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α-Properness and Axiom A

by

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Abstract. We show that under ZFC, for every indecomposable ordinal $\alpha < \omega_1$, there exists a poset which is β-proper for every $\beta < \alpha$ but not α-proper. It is also shown that a poset is forcing equivalent to a poset satisfying Axiom A if and only if it is α-proper for every $\alpha < \omega_1$.

0. Introduction. The notion of proper forcing was introduced by Shelah in [12]. He proved several very important results about this notion. For example, he showed that every proper poset preserves $\aleph_1$, and properness is preserved by countable support iteration. These are important theorems and indeed plenty of forcing arguments rely on them.

Shelah also introduced α-properness in the same book. It is defined for every countable ordinal α as a strengthening of properness; see Definition 1.2. This notion is used to establish further preservation theorems. For instance, Shelah gave a sufficient condition for a countable iteration to add no new reals, which requires a stronger property than properness and adding no new reals. Such an iteration is an effective tool to show the consistency of various principles with CH.

When Γ is some property of posets, let MA(Γ) denote Martin's Axiom for all posets with Γ. For example, MA(ccc) means the usual Martin's Axiom, and MA(proper) means the proper forcing axiom, which is often denoted by PFA. In [13], Shelah constructed a model of MA(ω-proper) + $\neg$ PFA. The key tool in his argument was a club guessing sequence on ω_1, defined in Definition 3.1. It was considered by Nyikos in [11] and has several applications to general topology (see e.g. [6]). Its generalization to larger cardinals was first considered by Shelah and has been very fruitfully used in many arguments including pcf theory. One of the most significant properties of club guessing sequences on a regular cardinal larger than ω_1 is their existence under ZFC. However, a club guessing sequence on ω_1 does not exist

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under PFA. Shelah showed that a club guessing sequence on ω_1 with a certain property is preserved by ω-proper forcing, which shows the difference between MA(ω-proper) and PFA.

Extending this result, Shelah mentioned the following facts in [13, pp. 838–839].

FACT 0.1. *Let α be an indecomposable ordinal.*

(i) *If $\vec{C} = \langle C_\gamma : \gamma < \omega_1$ and α divides $\gamma\rangle$ is a fully club guessing sequence such that each C_γ has order type α and for every $\xi < \omega_1$, $\{C_\gamma \cap \xi : \gamma < \omega_1$ and α divides $\gamma\}$ is countable, then $\vec{C}$ is preserved by every α-proper poset.*

(ii) *If β is an indecomposable ordinal larger than α and $\vec{C} = \langle C_\gamma : \gamma < \omega_1$ and β divides $\gamma\rangle$ is a fully club guessing sequence with $\mathrm{ot}(C_\gamma) = \beta$, then there exists an α-proper poset P which destroys $\vec{C}$, i.e. P forces that $\vec{C}$ is not a fully club guessing sequence.*

By the same argument, we can show that for every indecomposable $\alpha < \omega_1$, if there exists a fully club guessing sequence $\vec{C} = \langle C_\gamma : \gamma \in \omega_1 \cap \mathrm{Lim}\rangle$ such that for a club subset of γ, α divides $\mathrm{ot}(C_\gamma)$, then there exists a $<\alpha$-proper poset which destroys $\vec{C}$ and hence is not α-proper.

However, it was not known whether the existence of such a poset can be shown from ZFC for every indecomposable $\alpha < \omega_1$. One of the proper posets which destroy club guessing sequences is Baumgartner's poset to shoot a club with finite conditions. It is well known as an example of a proper poset which is not ω-proper. In Section 2, we shall extend this fact by proving the following theorem.

THEOREM A. *If α is a countable indecomposable ordinal, then there exists a poset which is $<\alpha$-proper but not α-proper.*

In Section 3, we shall show that our example of a $<\alpha$-proper but not α-proper poset destroys all club guessing sequences $\langle C_\gamma : \gamma \in S\rangle$ such that for a club set of γ in S, α divides $\mathrm{ot}(C_\gamma)$.

Another topic we deal with in this paper is the class of Axiom A posets. Axiom A was proposed by Baumgartner in [3]. It is a generalization of the property which is shared by various well known posets and used to show the preservation theorem under countable support iterations. All ccc posets and countably closed posets satisfy Axiom A, and a poset satisfying Axiom A is proper. Although this axiom works well once it is satisfied, it is not known what kind of forcing can be expressed by a poset satisfying Axiom A. In Section 4, we shall show the following theorem.

THEOREM B. *For every poset P, the following are equivalent:*

(i) *P is $<\omega_1$-proper.*

(ii) *There exists a pseudo partially ordered set Q which can be densely embedded in P and satisfies Axiom A.*

Several other equivalent conditions will also be established.

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1. Definitions. Basically we follow the standard notation in set theory. The following are a few non-standard symbols. Lim denotes the class of all limit ordinals and if X is a set of ordinals, $\lim(X)$ denotes the set of all limit points of X. $\lhd$ denotes a well-ordering on the underlying set.

For every ordinal α, we define the *indecomposable order type* of α, denoted by $\mathrm{iot}(\alpha)$, to be the indecomposable ordinal $\beta < \alpha$ such that $\alpha = \gamma + \beta$ for some $\beta < \alpha$. For example, $\mathrm{iot}(\omega_1 + \omega) = \omega$. If $\alpha = \beta_0 + \beta_1 + \cdots + \beta_{k-1}$ is the Cantor normal form, i.e. each β_i is indecomposable and $\beta_i \geq \beta_{i+1}$, then $\mathrm{iot}(\alpha) = \beta_{k-1}$. Moreover, $\mathrm{iot}(\alpha)$ is equal to the largest indecomposable ordinal which divides α. If X is a set of ordinals, the indecomposable order type of X, denoted by $\mathrm{iot}(X)$, is defined to be $\mathrm{iot}(\mathrm{ot}(X))$.

Let us introduce the following convenient term, which appears in [1].

