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Expanding repellers in limit sets for iterations of holomorphic functions

by

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Abstract. We prove that for Ω being an immediate basin of attraction to an attracting fixed point for a rational mapping of the Riemann sphere, and for an ergodic invariant measure μ on the boundary $\mathrm{Fr}\,\Omega$, with positive Lyapunov exponent, there is an invariant subset of $\mathrm{Fr}\,\Omega$ which is an expanding repeller of Hausdorff dimension arbitrarily close to the Hausdorff dimension of μ. We also prove generalizations and a geometric coding tree abstract version. The paper is a continuation of a paper in Fund. Math. 145 (1994) by the author and Anna Zdunik, where the density of periodic orbits in $\mathrm{Fr}\,\Omega$ was proved.

1. Introduction. Let Ω be a simply connected domain in $\overline{\mathbb{C}}$ and f be a holomorphic map defined on a neighbourhood W of $\mathrm{Fr}\,\Omega$ to $\overline{\mathbb{C}}$. Assume $f(W\cap\Omega)\subset\Omega$, $f(\mathrm{Fr}\,\Omega)\subset\mathrm{Fr}\,\Omega$ and $\mathrm{Fr}\,\Omega$ repells to the side of Ω, that is, $\bigcap_{n=0}^{\infty} f^{-n}(W\cap\overline{\Omega})=\mathrm{Fr}\,\Omega$. An important special case is where Ω is an immediate basin of attraction of an attracting fixed point for a rational function. This covers also the case of a component of the immediate basin of attraction to a periodic attracting orbit, as one can consider an iterate of f mapping the component to itself. Distances and derivatives are considered in the Riemann spherical metric on $\overline{\mathbb{C}}$.

Let $R:\mathbb{D}\to\Omega$ be a Riemann mapping from the unit disc onto Ω and let g be a holomorphic extension of $R^{-1}\circ f\circ R$ to a neighbourhood of the unit circle $\partial\mathbb{D}$. It exists and it is expanding on $\partial\mathbb{D}$ (see [P2, Section 7]). We prove the following

Theorem A. *Let ν be an ergodic g-invariant probability measure on $\partial\mathbb{D}$ such that for ν-a.e. $\zeta\in\partial\mathbb{D}$ the radial limit $\widehat{R}(\zeta):=\lim_{r\nearrow 1}R(r\zeta)$ exists. Assume that the measure $\mu:=\widehat{R}_*(\nu)$ has positive Lyapunov exponent $\chi_\mu(f)$.*

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Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a continuous real-valued function. Then for every $\varepsilon > 0$ there exist a g-invariant expanding repeller $Y \subset \partial\mathbb{D}$ and $C > 0$ such that for all positive integers n and all $\zeta \in Y$,

(i) $-\ln C + n(\int \varphi\, d\nu - \varepsilon) \le \sum_{j=0}^{n-1} \varphi(g^j(\zeta)) \le \ln C + n(\int \varphi\, d\nu + \varepsilon)$.

(ii) $\widehat{R}$ *is defined on all of* Y *and finite-to-one on* Y. *Moreover* $R(r\zeta) \to \widehat{R}(\zeta)$ *uniformly as* $r \nearrow 1$ *for* $\zeta \in Y$. *The set* $X := \widehat{R}(Y)$ *is an expanding repeller for* f *contained in* $\mathrm{Fr}\, \Omega$. *Both* Y *and* X *are Cantor sets.*

(iii) $C^{-1} \exp n(\chi_\mu(f) - \varepsilon) \le |(f^n)'(\widehat{R}(\zeta))| \le C \exp n(\chi_\mu(f) + \varepsilon)$.

(iv) $\mathrm{HD}(X) \ge \mathrm{HD}(\mu) - \varepsilon$.

The existence of an expanding repeller $X \subset \mathrm{Fr}\, \Omega$ *satisfying* (iii) *for all* $x \in X$ (*in place of* $\widehat{R}(\zeta)$) *and* (iv) *holds without the assumption that* Ω *is simply connected.*

Above, X being an *expanding repeller* for f means that X is compact, $f(X) \subset X$ and the map f restricted to X is open, topologically mixing and expanding, that is, there exist $C > 0$ and $\lambda > 1$, called an *expanding constant*, such that $|(f^n)'(x)| \ge C\lambda^n$ for every $x \in X$. The property that $f|_X$ is open is equivalent to the existence of a neighbourhood U of X in $\mathbb{C}$, called a *repelling neighbourhood*, such that every forward f-trajectory $x, f(x), \ldots, f^n(x), \ldots$ staying in U must be contained in X. The definition of an expanding repeller $Y \subset \partial\mathbb{D}$ for g is similar. $\mathrm{HD}(X)$ denotes the Hausdorff dimension of the set X, and $\mathrm{HD}(\mu)$ the Hausdorff dimension of the measure μ which is defined as the infimum of the Hausdorff dimensions of sets of full measure μ.

