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Genus sets and SNT sets of certain connective covering spaces

by

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Abstract. We study the genus and SNT sets of connective covering spaces of familiar finite CW-complexes, both of rationally elliptic type (e.g. quaternionic projective spaces) and of rationally hyperbolic type (e.g. one-point union of a pair of spheres). In connection with the latter situation, we are led to an independently interesting question in group theory: if f is a homomorphism from $\mathrm{Gl}(\nu, A)$ to $\mathrm{Gl}(n, A)$, $\nu < n$, $A = \mathbb{Z}$, resp. $\mathbb{Z}_p$, does the image of f have infinite, resp. uncountably infinite, index in $\mathrm{Gl}(n, A)$?

1. Introduction and statement of results. In this paper, we study the genus sets and SNT sets of certain m-connective covering spaces $X\langle m\rangle$, following the work initiated by McGibbon and Møller ([16]), and continued by McGibbon and Roitberg ([17]). Before stating our main results, we recall the basic notions; in the following definitions, X and Y are assumed to be spaces of the homotopy type of nilpotent, finite type CW-complexes.

Definition 1.

(i) $\widehat{\mathcal{G}}(X)$ is the set of homotopy types of spaces Y such that the profinite completion $\widehat{Y}$ of Y is homotopy equivalent to the profinite completion $\widehat{X}$ of X. (Note that $\widehat{X}$ is canonically homotopy equivalent to the product $\prod X_p$, where X_p is the p-completion of X.)

(ii) $\widehat{\mathcal{G}}_0(X)$ is the subset of $\widehat{\mathcal{G}}(X)$ for which the rationalizations $X_{(0)}$, $Y_{(0)}$ of X, Y are homotopy equivalent.

(iii) $\mathcal{G}(X)$ is the subset of $\widehat{\mathcal{G}}_0(X)$ for which the p-localizations $X_{(p)}$, $Y_{(p)}$ of X, Y are homotopy equivalent for all primes p.

Thus we have the set-theoretic inclusions

$$\mathcal{G}(X) \subset \widehat{\mathcal{G}}_0(X) \subset \widehat{\mathcal{G}}(X).$$

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$\widehat{\mathcal{G}}(X)$ is called the *completion genus* of X and $\mathcal{G}(X)$ is called the *localization genus* or *Mislin genus* of X.

DEFINITION 2. SNT(X) is the set of homotopy types of spaces Y such that the mth Postnikov approximations $P_m(X)$, $P_m(Y)$ are homotopy equivalent for all positive integers m.

In [30], Wilkerson proves that the completion genus set of a 1-connected, finite CW-complex, or finite Postnikov space, is finite. (Hence the same is true of the other two genus sets.) Wilkerson's result is certainly not valid for general 1-connected, finite type CW-complexes. As a simple example, let $C_\alpha = S^r \cup_\alpha e^n$, $r > 1$, be the mapping cone of a homotopy element α in the stable range and of order a prime $p > 3$, and let X be the one-point union $\bigvee_{i=0}^{\infty} \Sigma^{n_i} C_\alpha$, where $n_0 = 0$ and n_i is chosen so that $n_i + r > n_{i-1} + n$. As $\mathcal{G}(\Sigma^{n_i} C_\alpha)$ has cardinality $\frac{1}{2}(p-1) > 1$ ([10; III, Example 1.3]), it is not difficult to verify that $\mathcal{G}(X)$ is uncountably infinite.

An interesting example of an uncountably infinite Mislin genus set, due to Rector ([23]) for the case $G = S^3$ and to Møller ([20]) for all non-trivial, 1-connected, compact Lie groups G, is $\mathcal{G}(BG)$, where BG is the classifying space of G. Another striking, and somewhat surprising, example of an uncountably infinite Mislin genus set is given in the paper [16] by McGibbon and Møller. They prove that the Mislin genus set of $S^{2n}\langle 2n\rangle$, the $2n$-connective covering space of the $2n$-dimensional sphere S^{2n}, $n > 1$, is uncountably infinite, relying on the following corollary to a remarkable theorem of Neisendorfer ([22]): If X and Y are finite CW-complexes which are $1\frac{1}{2}$-connected (i.e. $\pi_1 = 0$ and π_2 is finite), then the induced map $[X_p, Y_p] \to [X\langle m\rangle_p, Y\langle m\rangle_p]$ on homotopy sets is a bijection for all primes p and all natural numbers m. Moreover, $\alpha \in [X_p, Y_p]$ is the homotopy class of a homotopy equivalence if and only if the same is true of $\alpha\langle m\rangle$.

We seek to extend the McGibbon–Møller result to a class of spaces containing S^{2n}. First observe that S^{2n} is a simple example of what rational homotopy theorists term a *rationally elliptic space* ([6]), i.e., a 1-connected, finite CW-complex with only finitely many non-zero rational homotopy groups. Indeed, S^{2n} is a "2-stage" rationally elliptic space, with $\pi_{2n} \otimes \mathbb{Q} = \mathbb{Q} = \pi_{4n-1} \otimes \mathbb{Q}$ the only non-zero rational homotopy groups. The cohomology ring has the form $H^*(S^{2n}; \mathbb{Q}) = \mathbb{Q}[a]/\langle a^2\rangle$, the truncated polynomial ring with $\deg(a) = 2n$, and the Sullivan minimal model of $S^{2n}_{(0)}$ is $(v, w : dw = v^2)$ with $\deg(v) = 2n$, $\deg(w) = 4n - 1$. We will consider more general 2-stage rationally elliptic spaces, namely spaces T for which $H^*(T; \mathbb{Q}) = \mathbb{Q}[a]/\langle a^k\rangle$ with $\deg(a) = 2n$, $k > 1$. This rational cohomology condition implies that $T_{(0)}$ has a Sullivan minimal model of the form $(v, w : dw = v^k)$ with $\deg(v) = 2n$, $\deg(w) = 2kn - 1$, hence that the two non-zero rational homotopy groups of T are $\pi_{2n} \otimes \mathbb{Q} = \mathbb{Q} = \pi_{2kn-1} \otimes \mathbb{Q}$.

Examples of such T are:

(i) S^{2n};
(ii) $J_{k-1}\left(S^{2n+1}\right)$, the $(k-1)$-st stage of the James reduced product construction on S^{2n+1}, $k<\infty$;
(iii) $\mathbb{C}P^{k-1}$, the complex projective space;
(iv) $\mathbb{H}P^{h-1}$, the quaternionic projective space;
(v) $\mathbb{O}P^2$, the Cayley (or octonionic) projective plane;
(vi) any finite stage of a homology decomposition of $K(\mathbb{Z},2n)$.

(Note that (i) is a special case of both (ii) and (vi).)

Our first main result on genus may be stated as follows.

Theorem A. *Let T be a $1\frac{1}{2}$-connected space as above. Then $\widehat{\mathcal{G}}_0(T\langle m\rangle)$ is uncountably infinite if $2n\le m\le 2nk-2$, and is trivial* (*the singleton set*) *if $m\ge 2nk-1$.*

The hypothesis that T is $1\frac{1}{2}$-connected implies that $n>1$. To see that this hypothesis is essential, note that Theorem A does not apply to the complex projective space $\mathbb{C}P^{k-1}$ since $\pi_2\left(\mathbb{C}P^{k-1}\right)$ is not finite; in fact, $\mathbb{C}P^{k-1}\langle m\rangle$ is homotopy equivalent to $S^{2k-1}\langle m\rangle$ for $m\ge 2$, and the Mislin genus of the latter space is easily seen to be finite.