Definition 1.1. Let $\alpha > 0$ be an ordinal and $\langle M_i : i < \alpha\rangle$ a sequence of countable elementary substructures of $\langle H(\lambda), \in, \lhd\rangle$ for some large regular cardinal λ. We say that $\langle M_i : i < \alpha\rangle$ is an *α-tower* if

(i) for every limit $\delta < \alpha$, $M_\delta = \bigcup_{i<\delta} M_i$,
(ii) for every $i < \alpha$, $\langle M_j : j \leq i\rangle \in M_{i+1}$.

If $\mathfrak{A}$ is a structure expanding $\langle H(\lambda), \in, \lhd\rangle$, then we can easily construct an ω_1-tower $\langle M_i : i < \omega_1\rangle$ such that each M_i is an elementary submodel of $\mathfrak{A}$. The following strengthening of properness is due to Shelah.

Definition 1.2. Let $\alpha > 0$ be a countable ordinal. A poset P is *α-proper* if for every sufficiently large λ and every α-tower $\langle M_i : i < \alpha\rangle$ of countable elementary substructures of $\langle H(\lambda), \in, \lhd\rangle$ such that $P \in M_0$, every $p \in P \cap M_0$ has an extension q which is M_i-generic for every $i < \alpha$. We say that P is *$<\alpha$-proper* if P is β-proper for all $\beta < \alpha$.

By the same trick as for proper forcing, for every fixed structure $\mathfrak{A}$ expanding $\langle H(\lambda), \in, \lhd\rangle$, we may assume that the α-tower $\langle M_i : i < \alpha\rangle$ satisfies $M_i \prec \mathfrak{A}$ for every $i < \alpha$. Clearly P is proper if and only if P is 1-proper. Moreover, if P is α-proper, then it is β-proper for all $\beta < \alpha$. Shelah also remarked that if P is α-proper and δ is the least indecomposable ordinal above α, then P is $<\delta$-proper. For proofs and further information, see [13].

Jech began the study of games played on Boolean algebras in [8]. There have been many interesting studies about various games on Booleans algebras and posets, such as [4], [15], [9], [14] and [16]. In particular, Gray [5] and Shelah [12] independently found game-theoretic proofs of several preservation theorems. The following definitions are variations of the game-theoretic characterization of properness used by them.

Definition 1.3 ([13, pp. 593–595]). Let P be a poset, $p \in P$ and α an ordinal. We define three games $\mathrm{P}\Game^{\alpha}_{l}(p, P)$ for $l = 0, 1$ or 2 as follows. All games are played by two players alternatingly. When $l = 0$, at stage β for every $\beta < \alpha$, player I chooses a P-name $\dot{\xi}_\beta$ for an ordinal and then player II chooses an ordinal ζ_β. Player II wins if and only if there exists a $q \le p$ such that $q \Vdash$ "for every $\beta < \alpha$, there exists an $n < \omega$ such that $\dot{\xi}_\beta = \zeta_{\beta+n}$".

When $l = 1$, player I plays in the same way and player II chooses a countable set Y_β of ordinals at each stage β. Player II wins if and only if there exists a $q \le p$ such that $q \Vdash$ "for every $\beta < \alpha$, $\dot{\xi}_\beta \in Y_\beta$".

When $l = 2$, at each stage β, player I chooses a P-name $\dot{X}_\beta$ for a countable set of ordinals and player II chooses a countable set Y_β of ordinals. Player II wins if and only if there exists a $q \le p$ such that $q \Vdash$ "for every $\beta < \alpha$, $\dot{X}_\beta \subseteq Y_\beta$".

Clearly if there exists a winning strategy for player II in $\mathrm{P}\Game^{\alpha}_{2}$, there exists one for player II in $\mathrm{P}\Game^{\alpha}_{1}$.

Shelah gave characterizations of α-properness in terms of games.

Theorem 1.4 (Shelah, [13, p. 594]). *Let P be a poset and $\alpha > 0$. Then P is α-proper if and only if player II wins $\mathrm{P}\Game^{\omega\alpha}_{0}(p, P)$ for every $p \in P$.*

We use a generalization of these characterizations in Section 4.

2. α-Properness. Baumgartner defined a poset to shoot a club subset of ω_1 with finite conditions, which appeared in [2]. It is known that the poset is proper but not ω-proper. In this section, we shall extend this result to every indecomposable ordinal $\alpha < \omega_1$.

Let α be a countable indecomposable ordinal and $\mathfrak{A}$ a structure expanding $\langle H(\lambda), \in, \lhd\rangle$ for some regular cardinal λ above $\aleph_1$. We shall define a poset $P(\omega_1, \mathfrak{A}, \alpha)$ by: $p \in P(\omega_1, \mathfrak{A}, \alpha)$ if and only if p is a function such that

(i) $\mathrm{dom}(p)$ is a subset of ω_1 with $\mathrm{ot}(\mathrm{dom}(p)) < \alpha$,
(ii) for every $\gamma \in \mathrm{dom}(p)$, $\gamma \le p(\gamma) < \omega_1$,
(iii) for any $\gamma < \delta$ both in $\mathrm{dom}(p)$, $p(\gamma) < \delta$,
(iv) for every $\gamma \in \mathrm{dom}(p)$, $p{\restriction}(\mathrm{dom}(p) \cap \gamma) \in \mathrm{Sk}^{\mathfrak{A}}(\gamma)$.

$P(\omega_1, \mathfrak{A}, \alpha)$ is ordered by extension.

Fix $\mathfrak{A}$ and α as above. We shall show that $P(\omega_1, \mathfrak{A}, \alpha)$ is not α-proper and that CH implies that $P(\omega_1, \mathfrak{A}, \alpha)$ is $<\alpha$-proper.

We shall use the following notation. For each $p \in P(\omega_1, \mathfrak{A}, \alpha)$, we say that γ is a *domain candidate* of p if $\gamma \notin \mathrm{dom}(p)$ and there exists a $q \le p$ such that $\gamma \in \mathrm{dom}(q)$.

