FUNDAMENTA
MATHEMATICAE
182 (2004)

F_σ-mappings and the invariance of absolute Borel classes

by

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Abstract. It is proved that F_σ-mappings preserve absolute Borel classes, which improves results of R. W. Hansell, J. E. Jayne and C. A. Rogers. The proof is based on the fact that any F_σ-mapping $f : X \to Y$ of an absolute Suslin metric space X onto an absolute Suslin metric space Y becomes a piecewise perfect mapping when restricted to a suitable F_σ-set $X_\infty \subset X$ satisfying $f(X_\infty) = Y$.

1. Introduction. We recall that a mapping $f : X \to Y$ of a metric space X to a metric space Y is called an *F_σ-mapping* if f maps F_σ-sets in X to F_σ-sets in Y and f^{-1} maps F_σ-sets in Y to F_σ-sets in X. A mapping f of a metric space X into a metric space Y is said to be *piecewise closed* if there is a sequence $\{X_n\}_{n\in\mathbb{N}}$ of closed subsets of X such that $X = \bigcup_{n\in\mathbb{N}} X_n$, and the restriction of f to X_n is a closed continuous mapping of X_n to Y for every $n \in \mathbb{N}$.

R. W. Hansell, J. E. Jayne and C. A. Rogers proved in [2, Theorem 3] that an F_σ-mapping f of an absolute Suslin metric space X onto an absolute Suslin metric space Y is in fact piecewise closed if

(a) Fleissner's axiom holds, or
(b) each point of X has a neighbourhood that is mapped by f onto a set in Y which is σ-locally of weight at most $\aleph_1$, or
(c) f is an open mapping, or
(d) $f^{-1}(y)$ is compact for each y in Y.

Each of these assumptions ensures that f maps discrete families to "almost σ-discretely decomposable families", which is the crucial point in the proof

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2000 *Mathematics Subject Classification*: 54H05, 54E40, 28A05.

Key words and phrases: F_σ-mapping, piecewise closed mapping, absolute Borel space.

The first author was supported in part by the grant GAČR 201/03/0931, in part by the grant GAČR 201/03/0933, and in part by the Research Project MSM 1132 00007. The second author was supported in part by the grant GAČR 201/03/0935, in part by the grant GAČR 201/03/D120, and in part by the Research Project MSM 1132 00007.

of [7, Theorem 1] used in the proof of [2, Theorem 3] (for the definition of almost σ-discretely decomposable families see below).

We are going to prove a weaker statement which does not need any of the assumptions (a)–(d) and which is still sufficient to get some significant corollaries of [2, Theorem 3]. The crucial point is that we deal with σ-discrete refinements instead of almost σ-discrete decomposability and we may use the following result [9, Theorem 3.5]:

Let $f : X \to Y$ be a mapping of a metric space X to an absolute Suslin metric space Y such that f maps F_σ-sets in X to F_σ-sets in Y. Then f maps any σ-discrete family of F_σ-sets in X onto a family admitting a σ-discrete refinement.

Our modification of [2, Theorem 3] reads:

Any F_σ-mapping f of an absolute Suslin metric space X onto an absolute Suslin metric space Y has a piecewise closed restriction to an F_σ-set $X_\infty \subset X$ satisfying $f(X_\infty) = Y$.

By [7, Theorem 3] there is a restriction which is even piecewise perfect. Our Theorems 4.1 and 4.2 appeared already as [5, Theorem 2] and [6, Theorem 2]. However, as explained in [2, Theorem 3], the authors needed at least one of the extra assumptions (a)–(d) mentioned above.

2. Preliminaries. By a *space* we mean a metrizable space without mentioning it explicitly. We write (X, ϱ) when a compatible metric ϱ is specified.

We use $\mathbb{N}^{<\mathbb{N}}$ to denote the space of finite sequences of positive integers. If $s \in \mathbb{N}^{<\mathbb{N}}$, then $l(s)$ stands for the length of s. As usual, for $s, t \in \mathbb{N}^{<\mathbb{N}}$, we write $s \prec t$ if t is an extension of s, i.e., $l(s) \le l(t)$ and $s_i = t_i$ if $1 \le i \le l(s)$. Using $\emptyset$ to denote the empty sequence, we adopt the convention that $l(\emptyset) = 0$. If $s \in \mathbb{N}^{<\mathbb{N}}$ and $n \in \mathbb{N}$, we write $s^\wedge n$ for the sequence $(s_1, \ldots, s_{l(s)}, n)$.