Property (iv) is a version of the fact that the hyperbolic Hausdorff dimension of the Julia set $J(f)$ for a rational mapping (= supremum of the Hausdorff dimensions of expanding repellers contained in $J(f)$) is equal to the hyperbolic dynamical dimension (= supremum of the Hausdorff dimensions of invariant probability measures on $J(f)$ of positive Lyapunov exponents); see for example [PU].

Theorem A, with property (v′) below added to the conclusions, extends the main theorem from the paper [PZ], where the density of periodic orbits in $\mathrm{Fr}\, \Omega$ was proved. The idea of the proof, as in [PZ], is to apply Pesin and Katok theories; see [HK, Suplement] for a general theory and [PU, Ch. 9] for its adaptation in holomorphic iteration. The problem is, as in [PZ], that the standard Katok method to produce a large hyperbolic (here expanding) set does not guarantee that the set is in $\mathrm{Fr}\, \Omega$. It does not give the set Y either.

We needed this theorem in [P3], applied to $\varphi = \ln |g'|$ and μ in the harmonic measure class, but it is of independent interest, so we have decided to put it in a separate paper.

2. Additional properties. The following additional properties of suitably constructed X in Theorem A will be proved:

(v) X can be arbitrarily close to the topological support $\operatorname{supp}\mu$ in the Hausdorff metric in the space of compact subsets of $\operatorname{Fr}\Omega$.

(vi) For any finite families of real-valued continuous functions $\varphi_1,\dots,\varphi_k$ on $\partial\mathbb{D}$, $\psi_1,\dots,\psi_{k'}$ on $\operatorname{Fr}\Omega$, for every $i=1,\dots,k$ and $i=1,\dots,k'$ respectively, for all $\zeta\in Y$, $x\in X$ and positive integers n,

$$-\ln C+n\Big(\int_{\partial\mathbb{D}}\varphi_i\,d\nu-\varepsilon\Big)\le\sum_{j=0}^{n-1}\varphi_i(g^j(\zeta))\le\ln C+n\Big(\int_{\partial\mathbb{D}}\varphi_i\,d\nu+\varepsilon\Big),$$

$$-\ln C+n\Big(\int_{\operatorname{Fr}\Omega}\psi_i\,d\mu-\varepsilon\Big)\le\sum_{j=0}^{n-1}\psi_i(f^j(x))\le\ln C+n\Big(\int_{\operatorname{Fr}\Omega}\psi_i\,d\mu+\varepsilon\Big).$$

(vii) For P denoting the topological pressure and $\mathrm{h}_{\mathrm{top}}$ the topological entropy,

$$\mathrm{P}(f|_X,\psi_i)\ge\mathrm{h}_\mu(f)+\int_{\operatorname{Fr}\Omega}\psi_i\,d\mu-\varepsilon,$$

$$\mathrm{P}(g|_Y,\varphi_i)\ge\mathrm{h}_\nu(g)+\int_{\partial\mathbb{D}}\varphi_i\,d\nu-\varepsilon,$$

in particular

(viii) $\mathrm{h}_{\mathrm{top}}(f|_X)\ge\mathrm{h}_\mu(f)-\varepsilon$ and $\mathrm{h}_{\mathrm{top}}(g|_Y)\ge\mathrm{h}_\nu(g)-\varepsilon$.

(xi) $\mathrm{HD}(Y)\ge\mathrm{HD}(\nu)-\varepsilon$.

REMARK 1. Property (v) implies

(v′) If $\operatorname{supp}\mu=\operatorname{Fr}\Omega$ then X is arbitrarily close to $\operatorname{Fr}\Omega$ in the Hausdorff metric.

The assumption $\operatorname{supp}\mu=\operatorname{Fr}\Omega$ holds for every $\mu=\widehat{R}_*(\nu)$ for ν being a g-invariant Gibbs state (measure) for a Hölder continuous potential function on $\partial\mathbb{D}$ (see [PZ]). In this case ν has positive entropy, hence the existence of the radial limit ν-a.e. holds automatically (see [PZ] and references there, in particular [P1]). This automatically implies $\chi_\mu(f)>0$, since $0<\mathrm{h}_\nu(g)=\mathrm{h}_\mu(f)\le 2\chi_\mu(f)$ (Ruelle inequality).