As mentioned earlier, the conclusion of [16] is that $\mathcal{G}(S^{2n}\langle 2n\rangle)$ is uncountably infinite, $n>1$. In fact, it is stated in [16; footnote], with scant indication of proof, that $\mathcal{G}(S^{2n}\langle 2n\rangle)=\widehat{\mathcal{G}}_0(S^{2n}\langle 2n\rangle)$. It is natural to wonder whether this equality remains true for the more general situation in Theorem A. We only offer the following partial result.

Addendum to Theorem A. *For T of the form* (i), (ii), (iv) *or* (v), $\mathcal{G}(T\langle m\rangle)=\widehat{\mathcal{G}}_0(T\langle m\rangle)$.

The proof of the Addendum is heavily dependent on Theorem C below.

In the so-called rational dichotomy, 1-connected, finite CW-complexes which are not rationally elliptic are termed *rationally hyperbolic*; such spaces have infinitely many non-zero rational homotopy groups and their rational homotopy groups "grow exponentially" ([6]). A simple example of such a space is the one-point union $B=S^k\vee S^l$, $2\le k\le l$. Our second main result on genus is centered on this example.

Theorem B. *Let B be as above*, $k>2$. *Then there exists an integer N_0 such that for all $N\ge N_0$, $\widehat{\mathcal{G}}_0(B\langle N\rangle)$ is uncountably infinite.*

Remark. For our choice of N_0, the condition $N\ge N_0$ is sufficient, but not necessary, for the conclusion of Theorem B to hold. Suitable examples will be given at the end of §3.

Note that the uncountability result in Theorem B is asserted for $\widehat{\mathcal{G}}_0$, not for $\mathcal{G}$. We do not settle the question of whether the Mislin genus of $B\langle N\rangle$ is uncountable for N as in Theorem B. Interestingly, it turns out that there are examples of 1-connected, finite type CW-complexes X such that $\mathcal{G}(X)$ is at most countably infinite (exactly countably infinite in many cases, possibly even finite in some cases) and $\widehat{\mathcal{G}}_0(X)$ is uncountably infinite. Basing ourselves on some computations of Møller ([20]) and McGibbon and Møller ([14]), we will present such examples in Appendix 1.

We next turn to SNT computations. With the help of the techniques in [14] and [15], we obtain the following results.

THEOREM C. *If T is as in the Addendum, then* $\mathrm{SNT}(T\langle m\rangle)$ *and* $\mathrm{SNT}(T\langle m\rangle_p)$ *are trivial for all m and all primes p.*

THEOREM D. *In the notation of Theorem B*, $\mathrm{SNT}(B\langle N\rangle)$ *is uncountably infinite for all $N \geq N_0$.*

In contrast to the situation for genus sets, the uncountability of $\mathrm{SNT}(Z)$ is actually equivalent to the non-triviality of $\mathrm{SNT}(Z)$, provided Z is of the homotopy type of a nilpotent, finite type CW-complex, or the P-localization of such a space, where P is a collection of primes; see [14; Corollary 2.1].

The proofs of Theorems A, B, C and D will be carried out in the next four sections. In the situation of Theorems B and D where $k = l$, our method of proof suggests a group-theoretic question which seems to have independent interest:

QUESTION. *Let* $f : \mathrm{Gl}(\nu, A) \to \mathrm{Gl}(n, A)$ *be a homomorphism*, $\nu < n$. *Is the coset space* $\mathrm{Gl}(n, A)/f(\mathrm{Gl}(\mu, A))$ *infinite*, *resp. uncountably infinite*, *when* $A = \mathbb{Z}$, *resp.* $A = \mathbb{Z}_p$?

The case $\nu \leq 2$ is pertinent to Theorems B and D and differs from the case $\nu > 2$. We discuss this question in Appendix 2.

Theorems A, B, C and D constitute an amended and expanded version of a portion of the first-named author's Ph.D. dissertation [11]. Some of the results contained in these theorems were also announced in [17] and [24].

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2. Proof of Theorem A and its Addendum. The proof of Theorem A (and also of Theorem B in the next section) rests on a generalization of the technique used in establishing [16; Example 4.2]. (A very brief description of this generalization was given in [17].) The starting point is the double coset formula of Wilkerson ([30; Theorem 3.8]), namely

$$\widehat{\mathcal{G}}_0(W) = (f.c)_* \operatorname{Aut}(W_{(0)})\backslash \operatorname{CAut}((W_{(0)})^-)/r_* \operatorname{Aut}(\widehat{W}). \tag{2.1}$$

In (2.1), W is a space of the homotopy type of a 1-connected, finite type CW-complex, and $(W_{(0)})^-$ is Sullivan's formal completion of $W_{(0)}$, which is homotopy equivalent to $(\widehat{W})_{(0)}$. Furthermore $\operatorname{CAut}((W_{(0)})^-)$ is the subgroup of the full automorphism group $\operatorname{Aut}((W_0)^-)$ (also known as the group of homotopy classes of self-homotopy equivalences of $(W_{(0)})^-$) consisting of those automorphisms that induce $\widehat{\mathbb{Q}}$-module automorphisms of the homotopy groups of $(W_{(0)})^-$, where $\widehat{\mathbb{Q}} = \mathbb{Q} \otimes \widehat{\mathbb{Z}}$, with $\widehat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$. Similarly we may define $\operatorname{CAut}((W_p)_{(0)})$, $\operatorname{CAut}(W_p)$ and $\operatorname{CAut}(\widehat{W})$, but $\operatorname{CAut}(W_p) = \operatorname{Aut}(W_p)$ and $\operatorname{CAut}(\widehat{W}) = \operatorname{Aut}(\widehat{W})$ by standard properties of p-completion and profinite completion for nilpotent, finite type CW-complexes. (All occurrences of $\operatorname{Aut}((W_p)_{(0)})$ in [16] and [17] should be replaced by $\operatorname{CAut}((W_p)_{(0)}$.) The homomorphisms $(f.c)_*$ and r_* are induced by formal completion $f.c$ and rationalization r, respectively. Analysis of $r_* \operatorname{Aut}(\widehat{W})$ is aided by noting the existence of a commutative diagram

$$\begin{array}{ccc} \operatorname{Aut}(\widehat{W}) & \longrightarrow & \prod \operatorname{Aut}(W_p) \\ \downarrow & & \downarrow \\ \operatorname{CAut}((\widehat{W})_{(0)}) & \longrightarrow & \prod \operatorname{CAut}((W_p)_{(0)}) \end{array} \tag{2.2}$$

where the vertical arrows are induced by rationalization and the horizontal arrows arise from the canonical homotopy equivalence $\widehat{W} \to \prod W_p$ mentioned in §1; moreover, the top horizontal arrow is an isomorphism.