LEMMA 2.1. *Let $p \in P(\omega_1, \mathfrak{A}, \alpha)$ and γ a domain candidate of p. Then for every δ, if $\gamma \le \delta < \min(\mathrm{dom}(p) \setminus (\gamma+1))$, then there exists a $q \le p$ such that $\gamma \in \mathrm{dom}(q)$ and $q(\gamma) = \delta$.*

Proof. Let $p' \le p$ be such that $\gamma \in \mathrm{dom}(p')$. Define q by: $\mathrm{dom}(q) = (\mathrm{dom}(p') \cap (\gamma+1)) \cup \mathrm{dom}(p)$ and

$$q(\xi) = \begin{cases} p'(\xi) & \text{if } \xi < \gamma, \\ \delta & \text{if } \xi = \gamma, \\ p(\xi) & \text{if } \xi > \gamma. \end{cases}$$

If $q \in P$, then clearly $q \le p$, $\gamma \in \mathrm{dom}(q)$, and $q(\gamma) = \delta$. Thus it suffices to show $q \in P$. The only condition which is not clear is (iv). To prove it, let $\xi \in \mathrm{dom}(q)$. If $\xi \le \gamma$, then $\xi \in \mathrm{dom}(p')$ and $q\restriction(\mathrm{dom}(q) \cap \xi) = p'\restriction(\mathrm{dom}(p') \cap \xi)$. Hence $q\restriction(\mathrm{dom}(q) \cap \xi) \in \mathrm{Sk}^{\mathfrak{A}}(\xi)$. If $\xi > \gamma$, then $\xi \in \mathrm{dom}(p)$, so $p\restriction(\mathrm{dom}(p) \cap \xi) \in \mathrm{Sk}^{\mathfrak{A}}(\xi)$. Moreover, by assumption, $\delta < \min(\mathrm{dom}(p) \setminus (\gamma+1)) \le \xi$. Thus, $\gamma, \delta \in \mathrm{Sk}^{\mathfrak{A}}(\xi)$ and hence $\langle \gamma, \delta \rangle \in \mathrm{Sk}^{\mathfrak{A}}(\xi)$. It follows that

$$q\restriction(\mathrm{dom}(q) \cap \xi) = p'\restriction(\mathrm{dom}(p') \cap \gamma) \cup \{\langle \gamma, \delta \rangle\} \cup p\restriction(\mathrm{dom}(p) \cap \xi) \in \mathrm{Sk}^{\mathfrak{A}}(\xi). \ \blacksquare$$

PROPOSITION 2.2. *If CH holds, then $P(\omega_1, \mathfrak{A}, \alpha)$ is $<\alpha$-proper.*

Proof. Let $P = P(\omega_1, \mathfrak{A}, \alpha)$ and $\beta < \alpha$. Since we assume CH, there exists a bijection f from ω_1 into the set of all functions from a countable subset of ω_1 into ω_1. Let f be the $\lhd$-least such bijection. Let $\langle M_i : i < \beta \rangle$ be a β-tower of elementary substructures of $\langle H(\lambda'), \in, \lhd \rangle$ for some regular cardinal $\lambda' > 2^{\lambda}$ such that $\mathfrak{A}, P \in M_0$. Set $\delta_i = M_i \cap \omega_1$ for every $i < \beta$.

Let $p \in P \cap M_0$ be arbitrary. Define $q = p \cup \{\langle \delta_i, \delta_i \rangle : i < \beta$ and i is 0 or a successor ordinal$\}$. We shall show that q is an M_i-generic condition for all $i < \beta$.

CLAIM 1. *$q \in P$.*

Proof. Since $p \in M_0$ and $|\mathrm{dom}(p)| = \aleph_0$, we have $\mathrm{dom}(p) \subseteq \delta_0$. Thus $\mathrm{ot}(\mathrm{dom}(q)) \le \mathrm{ot}(\mathrm{dom}(p)) + \beta$. Since α is indecomposable, $\mathrm{ot}(\mathrm{dom}(q)) < \alpha$. (ii) is trivial. Since $\mathrm{ran}(p) \subseteq \delta_0$, (iii) follows. For (iv), let $\gamma \in \mathrm{dom}(p)$. If $\gamma < \delta_0$, then $q\restriction(\mathrm{dom}(q) \cap \gamma) = p\restriction(\mathrm{dom}(p) \cap \gamma) \in \mathrm{Sk}^{\mathfrak{A}}(\gamma)$. Suppose $\gamma = \delta_0$. Since $p \in M_0$ and f is a bijection, there exists a $\xi < \delta_0$ such that $f(\xi) = p$. Thus we have $q\restriction(\mathrm{dom}(q) \cap \gamma) = p \in \mathrm{Sk}^{\mathfrak{A}}(\delta_0)$. If $\gamma > \delta_0$, then there exists an $i < \beta$ such that $\gamma = \delta_{i+1}$. Since $\langle M_j : j < \beta \rangle$ is a β-tower, $\langle M_j : j \le i \rangle \in M_{i+1}$. In particular, $A_i := \{\langle \delta_j, \delta_j \rangle : j \le i$ and j is 0 or a successor ordinal$\}$

is in M_{i+1}. Since f is a bijection, there exists a $\xi < \delta_{i+1}$ such that $f(\xi) = A_i$. Thus $A_i \in \mathrm{Sk}^{\mathfrak{A}}(\delta_{i+1})$. Therefore $q\restriction(\mathrm{dom}(q) \cap \gamma) = p \cup A_i \in \mathrm{Sk}^{\mathfrak{A}}(\delta_{i+1})$. ∎

Claim 2. *q is M_i-generic for all $i < \beta$.*

Proof. If i is a limit ordinal and q is M_j-generic for all $j < i$, then q is M_i-generic by the continuity of the tower. Thus it suffices to show that q is M_i-generic when $i = 0$ or i is a successor ordinal less than β.

Fix such an i and a dense open set D lying in M_i. Let $q' \le q$ be arbitrary. We need to find an $r \in D \cap M_i$ which is compatible with q'. Let $\overline{q} = q'\restriction(\mathrm{dom}(q') \cap \delta_i)$. By (iv) applied to q', $\overline{q} \in \mathrm{Sk}^{\mathfrak{A}}(\delta_i)$. In particular, $\overline{q} \in M_i$. Since D is a dense open set lying in M_i, there exists an $\overline{r} \le \overline{q}$ such that $\overline{r} \in D \cap M_i$. Let $r = \overline{r} \cup q'$. If $r \in P$, then r is a common extension of q' and $\overline{r} \in D \cap M_i$. Hence it suffices to show $r \in P$.