The space $\mathbb{N}^\mathbb{N}$ with the usual product topology will be denoted by $\mathbb{I}$. For $\sigma \in \mathbb{I}$, $\sigma = \{\sigma_k\}_{k=1}^\infty$, and $n \in \mathbb{N}$, we put $\sigma{\restriction}n = (\sigma_1, \ldots, \sigma_n)$. We adopt the convention that $\sigma{\restriction}0 = \emptyset$. For a given $s \in \mathbb{N}^n$, the *Baire interval* $\mathbb{I}(s)$ is defined by

$$\mathbb{I}(s) = \{\sigma \in \mathbb{I} : \sigma{\restriction}n = s\}.$$

A subset A of a space X is said to be *Suslin* if there exists a family $\{F_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$ of closed sets in X so that

$$A = \bigcup_{\sigma \in \mathbb{I}} \bigcap_{n=1}^{\infty} F_{\sigma{\restriction}n}.$$

A space X is called an *absolute Suslin space* if X is homeomorphic to a Suslin subset of some completely metrizable space. We use a well known fact that X is an absolute Suslin space if and only if X is a Suslin subset of every space containing X (this observation easily follows from the Lavrent'ev theorem [8, §35, II, Theorem]).

A set A in a space X is *discrete* if for every point $x \in X$ there exists a neighbourhood U of x which has at most one common point with A. If A can be written as a union of countably many discrete sets, it is said to be *σ-discrete.*

Let $\mathcal{A}$ be a family of sets in a space X. Then $\mathcal{A}$ is said to be *discrete* in X if each point $x \in X$ has a neighbourhood that meets at most one set from $\mathcal{A}$. If $\mathcal{A}$ is a countable union of discrete families $\mathcal{A}_n$, then it is said to be *σ-discrete.* It readily follows that $A \subset X$ is a discrete (respectively σ-discrete) set in X if and only if the family $\{\{x\} : x \in A\}$ is discrete (respectively σ-discrete) in X.

We say that $\mathcal{A}$ is *σ-discretely decomposable* if every set $A \in \mathcal{A}$ can be written as $A = \bigcup_{n\in\mathbb{N}} A(n)$, where $\{A(n) : A \in \mathcal{A}\}$ is a discrete family for every $n \in \mathbb{N}$.

A family $\mathcal{A}$ is said to be *almost σ-discretely decomposable* if it becomes σ-discretely decomposable when restricted to the complement of some σ-discrete set.

A family $\mathcal{R}$ is called a *refinement* of $\mathcal{A}$ if $\bigcup \mathcal{R} = \bigcup \mathcal{A}$ and for each $R \in \mathcal{R}$ there exists $A \in \mathcal{A}$ with $R \subset A$. We say that $\mathcal{A}$ has a *σ-discrete refinement* if there exists a refinement $\mathcal{R}$ of $\mathcal{A}$ which is a σ-discrete family. Clearly, any almost σ-discretely decomposable family has a σ-discrete refinement but the converse need not hold in general.

A mapping $f : X \to Y$ is *perfect* if f is closed, continuous, and the fiber $f^{-1}(y)$ is a compact subset of X for every $y \in Y$.

If X is a union of countably many closed sets X_n and the restriction of f to X_n is a perfect mapping of X_n to Y, then f is called *piecewise perfect.*

Without further reference we shall use the well known fact that any metrizable space has a σ-discrete base of open sets (see [8, §21, XVI, Corollary 1a]).

If ϱ is a metric on a space X and $A, B \subset X$, then $\operatorname{dist}_\varrho(A, B)$ stands for the distance of A and B, and $\operatorname{diam}_\varrho A$ for the diameter of A. For a sequence $\{A_n\}_{n\in\mathbb{N}}$ of nonempty sets in X and $x \in X$, we write $A_n \to \{x\}$ (as n tends to infinity) if for every neighbourhood U of x there exists $k \in \mathbb{N}$ so that $A_n \subset U$ for all $n \geq k$.

If $f : X \to Y$ is a mapping and $\mathcal{A}$ is a family of subsets of X, we write $f(\mathcal{A})$ for the family $\{f(A) : A \in \mathcal{A}\}$. Similarly we use $f^{-1}(\mathcal{B}) = \{f^{-1}(B) : B \in \mathcal{B}\}$ for a family $\mathcal{B}$ in Y.

3. Piecewise closed and piecewise perfect mappings. We are going to prove our main result on F_σ-mappings in Theorem 3.6. We use [4, Theorem 5] to reduce the problem to the case when f is, moreover, continuous. Then we prove Proposition 3.5 by modifying the inductive construction of [7, Lemma 2]. Here we essentially use the above-mentioned result [9, Theorem 3.5] instead of [7, Lemma 1].

We use the following easy fact without further reference (see, e.g., [9, Lemma 3.4]):

Let a family $\mathcal{A}$ of F_σ-sets in X have a σ-discrete refinement. Then $\mathcal{A}$ has a σ-discrete refinement consisting of F_σ-sets.