We apply the foregoing to $W = T\langle m\rangle$, beginning with the case $m \geq 2nk-1$. In this case, all the homotopy groups of W are finite. Thus $W_{(0)}$, and also $(W_{(0)})^-$, is trivial, and it follows immediately from (2.1) that $\widehat{\mathcal{G}}_0(W)$ is trivial. In the case $2n \leq m \leq 2nk-2$,

$$\begin{aligned} W_{(0)} &= K(\mathbb{Q}, 2nk-1), &&\text{so that} \quad \operatorname{Aut}(W_{(0)}) = \mathbb{Q}^*; \\ (W_{(0)})^- &= K(\widehat{\mathbb{Q}}, 2nk-1), &&\text{so that} \quad \operatorname{CAut}((W_{(0)})^-) = (\widehat{\mathbb{Q}})^*; \\ (W_p)_{(0)} &= K(\mathbb{Q}_p, 2nk-1), &&\text{so that} \quad \operatorname{CAut}((W_p)_{(0)}) = \mathbb{Q}_p^*. \end{aligned} \tag{2.3}$$

In (2.3), R^* denotes the multiplicative group of units of the ring R, and $\mathbb{Q}_p = \mathbb{Q} \otimes \mathbb{Z}_p$ is the field of p-adic numbers. Thus (2.1) reduces in this case to

$$\mathbb{Q}^*\backslash(\widehat{\mathbb{Q}})^*/r_* \operatorname{Aut}(\widehat{W}), \tag{2.4}$$

with $\mathbb{Q}^*$ canonically embedded in $(\widehat{\mathbb{Q}})^*$. By Neisendorfer's theorem, stated in detail in §1, any element of $\operatorname{Aut}(W_p)$ is induced by a (unique) element in $\operatorname{Aut}(T_p)$. But the image of $[T_p, T_p]$ in $[(W_p)_{(0)}, (W_p)_{(0)}] = \mathbb{Q}_p$ is contained in $(\mathbb{Z}_p)^k \subset \mathbb{Z}_p \subset \mathbb{Q}_p$ (as the image of $[T,T]$ in $[W_{(0)}, W_{(0)}]$ is contained in $\mathbb{Z}^k \subset \mathbb{Z} \subset \mathbb{Q}$). Here, R^k denotes the set consisting of the kth powers of elements of R; similarly, we will write $(R^*)^k$ for the group consisting of the kth powers of elements of R^*. Thus, in light of (2.2), we see that the double coset space in (2.4) maps surjectively to

$$\mathbb{Q}^* \backslash (\widehat{\mathbb{Q}})^* / \prod (\mathbb{Z}_p^*)^k. \tag{2.5}$$

To show that the latter double coset space is uncountably infinite, it suffices to check (see [16], [17]) that there are infinitely many primes p such that $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^k$ is non-trivial. But for p odd,

$$\mathbb{Z}_p^* \cong \mathbb{Z}_p \oplus \mathbb{Z}/p-1 \quad \text{(see, e.g., [26]).} \tag{2.6}$$

Hence

$$\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^k \cong \mathbb{Z}_p/k\mathbb{Z}_p \oplus (\mathbb{Z}/p-1)/k(\mathbb{Z}/p-1). \tag{2.7}$$

Now, for any $k \geq 2$, there are infinitely many primes p such that $p \equiv 1 \bmod k$ by Dirichlet's theorem (see, e.g., [26]). For such p, the second summand in (2.7) is non-trivial, and the proof of Theorem A is completed.

To prove the Addendum to Theorem A, we study the commutative diagram

$$\begin{array}{ccc} \mathcal{G}(W) & \longrightarrow & \varprojlim \mathcal{G}(P_r W) \\ \downarrow & & \downarrow \\ \widehat{\mathcal{G}}_0(W) & \longrightarrow & \varprojlim \widehat{\mathcal{G}}_0(P_r W) \end{array} \tag{2.8}$$

where the vertical arrows are induced by the inclusions $\mathcal{G}() \subset \widehat{\mathcal{G}}_0()$ and the horizontal arrows are the obvious natural maps. Since W, and therefore also $P_r W$, is a rational H-space, the inclusion $\mathcal{G}(P_r W) \subset \widehat{\mathcal{G}}_0(P_r W)$ is a bijection by a result of Belfi–Wilferson ([2; Theorem 1.1]). It follows that the right vertical arrow in (2.8) is a bijection. To prove that the left vertical arrow is also a bijection, it remains to prove that the two horizontal arrows are bijections. By [14; Lemma 6.1], the top horizontal arrow is injective provided $\mathrm{SNT}(V)$ is trivial for all V in $\mathcal{G}(W)$, and is surjective provided $\mathrm{SNT}(W_{(p)})$ is trivial for all primes p; the triviality of $\mathrm{SNT}(V)$ follows from the proof of Theorem C and the triviality of $\mathrm{SNT}(W_p)$ follows from Theorem C. Similarly, the bottom horizontal arrow is injective provided $\mathrm{SNT}(V)$ is trivial for all V in $\widehat{\mathcal{G}}_0(W)$ (which again follows from the proof of Theorem C) and is surjective provided $\mathrm{SNT}(W_p)$ is trivial for all primes p; the latter is a consequence of a general compactness argument—see Wilkerson ([29; Corollary II]). The proof of the Addendum to Theorem A is thereby achieved, modulo Theorem C.

3. Proof of Theorem B. We divide the proof into two cases, as follows.

CASE 1: $k < l$. We determine N_0 and the rational homotopy structure of both $W = B\langle N\rangle$ and W_p, $N \geq N_0$, in preparation for applying (2.1).

The inclusions $S^k \to B$ and $S^l \to B$ give rise to elements $u \in \pi_k(B)$ and $v \in \pi_l(B)$. Consider the Whitehead products

$$w_{r,s} = v \dots v.u \dots u.v,$$

with r occurrences of v ($r \geq 0$), followed by s occurrences of u ($s > 0$), followed by a single occurrence of v. Here we use abbreviated, bracket-free notation for Whitehead products, so that, for example,

$$v.u.u.v = [v, [u, [u, v]]].$$

Note that the $w_{r,s}$ are basic products in the sense of [9] provided we require that $u < v$, which we do. Now let P be any positive integer for which there are at least two distinct products $w_{r,s}$, $w_{\varrho,\sigma}$ having degree P; it is readily checked that such P exist. Taking P_0 to be the least such integer, for definiteness, set

$$N_0 = P_0 - (l - 1),$$

and more generally,

$$N_t = N_0 + t(l - 1), \qquad t \geq 0.$$

It is clear that the products $w_{r+t,s}$, $w_{\varrho+t,\sigma}$, both of degree N_{t+1}, are distinct for any $t \geq 0$.

By the Félix–Halperin mapping theorem (see, e.g., [6]), the rational category of W, $\mathrm{cat}_0(W)$, equals 1, i.e., W is a rational co-H-space. Thus there is a rational equivalence h_N from a one-point union of spheres $\bigvee S^i$ to W. Clearly, the induced map, $(h_N)_p$, from $(\bigvee S^i)_p$ to W_p is also a rational equivalence. For any $N \geq N_0$, there is a unique $t > 0$ such that $N_{t-1} \leq N < N_t$. Since N_t is certainly less than or equal to $2N$, the Hurewicz homomorphism $\pi_{N_t}(W) \to H_{N_t}(W)$ is a rational isomorphism. Thus, at least two of the spheres in the one-point union $\bigvee S^i$ are of dimension N_t and $h_N|S^{N_t} \vee S^{N_t}$ may be chosen to represent the elements of $\pi_{N_t}(W)$ mapping to the Whitehead products $w_{r+t,s}$, $w_{\varrho+t,\sigma}$ via the isomorphism induced by the N-connective covering map $W \to B$. We denote the latter elements by $z_{r+t,s}$, $z_{\varrho+t,\sigma}$ and their images in $\pi'_{N_t}(W) = \pi_{N_t}(W)/\mathrm{torsion}$ by $z'_{r+t,s}$, $z'_{\varrho+t,\sigma}$. Write $\bigvee S^i = S_1 \vee S_2$, where S_1 is the summand consisting of all the spheres of dimension N_t, and S_2 is the complementary summand. By a result of Bousfield and Kan ([3; Proposition VI.6.6]), the canonical map

$$(S_1)_p \vee (S_2)_p \to (S_1 \vee S_2)_p,$$

while not itself a homotopy equivalence, induces a map

$$[(S_1)_p \vee (S_2)_p]_p \to (S_1 \vee S_2)_p,$$

which is a homotopy equivalence. It will be convenient to regard the domains of h_N and $(h_N)_p$ as $S_1 \vee S_2$ and $[(S_1)_p \vee (S_2)_p]_p$, respectively.