We have $\mathrm{ot}(\mathrm{dom}(r)) \le \mathrm{ot}(\mathrm{dom}(\overline{r})) + \mathrm{ot}(\mathrm{dom}(q')) < \alpha$ and hence r satisfies (i). (ii) is trivial. (iii) is also trivial except for $\delta = \delta_i$. Since $\overline{r} \in M_i$ and $|\mathrm{ran}(\overline{r})| = \aleph_0$, we have $\mathrm{ran}(\overline{r}) \subseteq M_i$. Thus for every $\gamma \in \mathrm{dom}(r) \cap \delta_i$, $r(\gamma) < \delta_i$. To see (iv), let $\gamma \in \mathrm{dom}(r)$. If $\gamma < \delta_i$, the assertion follows directly from $\overline{r} \in P$. Suppose $\gamma \ge \delta_i$. Then we have $\overline{r} \in \mathrm{Sk}^{\mathfrak{A}}(\delta_i)$ as before and $q'\restriction(\mathrm{dom}(q') \cap \gamma) \in \mathrm{Sk}^{\mathfrak{A}}(\gamma)$. Thus $r\restriction(\mathrm{dom}(r) \cap \gamma) = \overline{r} \cup q'\restriction(\mathrm{dom}(q') \cap \gamma) \in \mathrm{Sk}^{\mathfrak{A}}(\gamma)$. ∎

This completes the proof of Proposition 2.2. ∎

Proposition 2.3. *$P(\omega_1, \mathfrak{A}, \alpha)$ is not α-proper. Moreover, for every tower $\langle M_i : i < \alpha\rangle$ of countable elementary substructures of $\mathfrak{A}$ with $P \in M_0$, there is no $q \le p$ which is M_i-generic for every $i < \alpha$.*

Proof. Let $\langle M_i : i < \alpha\rangle$ be any α-tower with $P \in M_0$. We shall show that there is no $q \in P$ such that q is M_i-proper for all $i < \alpha$. Let $\delta_i = M_i \cap \omega_1$ for each $i < \alpha$.

Suppose that q is M_i-proper for all $i < \alpha$. Since $\mathrm{ot}(\mathrm{dom}(q) \cup \mathrm{ran}(q)) = 2 \cdot \mathrm{ot}(\mathrm{dom}(q)) < \alpha$, there exists an $i < \alpha$ such that $(\mathrm{dom}(q) \cup \mathrm{ran}(q)) \cap [\delta_i, \delta_{i+2}) = \emptyset$. Let $D = \{r \in P : \mathrm{dom}(r) \not\subseteq \delta_i\}$. Then clearly D is a dense open subset lying in M_{i+1}. Since q is assumed to be M_{i+1}-generic, $D \cap M_{i+1}$ is predense below q. We shall derive a contradiction.

Claim 1. *There exists a domain candidate $\gamma \in [\delta_i, \delta_{i+1})$ of q.*

Proof. Let $\overline{r}$ be an element of $D \cap M_{i+1}$ which is compatible with q, and r a common extension of q and $\overline{r}$. Since $\overline{r} \in D \cap M_{i+1}$, there exists a $\gamma \in \mathrm{dom}(\overline{r})$ such that $\gamma \in [\delta_i, \delta_{i+1})$. Thus $\gamma \in \mathrm{dom}(r)$. Therefore γ is a domain candidate of q. ∎

Let γ_0 be the least domain candidate of q with $\gamma_0 \ge \delta_i$. By Lemma 2.1, there exists a $p \le q$ such that $\gamma_0 \in \mathrm{dom}(p)$ and $p(\gamma_0) = \delta_{i+1}$. Let

$\bar{r} \in D \cap M_{i+1}$ be compatible with p. Let r be a common extension of $\bar{r}$ and p. If $\gamma_0 \in \mathrm{dom}(\bar{r})$, then $\bar{r}(\gamma_0) = r(\gamma_0) = \delta_{i+1} \notin M_{i+1}$, which is a contradiction. Thus $\gamma_0 \notin \mathrm{dom}(\bar{r})$. If $\gamma_0 < \gamma \in \mathrm{dom}(\bar{r})$, then $\gamma \in \mathrm{dom}(r)$. Then $\gamma_0 < \gamma < \delta_{i+1} = r(\gamma_0)$. This is a contradiction. Hence $\mathrm{dom}(\bar{r}) \subseteq \gamma_0$. But since $\bar{r} \in D$, there exists a $\gamma \in \mathrm{dom}(\bar{r})$ with $\gamma \geq \delta_i$. Hence $\gamma \in \mathrm{dom}(r)$. Thus γ is a domain candidate of q such that $\delta_i \leq \gamma < \gamma_0$. This contradicts the minimality of γ_0. ∎

Thus CH implies that there exists a poset which is $<\alpha$-proper but not α-proper. In fact, we do not need CH to prove it. First, we shall prove the following lemma.

