mathcal{M}, b')$ is short for every $b' \in M^{n-1}$. By Lemma 4.2(ii), $S^{\phi}_{\bar{a}}(\mathcal{M}^n)$ has long dimension $< n$.

($\Rightarrow$) We prove the statement by induction on n. For $n = 1$, it is clear. Suppose it is proved for n. Let $\bar{a} = (a_1, \ldots, a_{n+1})$ be a tuple of rank, over A, equal to $n+1$ and assume, for a contradiction, that X is an A-definable set containing a with $\mathrm{lgdim}(X) < n+1$. By cell decomposition, we may assume that X is an A-definable cell. If X is the graph of a function, then a_{n+1} is in $\mathrm{dcl}(A \cup \{a_1, \ldots, a_n\})$ and hence in $\mathrm{scl}(A \cup \{a_1, \ldots, a_n\})$, a contradiction. Now assume that X is a cylinder. By the Refined Structure Theorem, we may assume that $X = (f, g)_{\pi(X)}$ is a cylinder whose domain is an A-definable long cone such that f and g are almost linear with respect to it. Since $\mathrm{rank}(\bar{a}/A) = n+1$, $g(a_1, \ldots, a_n) - f(a_1, \ldots, a_n)$ must be long. But by the inductive hypothesis, $\mathrm{lgdim}(\pi(X)) = n$. Hence, by Lemma 2.17, $\mathrm{lgdim}(X) = n+1$, a contradiction. ∎

Corollary 5.10. *For every definable $X \subseteq M^n$,*

$$\mathrm{lgdim}(X) = \mathrm{scl}\text{-}\mathrm{dim}(X).$$

Proof. It is easy to see that every A-definable k-long cone X contains a tuple a with $\mathrm{rank}(a/A) = k$. On the other hand, by Corollary 2.13 and Lemma 5.9, X cannot contain a tuple a with $\mathrm{rank}(a/A) > k$. ∎

5.1. Long-generics. For a treatment of the classical notion of generic elements, corresponding to the algebraic closure acl, see [Pi2]. Here we introduce the corresponding notion for scl.

Definition 5.11. Let $X \subseteq M^n$ be an A-definable set, and let $\bar{a} \in X$. We say that $\bar{a}$ is a *long-generic element of X over A* if it does not belong to any A-definable set of long dimension $< \mathrm{lgdim}(X)$. If $A = \emptyset$, we call $\bar{a}$ a *long-generic element of X*.

In a sufficiently saturated o-minimal structure, *long-generic elements always exist.* More precisely, every A-definable set X contains a long-generic element over A. Indeed, by compactness and Lemma 3.6(v), the collection of all formulas which express that x belongs to X but not to any A-definable set of long dimension $< \mathrm{lgdim}(X)$ is consistent.

A definable subset V of a definable set X is called *long-large in X* if $\operatorname{lgdim}(X \setminus V) < \operatorname{lgdim}(X)$. In a sufficiently saturated o-minimal structure, V is long-large in X if and only if for every A over which V and X are defined, V contains every long-generic element a in X over A.

Two long-generics are called *independent* if one (each) of them is long-generic over the other.

Let G be a definable abelian group. Let us recall the notion of a left generic *set* (not to be confused with the notion of a generic element). A subset $X \subseteq G$ is called *left n-generic* if n left translates of X cover G. It is called *left generic* if it is left n-generic for some n. We recall from [ElSt, Lemma 3.10] (see [PeS] for the notion of definable compactness):

Fact 5.12 (Generic Lemma). *Assume G is definably compact. Then, for every definable subset $X \subseteq G$, either X or $G \setminus X$ is left generic.*

The facts that (M, scl) is a pregeometry and that the scl-dim agrees with lgdim imply:

Claim 5.13. *Let $G = \langle G, \cdot \rangle$ be a definable group with* $\operatorname{lgdim}(G) = n$. *Then*

(1) *If $X \subseteq G$ is long-large in G, then X is left $(n+1)$-generic in G.*
(2) *If $a, g \in G$ are independent long-generics, then so are a and $a \cdot g$.*

Proof. The proof is standard. (1) is word-for-word the same as that of [Pet2, Fact 5.2] after replacing: a) the notion of a 'large' set by that of a 'long-large' set, b) the 'dimension' of a definable set by 'long dimension', and c) the 'dimension' of a tuple by 'rank'. (2) is straightforward using the Exchange property. ∎

Note that none of the notions 'generic element' and 'long-generic element' implies the other.