We first introduce our key auxiliary notion.

DEFINITION 3.1. Let $f : X \to Y$ be a mapping of a space X to a space Y. We say that a set $A \subset X$ is *covered by* f if there exist F_σ-sets $F, H \subset X$ such that $A \subset F$, the restriction of f to H is piecewise closed, and $f(H) \supset f(F)$.

REMARK 3.2. Note that $B \subset X$ is covered by f whenever $B \subset A$ and A is covered by f. Hence this notion is hereditary with respect to inclusion. Further, if $A \subset X$ is covered by f, we may demand without loss of generality that the set F from Definition 3.1 satisfies $F \subset \overline{A}$. It also easily follows from the definition that the union of countably many sets covered by f is covered by f as well. Let us point out that, if $A \subset X$ is not covered by a mapping $f : X \to Y$, then the restriction of f to $\overline{A}$ is not piecewise closed.

The following lemma indicates a situation in which the union of a σ-discrete family of sets covered by f is also covered by f.

LEMMA 3.3. *Let $f : X \to Y$ be a continuous mapping of a space X to a space Y such that*

(i) *f maps F_σ-sets in X to F_σ-sets in Y, and*
(ii) *$f(\mathcal{F})$ has a σ-discrete refinement in Y for any σ-discrete family $\mathcal{F}$ of F_σ-sets in X.*

If $\mathcal{A}$ is a σ-discrete family of sets covered by f, then $\bigcup \mathcal{A}$ is covered by f.

Proof. For every $A \in \mathcal{A}$, let F^A and H^A be F_σ-sets in X such that $A \subset F^A$, the restriction of f to H^A is piecewise closed, and $f(H^A) \supset f(F^A)$. By Remark 3.2, we may suppose that the family $\mathcal{F} = \{F^A : A \in \mathcal{A}\}$ is σ-discrete.

Find a σ-discrete refinement $\mathcal{R}$ of $f(\mathcal{F})$ consisting of F_σ-sets. For every $R \in \mathcal{R}$ find $A(R) \in \mathcal{A}$ with $R \subset f(F^{A(R)})$. Set

$$H(R) = f^{-1}(R) \cap H^{A(R)}, \quad R \in \mathcal{R}, \quad H = \bigcup_{R\in\mathcal{R}} H(R), \quad F = \bigcup \mathcal{F}.$$

Then F is an F_σ-set in X and $\bigcup \mathcal{A} \subset F$. Since f is continuous and $\mathcal{R}$ is a σ-discrete family of F_σ-sets, $\{H(R) : R \in \mathcal{R}\}$ is also a σ-discrete family of F_σ-sets and thus H is an F_σ-set in X.

It is sufficient to prove that $f(H) \supset f(F)$ and that the restriction of f to H is piecewise closed.

Since $\bigcup \mathcal{R} = \bigcup f(\mathcal{F}) = f(F)$ and $f(H(R)) = R$ for every $R \in \mathcal{R}$, we have $f(H) \supset f(F)$.

To show that $f\restriction_H$ is piecewise closed, write $\mathcal{R} = \bigcup_n \mathcal{R}_n$ so that $\mathcal{R}_n$ is discrete for $n \in \mathbb{N}$. For every $R \in \mathcal{R}$ find closed sets R_k, $k \in \mathbb{N}$, in Y and closed sets $H_m^{A(R)}$, $m \in \mathbb{N}$, in X such that

$$R = \bigcup_{k=1}^{\infty} R_k, \quad H^{A(R)} = \bigcup_{m=1}^{\infty} H_m^{A(R)},$$

and such that the restriction of f to each $H_m^{A(R)}$ is a closed mapping.

Fix $n, k, m \in \mathbb{N}$ and set

$$H_{n,k,m} = \bigcup \{f^{-1}(R_k) \cap H_m^{A(R)} : R \in \mathcal{R}_n\}.$$

Then $H_{n,k,m}$, as a discrete union of closed sets, is closed, and the restriction of f to $H_{n,k,m}$ is a closed mapping.

Indeed, let E be a closed set in $H_{n,k,m}$. Then

$$\begin{aligned} f\restriction_{H_{n,k,m}}(E) = f(E) &= \bigcup_{R \in \mathcal{R}_n} f(E \cap f^{-1}(R_k) \cap H_m^{A(R)}) \\ &= \bigcup_{R \in \mathcal{R}_n} f\restriction_{H_m^{A(R)}}(E \cap f^{-1}(R_k)). \end{aligned}$$

As the restriction of f to $H_m^{A(R)}$ is a closed mapping of $H_m^{A(R)}$ to Y, the latter set is a discrete union of closed sets in Y. Thus $f\restriction_{H_{n,k,m}}(E)$ is a closed set and the restriction of f to $H_{n,k,m}$ is a closed continuous mapping as required.

