FUNDAMENTA
MATHEMATICAE
182 (2004)

The return sequence of the Bowen–Series map for punctured surfaces

by

Manuel Stadlbauer (Göttingen)

Abstract. For a non-compact hyperbolic surface M of finite area, we study a certain Poincaré section for the geodesic flow. The canonical, non-invertible factor of the first return map to this section is shown to be pointwise dual ergodic with return sequence (a_n) given by

$$a_n = \frac{\pi}{4(\mathrm{Area}(M) + 2\pi)} \cdot \frac{n}{\log n}.$$

We use this result to deduce that the section map itself is rationally ergodic, and that the geodesic flow associated to M is ergodic with respect to the Liouville measure.

1. Introduction and statement of main results. The coding of the directed geodesics on a surface M of negative curvature by a finite alphabet was introduced by Artin and Morse (see [Ar], [Mor1], [Mor2]). An immediate application of this approach leads to a representation of the geodesic flow on M as a suspension flow over the two-sided shift (see [Se2]). In this context several problems of interest arise. The first is to determine whether the shift is of finite type, which can be done e.g. by giving a geometric construction of an invertible Markov map such that the geodesic flow is isomorphic to a suspension over the latter map (see [Se2], [AF]). Furthermore, one may be interested in the maximal non-invertible factor of this map, its dynamical properties, and their relation to the dynamics of the geodesic flow on the given surface.

A fundamental paper in this context is [BS]. There, a non-invertible Markov map $T : \partial\mathbb{H} \to \partial\mathbb{H}$ is introduced which is orbit equivalent to the action of the Fuchsian group G on $\partial\mathbb{H}$, where $\partial\mathbb{H}$ is the ideal boundary of the hyperbolic 2-space $\mathbb{H}$ and $\mathbb{H}/G$ is assumed to have finite hyperbolic area. If $\mathbb{H}/G$ is compact, it is shown that the map T is transitive and has the Rényi

2000 *Mathematics Subject Classification*: 30F35, 51M15.
The author was supported by the Graduate school "Gruppen & Geometrie" and the Forschergruppe "Zetafunktionen und lokalsymmetrische Räume".

property, which then implies that T admits an ergodic invariant probability measure which is absolutely continuous with respect to the Lebesgue measure. If $\mathbb{H}/G$ is not compact, Bowen and Series prove that a suitably chosen induced transformation has these properties, which implies that T itself is ergodic with respect to an infinite invariant measure. Furthermore, the geodesic flow is shown to be ergodic by using the orbit equivalence of the actions of T and G.

Later, it was shown in [Se2] that the map T also has the above mentioned property of being a maximal factor of the first return map to some Poincaré section for the geodesic flow on $\mathbb{H}/G$. Note that this gives an explicit construction of an invariant measure for T using the flow invariant Liouville measure on the unit tangent bundle of $\mathbb{H}/G$. Namely, since the Liouville measure induces an invariant measure for the first return map (see [AK]), the image measure under the factor map of the latter measure is T-invariant. By using this construction, this invariant measure was determined explicitly in [Se1] for the modular group and in [AF] for compact $\mathbb{H}/G$, leading to a new characterisation of the Gauß measure, and to a proof of the ergodicity of the geodesic flow, respectively.

In what follows, we refer to the map T as the *Bowen–Series map* or the *coding map*, and a Fuchsian group G is called *cocompact*, resp. *cofinite* if $\mathbb{H}/G$ is compact or of finite hyperbolic area.

We first consider the above construction for an arbitrary, cofinite, non-cocompact Fuchsian group G. In this situation, a result of Tukia (see [Tu]) shows that there exists a fundamental domain for G which is an ideal polygon. It then turns out that the T-invariant measure is infinite (see [BS], [Se1]). Therefore, we use methods from infinite ergodic theory (see [ADU], [Aa]) to show that the coding map is pointwise dual ergodic and to determine the associated return sequence (see [AD]). Then we adapt our methods to the coding map as introduced in [BS]. Note that this construction applies to any cofinite group.

More precisely, the paper is organised as follows. In Section 2, we construct a Poincaré section Y for the geodesic flow on $\mathbb{H}/G$, where G is non-compact and cofinite. Note that the construction of Y relies on the choice of a fundamental polygon (see [Se1], [AF], [St]) which can be taken to be an ideal polygon (see [Tu]). This then gives rise to a special flow representation (Proposition 2.1) of the geodesic flow. Therefore, the Liouville measure induces a measure m on Y which is invariant under the first return map S to Y.

In Section 3, the coding map T is introduced and is shown to be a non-invertible factor of S, that is, there is a surjective map $\pi : Y \to \partial\mathbb{H}$ such that $\pi \circ S = \pi \circ T$. Since m is S-invariant, the measure $\mu := m \circ \pi^{-1}$ is T-invariant. Moreover, it can be calculated explicitly (see p. 230). Also, we conclude that T is a topologically mixing Markov map (Proposition 3.1) and that S is the natural extension of T (Proposition 3.2). Combining this result with Corollary 2.2 then shows that T is conservative and ergodic if and only if the geodesic flow has these properties.

In Section 4, we first show that the measure μ is infinite (Proposition 4.1). Furthermore, by inducing T on a suitable set A of finite measure we deduce that T_A is an eventually hyperbolic dynamical system which has the Rényi property (Lemma 4.3). Using this result, we obtain the following result, where μ_A is the measure μ restricted to A.

THEOREM 1. *The induced map T_A has the Gibbs–Markov property with respect to the T_A-invariant measure μ_A.*

Note that the Gibbs–Markov property was introduced in [AD], where a similar result was obtained for a subgroup of the modular group by a different method. Moreover, recall that the continued fraction map is the classical example of a map with this property. By combining results from infinite ergodic theory (see [ADU], [Aa], [Z]) and the measure estimate in Proposition 4.1 we obtain the main result of this paper.

THEOREM 2. *The coding map T is conservative and ergodic with respect to μ. Moreover, for $K := \pi/(4(\mathrm{Area}(M) + 2\pi))$, the map T is pointwise dual ergodic with respect to the return sequence*

$$a_n = K \cdot \frac{n}{\log n}.$$

We recall that T is called *pointwise dual ergodic* (see also Section 4.3) with respect to the *return sequence* (a_n) if

$$\frac{1}{a_n}\sum_{i=0}^{n-1}\widehat{T}^i f \to \int_X f\,d\mu \quad \text{ a.e. as } n\to\infty\ \forall f\in L^1(\mu),$$

where $\widehat{T}$ is the dual of T. Note that the sequence (a_n) is unique up to asymptotic equality, that is, T is pointwise dual ergodic with respect to each sequence (b_n) with $\lim_{n\to\infty} a_n/b_n = 1$. Moreover, the measure μ can be normalised in the sense that with respect to the measure $(1/K)\mu$, the map T is pointwise dual ergodic with return sequence $(n/\log n)$. However, this normalisation corresponds to a change of the Riemannian metric on $\mathbb{H}$ and therefore to a change of the sectional curvature of $\mathbb{H}/G$.

In Section 5, the relation of the coding map which is constructed in Section 3 and the map T introduced in [BS] is discussed. It turns out that our analysis can be adapted to this more general situation as follows. Using the above construction of the invariant measure, a slight modification of Lemma 4.3 gives the following result, where $\mathcal{V}^*$ refers to the set of ideal vertices of the polygon involved in the construction of T.

Theorems 3 & 4. *Let G be a cofinite group and T the map introduced in* [BS]*. Then T is conservative and ergodic with respect to the invariant measure μ. Moreover, μ is finite if and only if G is cocompact. If G is not cocompact, then T is pointwise dual ergodic with respect to the return sequence*

$$a_n = K \cdot \frac{n}{\log n},$$

where $K = 1/(4\#\mathcal{V}^*)$.