REMARK 2. The radial convergence in (ii) automatically implies the nontangential convergence. This means the following: For every $\zeta\in\partial\mathbb{D}$, $0<\vartheta<\pi/2$ and $t>0$ define

$$S_{\vartheta,t}(\zeta)=\zeta\cdot(1+\{x\in\mathbb{C}\setminus\{0\}:\pi-\vartheta\le\operatorname{Arg}(x)\le\pi+\vartheta,\ |x|<t\}).$$

Such a set is called a *Stolz angle*. If t is irrelevant we skip it and write S_ϑ. Now (ii) can be written as (ii′) For every $0 < \vartheta < \pi/2$ the convergence $R(z) \to \widehat{R}(\zeta)$ is uniform for $\zeta \in X$ as $z \to \zeta$ and $z \in S_\vartheta$. The rate of convergence is exponential, more precisely, there exists $C > 0$ such that for $z \in S_{\vartheta,r}(\zeta)$,

$$C^{-1}(1-r)^{\chi_\mu(f)/(\chi_\nu(g)-\varepsilon)} \le \operatorname{dist}(R(z), \widehat{R}(\zeta)) \le C(1-r)^{\chi_\nu(g)/(\chi_\mu(f)+\varepsilon)}.$$

3. Geometric coding tree version. As in [PZ], we prove a more general, abstract version of these results, in the language of a geometric coding tree. We recall the definitions and notation:

Let U be an open connected subset of the Riemann sphere $\overline{\mathbb{C}}$. Consider any holomorphic mapping $f : U \to \overline{\mathbb{C}}$ such that $f(U) \supset U$ and $f : U \to f(U)$ is a proper map. Define $\operatorname{Crit}(f) = \{z : f'(z) = 0\}$, the set of *critical points* for f. Suppose that $\operatorname{Crit}(f)$ is finite. Consider any $z \in f(U)$. Let $z^1, \ldots, z^d$ be some of the f-preimages of z in U where $d \ge 2$. Consider continuous curves $\gamma^j : [0,1] \to f(U)$, $j = 1, \ldots, d$, joining z to z^j respectively (i.e. $\gamma^j(0) = z$, $\gamma^j(1) = z^j$) such that there are no critical values for the iterates of f in $\bigcup_{j=1}^d \gamma^j$, i.e. $\gamma^j \cap f^n(\operatorname{Crit}(f)) = \emptyset$ for every j and $n > 0$.

Let $\Sigma^d := \{1, \ldots, d\}^{\mathbb{Z}^+}$ denote the one-sided shift space and σ the shift to the left, i.e. $\sigma((\alpha_n)) = (\alpha_{n+1})$. For every sequence $\alpha = (\alpha_n)_{n=0}^\infty \in \Sigma^d$ we define $\gamma_0(\alpha) := \gamma^{\alpha_0}$. Suppose that for some $n \ge 0$, every $0 \le m \le n$, and all $\alpha \in \Sigma^d$, the curves $\gamma_m(\alpha)$ are already defined. Suppose that for $1 \le m \le n$ we have $f \circ \gamma_m(\alpha) = \gamma_{m-1}(\sigma(\alpha))$, and $\gamma_m(\alpha)(0) = \gamma_{m-1}(\alpha)(1)$.

Define the curves $\gamma_{n+1}(\alpha)$ so that the previous equalities hold by taking suitable f-preimages of γ_n. For every $\alpha \in \Sigma^d$ and $n \ge 0$ set $z_n(\alpha) := \gamma_n(\alpha)(1)$. Note that $z_n(\alpha)$ and $\gamma_n(\alpha)$ depend only on $(\alpha_0, \ldots, \alpha_n)$ so sometimes we consider z_n and γ_n as functions on blocks of symbols of length $n+1$. Sometimes it is convenient to denote z by z_{-1}.

The graph $\mathcal{T}(z, \gamma^1, \ldots, \gamma^d)$ with vertices z and $z_n(\alpha)$ and edges $\gamma_n(\alpha)$ is called a *geometric coding tree* with root at z. For every $\alpha \in \Sigma^d$ the subgraph composed of $z, z_n(\alpha)$ and $\gamma_n(\alpha)$ for all $n \ge 0$ is called a *geometric branch* and denoted by $b(\alpha)$.