Finally, $H = \bigcup_{n,k,m} H_{n,k,m}$ since

$$H(R) = f^{-1}(R) \cap H^{A(R)} = \bigcup_{k,m=1}^{\infty} (f^{-1}(R_k) \cap H_m^{A(R)}) \quad \text{for } R \in \mathcal{R}. \ \blacksquare$$

Lemma 3.4. *Let $f : X \to Y$ be a continuous mapping of (X, ϱ) to (Y, σ) such that f maps F_σ-sets in X to F_σ-sets in Y and $f(\mathcal{F})$ has a σ-discrete refinement in Y for any σ-discrete family $\mathcal{F}$ of F_σ-sets in X. Let $A \subset X$ be not covered by f and $\varepsilon > 0$ be arbitrary. Then there exist a sequence $\{L_k\}_{k \in \mathbb{N}}$ of subsets of A that are not covered by f and an element x of $\overline{A}$ such that*

(i) *$\{L_k : k \in \mathbb{N}\}$ is a discrete family in X;*
(ii) *$\operatorname{dist}_\varrho(L_k, L_l) > 0$ and $\operatorname{diam}_\varrho L_k < \varepsilon$ for $k, l \in \mathbb{N}$, $k \neq l$;*

(iii) $f(L_k) \to \{f(x)\}$ *as* k *tends to infinity*;
(iv) $\operatorname{dist}_\sigma(f(L_k), f(L_l)) > 0$ *and* $\operatorname{dist}_\sigma(f(L_k), \{f(x)\}) > 0$ *for* $k, l \in \mathbb{N}$, $k \neq l$.

Proof. Let $\mathcal{B}$ be a σ-discrete base of open sets in X. Set

$$\widehat{\mathcal{B}} = \{B \in \mathcal{B} : A \cap B \text{ is covered by } f\}, \quad G = A \cap \bigcup \widehat{\mathcal{B}}, \quad F = A \setminus G.$$

Since the family $\{A \cap B : B \in \mathcal{B}\}$ is σ-discrete, it follows from Lemma 3.3 that G is covered by f. The assumption that A is not covered by f implies that the restriction of f to $\overline{F}$ is not piecewise closed.

Since $f(\overline{F})$ is an F_σ-set in Y, we can write $f(\overline{F}) = \bigcup_n Y_n$, where Y_n are closed in Y. Since the restriction of f to $\overline{F}$ is not piecewise closed, there exists $n \in \mathbb{N}$ so that the restriction of f to $\overline{F} \cap f^{-1}(Y_n)$ is not a closed mapping. Thus we can find a closed set $\widehat{F} \subset \overline{F} \cap f^{-1}(Y_n)$ such that $f(\widehat{F})$ is not closed. Find pairwise distinct points y, y_k, $k \in \mathbb{N}$, so that

$$y \in Y_n \setminus f(\widehat{F}), \quad y_k \in f(\widehat{F}), \quad y_k \to y. \tag{1}$$

Since $y \in f(\overline{F}) \setminus f(\widehat{F})$, it is possible to select $x \in \overline{F} \setminus \widehat{F}$ with $f(x) = y$. We choose further a sequence $\{x_k\} \subset \widehat{F}$ with $f(x_k) = y_k$.

Find open balls C_k in Y centred at y_k so that

$$\operatorname{diam}_\sigma C_k < \varepsilon, \quad \operatorname{dist}_\sigma(C_k, C_l) > 0 \quad \text{for } k, l \in \mathbb{N},\ k \neq l,$$
$$\operatorname{dist}_\sigma(\{y\}, C_k) > 0, \quad C_k \to \{y\}.$$

Since $\mathcal{B}$ is a base of open sets and f is continuous, we can inductively find $B_k \in \mathcal{B}$ for $k \in \mathbb{N}$ so that

$$x_k \in B_k, \quad \operatorname{diam}_\varrho B_k < \varepsilon/k, \quad f(B_k) \subset C_k,$$
$$\operatorname{dist}_\varrho(B_k, B_l) > 0 \text{ for } k,\ l \in \mathbb{N},\ k \neq l.$$

We claim that the family $\{B_k : k \in \mathbb{N}\}$ is discrete in X.

Indeed, if we suppose the contrary, then there exists a point $z \in X$ an increasing sequence $\{k_n\}$ of positive integers such that $B_{k_n} \to \{z\}$ and so $x_{k_n} \to z$ as n tends to infinity. Then $z \in \widehat{F}$. The continuity of f implies $f(z) = y$, which contradicts (1).