Consider next the canonical homomorphisms

$$\begin{gathered} \mathrm{CAut}((W_{(0)})^-) \to \mathrm{CAut}(\pi_{N_t}((W_{(0)})^-)), \\ \mathrm{Aut}(W_{(0)}) \to \mathrm{Aut}(\pi_{N_t}(W_{(0)})), \\ \mathrm{Aut}(\widehat{W}) \to \mathrm{Aut}(\pi'_{N_t}(\widehat{W})). \end{gathered}$$

Denote the respective images of these homomorphisms by I, I' and I''. Also, denote the p-components of I, resp. I'', by $I(p)$, resp. $I''(p)$ (see (2.2)). The double coset space in (2.1) maps surjectively to the double coset space

$$(f.c)_* I' \backslash I / r_* I''. \tag{3.1}$$

From the rational structure of W and W_p described above, we conclude that

$$\begin{aligned} I &= \mathrm{CAut}(\pi_{N_t}((W_{(0)})^-)) \cong \mathrm{Gl}(n, \widehat{\mathbb{Q}}), \\ I' &= \mathrm{Aut}(\pi_{N_t}(W_{(0)})) \cong \mathrm{Gl}(n, \mathbb{Q}), \end{aligned} \tag{3.2}$$

where $n \geq 2$ denotes the torsion-free rank of $\pi_{N_t}(W)$. The isomorphisms in (3) may be chosen to be compatible with each other, depending on the selection of an ordered basis for the free abelian group $\pi'_{N_t}(W)$.

We compute I'' by using Neisendorfer's theorem in conjunction with (2.2), as in the proof of Theorem A. Any element $\alpha \in \mathrm{Aut}(W_p)$ is of the form $\beta\langle N \rangle$ for a (unique) element β in $\mathrm{Aut}(B_p)$. The induced homomorphism $\beta_\sharp$ on homotopy groups is determined by

$$\beta_\sharp(u) = a.u, \qquad \beta_\sharp(v) = d.v, \tag{3.3}$$

where u, v are now viewed as generators of $\pi_k(B_p) = \pi_k(B)_p$, $\pi_l(B_p) = \pi_l(B)_p$ qua $\mathbb{Z}_p$-modules, and $a, d \in \mathbb{Z}_p$. Thus,

$$\beta_\sharp(w_{r+t,s}) = a^s.b^{r+t+l}.w_{r+t,s}, \qquad \beta_\sharp(w_{\varrho+t,\sigma}) = a^\sigma.b^{\varrho+t+1}.w_{\varrho+t,\sigma}, \tag{3.4}$$

where $w_{r+t,s}$, $w_{\varrho+t,\sigma}$ are now viewed as elements of $\pi_{N_t}(B_p)$. Next, $z'_{r+t,s}$, $z'_{\varrho+t,\sigma}$ may be taken as the first two elements of an ordered basis $\mathcal{B}$ for $\pi'_{N_t}(W)$; viewing $\mathcal{B}$ as an ordered basis for the free $\mathbb{Z}_p$-module $\pi'_{N_t}(W_p)$, the automorphism $\alpha'_\sharp$ of $\pi'_{N_t}(W_p)$ induced by the automorphism $\alpha_\sharp$ of $\pi_{N_t}(W_p)$ is represented by a matrix $M = (m_{ij})$ with respect to $\mathcal{B}$, and it follows from (3.4) that

$$m_{21} = 0. \tag{3.5}$$

For x in $\mathbb{Z}_p$, consider the matrix

$$M_x = \begin{pmatrix} 1 & & & & & \\ x & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix},$$

all the non-displayed entries being 0. Since $M_x.M_y^{-1} = M_{x-y}$, it follows from (3.5) that for $x \neq y$, the matrices M_x, M_y determine distinct elements in the coset space $I(p)/r_*I''(p)$. Therefore, this coset space, and hence also the coset space in (3.1), is uncountably infinite. This completes the proof of Case 1 of Theorem B.

Case 2: $k = l$. We continue with the notation used in Case 1 and consider the three distinct Whitehead products $w_{0,3}$, $w_{1,2}$, $w_{2,1}$, each of degree $4k-3$. As in Case 1, we set

$$N_0 = 3k - 2,$$

more generally,

$$N_t = N_0 + t(k-1), \qquad t \geq 0,$$

and for $N_{t-1} \leq N < N_t$, $t > 0$, find a rational equivalence h_N from a one-point union of spheres $\bigvee S^i$ to W. In Case 2, at least three of the spheres in the one-point union $\bigvee S^i$ are of dimension N_t and $h_N|S^{N_t} \vee S^{N_t} \vee S^{N_t}$ may be chosen to represent the elements $z_{0,2+t}$, $z_{1,1+t}$, $z_{1+t,1}$ in $\pi_{N_t}(W)$ mapping to $w_{0,2+t}$, $w_{1,1+t}$, $w_{1+t,1}$. The isomorphisms of (3) in Case 1 hold also for Case 2 except that now $n \geq 3$, but the analog of (3.3) for Case 2 becomes

$$\beta_\sharp(u) = a.u + b.v, \qquad \beta_\sharp(v) = c.u + d.v, \tag{3.6}$$

where $a, b, c, d \in \mathbb{Z}_p$. The elements $z'_{0,2+t}, z'_{1,1+t}, z'_{1+t,1} \in \pi'_{N_t}(W)$ may be taken as the first, second and last elements of an ordered basis $\mathcal{B}$ for $\pi'_{N_t}(W)$. Viewing $\mathcal{B}$ as an ordered basis of the free $\mathbb{Z}_p$-module $\pi'_{N_t}(W_p)$, we study the matrix $M = (m_{ij})$ representing the automorphism $\alpha'_\sharp$ of $\pi'_{N_t}(W_p)$ with respect to $\mathcal{B}$. From (3.6), computation shows that

$$m_{11} = \Delta.a^{2+t}, \qquad m_{n1} = \Delta.c^{2+t}, \qquad m_{21} = \Delta.a^{1+t}.c, \tag{3.7}$$

where $\Delta = a.d + (-1)^k b.c$. Note that for k odd, $\Delta = \delta$, the determinant of the automorphism $\beta_\sharp$ of $\pi_k(B_p)$, hence is in $\mathbb{Z}_p^*$. We claim that, as in Case 1, the map sending x in $\mathbb{Z}_p$ to the coset of M_x in $I(p)/r_*I''(p)$ is injective.

Indeed, if $M_x.M_y^{-1} \in r_* I''(p)$, then (3.7) implies

$$1 = \Delta.a^{2+t},$$

$$0 = \Delta.c^{2+t}, \quad \text{hence} \quad c = 0, \tag{3.8}$$

$$x - y = a^{1+t}.c, \quad \text{hence} \quad x = y. \tag{3.9}$$

This completes the proof of Case 2 of Theorem B.