LEMMA 2.4. *Let α be a countable ordinal. Let P be a poset and $\dot{Q}$ a P-name for a poset such that P forces that for every α-tower $\langle M_i : i < \alpha\rangle$ of countable elementary substructures of $\langle H(\kappa), \in, \lhd\rangle$ where κ is a sufficiently large regular cardinal, there is no $q \in \dot{Q}$ which is M_i-generic for every $i < \alpha$. Then $P * \dot{Q}$ is not α-proper.*

Proof. Suppose that $P*\dot{Q}$ is α-proper. Let κ be a sufficiently large regular cardinal and λ be a regular cardinal with $\lambda > 2^{2^\kappa}$. Let $\langle M_i : i < \alpha\rangle$ be an α-tower of countable elementary substructures of $\langle H(\lambda), \in, \lhd, \kappa\rangle$ with $P * \dot{Q} \in M_0$. By assumption, there exists a $\langle p, \dot{q}\rangle \in P * \dot{Q}$ which is M_i-generic for every $i < \alpha$. Let $G \subseteq P$ be generic with $p \in G$. Work in $V[G]$. Let q be the interpretation of $\dot{q}$. Let $N_i = M_i[G] \cap H(\kappa)$ for every $i < \alpha$. Then for every $i < \alpha$, $\langle N_j : j \leq i\rangle \in M_{i+1}[G] \cap H(\kappa) = N_{i+1}$. Therefore, $\langle N_i : i < \alpha\rangle$ is an α-tower. It is easy to see that each N_i is a countable elementary substructure of $\langle H(\kappa), \in, \lhd\rangle$. Moreover, since $\langle p, \dot{q}\rangle$ is M_i-generic for every $i < \alpha$, q is N_i-generic for every $i < \alpha$. This contradicts the assumption. Therefore, $P * \dot{Q}$ is not α-proper. ∎

It is known that a countable support iteration of α-proper posets is α-proper. Using these facts, we can prove Theorem A.

THEOREM A. *If α is a countable indecomposable ordinal, then there exists a poset which is $<\alpha$-proper but not α-proper.*

Proof. Let P be the Levy collapse which collapses $2^{\aleph_0}$ to $\aleph_1$. In V^P, let λ be a sufficiently large regular cardinal and $\mathfrak{A} = \langle H(\lambda), \in, \lhd\rangle$. Let $Q = P(\omega_1, \mathfrak{A}, \alpha)$. Then Q is $<\alpha$-proper by Proposition 2.2. But by Proposition 2.3, for every α-tower $\langle M_i : i < \alpha\rangle$ of countable elementary substructures of $\langle H(\kappa), \in, \lhd\rangle$, there is no $q \in Q$ which is M_i-generic for every $i < \alpha$. Let $\dot{Q}$ be a P-name for Q.

Since P is $<\omega_1$-proper and P forces that $\dot{Q}$ is $<\alpha$-proper, $P * \dot{Q}$ is $<\alpha$-proper. However, by Lemma 2.4, $P * \dot{Q}$ is not α-proper. ∎

3. Destruction of club guessing sequences. In this section, we shall show the relationship between α-properness and the preservation of club guessing sequences. First of all we shall define a tail club guessing sequence which was investigated in various papers, such as [11] and [13, Chapter XVIII].

DEFINITION 3.1. Let $\vec{C} = \langle C_\gamma : \gamma \in S\rangle$ be a sequence on a stationary subset S of ω_1 such that each C_γ is an unbounded subset of γ. We say that $\vec{C}$ is a *tail* (resp. *fully*) *club guessing sequence* on S if and only if for every club subset D of ω_1, there exists a $\gamma \in S$ such that $C_\gamma \setminus \zeta \subseteq D$ for some $\zeta < \gamma$ (resp. $C_\gamma \subseteq D$).

We shall also define semiproperness and α-semiproperness because the method in this section works for these weaker properties. They were defined in [12] to handle posets which preserve $\aleph_1$ but add a countable set of ordinals which is not covered by a countable set in the ground model.

DEFINITION 3.2. Let P be a poset. For a set M and $p \in P$, we say that p is *M-semigeneric* if for every P-name for a countable ordinal $\dot{\xi} \in M$, $p \Vdash \dot{\xi} \in M$.

A poset P is *semiproper* if for every large enough regular cardinal λ and every countable elementary substructure $M \prec \langle H(\lambda), \in, \lhd\rangle$, whenever $P \in M$ and $p \in M \cap P$, there exists a $q \le p$ such that q is M-semigeneric.

Let α be a countable ordinal. P is said to be *α-semiproper* if for every large enough regular cardinal λ and every α-tower $\langle M_i : i < \alpha\rangle$ of countable elementary substructures of $\langle H(\lambda), \in, \lhd\rangle$, whenever $P \in M_0$ and $p \in M_0 \cap P$, there exists a $q \le p$ such that q is M_i-semigeneric for all $i < \alpha$.

Trivially every (α-)proper poset is (α-)semiproper. Shelah [12] showed that it is consistent that Namba forcing is a semiproper poset which is not proper.

The following lemma is standard.

LEMMA 3.3. *Let P be a poset and M a countable elementary substructure of $\langle H(\lambda), \in, \lhd\rangle$ with $P \in M$ for some sufficiently large regular cardinal λ. Suppose that $p \in P$ is M-semigeneric and $\dot{D} \in M$ is a P-name for a club. Then $p \Vdash M \cap \omega_1 \in \dot{D}$.*

Proof. Let $\delta = M \cap \omega_1$. It suffices to show that $p \Vdash$ "$\dot{D} \cap \delta$ is unbounded in δ". Let $\zeta < \delta$. Since $\dot{D}$ is a P-name for a club, we have $\Vdash \dot{D} \setminus \zeta \neq \emptyset$. Let $\dot{\alpha}$ be a P-name such that $\Vdash \dot{\alpha} \in \dot{D} \setminus \zeta$. Since M is an elementary substructure of $\langle H(\lambda), \in, \lhd\rangle$, we can assume $\dot{\alpha} \in M$. Since p is M-semigeneric, we have $p \Vdash \dot{\alpha} \in M \cap \omega_1 = \delta$. Hence, $p \Vdash$ "$\zeta \le \dot{\alpha} < \delta$ and $\dot{\alpha} \in \dot{D}$". Thus, $p \Vdash$ "$\dot{D} \cap \delta$ is unbounded in δ". ∎

The following are easy facts about the preservation of club guessing sequences. For the proof, see [7].

FACT 3.4. (i) *Every $<\omega_1$-semiproper poset preserves all fully club guessing sequences.*

(ii) *Every ω-semiproper poset preserves every fully club guessing sequence $\langle C_\gamma : \gamma \in \omega_1 \cap \mathrm{Lim}\rangle$ with $\mathrm{ot}(C_\gamma) = \omega$.*

(iii) *Baumgartner's poset to shoot a club with finite conditions destroys all tail club guessing sequences in the ground model.*

The following result extending Fact 3.4(ii) was mentioned by Shelah in [13]. We give the proof for the reader's convenience.