As $x_k \in \overline{F} \cap B_k$, the set $F \cap B_k$ is nonempty for every $k \in \mathbb{N}$. Put

$$L_k = F \cap B_k, \quad k \in \mathbb{N}.$$

So the point x and the sequence $\{L_k\}$ satisfy the required conditions (i)–(iv).

It remains to verify that L_k is not covered by f for every $k \in \mathbb{N}$. Suppose that $L_k = F \cap B_k$ is covered by f for some $k \in \mathbb{N}$. Since G is covered by f,

$$A \cap B_k = (G \cap B_k) \cup (F \cap B_k)$$

is covered by f likewise (see Remark 3.2). Thus $B_k \in \widehat{\mathcal{B}}$ and $A \cap B_k \subset G$, which contradicts the fact that $B_k \cap F \neq \emptyset$. Hence no L_k is covered by f and the proof is finished. ■

PROPOSITION 3.5. *Let $f : X \to Y$ be a continuous mapping of an absolute Suslin space X onto an absolute Suslin space Y which maps F_σ-sets in X to F_σ-sets in Y. Then X is covered by f, i.e., there exists an F_σ-subset X_∞ of X so that $f(X_\infty) = Y$ and the restriction of f to X_∞ is piecewise closed.*

Proof. We may and do suppose that $X \subset (\widehat{X}, \varrho)$ and $Y \subset (\widehat{Y}, \sigma)$, where $\widehat{X}$ and $\widehat{Y}$ are completions of (X, ϱ) and (Y, ϱ), respectively, $\operatorname{diam}_\varrho \widehat{X} < 1$, and $\operatorname{diam}_\sigma \widehat{Y} < 1$. Let

$$X = \bigcup_{\sigma \in \mathbb{I}} \bigcap_{n=1}^{\infty} F(\sigma{\restriction}n),$$

where $\{F(s)\}_{s \in \mathbb{N}^{<\mathbb{N}}}$ are closed sets in $\widehat{X}$. We write $\overline{A}^X$ (respectively $\overline{A}^{\widehat{X}}$) for the closure of a set $A \subset X$ in X (respectively in $\widehat{X}$). Similarly we use $\overline{B}^Y$ and $\overline{B}^{\widehat{Y}}$ for $B \subset Y$. For $s \in \mathbb{N}^{<\mathbb{N}}$, set

$$X(s) = \bigcup_{\tau \in \mathbb{I}(s)} \bigcap_{n=1}^{\infty} F(\tau{\restriction}n). \tag{2}$$

We want to prove that the space X is covered by f. Suppose that this is not the case. We shall construct by induction nonempty sets $L_s \subset X$, $s \in \mathbb{N}^{<\mathbb{N}}$, that are not covered by f, points $x_s \in \overline{L_s}^X$, and finite sequences $\sigma_s \in \mathbb{N}^{<\mathbb{N}}$, so that, for every finite sequence $s \in \mathbb{N}^{<\mathbb{N}}$ of length n (including the empty sequence $s = \emptyset$), we have:

(i) $\{L_{s^\wedge k} : k \in \mathbb{N}\}$ is a discrete family in X;
(ii) $\overline{L_{s^\wedge k}}^{\widehat{X}}$, $k \in \mathbb{N}$, are pairwise disjoint and $\operatorname{diam}_\varrho L_s < 2^{-n}$;
(iii) $f(L_{s^\wedge k}) \to \{f(x_s)\}$ as k tends to infinity;
(iv) the sets $\{f(x_s)\}$, $\overline{f(L_{s^\wedge k})}^{\widehat{Y}}$, $k \in \mathbb{N}$, are pairwise disjoint;
(v) σ_s is of length $n+1$ and $\sigma_s \prec \sigma_{\widehat{s}}$ if $s \prec \widehat{s}$; and
(vi) $L_{s^\wedge k} \subset L_s \cap X(\sigma_s)$.

We first find $L_\emptyset$, $x_\emptyset$, $\sigma_\emptyset$, and L_k for $k \in \mathbb{N}$ so that (i)–(vi) are satisfied for them. Put $L_\emptyset = X$. As $X = \bigcup_{j \in \mathbb{N}} X(j)$, there is a $\sigma_\emptyset \in \mathbb{N}$ such that $X(\sigma_\emptyset)$ is not covered by f (cf. Remark 3.2). The assumptions on the mapping f in Lemma 3.4 are satisfied due to [9, Theorem 3.5] recalled above. Applying it with $A = X(\sigma_\emptyset)$ and $\varepsilon = 2^{-1}$, we obtain sets $L_k \subset A$, $k \in \mathbb{N}$, which are not covered by f, and a point $x_\emptyset \in \overline{A}^X$ satisfying conditions (i)–(iv) of

Lemma 3.4. So $L_\emptyset$, $x_\emptyset$, $\sigma_\emptyset$, and L_k for $k \in \mathbb{N}$ satisfy (i)–(vi), which concludes the first step of the inductive construction.