Note that the latter result comprises the assertion of Theorem 2. This is due to the fact that the hyperbolic area of an ideal polygon P with n vertices is $\mathrm{Area}(P) = \pi(n-2)$.

These results have the following consequences. Since the natural extension of T is the first return map to a Poincaré section (see Proposition 3.2 for the construction given here and [Se2], [AF] for the construction in [BS]), the geodesic flow is isomorphic to a special flow over a conservative and ergodic section (Proposition 2.1). Hence the classical result of Hopf (see e.g. [Ho]) which states that the geodesic flow on a surface of finite area is conservative and ergodic with respect to the Liouville measure follows from a result in [AK]. Furthermore, if the invariant measure for this section is infinite, the fact that the first return map is conservative implies that S_A is well defined for each subset A of positive measure of the Poincaré section. Hence for A suitably chosen, the induced map $S_A : A \to A$ is the first return map to the alternative Poincaré section A, and is an ergodic, finite measure preserving Markov map with respect to a countable partition (see [AD]).

2. The special flow representation. The aim of this section is to construct a special flow over some measure preserving transformation which is isomorphic to the geodesic flow on the underlying surface. Recall that the *Poincaré model* $\mathbb{H}$ of the hyperbolic plane is $\mathbb{H} := \{z \in \mathbb{C} : |z| < 1\}$, where the *hyperbolic metric* and *hyperbolic area* are given by

$$ds(z) = \frac{2|dz|}{1-|z|^2} \quad \text{and} \quad dA(z) = \frac{4dz}{(1-|z|^2)^2},$$

respectively. Also recall that an oriented *geodesic* is an isometry $\gamma : \mathbb{R} \to \mathbb{H}$ and corresponds to a circle segment which is perpendicular to the boundary at infinity $\partial\mathbb{H} = S^1$ at the two endpoints $\lim_{t\to\pm\infty} \gamma(t)$. Note that the set of oriented geodesics corresponds to the unit tangent bundle $T^1(\mathbb{H})$ of $\mathbb{H}$, where the usual representation of $T^1(\mathbb{H})$ is given by

$$T^1(\mathbb{H}) = \{(\xi, \eta, s) : \xi, \eta \in \partial\mathbb{H},\ \xi \neq \eta,\ s \in \mathbb{R}\}.$$

Moreover, the *geodesic flow* $(\phi_t)_{t\in\mathbb{R}}$ on $T^1(\mathbb{H})$ is defined by the canonical $\mathbb{R}$-action $\phi_t : (\xi, \eta, s) \mapsto (\xi, \eta, s+t)$, and the *Liouville measure* m_L given by $dm_L = d|\xi|d|\eta|dt/(|\xi-\eta|^2)$ is invariant under the action of $(\phi_t)_{t\in\mathbb{R}}$.

Recall that each orientable hyperbolic surface is isometric to the quotient $\mathbb{H}/G$, where G is a Fuchsian group without torsion. A group G is *torsion-free* if there exists no non-trivial element $g \in G$ such that $g^n = \mathrm{id}$ for some $n \neq 0$. Furthermore, a group is called a *Fuchsian group* if G is a discrete subgroup of the group $\mathrm{Iso}^+(\mathbb{H})$ of orientation-preserving isometries of $(\mathbb{H}, s)$. Also note that m_L is invariant under the usual action of G on $T^1(\mathbb{H})$, and that the actions of G and $(\phi_t)_{t\in\mathbb{R}}$ on $T^1(\mathbb{H})$ commute. These observations give rise to the definition of the geodesic flow on the quotient $T^1(\mathbb{H}/G)$ with respect to the projected Liouville measure which, for ease of notation, will also be denoted by m_L (see [Ni]).

We will extensively use a specially shaped fundamental polygon for a given group G. Recall that, in general, the set $\mathcal{S}$ of *sides* of a fundamental polygon P consists of geodesic segments, geodesic rays and geodesics, and that the set $\mathcal{V}$ of *vertices* of P is a subset of $\overline{\mathbb{H}}$. Also, the elements of $\mathcal{S}$ are equivalent in pairs, and for each pair $\{s,t\}$ the element $g_s \in G$ is uniquely determined by $g_s(s) = t$. This gives rise to the following notation. Denote by s' the side $g_s(s)$ of P. As is easily seen, $s'' = s$ and $g_{s'} = g_s^{-1}$.

So assume that G is a torsion-free, non-cocompact Fuchsian group. In this situation, a result of Tukia ([Tu, p. 15]) states that there exists a fundamental polygon P for G with the following properties. The sides of P are geodesics. For each $s \in \mathcal{S}$, denote by $G_{\{s,s'\}}$ the subgroup of G generated by the element g_s. Then G is the free product of the groups $G_{\{s,s'\}}$ for all pairs $\{s, s'\}$.

If G corresponds to a non-compact surface of finite area, the above result shows that there exists P whose sides are geodesics. Therefore, since P is of finite area, P has to be an *ideal polygon*, that is, P is the (hyperbolically) convex hull of a finite subset $\mathcal{V}$ of $\partial\mathbb{H}$ (see Figure 1). Note that this implies that $\mathcal{S}$ is finite. Moreover, by [Be, Theorem 10.5.1], for each fundamental polygon for G we have $\#\mathcal{S} \leq 4g + 2n - 2$, where g is the genus and n the number of cusps of $\mathbb{H}/G$. Using the Gauß–Bonnet formula (e.g. [Be, Theorem 10.4.3]) it is not hard to deduce that this inequality is sharp for each ideal fundamental polygon. In other words, this choice of P has the fewest number of sides and therefore gives a minimal set of generators.

In addition, since $\mathcal{V} \subset \partial\mathbb{H}$, the polygon P gives rise to a partition of $\partial\mathbb{H}$ as follows (see Figure 1). Denote by $H(s)$, for $s \in \mathcal{S}$, the open hyperbolic half-space for which $\partial H(s) = s$ and $P \cap H(s) = \emptyset$, and by $a_s \subset \partial\mathbb{H}$ the open interval which is adjacent to $H(s)$. As is easily seen, $g_s(a_s) = \mathrm{Int}(a_{s'}^c)$, where $\mathrm{Int}(\cdot)$ refers to the interior with respect to the topology of $\partial\mathbb{H}$. Furthermore, $a_s \cap a_t = \emptyset$ for all distinct $s, t \in \mathcal{S}$ and $\bigcup_{s\in\mathcal{S}} a_s = \partial\mathbb{H} \setminus \mathcal{V}$.

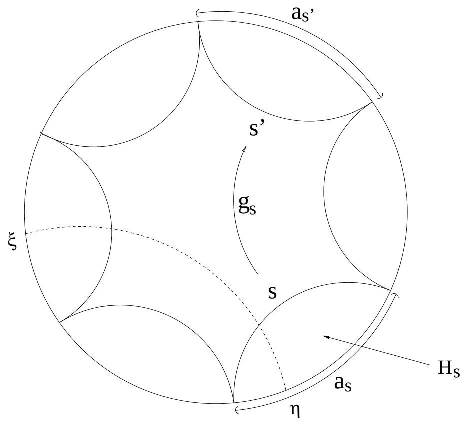

Fig. 1. The first return map S for a free Fuchsian group of first kind

Set

$$\mathcal{G}_P := \{(\xi,\eta,t) \in T^1(\mathbb{H}) : \xi \in G\mathcal{V} \text{ or } \eta \in G\mathcal{V}\}.$$