PROPOSITION 3.5 (Shelah). *Let α be a countable indecomposable ordinal and P an α-semiproper poset. Let $\langle C_\gamma : \gamma \in S\rangle$ be a fully club guessing sequence on a stationary subset S of ω_1 such that C_γ is closed and $\mathrm{ot}(C_\gamma) \le \alpha$ for every $\gamma \in S$. Suppose also that for every $\xi < \omega_1$, $|\{C_\gamma \cap \xi : \gamma \in S\}| = \aleph_0$. Then P preserves $\langle C_\gamma : \gamma \in S\rangle$.*

Proof. Let $p \in P$ and $\dot{D}$ be a name for a club subset of ω_1. Let $\mathcal{A}_\xi = \{C_\gamma \cap \xi : \gamma \in S\}$ and $\mathfrak{A} = \langle H(\lambda), \in, \lhd, P, p, \dot{D}, \langle \mathcal{A}_\xi : \xi \in S\rangle\rangle$. Then if $\gamma \in S$ and $\xi \in C_\gamma \setminus \lim(C_\gamma)$, then $C_\gamma \cap \xi \in \mathrm{Sk}^{\mathfrak{A}}(\xi)$.

We can construct an ω_1-tower $\langle M_\gamma : \gamma < \omega_1\rangle$ of countable elementary substructures of $\mathfrak{A}$. Let $\widetilde{D} = \{M_\gamma \cap \omega_1 : \gamma < \omega_1\}$. Clearly $\widetilde{D}$ is a club subset of ω_1. Thus there exists a $\delta < \omega_1$ such that $C_\delta \subseteq \widetilde{D}$. Let $\{\xi_i : i < \eta\}$ be the increasing enumeration of C_δ. Then for each $i < \eta$, there exists a $\delta_i < \omega_1$ such that $M_{\delta_i} \cap \omega_1 = \xi_i$. Note that $\eta = \mathrm{ot}(C_\delta) \le \alpha$.

We claim that $\langle M_{\delta_i} : i < \eta\rangle$ is an η-tower of countable elementary substructures of $\mathfrak{A}$. Let $i < \eta$. By definition, $\langle M_\gamma : \gamma \le \delta_i\rangle \in M_{\delta_i+1}$. By the assumption on C_δ, $C_\delta \cap \xi_{i+1} \in \mathrm{Sk}^{\mathfrak{A}}(\xi_{i+1})$. In particular, $\{\xi_j : j \le i\} = C_\delta \cap \xi_{i+1} \in M_{\delta_{i+1}}$. But M_{δ_j} is definable from ξ_j and $\langle M_\gamma : \gamma \le \delta_i\rangle$. Thus $\langle M_{\delta_j} : j \le i\rangle \in M_{\delta_{i+1}}$.

Since P is α-semiproper, there exists a $q \le p$ such that q is M_{δ_i}-semigeneric for all $i < \eta$. In particular, $q \Vdash \xi_i \in \dot{D}$ for all $i < \eta$. By the definition of ξ_i, it follows that $q \Vdash C_\delta \subseteq \dot{D}$. ∎

It was shown in [7] that if $\langle C_\gamma : \gamma \in S\rangle$ is a tail club guessing sequence, then there exists a $\zeta < \omega_1$ such that $\langle C_\gamma \setminus \zeta : \gamma \in S \setminus (\zeta + 1)\rangle$ is a fully club guessing sequence. Thus the previous proposition can be easily modified to cover tail club guessing sequences.

Notice that the coherence condition of the previous proposition is not always satisfied. For example, although a $\diamondsuit$-sequence is trivially a fully club guessing sequence, it does not satisfy the condition. In [17], Zapletal constructed a poset which adds a fully club guessing sequence and satisfies that condition.

We do not know if the coherence condition is necessary. But the order type restriction is: this is witnessed by the poset Shelah introduced in [13] as well as the poset defined in Section 2.

PROPOSITION 3.6. *Let $\alpha < \omega_1$ be an indecomposable ordinal. Suppose that $\vec{C} = \langle C_\gamma : \gamma \in S\rangle$ is a tail club guessing sequence on ω_1 such that for every $\gamma \in S$, $\mathrm{iot}(C_\gamma) \geq \alpha$. Then $P(\omega_1, \mathfrak{A}, \alpha)$ forces that $\vec{C}$ is not a tail club guessing sequence.*

Proof. Let $P = P(\omega_1, \mathfrak{A}, \alpha)$ and let $G \subseteq P$ be generic. Let $f = \bigcup G$ and define D to be the set of all limit points of $\mathrm{dom}(f)$ less than ω_1.

We claim that D is not guessed by $\vec{C}$. Suppose otherwise. Let $\dot{f}$ and $\dot{D}$ be P-names for f and D respectively. Then there exist a $p \in G$, a $\gamma \in S$ and a $\zeta < \gamma$ such that $p \Vdash C_\gamma \backslash \zeta \subseteq \dot{D}$. Let $\{\xi_i : i < \eta\}$ be an increasing enumeration of $C_\gamma \setminus \zeta$. Since $\mathrm{iot}(C_\gamma) \geq \alpha$, we have $\eta \geq \alpha$. Since $\mathrm{ot}(\mathrm{dom}(p)) < \alpha$, there exists an $i < \eta$ such that $[\xi_i, \xi_{i+2}) \cap \mathrm{dom}(p) = \emptyset$. Since $p \Vdash \xi_{i+1} \in \dot{D}$, i.e. ξ_{i+1} is a limit point of $\mathrm{dom}(\dot{f})$, there exists a $\nu \in [\xi_i, \xi_{i+1})$ such that ν is a domain candidate of p. By Lemma 2.1, we can get $q \leq p$ such that $q(\nu) = \xi_{i+1}$. But then $q \Vdash$ "ξ_{i+1} is not a limit point of $\mathrm{dom}(\dot{f})$". This is a contradiction. ∎

4. The forcing-theoretic equivalence of $<\omega_1$-properness and Axiom A.

In this section, we shall show Theorem B, which asserts that a forcing notion is describable by a pseudo partially ordered set satisfying Axiom A if and only if it is $<\omega_1$-proper. It determines the strength of Axiom A in terms of forcing. It is well known that we may adopt a pseudo partially ordered set as a forcing notion. We remark that this convention is essential in our proof. In terms of forcing, a pseudo partially ordered set can be identified as a poset by taking equivalence classes. But when we build a forcing notion satisfying Axiom A, we may treat conditions in the same equivalence class differently. It is not known if we can prove this result without using a pseudo partial ordering.