Let x_s, σ_s, $L_{s^\wedge k}$ with $s \in \mathbb{N}^{<\mathbb{N}}$, $l(s) < n$, $k \in \mathbb{N}$ satisfying (i)–(vi) be already constructed for an $n \geq 1$. Pick a finite sequence s of length n and an $i \in \mathbb{N}$. As $L_{s^\wedge i} \subset X(\sigma_s) = \bigcup_{j\in\mathbb{N}} X({\sigma_s}^\wedge j)$ and $L_{s^\wedge i}$ is not covered by f, using Remark 3.2 we may find $j \in \mathbb{N}$ so that $L_{s^\wedge i} \cap X({\sigma_s}^\wedge j)$ is not covered by f. Define

$$\sigma_{s^\wedge i} = {\sigma_s}^\wedge j.$$

If we put $A = L_{s^\wedge i} \cap X(\sigma_{s^\wedge i})$ and $\varepsilon = 2^{-n-1}$ in Lemma 3.4 (again [9, Theorem 3.5] allows us to use it), we obtain a point $x_{s^\wedge i} \in \overline{A}^X$ and sets

$$L_{s^\wedge i^\wedge k} \subset A, \quad k \in \mathbb{N},$$

so that all the properties (i)–(iv) in Lemma 3.4 are satisfied. This finishes the construction.

Set

$$Q = \bigcap_{k=1}^{\infty} \Big(\bigcup_{s\,:\,0\leq l(s)\leq k-1} \{f(x_s)\} \cup \bigcup_{s\,:\,l(s)=k} \overline{f(L_s)}^{\widehat{Y}} \Big),$$

$$Q_0 = \{f(x_s) : s \in \mathbb{N}^{<\mathbb{N}}\}, \qquad Q_1 = \bigcap_{k=1}^{\infty} \bigcup_{s\,:\,l(s)=k} \overline{f(L_s)}^{\widehat{Y}},$$

$$P = \bigcap_{k=1}^{\infty} \bigcup_{s\,:\,l(s)=k} \overline{L}_s^{\widehat{X}}.$$

Note that $Q = Q_0 \cup Q_1$ and that $Q_0 \cap Q_1 = \emptyset$ by (iv). According to (iii), (iv), and the fact that $f(x_{s^\wedge k}) \in \overline{f(L_{s^\wedge k})}^{\widehat{Y}}$, the countable set Q_0 is dense-in-itself. Claims 1 and 2 below imply that $Q_1 = f(P)$ is an F_σ-set in Y. Using (iii) inductively, we deduce that Q is closed in $\widehat{Y}$. As $Q \subset Y$, the set Q_1 is an F_σ-subset of $\widehat{Y}$. Finally, Q_0 is a countable dense-in-itself G_δ-subset of the complete space $\widehat{Y}$, which is a contradiction.

It remains to prove the following two claims.

Claim 1. *The set P is a closed subset of X.*

Claim 2. *$f(P) = Q_1$ and so $Q \subset Y$.*

Proof of Claim 1. First of all we show that $P \subset X$. Indeed, if $s, t \in \mathbb{N}^{<\mathbb{N}}$ are given, we know from (ii) and (vi) that

$$\overline{L}_s^{\widehat{X}} \cap \overline{L}_t^{\widehat{X}} \neq \emptyset$$

if and only if either $s \prec t$ or $t \prec s$. Thus, for a given $x \in P$, there exists $\varrho \in \mathbb{I}$ so that

$$x \in \overline{L}_{\varrho\restriction n}^{\widehat{X}} \qquad \text{for every } n \geq 1.$$

Using (v) we find a sequence $\sigma(\varrho) \in \mathbb{I}$ so that

$$\sigma_{\varrho\restriction n-1} = \sigma(\varrho)\restriction n, \quad n \geq 1.$$

Then equality (2) and condition (vi) give

$$x \in \bigcap_{n=1}^{\infty} \overline{L}_{\varrho\restriction n}^{\widehat{X}} \subset \bigcap_{n=1}^{\infty} F(\sigma(\varrho)\restriction n) \subset X$$

and so $P \subset X$. To check that P is closed, note that a use of (i) entails that the set

$$\bigcup_{s\,:\,l(s)=n} \overline{L}_s^X = \bigcup_{s\,:\,l(s)=n} (\overline{L}_s^{\widehat{X}} \cap X),$$

as the union of a discrete family of closed sets in X, is closed in X. From the equalities

$$P = \Big(\bigcap_{n=1}^{\infty} \bigcup_{s\,:\,l(s)=n} \overline{L}_s^{\widehat{X}}\Big) \cap X = \bigcap_{n=1}^{\infty} \bigcup_{s\,:\,l(s)=n} \overline{L}_s^X$$

it follows that P is closed in X.