Clearly, $\mathcal{G}_P$ is invariant under the actions of (ϕ_t) and G. Moreover, since $\mathcal{V}$ is finite, $\mathcal{G}_P$ is of Liouville measure zero. Let $\gamma_{\xi,\eta}$ be the oriented geodesic from $\eta \in \partial\mathbb{H}$ to $\xi \in \partial\mathbb{H}$, where $\gamma_{\xi,\eta}$ is normalised so that the Euclidean distances $d_E(\eta,\gamma(0))$ and $d_E(\xi,\gamma(0))$ coincide. Define

$$Y := \{(\xi,\eta) \in \partial\mathbb{H}\times\partial\mathbb{H} : \exists t \in \mathbb{R} \text{ such that } (\xi,\eta,t) \notin \mathcal{G}_P \text{ and } \gamma_{\xi,\eta}(t) \in P\}.$$

Furthermore, observe that, for distinct $\xi,\eta \in \partial\mathbb{H}\setminus\mathcal{V}$, there exists $t \in \mathbb{R}$ such that $\gamma_{\xi,\eta}(t) \in \mathrm{Int}(P)$ if and only if there exist distinct $s,t \in \mathcal{S}$ such that $\xi \in a_s$ and $\eta \in a_t$. Thus,

$$Y \stackrel{m}{=} \{(\xi,\eta) \in \partial\mathbb{H}\times\partial\mathbb{H} : \exists s \in \mathcal{S} \text{ such that } \xi \in a_s,\ \eta \notin a_s\},$$

where m is given by $dm(\xi,\eta) = d|\xi|d|\eta|/(|\xi-\eta|^2)$ and $\stackrel{m}{=}$ denotes equality up to a set of measure zero. Let

$$S : Y \to Y, \quad S|_{(a_s\times a_s^c)}(\xi,\eta) = (g_s\xi, g_s\eta).$$

Moreover, since we have excluded the set $\mathcal{G}_P$, the two maps $t^{\pm}_{\xi,\eta} : Y \to \mathbb{R}$ defined by

$$t^+_{\xi,\eta} := \sup\{t : \gamma_{\xi,\eta}(t) \in P\} \le \infty, \qquad t^-_{\xi,\eta} := \inf\{t : \gamma_{\xi,\eta}(t) \in P\} \ge -\infty$$

satisfy $|t^{\pm}_{\xi,\eta}(\xi,\eta)| < \infty$ for all $(\xi,\eta) \in Y$.

Recall that (see [AK]) the special flow $(Y_h, \mathcal{B}_h, m\times\lambda, (\varphi_t^{Y_h})_{t\in\mathbb{R}})$ over $S : (Y,\mathcal{B},m) \to (Y,\mathcal{B},m)$ with *height function* $h(\xi,\eta) := t^+_{\xi,\eta} - t^-_{\xi,\eta}$ is defined by

$$Y_h := \{(\xi,\eta,\theta) : (\xi,\eta) \in Y,\ 0 \le y < h((\xi,\eta))\},$$
$$\varphi_t^{Y_h}(\xi,\eta,\theta) := (S^n(\xi,\eta), \theta + t - h_n(x)),$$

where $n \in \mathbb{Z}$ is such that $h_n(\xi,\eta) \le \theta + t < h_{n+1}(\xi,\eta)$ for

$$h_n(\xi,\eta) := \begin{cases} 0, & n = 0, \\ \sum_{k=0}^{n-1} h(T^k(\xi,\eta)), & n \ge 1, \\ -\sum_{k=n}^{-1} h_k(T^k(\xi,\eta)), & n < 0. \end{cases}$$

Here, $m \times \lambda$ is the product measure of m and the Lebesgue measure λ restricted to the Borel σ-field $\mathcal{B}_h$ of Y_h.

In this context, S is also referred to as the *first return map* to the *Poincaré section* Y. Note that, by the results of [AK], the measure m is S-invariant if and only if $m \times \lambda$ is invariant under $(\varphi_t^{Y_h})_{t\in\mathbb{R}}$, and that S is ergodic and conservative if and only if $(\varphi_t^{Y_h})_{t\in\mathbb{R}}$ is ergodic and conservative.

PROPOSITION 2.1. *The geodesic flow* $(T^1(\mathbb{H}/G), \mathcal{B}, m_L, (\phi_t))$ *is measure theoretically isomorphic to the special flow* $(Y_h, \mathcal{B}_h, m \times \lambda, (\varphi_t^{Y_h}))$.

Proof. We only give the sketch of the proof since similar arguments can be found in [Se1] and [AF]. As is easily seen, the set $\mathcal{Y}$ is a fundamental domain for the action of G on $T^1(\mathbb{H}) \setminus \mathcal{G}_P$, where

$$\mathcal{Y} := \{(\xi,\eta,t) \in Y \times \mathbb{R} : t^-_{\xi,\eta} \le \theta < t^+_{\xi,\eta}\}.$$

In addition, for $g \in G$ we have $g\mathcal{Y} \cap \mathcal{Y} = \emptyset$ if and only if $g = \mathrm{id}$. Hence by the product structure of m_L and the flow invariance of $\mathcal{G}_P$ we find that the geodesic flow (ϕ_t) on $T^1(\mathbb{H}/G)$ with respect to the Liouville measure and the flow (ψ_t) on $\mathcal{Y}$ with respect to $m \times \lambda$ are measure theoretically isomorphic, where $\psi_t(\xi,\eta,\theta) = G(\phi_t(\xi,\eta,\theta)) \cap \mathcal{Y}$.

Moreover, observe that $g_s(\xi,\eta,t^+_{\xi,\eta}) = (g_s\xi, g_s\eta, t^-_{g_s\xi,g_s\eta})$ for $\xi \in a_s$. This essentially gives the assertion. ∎

Note that $h \in L^1(Y,m)$, since

$$\int_Y h\,dm = m_L(T^1(\mathbb{H}/G)) = \mathrm{Area}(\mathbb{H}/G).$$

Furthermore, since the Liouville measure is flow-invariant we immediately obtain the following.

COROLLARY 2.2. *The map* $S : Y \to Y$ *is the first return map of the Poincaré section* Y. *Moreover*, *the measure* m *is* S-*invariant*, *and* S *is conservative and ergodic if and only if the geodesic flow is conservative and ergodic.*

3. The coding map. The coding map is an endomorphism defined on $\partial\mathbb{H}$ which is defined piecewise by a set of generators given by a fundamental polygon. Note that

$$\alpha := \{a_s : s \in \mathcal{S}\}$$

is a partition of $\partial\mathbb{H}$ up to a set of Lebesgue measure zero. The map $T : \partial\mathbb{H} \to \partial\mathbb{H}$ is now defined by

$$T|_{a_s} := g_s|_{a_s},$$

where a_s is an arbitrary atom of α. Observe that $\mathrm{pr}_1 \circ S = T \circ \mathrm{pr}_1$, where pr_1 is the projection onto the first coordinate. Hence T is a factor of S and the measure $\mu := m \circ \mathrm{pr}^{-1}$ is T-invariant. Moreover, for a countable collection $\{\beta_i : i \in I\}$ of partitions, denote by $\bigvee_{i\in I} \beta_i$ the common refinement of the β_i $(i \in I)$. Let

$$\alpha_{n+1} := \bigvee_{i=0}^{n} T^{-i}\alpha.$$

Proposition 3.1. *The coding map T is a topologically mixing Markov map with respect to the partition α and the measure μ* (*and with respect to the Lebesgue measure*).

Proof. T restricted to an element of α is a Möbius transformation, whence $T|_{a_s}$ is injective. Moreover, since $T|_{a_s} = g_s|_{a_s}$, we have $T(a_s) = (a_{s'})^c \bmod \mu$. To verify the Markov property it remains to show that $\sigma(\bigvee_{i=0}^{\infty} T^{-i}\alpha) = \mathcal{B} \bmod \mu$.