Lemma 5.14. *Let X, W be definable subsets of a definable group G. Assume that X is a long-large subset of W and that W is left generic in G. Then X is left generic in G.*

Proof. This is similar to the proof of [PePi, Lemma 3.4(ii)]. Since W is left generic we can write $G = g_1 W \cup \cdots \cup g_m W$. Let $Y = W \setminus X$. Then $Z = g_1 Y \cup \cdots \cup g_m Y$ has long dimension $< \operatorname{lgdim}(G)$. So, by Claim 5.13, finitely many left translates of $G \setminus Z$ cover G. It then follows that finitely many left translates of X cover G. ∎

We record one more lemma, which however will not be used in this paper:

Lemma 5.15. *Let G be a definable group and X a definable subset of G. If X is left generic in G then* $\operatorname{lgdim}(X) = \operatorname{lgdim}(G)$.

Proof. Since the group conjugation is a definable bijection, the statement follows from Lemma 3.6(v) and Corollary 3.11. ∎

6. The local structure of semi-bounded groups. *In this section, we assume that $\mathcal{M}$ is sufficiently saturated, and we fix a $\emptyset$-definable group $G = \langle G, \oplus, e_G\rangle$ with $G \subseteq M^n$ and long dimension k.*

By [Pi2], we know that every group definable in an o-minimal structure can be equipped with a unique definable manifold topology that makes it into a topological group, called the *t-topology*.

REMARK 6.1. By the Refined Structure Theorem and Corollaries 3.7 and 5.10, for any two independent long-generic elements a and b of G and any $\emptyset$-definable function $f : G \times G \to G$, there are k-long cones C_a and C_b in G containing a and b, respectively, such that for all $x \in C_a$ and $y \in C_b$

$$f(x, y) = \lambda x + \mu y + d,$$

for some fixed $\lambda, \mu \in \mathbb{M}(n, \Lambda)$ and $d \in M^n$. If $f(x, y) = x \oplus y$ is the group operation of G, the λ and μ have to be moreover invertible matrices (for example, if we set $y = b$, then $x \oplus b = \lambda x + \mu b + d$ is invertible, showing that λ is invertible).

LEMMA 6.2. *For every two independent long-generics $a, b \in G$, there is a k-long cone C_a containing a, invertible λ, $\lambda' \in \mathbb{M}(n, \Lambda)$ and $c, c' \in M^n$ such that for all $x \in C_a$,*

$$x \ominus a \oplus b = \lambda x + c \quad \textit{and} \quad \ominus a \oplus b \oplus x = \lambda' x + c'.$$

Proof. By Claim 5.13, since a and b are independent long-generics of G, a and $\ominus a \oplus b$ are independent long-generics of G as well. Therefore, by Remark 6.1, there are long cones C_a of a and $C_{\ominus a \oplus b}$ of $\ominus a \oplus b$ in G, as well as invertible $\lambda, \mu \in \mathbb{M}(n, D)$ and $d \in M^n$, such that for all $x \in C_a$ and $y \in C_{\ominus a \oplus b}$,

$$x \oplus y = \lambda x + \mu y + d.$$

In particular, for all $x \in C_a$, $x \ominus a \oplus b = \lambda x + \mu(\ominus a \oplus b) + d$. Letting $c = \mu(\ominus a \oplus b) + d$ shows the first equality. The second equality can be shown similarly. ∎

We are now ready to prove the local theorem for semi-bounded groups.

THEOREM 6.3. *Let a be a long-generic element of G. Then there is a k-long cone $C_a \subseteq G$ containing a and such that for every $x, y \in C_a$,*

$$x \ominus a \oplus y = x - a + y.$$

Proof. We first prove:

CLAIM. *There is a k-long cone $C_a \subseteq G$ containing a, and $\lambda, \mu \in \mathbb{M}(n, \Lambda)$ and $d \in M^n$, such that for all $x, y \in C_a$,*

$$x \ominus a \oplus y = \lambda x + \mu y + d.$$

Proof. Let a_1 be a long-generic element of G independent of a. Then $a_2 = a \ominus a_1$ is also a long-generic element of G independent of a. By Lemma 6.2, we can find k-long cones C and C' in G containing a, as well as $\lambda_1, \lambda_2 \in \mathbb{M}(n, \Lambda)$ and $c_1, c_2 \in M^n$, such that for all $x \in C$ and $y \in C'$,

$$x \ominus a_2 = \lambda_1 x + c_1 \quad \text{and} \quad \ominus a_1 \oplus y = \lambda_2 y + c_2. \tag{6.1}$$

We let V_{a_1} be the image of C under the map $x \mapsto x \ominus a_2$, and V_{a_2} the image of C' under $y \mapsto \ominus a_1 \oplus y$. Then V_{a_1} and V_{a_2} are open neighborhoods in G of a_1 and a_2, respectively. Also, since a is long-generic, it must be contained in a k-long cone $C_a \subseteq C \cap C'$, on which, of course, every x and every y satisfy (6.1).