Proof of Claim 2. Pick $x \in P$. By the reasoning in Claim 1, there exists $\varrho \in \mathbb{I}$ so that

$$x \in \bigcap_{n=1}^{\infty} \overline{L}_{\varrho\restriction n}^{\widehat{X}}.$$

Choose $x_n \in L_{\varrho\restriction n}$. Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x according to (ii). Since f is continuous, we get

$$f(x) \in \bigcap_{k=1}^{\infty} \overline{f(L_{\varrho\restriction k})}^{\widehat{Y}} \subset Q_1.$$

Conversely, let $y \in Q_1$ be given. Due to condition (iv),

$$Q_1 = \bigcap_{n=1}^{\infty} \bigcup_{s\,:\,l(s)=n} \overline{f(L_s)}^{\widehat{Y}} = \bigcup_{\varrho\in\mathbb{I}} \bigcap_{n=1}^{\infty} \overline{f(L_{\varrho\restriction n})}^{\widehat{Y}}.$$

Hence there exists $\varrho \in \mathbb{I}$ so that

$$y \in \bigcap_{n=1}^{\infty} \overline{f(L_{\varrho\restriction n})}^{\widehat{Y}}.$$

Since $\widehat{X}$ is a complete space and $\{\overline{L}_{\varrho\restriction n}^{\widehat{X}}\}_{n\in\mathbb{N}}$ is a decreasing sequence of nonempty closed sets with diameters converging to zero, there exists a point $x \in \widehat{X}$ with

$$\{x\} = \bigcap_{n=1}^{\infty} \overline{L}_{\varrho\restriction n}^{\widehat{X}}.$$

Obviously $x \in P$. Due to Claim 1, $x \in X$, and from the continuity of f it follows that $f(x) = y$, which concludes the proof. ∎

THEOREM 3.6. *Let $f : X \to Y$ be an F_σ-mapping of an absolute Suslin space X onto an absolute Suslin space Y. Then there is an F_σ-set $X_\infty \subset X$ such that $f(X_\infty) = Y$ and the restriction of f to X_∞ is piecewise perfect.*

Proof. According to [4, Theorem 5], the mapping f is piecewise continuous, i.e., X can be written as a union of closed sets Z_n such that f is continuous on every Z_n. Since f preserves F_σ-sets, we can find closed sets $Y_{n,k} \subset Y$, $n, k \in \mathbb{N}$, with

$$f(Z_n) = \bigcup_{k=1}^{\infty} Y_{n,k}, \qquad n \in \mathbb{N}.$$

Then f is a continuous F_σ-mapping on each closed set

$$Z_{n,k} = Z_n \cap f^{-1}(Y_{n,k}), \qquad n, k \in \mathbb{N}.$$

By Proposition 3.5, for each couple $n, k \in \mathbb{N}$ there exists an F_σ-set $H_{n,k}$ in $Z_{n,k}$ such that $f(Z_{n,k}) = f(H_{n,k})$ and the restriction of f to $H_{n,k}$ is a piecewise closed mapping of $H_{n,k}$ to Y. Then $H = \bigcup_{n,k} H_{n,k}$ is an F_σ-subset of X, the restriction of f to H is a piecewise closed mapping of H to Y and $f(H) = Y$.

By [7, Theorem 3] it is possible to find a sequence $\{F_m\}$ of closed sets in H so that the restriction of f to F_m is a perfect mapping of F_m to Y and $f(\bigcup_m F_m) = f(H) = Y$. Write $H = \bigcup_j H_j$, where every H_j is closed in X, and set

$$X_{m,j} = F_m \cap H_j, \quad m, j \in \mathbb{N}, \qquad X_\infty = \bigcup_{m,j=1}^{\infty} X_{m,j}.$$

Then $f(X_\infty) = Y$ and the restriction of f to $X_{m,j}$ is a perfect mapping of $X_{m,j}$ to Y for every $m, j \in \mathbb{N}$, which concludes the proof. ∎

4. The invariance of Borel sets and absolute Borel spaces. We recall the definition of the Borel hierarchy in metrizable spaces. For a space X, the sets of additive, or multiplicative, class zero are just the open, or closed, sets in X. If $1 \le \alpha < \omega_1$, the sets of additive, or multiplicative, class α are just the unions, or intersections, of sets each being contained in some lower additive or multiplicative class.