As is easily seen, the inverse branches of T correspond to elements of the group G. Hence any element of α_n corresponds to a side of a copy of P under an element of G. Note that, since the tessellation GP is locally finite, the Euclidean distances of the endpoints of the sides of $g_n P$ tend to zero, where g_n is a sequence of distinct elements in G. This gives the Markov property with respect to α.

In order to complete the proof, it remains to show that T is topologically mixing, which is equivalent to the aperiodicity of the underlying incidence graph (see [Aa, §4.2]). Recall that the set of vertices of this graph consists of the elements of α, and that the set of (directed) edges consists of the pairs (a, b) with $T(a) \supset b \bmod \mu$. Since $T(a) \supset b$ is equivalent to $b \neq a'$ there are edge cycles $((a_0, a_1), (a_1, a_2), \ldots, (a_{k-1}, a_k), (a_k, a_0))$ of arbitrary length for all $a = a_0$. Hence the incidence graph is aperiodic and T is topologically mixing. ∎

The Markov property now allows us to introduce the following notions. A word $(s_1 \ldots s_n)$ is called *admissible* whenever $s_i \neq s'_{i+1}$ for $1 \leq i < n$ (which is equivalent to $T(a_{s_i}) \supset a_{s_{i+1}}$). Note that each admissible word $\omega = (s_1 \ldots s_n)$ defines an element $[\omega]$ of α_n by

$$[\omega] := \{\xi \in \partial\mathbb{H} : T^i(\xi) \in a_{s_i} \text{ for all } 0 \leq i < n-1\}.$$

Moreover, if $g_\omega = g_{s_n} \cdots g_{s_1} \in G$, then $T^n|_{[\omega]} : \omega \to T([a_{s_1}])$ is injective and $T^n|_{[\omega]} = g_\omega$. This gives the following well known relation between admissible words and inverse branches of T: for $\mathcal{D}(\nu_\omega) := T^n([\omega])$ the map

$$\nu_\omega : \mathcal{D}(\nu_\omega) \to [\omega], \qquad \nu_\omega := g_\omega^{-1}|_{T^n([\omega])},$$

satisfies

$$T^n \circ \nu_\omega|_{\mathcal{D}(\nu_\omega)} = \mathrm{id}|_{\mathcal{D}(\nu_\omega)}.$$

We are now in a position to show that the first return map S also has the Markov property. Recall that S is the *natural extension* of T if $\mathrm{pr}_1 \circ S = T \circ \mathrm{pr}_1$, $m \circ \pi^{-1} = \mu$, and $\bigvee_{n=1}^{\infty} S^n \mathrm{pr}_1^{-1} \mathcal{B}_{\partial\mathbb{H}} \overset{m}{=} \mathcal{B}_Y$, where $\mathcal{B}_{\partial\mathbb{H}}$ and $\mathcal{B}_Y$ denote the respective σ-fields of Borel subsets (see e.g. [Aa]).

PROPOSITION 3.2. *The map* $(Y, \mathcal{B}_Y, m, S)$ *is the natural extension of* $(\partial\mathbb{H}, \mathcal{B}_{\partial\mathbb{H}}, \mu, T)$.

Proof. Assume that $(s_1 \ldots s_m)$ is an admissible word and that $0<n<m$. Then

$$\begin{aligned} S^n \circ \mathrm{pr}_1^{-1}[s_1 \ldots s_m] &= g_{s_1 \ldots s_n}([s_1 \ldots s_m]) \times g_{s_1 \ldots s_n}([s_1]^c) \\ &= [s_{n+1} \ldots s_m] \times (g_{s_1'} \cdots g_{s_n'})^{-1} T([s_1']) \\ &= [s_{n+1} \ldots s_m] \times \nu_{s_1' \ldots s_n'} T([s_1']) \\ &= [s_{n+1} \ldots s_m] \times [s_1' \ldots s_n']. \end{aligned}$$

As α is a generating partition with respect to T the last equality proves that the Borel subsets of Y are generated by

$$\bigvee_{m>n>0} S^n \circ \mathrm{pr}_1^{-1}(\alpha_m).$$

By definition of T and μ the other two criteria are satisfied and hence the assertion is proven. ∎

The following corollary is an immediate consequence of the latter proposition.

COROLLARY 3.3. *The first return map S has the Markov property with respect to the partition* $\{a_s \times a_t : s, t \in \mathcal{S},\ s \neq t\}$ *of* Y.

4. Ergodic properties of the coding map. By Proposition 3.1 the map T is a measure preserving Markov map with respect to a partition with finitely many atoms. However, the fact that $\mathbb{H}/G$ is a surface with cusps implies that there exist indifferent periodic orbits of T. This then will give rise to the observation that μ is an infinite measure (see [BS], [Se1]), and therefore we will describe the dynamical behaviour of T in terms of infinite ergodic theory (see [Th], [ADU], [Aa]).

Denote by $\mathbb{U} := \{z \in \mathbb{C} : \Im z > 0\}$ the upper half-space model of the hyperbolic plane. Recall that the Liouville measure m_L on $T^1(\mathbb{U}) = \{(\xi, \eta, t) :$

$\xi, \eta \in \mathbb{R} \cup \{\infty\}$, $\xi \neq \eta$, $t \in \mathbb{R}\}$ is given by

$$dm_L = \frac{2d\xi d\eta ds}{(\xi - \eta)^2}.$$

Hence, the invariant measure m for the first return map S with respect to $\mathbb{U}$ is given by

$$dm = \frac{2d\xi d\eta}{(\xi - \eta)^2},$$

and as in the Poincaré model, the invariant measure μ for T is the image measure of the projection onto the first coordinate. We now consider an arbitrary atom a_s of α and assume without loss of generality that $a_s = (a, \infty)$ for some $a \in \mathbb{R}$. Hence, for each measurable set $A \subset a_s$, we have

$$\mu(A \times a_s^c) = \int_{A \times a_s^c} \frac{2\, d\xi\, d\eta}{(\xi - \eta)^2} = \int_A \left(\int_{a_s^c} \frac{2\, d\eta}{(\xi - \eta)^2} \right) d\xi. \tag{1}$$

Hence,

$$\frac{d\mu}{d\eta}(\xi) = \int_{-\infty}^{a} \frac{2\, d\eta}{(\xi - \eta)^2} = 2\frac{1}{\eta - a}\, d\eta \qquad \text{for all } \xi \in a_s, \tag{2}$$

and μ is an infinite measure which is equivalent to the Lebesgue measure on $\mathbb{R}$.

4.1. *The wandering rate.* The wandering rate of a set A of finite measure with respect to a measure preserving transformation is given by the asymptotic type of $\mu(\bigcup_{i=1}^{n} T^{-i}(A))$ as $n \to \infty$. Note that T is conservative if $\mu(\partial\mathbb{U} \setminus \bigcup_{i=1}^{\infty} T^{-i}(A)) = 0$. To construct a set with this property the indifferent periodic orbits of T have to be characterised. Recall the definition of a cycle for an ideal vertex of a fundamental polygon P for the group G. Assume that, for ideal vertices $v_1, \ldots, v_n \in \partial\mathbb{H}$, sides $s_1, \ldots, s_n \in \mathcal{S}$, and boundary identifications $g_{s_1}, \ldots, g_{s_n}$, the following holds, where the indices are taken mod n:

- $g_{s_i}(v_i) = v_{i+1}$ for $0 < i \leq n$,
- v_i is adjacent to s_i and v_{i+1} is adjacent to $g_{s_i}(s_i)$ for all $0 < i \leq n$.

If n is minimal with respect to these properties, then $(v_1, \ldots, v_n)$ is referred to as a *vertex cycle* of v_1. For the case of a cofinite group, it is well known that each ideal vertex v of P is contained in a vertex cycle. Moreover, for the associated boundary identifications $g_{s_1}, \ldots, g_{s_n}$, we find that $g_{s_n} \cdot g_{s_{n-1}} \cdots g_{s_1}$ is a parabolic transformation with fixed point v.