We conclude this section with some remarks regarding the proof of Case 2 of Theorem B. If we fix ordered bases for $\mathrm{Aut}(B_p) = \mathrm{Aut}(\pi_k(B_p))$ and $\mathrm{Aut}(\pi'_{N_t}(W_p)) = \mathrm{Aut}(\pi'_{N_t}(B_p))$, the homomorphism

$$\mathrm{Aut}(\pi_k(B_p)) \to \mathrm{Aut}(\pi'_{N_t}(W_p)),$$

implicit in the proof of Case 2 of Theorem B, is represented by a homomorphism

$$f : \mathrm{Gl}(2, \mathbb{Z}_p) \to \mathrm{Gl}(n, \mathbb{Z}_p), \qquad n > 2.$$

Our explicit computations lead to the conclusion that the coset space $\mathrm{Gl}(n, \mathbb{Z}_p)/f(\mathrm{Gl}(2, \mathbb{Z}_p))$ is uncountably infinite. Also, in the proof of Theorem D below, a similar homomorphism

$$f : \mathrm{Gl}(2, \mathbb{Z}) \to \mathrm{Gl}(n, \mathbb{Z}), \qquad n > 2,$$

appears implicitly, with the property that the coset space $\mathrm{Gl}(n,\mathbb{Z})/f(\mathrm{Gl}(2,\mathbb{Z}))$ is (countably) infinite. The question raised near the end of the introduction asks whether the conclusions about the size of the coset spaces are valid for general homomorphisms f.

Here are three examples of the foregoing. In contrast with the first example, the latter two are not strict illustrations of the recipe used in the proof of Case 2 of Theorem B, but are rather variations of that recipe. In all three examples, we fix the ordered basis $\{u, v\}$ for $\pi_k(B_p)$.

Example 1. Let $k = 3$, so that $N_0 = 7$, and set $N = 7$, so that $N_1 = 9$. Fixing the ordered basis $\{z'_{0,3}, z'_{1,2}, z'_{2,1}\}$ for $\pi'_9(W_p)$, we compute

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = M = \delta. \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix},$$

whose determinant is δ^6.

Example 2. Let $k = 3$ again, but now set $N = 3$. Fixing the ordered basis $\{z'_{0,2}, z'_{1,1}\}$ for $\pi'_7(W_p)$, we consider the homomorphism

$$\mathrm{Aut}(\pi_3(B_p)) \to \mathrm{Aut}(\pi'_7(W_p))$$

and compute

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \delta.\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

whose determinant is δ^2. In this example, the coset space $\mathrm{Gl}(2,\mathbb{Z}_p)/f(\mathrm{Gl}(2,\mathbb{Z}_p))$ is non-trivial since the determinant homomorphism induces a surjection

$$\mathrm{Gl}(2,\mathbb{Z}_p)/f(\mathrm{Gl}(2,\mathbb{Z}_p)) \to \mathbb{Z}_p^*/(\mathbb{Z}_p^*)^2.$$

The method of proof of Theorem A then shows that $\widehat{\mathcal{G}}_0(W)$ is uncountably infinite.

Example 3. Let $k = 4$, so that $N_0 = 10$, but set $N = 4$. Denote by $(uu)'$, $(uv)'$, $(vv)'$ the canonical images in $\pi_7'(B_p)$ of the Whitehead products uu, uv, vv. (Of course, uu and vv are not basic products.) Note that $\{(uu)', (uv)', (vv)'\}$ is an ordered basis for $\pi_7'(B_p)$, at least for p odd. Fixing this basis, we consider the homomorphism

$$\mathrm{Aut}(\pi_4(B_p)) \to \mathrm{Aut}(\pi_7'(B_p))$$

and compute

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix},$$

whose determinant is δ^3.

The fact that in all three examples, the determinant of $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ is of the form δ^e, $e > 1$, is no accident. It can be shown that in the context of the proof of Case 2 of Theorem B, the determinant of $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ is always of this form.

4. Proof of Theorem C. First suppose $m < 2n$. Then, reverting to the notation of §2, $W = T\langle m\rangle = T$, a finite CW-complex, and $W_{(p)} = T\langle m\rangle_{(p)} = T_{(p)}$, a finite-dimensional CW-complex. It is thus clear that the conclusion of Theorem C holds in this case.

Next suppose $m > 2nk - 2$. Then all the homotopy groups of W and $W_{(p)}$ are finite. The conclusion of Theorem C follows in this case from [29; Corollary II].

For the remainder of this section, we focus on the interesting range $2n \le m \le 2nk - 2$, and utilize the following criterion of McGibbon and Møller ([14; Theorem 3]): If Z is the P-localization of a 1-connected, finite type, rational H-space, where P is a collection of primes, then $\mathrm{SNT}(Z)$ is trivial if and only if the image of the canonical (anti)homomorphism from $\mathrm{Aut}(Z)$ to $\mathrm{Aut}(H^{\le\mu}(Z;\mathbb{Z}_{(P)}))$ is of finite index for all μ. By the latter automorphism group, we mean the group of graded ring automorphisms of the graded ring obtained from $H^*(Z;\mathbb{Z}_{(P)})$ by replacing the cohomology groups in degrees $> \mu$ by 0. This criterion applies to the situation in Theorem C since W and $W_{(p)}$ are 1-connected, finite type, rational H-spaces.

Since $H^i(W;\mathbb{Z})$ is finite if $i \neq 2nk-1$, and $H^{2nk-1}(W;\mathbb{Z})$ is finitely generated abelian with torsion-free rank 1, $\mathrm{Aut}(H^{\le\mu}(W;\mathbb{Z}))$ is itself finite for all μ and the criterion for triviality of $\mathrm{SNT}(W)$ is satisfied, a fortiori. It then remains to show that the image of the canonical (anti)homomorphism from $\mathrm{Aut}(W_{(p)})$ to $\mathrm{Aut}(H^{2nk-1}(W_{(p)};\mathbb{Z}_{(p)})/\mathrm{torsion}) = \mathbb{Z}^*_{(p)}$ is of finite index. Equivalently, it suffices to show that the image of the canonical homomorphism from $\mathrm{Aut}(W_{(p)})$ to $\mathrm{Aut}(\pi'_{2nk-1}(W_{(p)})) \cong \mathrm{Aut}(H_{2nk-1}(W_{(p)};\mathbb{Z})/$ torsion) is of finite index, since these last two automorphism groups are (anti)isomorphic to $\mathrm{Aut}(H^{2nk-1}(W_{(p)};\mathbb{Z}_{(p)})/\mathrm{torsion})$. Our strategy will be to show that the image of the canonical map from $\mathrm{Aut}(W_p)$ to $\mathrm{Aut}(\pi'_{2nk-1}(W_p))$ is sufficiently large in an appropriate sense, and then to pass from W_p to $W_{(p)}$ using a local arithmetic square argument. The details follow.