Now we are ready to prove [5, Theorem 6] and [6, Theorem 2] without the assumptions (a)–(d) of [2, Theorem 3] that were implicitly used in [5] and in [6] as mentioned above.

THEOREM 4.1. *Let f be an F_σ-mapping of an absolute Suslin space X onto an absolute Suslin space Y. Let $\alpha \ge 1$ be a countable ordinal. If B is* *a subset of Y such that $f^{-1}(B)$ is a set of additive, or multiplicative, class α in X, then B is of the same class in Y. Similarly, if $f^{-1}(B)$ is a Suslin set in X, then B is a Suslin set in Y.*

Proof. According to Theorem 3.6, there exists a sequence $\{X_n\}$ of closed sets in X such that, for every $n \in \mathbb{N}$, $f_n = f\restriction_{X_n}$ is a perfect mapping of X_n to Y and $f(\bigcup_n X_n) = Y$.

Let $\alpha \geq 1$ and $B \subset Y$ be such that $f^{-1}(B)$ is of additive class α in X. Fix $n \in \mathbb{N}$ and set $Y_n = f(X_n)$.

Then $f^{-1}(B) \cap X_n$ is of additive class α in X_n. Since f_n is perfect on X_n and $f^{-1}(B) \cap X_n = f_n^{-1}(B \cap Y_n)$, the assumptions of [6, Lemma 3] are satisfied for $f_n : X_n \to Y_n$ and $B \cap Y_n$. Thus $B \cap Y_n$ is of additive class α in Y.

Since $B = \bigcup_n (B \cap Y_n)$ and every Y_n is closed in Y, we see that B is of additive class α in Y as needed.

The case of a multiplicative class α follows from the previous argument by taking the complements.

If $f^{-1}(B)$ is a Suslin set in X for $B \subset Y$, we can use the same considerations as above and the fact that the image of an absolute Suslin space under a closed mapping is an absolute Suslin space ([1, Theorem 3.3]). ∎

The next Theorem 4.2 asserts that F_σ-mappings preserve absolute Borel classes. We recall that a space X is said to be of *absolute additive*, or *absolute multiplicative*, *class* α, $1 \leq \alpha < \omega_1$, if X is of the same class whenever it is embedded in a space.

Note that X is of absolute multiplicative class one, i.e., an absolute G_δ-space, if and only if X is completely metrizable.

Let $\alpha \geq 2$ (respectively $\alpha \geq 1$) be a countable ordinal. We note that X is of absolute additive (respectively multiplicative) class α if X is of the same class in some completely metrizable space. This easily follows from the Lavrent'ev theorem (see [8, §35, II, Theorem]).

Spaces of absolute additive class one, i.e., absolute F_σ-spaces, were characterized by Stone in [10, Theorem 2].

THEOREM 4.2. *Let $f : X \to Y$ be an F_σ-mapping of a space X onto an absolute Suslin space Y. If X is of absolute additive class α, $1 \leq \alpha < \omega_1$, or of absolute multiplicative class α, $2 \leq \alpha < \omega_1$, then Y is of the same absolute class.*

Proof. Since X is assumed to be a Borel set in some completely metrizable space and Borel sets are Suslin, X is an absolute Suslin space. By Theorem 3.6 we can find an increasing sequence $\{X_n\}$ of closed sets in X so that f is a perfect mapping on each X_n, every $f(X_n)$ is closed in Y and $f(X_\infty) = f(\bigcup_n X_n) = Y$.

If X is of absolute additive, or multiplicative, class α, $\alpha \geq 2$, then X_∞ is an absolute Borel space of the same class. This follows from the remark preceding the theorem. Now we can use [5, Corollary to Theorem 2] to deduce that Y is of the same absolute class.

Since the assertion for spaces of absolute Borel class one is not proved in the aforementioned [5, Corollary to Theorem 2], we briefly indicate its proof. Let us assume that X is of absolute additive class one, i.e., X is an absolute F_σ-space. Let X_n, $n \in \mathbb{N}$, be as above. Fix $n \in \mathbb{N}$. It follows from [3, Claim] that $X_n = \bigcup_k (F_k \cap G_k)$, where every F_k (respectively G_k) is a closed (respectively open) set in the Stone–Čech compactification of X. According to [3, Corollary 14], X_n is a countable union of intersections of closed and open sets in every Tikhonov topological space. Consecutive use of [3, Corollary 15 and Claim] shows that $f(X_n)$ is an absolute F_σ-space. Thus $Y = \bigcup_n f(X_n)$ is an absolute F_σ-space as well. ∎

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Received 20 May 2003;
in revised form 24 June 2004