Let N be the least common multiple of the lengths of all vertex cycles. Hence each ideal vertex v is a parabolic fixed point of $\overline{T}^N$, where $\overline{T}^N|_{[w]}$ is the continuous extension of T^N to $[w] \in \alpha_N$. As is easily seen, if $(s_1, \ldots, s_n)$

is the cycle of sides associated to v, then

$$U(v) := [\underbrace{s_1 \dots s_n \dots s_1 \dots s_n}_{N/n \text{ times}}] \cup [\underbrace{s'_n \dots s'_1 \dots s'_n \dots s'_1}_{N/n \text{ times}}] \cup \{v\}$$

is a neighbourhood of v. Define

$$w(v) := [\underbrace{s_1 \dots s_n \dots s_1 \dots s_n}_{N/n \text{ times}}],$$

$$w'(v) := [\underbrace{s'_n \dots s'_1 \dots s'_n \dots s'_1}_{N/n \text{ times}}].$$

Since $w(v)w(v)$ is admissible, we have

$$\bigcup_{i=1}^{n} T^{-iN}([w(v)])^c = [\underbrace{w(v) \dots w(v)}_{n+1 \text{ times}}]^c.$$

For $A := (\bigcup_{v\in\mathcal{V}} U(v))^c$ this gives

$$\begin{aligned}\bigcup_{i=0}^{n} T^{-iN} A &\overset{\mu}{=} \bigcup_{i=0}^{n} T^{-iN}\Big(\bigcup_{v\in\mathcal{V}}([w(v)] \cup [w'(v)])\Big)^c \\ &= \Big(\bigcup_{v\in\mathcal{V}} [\underbrace{w(v) \dots w(v)}_{n+1 \text{ times}}] \cup [\underbrace{w'(v) \dots w'(v)}_{n+1 \text{ times}}]\Big)^c.\end{aligned}$$

Without loss of generality, assume that $v = \infty$ and $T^N|_{[w(v)]}(z) = z - 1$. Then there exist $a, b \in \mathbb{R}$, $b < a$, such that $w(v) = (a, \infty)$ and $(b, \infty) = [s]$, where $[s] \in \alpha$ is the atom in α such that $w(v) \subset [s]$. We now have

$$\begin{aligned}\mu([w(v)] \setminus [\underbrace{w(v) \dots w(v)}_{n \text{ times}}]) = \mu((a, a+n]) &= \int_a^{a+n} \frac{2}{x-b}\,dx \\ &= 2(\log(a+n-b) - \log(a-b)))\end{aligned}$$

and so

$$\frac{\mu([w(v)] \setminus [\overbrace{w(v) \dots w(v)}^{n \text{ times}}])}{\log n} \xrightarrow{n\to\infty} 2.$$

This implies that $\mu(A) < \infty$, and that for the wandering rate of A with respect to T^N, we have

$$\frac{\mu(\bigcup_{i=0}^{n} T^{-iN} A)}{\log n} \xrightarrow{n\to\infty} 4\#\mathcal{V},$$

where $\#\mathcal{V}$ denotes the cardinality of $\mathcal{V}$. Since $\mu(\{\bigcup_{i=0}^{n} T^{-i} A\}_{i\in\mathbb{N}})$ increases monotonically, we have the following.

PROPOSITION 4.1.

$$\frac{\mu(\bigcup_{i=0}^{n} T^{-i}A)}{\log n} \xrightarrow{n\to\infty} 4\#\mathcal{V}.$$

In addition, since $(a, a+n] \xrightarrow{n\to\infty} (a,\infty)$, we have $\mu(\partial\mathbb{U}\setminus\bigcup_{i=1}^{\infty} T^{-i}(A)) = 0$.

PROPOSITION 4.2. *The first return map $T_A : A \to A$ is well defined and preserves the finite measure μ restricted to A. Furthermore, T_A and T are conservative.*

Proof. As $\mu(\partial\mathbb{U} \setminus \bigcup_{i=1}^{\infty} T^{-i}(A)) = 0$, T_A is defined almost everywhere. As $\mu(A)$ is finite, T_A preserves a finite measure and hence is conservative. By standard arguments T has to be conservative as well. ■

4.2. *Distortion properties.* In order to derive distortion properties like the Rényi or Gibbs property for the induced transformation T_B for some measurable set B of finite measure, recall the following. For each Möbius transformation g which does not fix ∞ there exists a unique circle I_g, the *isometric circle*, on which g acts as a Euclidean isometry. If g is in particular an isometry of the Poincaré model and $g(0) \neq 0$ then there exists a reflection τ in a straight line through the origin such that $g = \tau\sigma$, where σ is the inversion in I_g. In addition, I_g is perpendicular to S^1 and hence corresponds to a geodesic (see [Ra, §4.3]).

Denote by m_g the centre and by r_g the radius of I_g. Since I_g is perpendicular to S^1, we have $|m_g|^2 = r_g^2 + 1$. The inversion σ in I_g is given by

$$\sigma(z) = \frac{m_g\overline{z} - |m_g|^2 + r_g^2}{\overline{z} - \overline{a}} = \frac{m_g\overline{z} - 1}{\overline{z} - \overline{m}_g}.$$

Hence, for $D(\cdot)$ being the derivative of a holomorphic function, we have

$$|Dg(z)| = \left|\frac{r_g^2}{|z - m_g|^2}\right|, \qquad |D^2g(z)| = \left|\frac{2r_g^2}{(z - m_g)^3}\right|.$$

If ψ_g is the repelling (resp. indifferent) fixed point of the hyperbolic (resp. parabolic) transformation g then $|g'(z)| \leq 1$ if and only if $|z - m_g| \leq r_g$. Thus $|\psi_g - m_g| \leq r_g$. Furthermore,

$$\left|\frac{D^2g(z)}{(Dg(z))^2}\right| = 2\,\frac{|z - m_g|}{r_g^2}. \tag{3}$$

Recall that any Möbius transformation leaves invariant the *cross ratio* $[u, v, x, y]$, where u, v, x, y are four different points in $\mathbb{C}\cup\{\infty\}$ and the cross ratio is given by

$$[u, v, x, y] := \frac{|u - x|\,|u - v|}{|u - v|\,|x - y|}.$$

Lemma 4.3. *With respect to the disc model, for each measurable B with $d(B,\mathcal{V}) > \varepsilon$ for some $\varepsilon > 0$ (e.g. $B = A$ as in Proposition 4.2), there exists $0 < C < \infty$ such that, for all $n \in \mathbb{N}$ and for Lebesgue-a.e. z with $T^n(z) \in B$,*

$$\left|\frac{D^2T^n(z)}{(DT^n(z))^2}\right| < C.$$

Proof. Fix $\omega = (s_1 \dots s_n) \in \alpha^n$. Then $T^n|_{[\omega]} = g_\omega = g$. Assume that η_g is an element of the isometric circle I_g of g with centre m_g and radius r_g. Since $g(I_g) = I_{g^{-1}}$ it follows that $r_g = r_{g^{-1}}$ and $m_{g^{-1}} = g(\infty)$. Hence

$$\frac{|m_g - z|}{r_g}[m_g, \eta_g, z, \infty] = [g(m_g), g(\eta_g), g(z), g(\infty)],$$

which implies

$$\frac{|m_g - z|}{|m_g - \eta_g|} = \frac{|m_{g^{-1}} - g(\eta_g)|}{|m_{g^{-1}} - g(z)|}$$

and so

$$\frac{|m_g - z|}{r_g} = \frac{r_g}{|m_{g^{-1}} - g(z)|}.$$

Therefore, by (3),

$$\left|\frac{D^2g(z)}{(Dg(z))^2}\right| = \frac{2}{|m_{g^{-1}} - g(z)|}.$$

Hence it remains to derive an estimate of $|m_{g^{-1}} - g(z)|$ from below for all ω and z with $z \in B \cap [\omega]$ and $T^n(z) \in B$.