First of all, let us define Axiom A and uniform Axiom A, which is a stronger notion.

DEFINITION 4.1. A pseudo partially ordered set P *satisfies Axiom A* if there exists a sequence $\langle \leq_n : n < \omega\rangle$ of pseudo partial orderings on P such that

(i) $p \leq_0 q$ implies $p \leq q$,
(ii) $p \leq_{n+1} q$ implies $p \leq_n q$ for every $n < \omega$,
(iii) if $\langle p_n : n < \omega\rangle$ is a sequence such that $p_{n+1} \leq_n p_n$ for all $n < \omega$, then there exists a $q \in P$ such that $q \leq_n p_n$ for all $n < \omega$,
(iv) for every $p \in P$ and $n < \omega$, if $\dot{\alpha}$ is a P-name for an ordinal, then there exist a $q \leq_n p$ and a countable set X of ordinals such that $q \Vdash \dot{\alpha} \in X$.

If there exists a sequence $\langle \leq_n : n < \omega \rangle$ which is constant and witnesses Axiom A, then we say that P satisfies *uniform Axiom A*. We usually denote the constant value by $\leq_\infty$.

It is easy to see that ccc posets and countably closed posets satisfy uniform Axiom A. It is well known that if P satisfies Axiom A, then P is $<\omega_1$-proper, which can be easily shown by induction.

We need the following natural extension of the games which characterize properness properties.

DEFINITION 4.2. Let P be a (pseudo) partially ordered set and $p \in P$. We shall define a game $\mathrm{P}\Game_l^{<\omega_1}(P,p)$ for each $l = 0, 1$ or 2. In each game, players play in the same way as in $\mathrm{P}\Game_l^{\alpha}(P,p)$ of length ω_1. Player II wins if and only if at every limit stage $\alpha < \omega_1$, the sequence satisfies player II's winning condition in $\mathrm{P}\Game_l^{\alpha}(P,p)$.

In order to prove Theorem B, we shall show the following stronger theorem.

THEOREM 4.3. *For every poset P, the following are equivalent*:

(i) *P is $<\omega_1$-proper.*
(ii) *Player II has a winning strategy in $\mathrm{P}\Game_0^{<\omega_1}(P,p)$ for every $p \in P$.*
(iii) *Player II has a winning strategy in $\mathrm{P}\Game_2^{<\omega_1}(P,p)$ for every $p \in P$.*
(iv) *Player II has a winning strategy in $\mathrm{P}\Game_1^{<\omega_1}(P,p)$ for every $p \in P$.*
(v) *There exists a pseudo partially ordered set Q which can be densely embedded in P and satisfies uniform Axiom A.*
(vi) *There exists a pseudo partially ordered set Q which can be densely embedded in P and satisfies Axiom A.*

Proof. Let P be a poset and fix a large regular cardinal λ. Define $\mathfrak{A} = \langle H(\lambda), \in, \lhd \rangle$.

First we shall show (i)⇒(ii). Let $n \mapsto (k_n, l_n)$ be a bijection from ω onto $\omega \times \omega$ with $k_n \leq n$. Suppose that P is $<\omega_1$-proper. Let $p \in P$. We shall describe a winning strategy for player II. We also define an increasing continuous sequence $\langle M_\beta : \beta < \omega_1 \rangle$ of countable elementary substructures of $H(\lambda)$ such that $\langle M_\gamma : \gamma \leq \beta \rangle \in M_{\beta+1}$. Let $M_0 = \mathrm{Sk}^{\mathfrak{A}}(\{p\})$. At stage β, suppose that player I plays $\dot\alpha_\beta$. Let $M_{\beta+1} = \mathrm{Sk}^{\mathfrak{A}}(\{p, \langle M_\gamma : \gamma \leq \beta \rangle, \langle \dot\alpha_\gamma : \gamma \leq \beta \rangle\})$. Then clearly $\langle M_\gamma : \gamma \leq \beta \rangle \in M_{\beta+1}$ and $M_{\beta+1}$ is countable. Let $\{\xi_i^{\beta+1} : i < \omega\}$ enumerate $M_{\beta+1} \cap \mathbf{ON}$ and let player II play $\zeta_\beta = \xi_{l_n}^{\delta+k_n+1}$ where a limit ordinal δ and $n < \omega$ are such that $\beta = \delta + n$.

If β is a non-zero limit ordinal, let $M_\beta = \bigcup_{\gamma<\beta} M_\gamma$. We shall show that player II does not lose at this point. Since P is $<\omega_1$-proper, there exists a $q \leq p$ which is M_γ-generic for all $\gamma \leq \beta$. Let $\gamma < \beta$. Write $\gamma = \delta + k$ where δ is a limit ordinal and $k < \omega$. Since q is $M_{\gamma+1}$-generic and $\dot\alpha_\gamma \in M_{\gamma+1}$ as a name,

$q \Vdash \dot{\alpha}_\gamma \in M_{\gamma+1}$. But $M_{\gamma+1} \cap \mathbf{ON} = \{\xi_l^{\gamma+1} : l < \omega\} = \{\xi_{l_n}^{\delta+k_n+1} : n < \omega$ and $k_n = k\} \subseteq \{\zeta_i : \gamma \leq i < \gamma+\omega\}$. Therefore $q \Vdash \dot{\alpha}_\gamma = \zeta_{\gamma+n}$ for some $n < \omega$. If player II follows this strategy, he wins the game, because he does not lose at any stage. Thus this is a winning strategy.