LEMMA 4.1. *There exists a positive integer* e, *depending only on* T, *with the following property*: *If* d *is any integer*, *there exists an* α_d *in* $[T,T]$ *such that*

$$(4.1)\qquad (\alpha_d)_\sharp : \pi_{2n}(T) \to \pi_{2n}(T) \text{ is multiplication by } d^e,$$

and consequently,

$$(4.2)\qquad (\alpha_d)_\sharp : \pi'_{2nk-1}(T) \to \pi'_{2nk-1}(T) \text{ is multiplication by } d^E, \text{ where } E = ke.$$

Proof. The result is clear, with $e = 1$, in case (i) and therefore also in case (ii) (even if $k = \infty$). For case (iv), a theorem of Sullivan ([27; pp. 58–59, Remark IV]) asserts that (4.1) holds for d odd, with $e = 2$ (even if $k = \infty$), and a theorem of McGibbon ([13; Proposition 2.4]) asserts that (4.1) holds for d even. (A classical homotopy theory calculation shows that, already for $\mathbb{H}P^2$, an α_d satisfying (4.1) exists precisely when

$$d^e(d^e-1) \equiv 0 \bmod 24.$$

Hence, for $d = 2$, we see that $e \ge 4$.) Finally, for case (v), an argument similar to the one referred to in the previous sentences shows an α_d satisfying (4.1) exists precisely when

$$d^e(d^e-1) \equiv 0 \bmod 240.$$

But this congruence holds for $d = 2, 3$ or 5, with $e = 4$, and for d relatively prime to 240, with $e = 64$, by the Euler–Fermat theorem. ∎

LEMMA 4.2. *For any* x *in* $\mathbb{Z}_p$, *there exists a* β_x *in* $[T_p, T_p]$ *such that*

$$(4.3)\qquad (\beta_x)_\sharp : \pi_{2n}(T_p) \to \pi_{2n}(T_p) \text{ is multiplication by } x^e,$$

and consequently

$$(4.4)\qquad (\beta_x)_\sharp : \pi'_{2nk-1}(T_p) \to \pi'_{2nk-1}(T_p) \text{ is multiplication by } x^E.$$

Moreover, *if* x *is in* $\mathbb{Z}_p^*$, *then any such* β_x *actually lies in* $\mathrm{Aut}(T_p)$.

Proof. Let (d_i) be a sequence of integers converging to x in the p-adic topology. For each d_i, let α_{d_i} be as in Lemma 4.1 and let

$$\beta_{d_i} = (\alpha_{d_i})_p,$$

the p-completion of α_{d_i}. The homotopy set $[T_p, T_p]$ has a natural compact, Hausdorff topology ([27]) and so the sequence (β_{d_i}) admits a convergent subsequence. If β_x denotes the limit of such a subsequence, then (4.3) is readily verified for this choice of β_x.

Suppose now that x is in $\mathbb{Z}_p^*$. We will show that β_x induces automorphisms on all homotopy groups, hence lies in $\mathrm{Aut}(T_p)$. First, from (4.3), β_x induces an automorphism on $H_{2n}(T_p)$, hence, a fortiori, on $H_{2n}(T_p; \mathbb{Z}/p^r)$ (and $H^{2n}(T_p; \mathbb{Z}/p^r)$), $r > 0$. From the homological structure of T_p with coefficients in $\mathbb{Z}_{p^r}$, we see that β_x induces an automorphism on $H_j(T_p; \mathbb{Z}/p^r)$ for all $j \geq 0$ and all $r > 0$. It then follows from [21; Corollary 3.10] that β_x induces an automorphism on $\pi_j(T_p; \mathbb{Z}/p^r)$, the homotopy groups with coefficients in $\mathbb{Z}/p^r$, for all $j \geq 0$ and all $r > 0$. Finally, using the functorial short exact sequence (universal coefficient theorem; see, e.g., [21; Proposition 1.4])

$$0 \to \pi_j(T_p) \otimes \mathbb{Z}/p^r \to \pi_j(T_p; \mathbb{Z}/p^r) \to \mathrm{Tor}(\pi_{j-1}(T_p), \mathbb{Z}/p^r) \to 0,$$

in conjunction with (4.3), (4.4) and the fact that $\pi_j'(T_p) = 0$ for all $j \neq 2n$, $2nk-1$, we conclude that β_x induces automorphisms on $\pi_j(T_p)$ for all $j \geq 0$, as desired. ∎

Next, let $\eta \in \mathbb{Z}_{(p)}^*$ be such that

$$C(\eta) = x^E \quad \text{for some } x \in \mathbb{Z}_p^*, \tag{4.5}$$

where $C : \mathbb{Z}_{(p)} \subset \mathbb{Z}_p$ is the p-completion homomorphism, and let $\beta = \beta_x \in \mathrm{Aut}(T_p)$ be as in Lemma 4.2. Clearly, $\beta\langle m\rangle \in \mathrm{Aut}(W_p)$ and, by (4.4),

$$(\beta\langle m\rangle)_\sharp : \pi_{2nk-1}'(W_p) \to \pi_{2nk-1}'(W_p) \text{ is multiplication by } x^E.$$

We also have $\gamma \in \mathrm{Aut}(W_{(0)})$ (uniquely) defined by the condition that

$$\gamma_\sharp : \pi_{2nk-1}(W_{(0)}) \to \pi_{2nk-1}(W_{(0)}) \text{ is multiplication by } R(\eta),$$

where $R : \mathbb{Z}_{(p)}^* \subset \mathbb{Q}^*$ is the rationalization homomorphism. From the homotopy-pullback diagram

$$\begin{array}{ccc} W_{(p)} & \longrightarrow & W_p \\ \downarrow & & \downarrow \\ W_{(0)} & \longrightarrow & (W_p)_{(0)} \end{array}$$

(local arithmetic square), we readily infer the existence of an ϵ in $[W_{(p)}, W_{(p)}]$ whose images in $\mathrm{Aut}(W_p)$ and $\mathrm{Aut}(W_{(0)})$ are, respectively, $\beta\langle m\rangle$ and γ. By a homotopical Mayer–Vietoris argument, we see that ϵ is in $\mathrm{Aut}(W_{(p)})$ and that

$$\epsilon_\sharp : \pi'_{2nk-1}(W_{(p)}) \to \pi'_{2nk-1}(W_{(p)}) \text{ is multiplication by } \eta.$$

We have thus shown that the canonical homomorphism from $\mathrm{Aut}(W_{(p)})$ to $\mathrm{Aut}(\pi'_{2nk-1}(W_{(p)}))$ contains (in fact equals) $C^{-1}((\mathbb{Z}_p^*)^E)$. But since $(\mathbb{Z}_p^*)^E$ has finite index in $\mathbb{Z}_p^*$ by (2.6), $C^{-1}((\mathbb{Z}_p^*)^E)$ has finite index in $\mathbb{Z}_{(p)}^*$. This completes the proof of Theorem C.

Remark. While $(\mathbb{Z}_p^*)^E$ has finite index in $\mathbb{Z}_p^*$, it is not true that $(\mathbb{Z}_{(p)}^*)^E$ has finite index in $\mathbb{Z}_{(p)}^*$ if $E > 1$. In fact, $\mathbb{Z}_{(p)}^*/(\mathbb{Z}_{(p)}^*)^E$ is isomorphic to a direct sum of countably many copies of $\mathbb{Z}/E$; generators are provided by the cosets determined by the primes $q \neq p$. As a consequence, there is no analog of Neisendorfer's theorem for p-localization; that is, not every element of $[W_{(p)}, W_{(p)}]$ "comes from" an element of $[T_{(p)}, T_{(p)}]$.