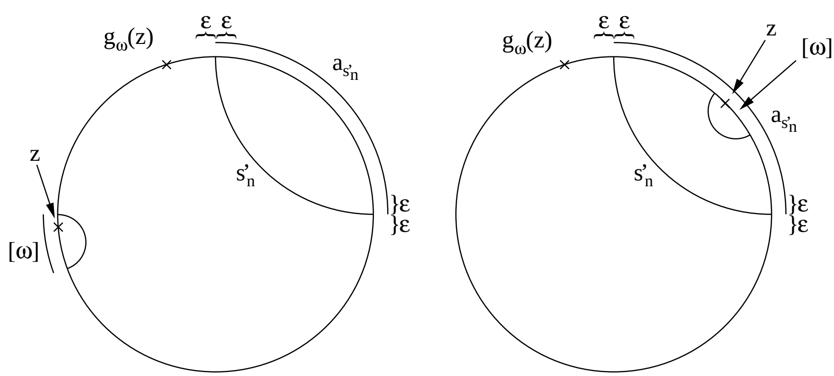

Fig. 2. The two cases $s_1 \neq s'_n$ resp. $s_1 = s'_n$

Case 1: Assume that $s_1 \neq s'_n$. Then $[\omega] \subset T^n[\omega] = g([\omega]) \overset{\mu}{=} [s'_n]^c$. Fix $z \in [\omega] \cap B$ with $g(z) \in B$. By the intermediate value theorem for continuous functions it follows that $\mathrm{Clos}([\omega])$ contains a fixed point of g. By the same argument $\mathrm{Clos}([s'_n]) = g([\omega])^c$ contains a fixed point of g^{-1}. Clearly, the latter is a repelling or indifferent fixed point of g^{-1}. Hence $\psi_{g^{-1}} \in \mathrm{Clos}([s'_n])$. Since $g(z) \in (a_{s'_n})^c \cap B$ and $d(B, \mathcal{V}) > \varepsilon$, the inequality $|\psi_{g^{-1}} - m_{g^{-1}}| \le r_{g^{-1}}$ implies that $|m_{g^{-1}} - g(z)| \ge \varepsilon - r_{g^{-1}}$. Now assume that $(g_1, g_2, \ldots)$ is a sequence of distinct elements of G. Then by Theorem 3.3.7 in [Ka], $r_{g_k} \to 0$ as $k \to \infty$. Hence

$$\left| \frac{D^2 g(z)}{(Dg(z))^2} \right| = \frac{2}{|m_{g^{-1}} - g(z)|} \le \frac{4}{\varepsilon}$$

for at most finitely many $g \in G$.

Case 2: Assume that $s_1 = s'_n$. In this case $[\omega]$ and $g([\omega])$ are disjoint. Hence neither $[\omega]$ nor $g([\omega])$ contain any fixed point of g^{-1}. Hence by the same arguments as above for $z \in B$ with $g(z) \in B$, we have

$$\left| \frac{D^2 g(z)}{(Dg(z))^2} \right| = \frac{2}{|m_{g^{-1}} - g(z)|} \le \frac{4}{\varepsilon}$$

for at most finitely many $g \in G$. ∎

Recall that the set A of Proposition 4.2 is defined by

$$A = \partial\mathbb{H} \setminus \bigcup_{v \in \mathcal{V}} U(v) = \partial\mathbb{H} \setminus \bigcup_{v \in \mathcal{V}} ([w(v)] \cup [w'(v)]),$$

where N is the least common multiple of the lengths of the edge cycles. Define $\widetilde{\alpha} := \bigcup_{n=0}^{\infty} \alpha^n$. We now introduce the following two partitions of $\partial\mathbb{H}$:

$$\beta^* := \{a \in \alpha^N : a \neq [w(v)],\, a \neq [w'(v)]\ \forall v \in \mathcal{V}\},$$

$$\beta := \{b \in \widetilde{\alpha} : \exists a_1, a_2 \in \beta^* \text{ such that } b \subset a_1,\, T_A(b) = a_2,\\ T_A : b \to a_2 \text{ is injective}\}.$$

By definition, $T_A(b) \in \beta^*$ for all $b \in \beta$. Since β^* is a finite partition of A there exists a constant $C > 0$ such that $\lambda(T_A(b)) > C$ for all $b \in \beta$, or in other words, T_A has the *big image property* with respect to β and the Lebesgue measure λ restricted to A.

Proposition 4.4. *$(A, \mathcal{B}, \lambda, T_A, \beta)$ is a topologically mixing Markov map which is eventually expanding (i.e. there are $\Lambda > 1$ and $n_0 \in \mathbb{N}$ such that $|DT_A^n(z)| > \Lambda$ for all $n > n_0$ and λ-a.e. $z \in \partial\mathbb{H}$). Furthermore, T_A has the Rényi property, i.e. there is $C > 0$ such that*

$$\left| \frac{D^2 T_A^n(z)}{(DT_A^n(z))^2} \right| < C \quad \textit{for all } n \in \mathbb{N} \textit{ and for Lebesgue-a.e. } z.$$

Proof. Clearly, β is a Markov partition and T_A is topologically mixing. Since the Rényi property follows immediately from Lemma 4.3, it remains to show that T_A is eventually expanding. Moreover, the Rényi property gives rise to an estimate of the diameter of an element $b \in \beta^n$. By a straightforward calculation (which can be found e.g. in [Aa, p. 145]) there is $M > 0$ such that, if ν_b denotes the inverse branch of T_A on $b \in \beta^n$ for arbitrary $n \in \mathbb{N}$, then

$$\exp(-M)\,\frac{\lambda(b)}{\lambda(T_A(b))} \le |\nu_b'(z)| \le \exp(M)\,\frac{\lambda(b)}{\lambda(T_A(b))}.$$

Since $\sup\{\lambda(b) : b \in \beta^n\}$ tends to zero as $n \to \infty$ and since β has the big image property, it follows that T_A is eventually expanding. ∎

From a general point of view, a distortion property is a feature of the multiplicative variation of the Radon–Nikodym derivative $\nu_\omega' := d(\mu \circ \nu_\omega)/d\mu$, where $\nu_\omega : \mathcal{D}(\nu_\omega) \to [\omega]$ is the inverse branch determined by the admissible word ω. Define

$$\widetilde{\beta}_+ := \{a \in \widetilde{\beta} : \mu(a) > 0\}, \quad \text{where} \quad \widetilde{\beta} := \bigcup_{n=0}^{\infty} \beta^n.$$

Recall that the Markov map $(X, \mathcal{B}, \mu, T, \beta)$ has the *Gibbs property* if there are $C > 1$ and $0 < r < 1$ such that $\mathfrak{g}_r(C,T) \stackrel{\mu}{=} \widetilde{\beta}_+$, where

$$\mathfrak{g}_r(C,T) := \left\{ a \in \widetilde{\beta}_+ : \left| \log \frac{v_a'(x)}{v_a'(y)} \right| \le C r^{t(x,y)} \text{ for } \mu \times \mu\text{-a.e. } (x,y) \in (\mathcal{D}(v_a))^2 \right\}.$$

Here $t : \bigcup_{a,b\in\beta} a \times b \to \mathbb{N} \cup \{0\}$ is defined by

$$t(x,y) := \min\{n \ge 0 : T^n x \in a \in \alpha,\ T^n y \in b \in \alpha,\ a \ne b\}.$$

THEOREM 1. *Let A be defined as in Proposition* 4.1. *Then T_A has the Gibbs property with respect to μ_A, where μ_A is the measure μ restricted to A.*

Proof. Combining the observations that T_A has the Rényi property and that T_A is eventually expanding, it immediately follows that T_A has the Gibbs property with respect to λ (for details see for instance [Aa, Proposition 4.3.3]). Moreover, a straightforward calculation shows that $\log(d\mu/d\lambda)$ is a bounded, continuous function on A. Hence it is Lipschitz continuous on A, which implies by Proposition 4.7.1 of [Aa] that T_A also has the Gibbs property with respect to the invariant measure μ. ∎

4.3. *Ergodic properties of the coding map.* By a result of Aaronson, Denker and Urbański (see [ADU, Theorem 3.2] and [Aa, Theorem 4.4.7]), a topologically mixing, conservative Gibbs–Markov map is exact. Thus by Propositions 4.2, 4.4 and Theorem 1, T_A is exact (and hence ergodic). Since T_A is conservative, the map T is exact (and ergodic) as well. For systems of this type there is a further classification (see [Aa]).