Now, by Remark 6.1 and since a_1 and $a_2 = a \ominus a_1$ are independent long-generics of G, there are k-long cones C_{a_1} and C_{a_2} containing a_1 and a_2, respectively, such that for some fixed $\nu, \xi \in \mathbb{M}(n, \Lambda)$ and $\varepsilon \in M^n$, we have $x \oplus y = \nu x + \xi y + \varepsilon$ for all $x \in C_{a_1}$ and $y \in C_{a_2}$. By continuity of $\oplus$, we could choose C_a, V_{a_1}, V_{a_2} so that $V_{a_1} \subseteq C_{a_1}$ and $V_{a_2} \subseteq C_{a_2}$, and still all $x, y \in C_a$ satisfy (6.1). Then for all $x, y \in C_a$, we have

$$\begin{aligned} x \ominus a \oplus y &= x \ominus a \oplus a_1 \ominus a_1 \oplus y = (x \ominus a_2) \oplus (\ominus a_1 \oplus y) \\ &= \nu(\lambda_1 x + c_1) + \xi(\lambda_2 y + c_2) + \varepsilon = \nu\lambda_1 x + \xi\lambda_2 y + \nu c_1 + \xi c_2 + \varepsilon. \end{aligned}$$

Setting $\lambda = \nu\lambda_1$, $\mu = \xi\lambda_2$, and $d = \nu c_1 + \xi c_2 + o$ finishes the proof of the claim. ∎

By the Claim, for all $x, y \in C_a$,

$$\begin{aligned} y &= a \ominus a \oplus y = \lambda a + \mu y + d, \\ x &= x \ominus a \oplus a = \lambda x + \mu a + d, \\ a &= a \ominus a \oplus a = \lambda a + \mu a + d. \end{aligned}$$

Hence, $x - a + y = \lambda x + \mu y + d = x \ominus a \oplus y$. ∎

We conclude with a stronger version of the local theorem that we expect to be useful in a future global analysis for semi-bounded groups. By [Pi2], we know that the t-topology of G coincides with the subspace topology on a large open definable subset W^G. The proof of the following proposition involves all machinery developed so far.

PROPOSITION 6.4. *Assume G is definably compact. There is a left generic k-long cone C contained in G, on which the t-topology agrees with the* *subspace topology, and for every $a \in C$, there is a relatively open subset V_a of $a + \langle C \rangle$, containing a, such that for all $x, y \in V_a$,*

$$x \ominus a \oplus y = x - a + y. \tag{6.2}$$

Proof. By the Refined Structure Theorem, W^G is the union of finitely many long cones $C_1, \ldots, C_l$. Let $\bar{v}_j = (v_{j1}, \ldots, v_{jk_j})$ be the direction of each C_j. By the Generic Lemma, one of the C_j's, say C_1, is a left generic k-long cone.

Claim. *Every long-generic element a of W^G is contained in some k-long cone contained in G with direction some $\bar{v}_j$ on which* (6.2) *holds.*

Proof. Since a is long-generic, Theorem 6.3 implies that a is contained in some k-long cone D on which (6.2) holds. Since a is in W^G, it is contained in some C_j. By Corollary 3.5, it is not hard to see that some k-long cone with direction $\bar{v}_j$ must be contained in D and contain a. ∎

Consider now the following property, for an element $a \in C_1$:

(∗) there is a relatively open subset V_a of $a + \langle C_1 \rangle$ containing a and such that for all $x, y \in V_a$, (6.2) holds.

The set X of elements of C_1 that satisfy (∗) is clearly definable. We claim that it is also long-large in C_1.

Clearly, it suffices to prove that every long-generic element of C_1 satisfies (∗). Let a be a long-generic element of C_1. If a belongs to a k-long cone of direction $\bar{v}_1$ on which (6.2) holds, then we are done. Hence, by the Claim, it clearly suffices to show that for every $j \neq 1$, the set A_j of all elements of C_1 that belong to a k-long cone of direction $\bar{v}_j$ but do not satisfy (∗), is contained in a definable set of long dimension $< k$. To see this, note that if $a \in A_j$, then by Corollaries 2.14 and 3.5, one of the $v_{j1}, \ldots, v_{jk_j}$, say v_{j1}, must be so that for every positive $t \in M$, $v_{j1}t \notin \langle C_1 \rangle$. Let κ be a tall element and $D_j = \{v_{j1}t : t \in (0, \kappa)\}$. The set

$$K_j = (C_1 + D_j) \cap G$$

has long dimension $\leq k$, as a subset of G. Hence, by Lemma 4.3, and since each D_j has long dimension 1, A_j is contained in a definable set of long dimension $\leq \operatorname{lgdim}(K_j) - 1$.