5. Proof of Theorem D. We now revert to the notation of §3, writing W for the N-connective covering $B\langle N\rangle = (S^k \vee S^l)\langle N\rangle$. In order to prove Theorem D, we will utilize a criterion of McGibbon and Møller ([15; Theorem 1]) dual to the criterion used in §4, namely: If Z is a 1-connected, finite type, rational co-H-space, then $\mathrm{SNT}(Z)$ is trivial if and only if the image of the canonical homomorphism from $\mathrm{Aut}(Z)$ to $\mathrm{Aut}(\pi_{\le\mu}(Z))$ is of finite index for all μ. By the latter automorphism group, we mean the group of automorphisms of the graded Lie ring obtained from $\pi_*(Z)$ by replacing the homotopy groups in degrees $> \mu$ by 0. This criterion applies to the situation in Theorem D since W is, as noted in §3, a 1-connected, finite type, rational co-H-space.

We will prove Theorem D by showing that the image of the canonical homomorphism from $\mathrm{Aut}(W)$ to $\mathrm{Aut}(\pi_{\le N_t}(W))$ is of infinite index. Since N_t is certainly less than or equal to $2N-2$, the Lie ring structure on $\pi_{\le N_t}(W)$ is trivial and it therefore suffices to show that the image of the canonical homomorphism from $\mathrm{Aut}(W)$ to $\mathrm{Aut}(\pi_{N_t}(W))$, or from $\mathrm{Aut}(W)$ to $\mathrm{Aut}(\pi'_{N_t}(W))$, is of infinite index. To that end, pick a prime p arbitrarily and consider the commutative square

$$\begin{array}{ccc} \mathrm{Aut}(W) & \longrightarrow & \mathrm{Aut}(\pi'_{N_t}(W)) \\ \downarrow & & \downarrow \\ \mathrm{Aut}(W_p) & \longrightarrow & \mathrm{Aut}(\pi'_{N_t}(W_p)) \end{array} \tag{5.1}$$

with vertical arrows induced by p-completion $W \to W_p$. With respect to the ordered bases $\mathcal{B}$ for $\pi'_{N_t}(W)$ (or for $\pi'_{N_t}(W_p)$) described in §3, $\mathrm{Aut}(\pi'_{N_t}(W))$ and $\mathrm{Aut}(\pi'_{N_t}(W_p))$ may be identified with $\mathrm{Gl}(n,\mathbb{Z})$ and $\mathrm{Gl}(n,\mathbb{Z}_p)$, respectively, and the right vertical arrow may be identified with the homomorphism from $\mathrm{Gl}(n,\mathbb{Z})$ to $\mathrm{Gl}(n,\mathbb{Z}_p)$ induced by p-completion $\mathbb{Z} \subset \mathbb{Z}_p$. It follows that an element in the image of the top horizontal arrow in (5.1) has matrix representative M satisfying the properties derived in §3 ((3.5), (3.7)) except that the entries of M are in $\mathbb{Z}$. The computations of §3 (see especially (3.8) in Case 2: $k = l$) then show that the matrices M_x of §3, $x \in \mathbb{Z}$, determine mutually distinct cosets modulo the image of $\mathrm{Aut}(W)$ in $\mathrm{Gl}(n,\mathbb{Z})$. This completes the proof of Theorem D.

Appendix 1. We describe here a family of spaces with each X in the family having $\mathcal{G}(X)$ at most countably infinite and $\widehat{\mathcal{G}}_0(X)$ uncountably infinite. We begin with the spaces $\mathrm{BSU}(3)$, the classifying space of the special unitary group $\mathrm{SU}(3)$, and $K = K(\mathbb{Z},4) \times K(\mathbb{Z},6)$. Observe that these spaces are rationally equivalent and write $\mathcal{R}$ for a common rationalization. Let P be a finite set of primes, each > 3, let Q be the set of all primes not in P, and choose rationalization maps

$$\mathrm{BSU}(3)_{(P)} \to \mathcal{R}, \qquad K_{(Q)} \to \mathcal{R},$$

which we may assume to be fibrations. We then define X to be the pullback of these two maps. Thus X is a "Zabrodsky mix" of $\mathrm{BSU}(3)$ and K; that is,

$$X_{(P)} \simeq \mathrm{BSU}(3)_{(P)}, \qquad X_{(Q)} \simeq K_{(Q)}. \tag{A1.1}$$

According to [20; Theorem 2.1], if P contains at least 2 primes, there are, up to homotopy, exactly countably (infinitely) many nilpotent, P-local spaces of finite type over $\mathbb{Z}_{(P)}$, $\{U_1, U_2, \ldots\}$, satisfying

$$(U_i)_{(p)} \simeq \mathrm{BSU}(3)_{(p)}, \qquad i \geq 1,\ j \in P. \tag{A1.2}$$

Of course, if P is the singleton set $\{p\}$, there is only one U_i as in (A1.2), namely $\mathrm{BSU}(3)_{(p)}$ itself. Furthermore, up to homotopy, the only nilpotent Q-local space of finite type over $\mathbb{Z}_{(Q)}$ whose p-localization is homotopy equivalent to $K_{(p)}$, for all $p \in Q$, is $K_{(Q)}$ itself.

Suppose now that Y (more accurately, the homotopy type of Y) is in $\mathcal{G}(X)$. Then

$$Y_{(P)} \simeq U_{i_0} \quad \text{for some } i_0, \tag{A1.3}$$

by (A1.1) and (A1.2). Similarly,

$$Y_{(Q)} \simeq K_{(Q)}. \tag{A1.4}$$

By [10; II, Theorem 5.9], Y is homotopy equivalent to the pullback of the rationalization maps

$$R_{(P)} : Y_{(P)} \to Y_{(0)}, \qquad R_{(Q)} : Y_{(Q)} \to Y_{(0)}$$

induced by the rationalization map $R : Y \to Y_{(0)}$ (assuming $R_{(P)}$ and $R_{(Q)}$ to be fibrations). It follows from this, together with (A1.3) and (A1.4), that Y

is homotopy equivalent to the pullback of appropriate rationalization maps

$$r_{(P)} : U_{i_0} \to Y_{(0)}, \quad r_{(Q)} : K_{(Q)} \to Y_{(0)}$$

(once again assuming $r_{(P)}$ and $r_{(Q)}$ are fibrations). In particular, the homotopy type of Y is completely determined by the homotopy classes of $r_{(P)}$ and $r_{(Q)}$. But the number of choices for the homotopy classes of $r_{(P)}$ and $r_{(Q)}$ is countably infinite since the full homotopy sets

$$[U_{i_0}, Y_{(0)}] \cong H^4(U_{i_0};\mathbb{Q}) \times H^6(U_{i_0};\mathbb{Q}),$$
$$[K_Q, Y_{(0)}] \cong H^4(K_{(Q)};\mathbb{Q}) \times H^6(K_{(Q)};\mathbb{Q})$$

are clearly countably infinite. Thus $\mathcal{G}(X)$ is at most countably infinite, and exactly countably infinite when P contains at least two primes.

To see that $\widehat{\mathcal{G}}_0(X)$ is uncountably infinite, we use the double coset formula (2.1), which in the present situation reduces to

$$\text{(A1.5)} \qquad \mathbb{Q}^* \times \mathbb{Q}^* \backslash (\widehat{\mathbb{Q}})^* \times (\widehat{\mathbb{Q}})^* / r_* \operatorname{Aut}(\widehat{X}).$$

By examining the computation in [14; Ex. H], we find that the p-component of $r_* \operatorname{Aut}(\widehat{X})$ is

$$\text{(A1.6)} \qquad J(p) = \{(x^2, x^3) \mid x \in \mathbb{Z}_p^*\}, \quad \text{provided } p \in P.$$

(For $p \in Q$, the p-component of $r_* \operatorname{Aut}(\widehat{X})$ is all of $\mathbb{Z}_p^* \times \mathbb{Z}_p^*$.)