A conservative, ergodic, measure preserving transformation T of $(X, \mathcal{B}, \mu)$ is called *rationally ergodic* if there is a set $A \in \mathcal{B}$ with $0 < \mu(A) < \infty$ and a constant $M > 0$ such that

$$\int_A \Big(\sum_{i=0}^{n-1} 1_A \circ T^i\Big)^2 d\mu \le M\Big(\int_A \sum_{i=0}^{n-1} 1_A \circ T^i \, d\mu\Big)^2 \quad \forall n \ge 1. \tag{4}$$

Furthermore, there is a sequence (a_n), $a_n \nearrow \infty$, associated to T such that for a set A which satisfies (4), we have

$$\frac{1}{a_n}\sum_{i=0}^{n-1} \mu(B \cap T^{-i}C) \xrightarrow{n\to\infty} m(B)m(C) \quad \forall B, C \in \mathcal{B} \cap A.$$

This sequence is unique up to asymptotic equality and is called the *return sequence* of T.

A stronger ergodic property of T is defined via the *transfer operator* $\widehat{T} : L^1(\mu) \to L^1(\mu)$ given by

$$\int_X \widehat{T}f \cdot g \, d\mu = \int_X f \cdot g \circ T \, d\mu \quad \forall f \in L^1(\mu),\ g \in L^\infty(\mu).$$

Namely, a conservative, ergodic, measure preserving transformation T is *pointwise dual ergodic* if there is a sequence (b_n), $b_n \nearrow \infty$, such that

$$\frac{1}{b_n}\sum_{i=0}^{n-1} \widehat{T}^i f \to \int_X f \, d\mu \quad \text{a.e. as } n \to \infty\ \forall f \in L^1(X).$$

Any pointwise dual ergodic transformation is rationally ergodic and the sequences (a_n) and (b_n) coincide up to asymptotic equality (see [Aa, Proposition 3.7.1]). Therefore (b_n) is also referred to as the return sequence of T.

Theorem 2. *Let G be a cofinite Fuchsian group which is not cocompact. Then the coding map T is pointwise dual ergodic with respect to μ. The return sequence (a_n) of T is given by*

$$a_n = \frac{\pi}{4(\mathrm{Area}(\mathbb{H}/G) + 2\pi)} \cdot \frac{n}{\log n}.$$

Proof. Since T_A has the Gibbs–Markov property, Theorem 4.8.1 in [Aa] shows that T is pointwise dual ergodic and A is a *Darling–Kac set*, that is, there exists a sequence $(a_n), a_n \nearrow \infty$, such that

$$\frac{1}{a_n}\sum_{i=0}^{n-1} \widehat{T}^i 1_A \to \mu(A) \quad \text{almost uniformly on } A.$$

Hence the Chacon–Ornstein theorem implies that (a_n) is a return sequence for T. Furthermore, since the wandering rate $L_A(n)$ of A is proportional to $\log n$ (Proposition 4.1), $L_A(n)$ is regularly varying at ∞ with index $\alpha = 0$. Using Proposition 3.8.7 of [Aa], we obtain

$$a_n \sim \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)} \cdot \frac{n}{L_A(n)} = \frac{n}{\log n} \cdot \frac{1}{4\#\mathcal{V}}.$$

Since $\mathrm{Area}(P) = (\#\mathcal{V} - 2)\pi$, the return sequence is given by

$$a_n = \frac{\pi}{4(\mathrm{Area}(\mathbb{H}/G) + 2\pi)} \cdot \frac{n}{\log n}. \ \blacksquare$$

COROLLARY 4.5. *The first return map S is rationally ergodic with return sequence (a_n).*

Proof. By the last theorem, T is rationally ergodic. The assertion follows from the fact that S is the natural extension of T (see Proposition 3.2). We note that no invertible transformation can be pointwise dual ergodic. $\blacksquare$

5. A different choice of the fundamental domain. The construction of the coding map T presented here relies on the choice of a specially shaped fundamental polygon. Namely, we use the fact that for a cofinite Fuchsian group G with parabolic elements there exists a fundamental polygon P such that P is an ideal polygon, that is, all vertices of P are contained in $\partial\mathbb{H}$. This shape of P then immediately implies that the section (Y, S) has the Markov property and that the invariant measure induced by the Liouville measure is infinite. Moreover, the combinatorial structure of the canonical factor T is less complicated than that of the map introduced in [BS].

Recall that in [BS], the construction of the coding map T relies on the choice of a fundamental polygon which satisfies the so-called *even corner property* or *net condition*. Namely, a fundamental domain P for a given group G has the even corner property if $G(\partial P)$ is the union of geodesics. Note that such a P exists for any cofinite Fuchsian group G (see [BS]), and that the class of ideal polygons is included in the class of fundamental polygons with that property.

Moreover, by a result of [Se2] the first return map of the Poincaré section $\{(\xi, \eta) : \exists t \in \mathbb{R} \text{ such that } \gamma_{\xi,\eta}(t) \in P\}$ is conjugate to a Markov map S acting on some subset Y of $\partial\mathbb{H} \times \partial\mathbb{H}$. Combining the fact that the conjugating map is piecewise defined via elements of G and that the measure m given by $dm(\xi, \eta) = d|\xi| d|\eta| / |\xi - \eta|^2$ is invariant under the action of $\mathrm{Iso}^+(\mathbb{H})$, one clearly deduces that $(Y, \mathcal{B}, m|_Y, S)$ is a measure preserving Markov map. Due to the construction of Y it follows immediately that $m(Y)$ is infinite if and only if there are some vertices of P in $\partial\mathbb{H}$, which is equivalent to the non-compactness of $\mathbb{H}/G$.

Following [BS], [Se1] and [AF] one infers that the canonical non-invertible factor $(\partial\mathbb{H}, \mathcal{B}, \mu, T)$ is a topologically mixing, measure preserving Markov map, where μ is the image measure of the factor map as in Proposition 3.1. Recall that this factor is defined as follows.