We have proved that X is long-large in C_1. By Lemma 5.14, X is left generic. By the Refined Structure Theorem, the Generic Lemma and the Lemma on Subcones, there is a left generic k-long cone C contained in X with the desired property. ∎

7. Appendix. If we tried to prove Lemma 2.8 by induction on n, then in the inductive step we would run into a system whose form is more general than that of the original one. Thus, we prove the following, more general statement.

LEMMA 7.1. *Let $w_1, \dots, w_n, w_{n+1}, \dots, w_{n+k} \in \Lambda^n$ be M-independent and $\lambda_1, \dots, \lambda_n \in \Lambda^n$. Let $t_1, \dots, t_n \in M$ be non-zero elements and, for every $i = 1, \dots, n$, let $r_i^1, \dots, r_i^k \in M$ be such that*

$$w_1 t_1 + \sum_{j=1}^{k} w_{n+j} r_1^j = \lambda_1 s_1^1 + \dots + \lambda_n s_1^n,$$
$$\vdots$$
$$w_n t_n + \sum_{j=1}^{k} w_{n+j} r_n^j = \lambda_1 s_n^1 + \dots + \lambda_n s_n^n$$

for some $s_i^j \in M$. Then there are non-zero $a_1, \dots, a_n \in M$ and $b_i^j \in M$, $i = 1, \dots, n$, $j = 1, \dots, n+k$, such that,

$$\lambda_1 a_1 = w_1 b_1^1 + \dots + w_{n+k} b_1^{n+k},$$
$$\vdots$$
$$\lambda_n a_n = w_1 b_n^1 + \dots + w_{n+k} b_n^{n+k}.$$

Proof. By induction on n. For $n = 1$, the result is trivial. Assume that $n > 1$ and we know the statement for $< n$ equations. Since $w_1, \dots, w_{n+k} \in \Lambda^n$ are M-independent and $t_1 \neq 0$, $w_1 t_1 + \sum_{j=1}^{k} w_{n+j} r_1^j \neq 0$. Hence there is some s_1^j, say s_1^1, which is not zero. By switching the equations if necessary, we may also assume that $s_i^1 < s_1^1$ for every $i = 2, \dots, n$. Since

$$\lambda_1 s_1^1 = w_1 t_1 + \sum_{j=1}^{k} w_{n+j} r_1^j - (\lambda_2 s_1^2 + \dots + \lambda_n s_1^n), \tag{7.1}$$

Lemma 2.7 gives, for every $i = 2, \dots, n$,

$$\lambda_1 s_i^1 = w_1 T_i + \sum_{j=1}^{k} w_{n+j} R_i^j - (\lambda_2 S_i^2 + \dots + \lambda_n S_i^n)$$

for some $S_i^2, \dots, S_i^n, T_i, R_i^1, \dots R_i^k \in M$. By substituting into the original system of equations, we obtain

$$w_2 t_2 - w_1 T_1 + \sum_{j=1}^{k} w_{n+j} (r_2^j - R_2^j) = \lambda_2 (s_2^2 - S_2^2) + \dots + \lambda_n (s_2^n - S_2^n),$$
$$\vdots$$
$$w_n t_n - w_1 T_1 + \sum_{j=1}^{k} w_{n+j} (r_n^j - R_n^j) = \lambda_2 (s_n^2 - S_n^2) + \dots + \lambda_n (s_n^n - S_n^n).$$

Now apply the inductive hypothesis to find $a_2, \ldots, a_n$ such that

(7.2) each of $\lambda_2 a_2, \ldots, \lambda_n a_n$ can be expressed in terms of $w_1, \ldots, w_{n+k}$.

By Lemma 2.7, we can replace the elements of M that appear in (7.1) by arbitrarily small positive ones; that is, there are arbitrarily small $a_1, p_1, q_1^j, u_1^j \in M$ such that

$$\lambda_1 a_1^1 = w_1 p_1 + \sum_{j=1}^{k} w_{n+j} q_1^j - (\lambda_2 u_1^2 + \cdots + \lambda_n u_1^n). \tag{7.3}$$

In particular, we may choose $0 < u_1^j < a_j$. Hence, by Lemma 2.7 again and (7.2), we can express each of $\lambda_2 u_1^2, \ldots, \lambda_n u_1^n$ in terms of $w_1, \ldots, w_{n+k}$. Hence $\lambda_1 a_1^1$ is now also expressed in terms of $w_1, \ldots, w_{n+k}$, finishing the proof. ■

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