We next observe that the quotient $\mathbb{Z}_p^* \times \mathbb{Z}_p^* / J(p)$ is uncountably infinite if $p \in P$. In fact, by (2.6), this quotient contains a summand isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p / \{(2x, 3x) \mid x \in \mathbb{Z}_p\}$, which is itself isomorphic to $\mathbb{Z}_p$. It follows that the double coset in (A1.5), and hence $\widehat{\mathcal{G}}_0(X)$, is uncountably infinite.

Appendix 2. This appendix consists largely of speculative remarks, which we hope to develop on a future occasion. However, we do include the following partial answer to the Question raised in §1, whose formulation and proof owe much to suggestions of Pierre de la Harpe.

THEOREM E. *Let* $f : \mathrm{Gl}(\nu, \mathbb{Z}) \to \mathrm{Gl}(n, \mathbb{Z})$ *be a homomorphism*, $\nu < n$. *If* $\ker(f)$ *is finite*, *then the coset space* $\mathrm{Gl}(n, \mathbb{Z})/f(\mathrm{Gl}(\nu, \mathbb{Z}))$ *is infinite.*

We point out that the condition that $\ker(f)$ be finite is satisfied in the situation considered in §5, as can be verified by making matrix computations similar to those carried out in §3.

Proof of Theorem E. Our argument relies on the formula

$$\text{(A2.1)} \qquad \operatorname{vcd}(\mathrm{Sl}(k, \mathbb{Z})) = \frac{k(k-1)}{2},$$

where $\operatorname{vcd}(G)$ stands for the "virtual cohomological dimension" of the group G, that is, the cohomological dimension, $\operatorname{cd}(H)$, of any torsion-free subgroup H of finite index in G (provided such subgroups exist); see [25] or [4] for a discussion of vcd, and [4] for a proof of (A2.1). Since $\mathrm{Sl}(k,\mathbb{Z})$ is of index 2 in $\mathrm{Gl}(k,\mathbb{Z})$, (A2.1) is also valid for $\mathrm{Gl}(k,\mathbb{Z})$.

Let then H be a torsion-free subgroup of finite index in $\mathrm{Gl}(\nu,\mathbb{Z})$. By (A2.1), we have

$$\mathrm{cd}(H) = \frac{\nu(\nu-1)}{2}. \tag{A2.2}$$

Assume, for a contradiction, that $\mathrm{Gl}(n,\mathbb{Z})/f(\mathrm{Gl}(\nu,\mathbb{Z}))$ is finite, and consider the homomorphism ϕ from H to $f(H)$ induced by f. Since $\ker(f)$ is finite and H is torsion-free, ϕ is an isomorphism, so that

$$\mathrm{cd}(f(H)) = \frac{\nu(\nu-1)}{2}.$$

But $f(H)$ is of finite index in $f(\mathrm{Gl}(\nu,\mathbb{Z}))$, which is, by assumption, of finite index in $\mathrm{Gl}(n,\mathbb{Z})$. Hence, again by (A2.1),

$$\mathrm{cd}(f(H)) = \frac{n(n-1)}{2},$$

and we have arrived at our contradiction.

In the case $\nu = 2$, there are alternative approaches to proving Theorem E based on [1], [7] or [18] rather than (A2.1).

If the assumption that $\ker(f)$ be finite is dropped, then the technique of proof of Theorem E fails. As de la Harpe points out, there is a substantial difference between the case $\nu > 2$, where the conclusion of Theorem E is probably true, and the case $\nu = 2$, where the conclusion of Theorem E is probably false. We will discuss only the case $\nu = 2$ here. In that case, there is an example of Conder (implicit in [5]) of a homomorphism from $\mathrm{Sl}(2,\mathbb{Z})$ (actually from $\mathrm{PSl}(2,\mathbb{Z})$) to $\mathrm{Sl}(3,\mathbb{Z})$ such that $\mathrm{Sl}(3,\mathbb{Z})/f(\mathrm{Sl}(2,\mathbb{Z}))$ is finite, and examples of Tamburini, Wilson and Gavioli ([28]) of epimorphisms from $\mathrm{Sl}(2,\mathbb{Z})$ to $\mathrm{Sl}(n,\mathbb{Z})$, $n \geq 28$; see also [8; III.39] for further discussion and references. It therefore seems plausible that there should exist homomorphisms from $\mathrm{Gl}(2,\mathbb{Z})$ to $\mathrm{Gl}(n,\mathbb{Z})$, for various $n > 2$, with $\mathrm{Gl}(n,\mathbb{Z})/f(\mathrm{Gl}(2,\mathbb{Z}))$ finite, although Marston Conder has pointed out to us that the particular homomorphism he constructs in [5] does not extend to a homomorphism from $\mathrm{Gl}(2,\mathbb{Z})$ to $\mathrm{Gl}(n,\mathbb{Z})$.

The examples in [5] and [28] are presumably more "exotic" than the homomorphisms which arise in §3 and §5. These latter homomorphisms are of a rather specialized sort; for any such f, and any $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{Gl}(2,A)$, $A = \mathbb{Z}$ or $\mathbb{Z}_p$, computation shows that the entries of $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ are polynomial functions, with coefficients in $\mathbb{Z}$, of a, b, c and d. For such f, one might expect to be able to approach the Question using the theory of algebraic groups. However, we have been warned by experts (Moskowitz, Hoobler) to tread carefully since: (1) the coefficient rings $\mathbb{Z}$, $\mathbb{Z}_p$ are not fields; (2) the traditional coset spaces are not the natural ones arising in the theory of algebraic groups.

Finally, we make a couple of remarks about the Question when the coefficient ring is $\mathbb{Z}_p$. First, there is a version of (A2.1) in that case. Indeed, according to Henn, classical work of Lazard ([12]) may be used to prove the formula

$$\text{(A2.3)} \qquad \operatorname{vcd}(\mathrm{Sl}(k, \mathbb{Z}_p)) = k(k-1),$$

provided we work with "continuous cohomology". However, since $\mathrm{Sl}(k, \mathbb{Z}_p)$ does not have finite index in $\mathrm{Gl}(k, \mathbb{Z}_p)$, it is not clear how to use (A2.3) to derive a version of Theorem E in the case of $\mathbb{Z}$ replaced by $\mathbb{Z}_p$. Secondly, we wonder whether there is a "differentiable" approach to the Question. To explain, first note that for any differentiable (i.e., C^∞) homomorphism $f : \mathrm{Gl}(\nu, \mathbb{R}) \to \mathrm{Gl}(n, \mathbb{R})$, $\nu < n$, $\mathbb{R}$ the reals, $f(\mathrm{Gl}(\nu, \mathbb{R}))$ has Lebesgue measure zero in $\mathrm{Gl}(n, \mathbb{R})$, by Sard's lemma (see, e.g., [19]), from which it easily follows that $\mathrm{Gl}(n, \mathbb{R})/f(\mathrm{Gl}(\nu, \mathbb{R}))$ is uncountably infinite. Now one can also do analysis over $\mathbb{Q}_p$, and $\mathbb{Z}_p$ is an open, dense subspace of $\mathbb{Q}_p$. We ask: Is there a p-adic version of Sard's lemma? If so, can we thereby deduce an affirmative answer to the Question in case $f : \mathrm{Gl}(\nu, \mathbb{Z}_p) \to \mathrm{Gl}(n, \mathbb{Z}_p)$ is a differentiable homomorphism?

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