Let the sides $\mathcal{S} = \{s_1, \ldots, s_k\}$ of P be labelled in anticlockwise order and denote by $\gamma(i)$, for $s_i \in \mathcal{S}$, the geodesic containing s_i. Furthermore, let P_i, Q_i be the elements of $\partial\mathbb{H}$ such that $\{P_i, Q_i\}$ is the set of endpoints of

$a(\gamma(i))$, and P_i comes before Q_i in anticlockwise order. Now $T : \partial\mathbb{H} \to \partial\mathbb{H}$ is partially defined as follows (see Figure 3):

$$x \mapsto g_{s_i}(x) \text{ for } x \in [P_i, P_{i+1}),\ i = 1, \dots, k-1, \quad x \mapsto g_{s_k}(x) \text{ for } x \in [P_k, P_1).$$

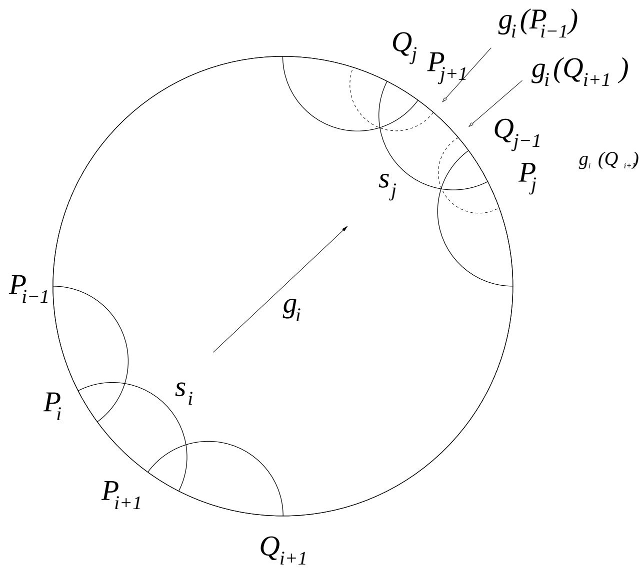

Fig. 3. The Bowen–Series construction

We are now in a position to adapt our analysis to this situation. For $s_i \in \mathcal{S}$ with endpoints in $\mathbb{H}$, $g_i(P_{i-1})$ and $g_i(Q_{i+1})$ lie in the interval $[Q_{j-1}, P_{j+1}]$, where j is given by $g_i(s_i) = s_j$ and the indices are taken modulo $k = \#\mathcal{S}$ (see Figure 3). For a cocompact group G the proof of Lemma 4.3 can now be easily adapted as follows.

If $T^n|_{[\omega]} = g|_{[\omega]}$ for some interval $[\omega] \subset \partial\mathbb{H}$ and $g \in G$ such that $[\omega] \subset T^n([\omega])$ then by the last observation the repelling fixed point $m_{g^{-1}}$ of g^{-1} is in $[Q_{j-1}, P_{j+1}]$ for some $j \in \{1, \dots, k\}$. Hence, for $\varepsilon := \min\{|Q_{j-1} - P_{j+1}| : j \in \{1, \dots, k\}\}$, the same arguments as in the first case of the proof of Lemma 4.3 yield $|D^2 g(z)/(Dg(z))^2| < 4/\varepsilon$ for all $z \in [\omega]$. If $[\omega] \not\subset T^n([\omega])$, a similar argument applies.

Following the arguments of Sections 4.2 and 4.3 we deduce the following well known result (see [BS], [AF]).

Theorem 3. *Let G be a cocompact Fuchsian group. Then the map T is an ergodic, measure preserving Gibbs–Markov map with respect to the finite measure μ.*

Note that in this case the finiteness and invariance of μ immediately imply that T is conservative. So assume from now on that μ is infinite or equivalently that G is not cocompact. In this situation the conservativity of T can be deduced as in Proposition 4.2 from the existence of a set A of finite measure such that $\bigcup_{i=1}^{\infty} T^{-i}(A) = \partial\mathbb{H} \bmod \mu$. Moreover, the wandering rate can be specified precisely as in Proposition 4.1. Namely, for A bounded away from the set of ideal vertices $\mathcal{V}^* \subset \partial\mathbb{H}$ of P we have

$$\frac{\mu(\bigcup_{i=0}^{n} T^{-i}A)}{\log n} \xrightarrow{n\to\infty} 4\#\mathcal{V}^*.$$

By using the above observation the analogue of Lemma 4.3 can be easily obtained in this situation. Hence, by the arguments of Sections 4.2 and 4.3, we obtain the following analogue of Theorem 2.

THEOREM 4. *Let G be a cofinite Fuchsian group which is not cocompact. Then the map T is an ergodic, conservative, measure preserving Markov map with respect to the infinite measure μ. Moreover, T is pointwise dual ergodic and the associated return sequence is given by*

$$a_n = \frac{n}{\log n} \cdot \frac{1}{4\#\mathcal{V}^*}.$$

References

[Aa] J. Aaronson, *An Introduction to Infinite Ergodic Theory*, Math. Surveys Monogr. 50, Amer. Math. Soc., 1997.

[AD] J. Aaronson and M. Denker, *The Poincaré series of* $\mathbb{C} \setminus \mathbb{Z}$, Ergodic Theory Dynam. Systems 19 (1999), 1–20.

[ADU] J. Aaronson, M. Denker and M. Urbański, *Ergodic theory for Markov fibred systems and parabolic rational maps*, Trans. Amer. Math. Soc. 337 (1993), 495–548.

[AF] R. Adler and L. Flatto, *Geodesic flows, interval maps and symbolic dynamics*, Bull. Amer. Math. Soc. 25 (1991), 229–334.

[AK] W. Ambrose and S. Kakutani, *Structure and continuity of measurable flows*, Duke Math. J. 9 (1942), 25–42.

[Ar] E. Artin, *Ein mechanisches System mit quasiergodischen Bahnen*, Abh. Math. Sem. Hamburg. Univ. 3 (1924), 170–175.

[Be] A. F. Beardon, *The Geometry of Discrete Groups*, Grad. Texts in Math. 91, Springer, New York, 1983.

[BS] R. Bowen and C. Series, *Markov maps associated with Fuchsian groups*, Publ. Math. Inst. Hautes Études Sci. 50 (1979), 153–170.

[Ho] E. Hopf, *Fuchsian groups and ergodic theory*, Trans. Amer. Math. Soc. 39 (1936), 299–314.

[Ka] S. Katok, *Fuchsian Groups*, Chicago Lectures in Math. Ser., Univ. of Chicago Press, 1992.

[Ma] B. Maskit, *On Poincaré's theorem for fundamental polygons*, Adv. Math. 7 (1971), 219–230.

[Mor1] H. M. Morse, *A one-to-one representation of geodesics on a surface of negative curvature*, Amer. J. Math. 43 (1921), 33–51.

[Mor2] —, *Recurrent geodesics on a surface of negative curvature*, Trans. Amer. Math. Soc. 22 (1921), 84–100.

[Ni] P. Nicholls, *The Ergodic Theory of Discrete Groups*, London Math. Soc. Lecture Note Ser. 143, Cambridge Univ. Press, Cambridge, 1989.

[Ra] J. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Grad. Texts in Math. 149, Springer, New York, 1994.

[Se1] C. Series, *The modular surface and continued fractions*, J. London Math. Soc. (2) 31 (1985), 69–80.

[Se2] —, *Geometrical Markov coding of geodesics on surfaces of constant negative curvature*, Ergodic Theory Dynam. Systems 6 (1986), 601–625.

[St] M. Stadlbauer, *The Bowen–Series map for some free groups*, Dissertation, Univ. of Göttingen, 2002, preprint in Mathematica Gottingensis 5 (2002), 1–53.

[Th] M. Thaler, *Transformations on* $[0,1]$ *with infinite invariant measures*, Israel J. Math. 46 (1983), 67–96.

[Tu] P. Tukia, *Extension of boundary homeomorphisms of discrete groups of the unit disk*, Ann. Acad. Sci. Fenn. Ser. A I 548 (1973), 1–16.

[Z] R. Zweimüller, *Ergodic properties of infinite measure-preserving interval maps with indifferent fixed points*, Ergodic Theory Dynam. Systems 20 (2000), 1519–1549.

Institut für Mathematische Stochastik
Maschmühlenweg 8–10
37073 Göttingen, Germany
E-mail: stadelba@math.uni-goettingen.de

Received 30 September 2003;
in revised form 5 September 2004