For (ii)⇒(iii), suppose that σ is a winning strategy for player II in $\mathrm{P}\eth_0^{<\omega_1}(P,p)$, i.e. σ is a function from the set of all sequences of P-names for ordinals of countable length into the set of ordinals such that when $\langle \dot{\alpha}_\gamma : \gamma < \omega_1 \rangle$ is a sequence of moves of player I and player II plays $\sigma(\langle \dot{\alpha}_i : i \leq \gamma \rangle)$ at each stage γ, then player II wins. We shall describe a winning strategy for player II in $\mathrm{P}\eth_2^{<\omega_1}(P,p)$. At stage β, suppose that player I chooses $\dot{X}_\beta$. Then there exists a countable set $\{\dot{\eta}_n^\beta : n < \omega\}$ of P-names for ordinals such that $\Vdash_P \dot{X}_\beta = \{\dot{\eta}_n^\beta : n < \omega\}$. Define $\dot{\xi}_{\omega\beta+n} = \dot{\eta}_n^\beta$ for all $n < \omega$. Let $Y_\beta = \{\sigma(\langle \dot{\xi}_i : i \leq \omega\beta+n \rangle) : n < \omega\}$.

We need to show that it is a winning strategy. Let $\delta < \omega_1$ be a limit ordinal. Since σ is a winning strategy, there exists a $q \leq p$ such that $q \Vdash$ "for all $\beta < \delta$, there exists an $n < \omega$ such that $\dot{\xi}_\beta = \sigma(\langle \dot{\xi}_i : i \leq \beta+n \rangle)$". But then $q \Vdash$ "$\dot{X}_\beta = \{\dot{\eta}_n^\beta : n < \omega\} = \{\dot{\xi}_{\omega\beta+n} : n < \omega\} \subseteq \{\sigma(\langle \dot{\xi}_i : i \leq \omega\beta+m \rangle) : m < \omega\} = Y_\beta$".

(iii)⇒(iv) is trivial.

(iv)⇒(v) is proved in the author's master thesis. We present it for the reader's convenience. Suppose that player II has a winning strategy σ_p in $\mathrm{P}\eth_1^{<\omega_1}(P,p)$ for every $p \in P$. Let $\mathcal{B}(P)$ be the Boolean completion of P, i.e. $\mathcal{B}(P)$ is a complete Boolean algebra which has a dense subset $\mathcal{D}$ isomorphic to a dense subset of P. Let Q be defined by: $q \in Q$ if and only if q is of the form $\langle p, \langle \dot{\alpha}_i : i < \eta \rangle \rangle$ where $p \in P$, $\eta < \omega_1$ and $\dot{\alpha}_i$ is a P-name for an ordinal for every $i < \eta$. Define $\tau : Q \to \mathcal{B}(P)$ by $\tau(\langle p, \langle \dot{\alpha}_i : i < \eta \rangle \rangle) = \bigwedge_{i<\eta} [\![\dot{\alpha}_i \in \sigma(p, \langle \dot{\alpha}_j : j \leq i \rangle)]\!]$. Here we identify each $\dot{\alpha}_i$ as a $\mathcal{B}(P)$-name in an obvious way and $[\![\varphi]\!]$ denotes the truth value of φ. Note that since σ is a winning strategy, $\tau(q) > 0$ for every $q \in Q$. Define $q \leq q'$ if and only if $\tau(q) \leq \tau(q')$. Then $\tau(\langle p, \langle \dot{\alpha}_i : i < \eta \rangle \rangle) \leq p$ for every $\langle p, \langle \dot{\alpha}_i : i < \eta \rangle \rangle \in Q$. In particular, τ is a dense embedding. For every $q = \langle p, \langle \dot{\alpha}_i : i < \eta \rangle \rangle$ and $q' = \langle p', \langle \dot{\alpha}'_i : i < \eta' \rangle \rangle$, we define $q \leq_\infty q'$ if and only if $p = p'$, $\eta \geq \eta'$ and $\dot{\alpha}_i = \dot{\alpha}'_i$ for all $i < \eta'$. Then $(Q, \leq_\infty)$ is clearly countably closed. Suppose that A is a maximal antichain of Q and $q = \langle p, \langle \dot{\alpha}_i : i < \eta \rangle \rangle \in Q$. We need to show that there exist a $q' \leq_\infty q$ such that $|\{a \in A : a$ and q' are compatible$\}| \leq \aleph_0$. Let $A = \{a_\gamma : \gamma < |A|\}$ be an enumeration. Since τ is a dense embedding, $\{\tau(a_\gamma) : \gamma < |A|\}$ is a maximal antichain. Define a P-name $\dot{\alpha}_\eta$ so that $\tau(a_\gamma) \Vdash \dot{\alpha}_\eta = \gamma$ for every $\gamma < |A|$. Let $q' = \langle p, \langle \dot{\alpha}_i : i \leq \eta \rangle \rangle$. Then $\tau(q') \Vdash \dot{\alpha}_\eta \in \sigma(\langle p, \langle \dot{\alpha}_j : j \leq \eta \rangle \rangle)$. Let $Y = \{a_\gamma : \gamma \in \sigma(\langle p, \langle \dot{\alpha}_j : j \leq \eta \rangle \rangle)\}$. If q'' is a common extension of q' and a_γ, then $q'' \Vdash \gamma = \dot{\alpha}_\eta \in \sigma(\langle p, \langle \dot{\alpha}_j : j \leq \eta \rangle \rangle)$. Thus $a_\gamma \in Y$. This implies that $|\{a_\gamma : a_\gamma$ and q' are compatible$\}| \leq |Y| = \aleph_0$.

(v)$\Rightarrow$(vi) is again trivial. (vi)$\Rightarrow$(i) has already been remarked. ∎

In [10], Miyamoto proposed a generalization of Axiom A, named Axiom C. Since he showed that Axiom A implies Axiom C and Axiom C implies $<\omega_1$-properness in the same paper, the previous theorem shows that every pseudo partially ordered set satisfying Axiom C is equivalent to a pseudo partially ordered set satisfying Axiom A.

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