For each $j = 1, \ldots, d$ we define f_j^{-1} on a small neighbourhood of z as the branch of f^{-1} mapping z to z^j. For each $\alpha \in \Sigma^d$ the branch f_j^{-1} has an analytic continuation $f_{j,\alpha}^{-1}$ along the curve $b(\alpha)$. Note that by construction $f_{j,\alpha}^{-1}(b(\alpha)) = b(j\alpha)$, where $j\alpha$ is the concatenation of the symbol j and the sequence α. By induction, for any block w of k symbols in $\{1, \ldots, d\}$, for f_w^{-k} being the branch of f^{-k} mapping z to $z_{k-1}(w)$ and for $f_{w,\alpha}^{-k}$ being the analytic continuation along $b(\alpha)$, we get

$$f_{w,\alpha}^{-k}(b(\alpha)) = b(w\alpha). \tag{1}$$

Similar notation is used and properties hold for finite sequences α, where for $\alpha = (\alpha_0, \ldots, \alpha_n)$, $b(\alpha)$ is the path in $\mathcal{T}$ from z to $z_n(\alpha)$.

For infinite α the branch $b(\alpha)$ is called *convergent* if the sequence $\gamma_n(\alpha)$ is convergent to a point in $\mathrm{cl}\, U$ in the Hausdorff metric. We define the *coding map* $z_\infty : \mathcal{D}(z_\infty) \to \mathrm{cl}\, U$ by $z_\infty(\alpha) := \lim_{n\to\infty} z_n(\alpha)$ on the domain $\mathcal{D} = \mathcal{D}(z_\infty)$ of all α's for which $b(\alpha)$ is convergent.

For each geometric branch $b(\alpha)$ denote by $b_m(\alpha)$ the part of $b(\alpha)$ starting from $z_m(\alpha)$, i.e. consisting of the vertices $z_k(\alpha)$, $k \geq m$, and of the edges $\gamma_k(\alpha)$, $k > m$.

If the map f extends holomorphically to a neighbourhood of the closure of the limit set Λ of a geometric coding tree, $\Lambda = z_\infty(\mathcal{D}(z_\infty))$, then Λ is called a *quasi-repeller* (see [PUZ]). Note that $f(\Lambda) \subset \Lambda$ and $f z_\infty = z_\infty \sigma$.

THEOREM B. *Let Λ be a quasi-repeller for a geometric coding tree $\mathcal{T}(z, \gamma^1, \ldots, \gamma^d)$ for a holomorphic map $f : U \to \overline{\mathbb{C}}$. Let ν be an ergodic σ-invariant probability measure on Σ^d such that for ν-a.e. $\alpha \in \Sigma^d$ the limit $z_\infty(\alpha)$ exists. Assume that the measure $\mu := z_\infty(\nu)$ has positive Lyapunov exponent $\chi_\mu(f)$. Let $\varphi, \varphi_j, \psi_j$ be continuous real-valued functions on Σ^d or* $\mathrm{cl}\, \Lambda$ *respectively. Then all the properties* (i)–(ix) *hold, with $\widehat{R} : \partial\mathbb{D} \to \mathrm{Fr}\, \Omega$ replaced by $z_\infty : \Sigma^d \to \mathrm{cl}\, \Lambda$ defined ν-a.e. and $R(r\zeta) \to \widehat{R}(\zeta)$ replaced by $\gamma_n(\alpha) \to z_\infty(\alpha)$ as $n \to \infty$.*

The assumption that $z_\infty(\alpha)$ exists for ν-a.e. $\alpha \in \Sigma^d$, i.e. $\nu(\mathcal{D}) = 1$, holds for every ν of positive entropy (compare Remark 1; see [PZ, Convergence Theorem], where further references are given). As in the Riemann mapping case, $\chi_\mu(f) > 0$ then holds automatically.

In the setting of Theorem B property (v′) also holds, with $\mathrm{Fr}\, \Omega$ replaced by $\mathrm{cl}\, \Lambda$, which immediately follows from (v).

The assumption $\mathrm{supp}\, \mu = \mathrm{cl}\, \Lambda$ holds whenever ν is a σ-invariant Gibbs state for a Hölder continuous function on Σ^d (cf. Remark 1), and if additionally the tree $\mathcal{T}$ satisfies $\gamma^j \cap \mathrm{cl}(\bigcup_{n\geq 0} f^n(\mathrm{Crit} f)) = \emptyset$ for all $j = 1, \ldots, d$ and there exists a neighbourhood $U^j \subset f(U)$ of γ^j such that $\mathrm{area}(f^{-n}(U^j)) \to 0$, where area denotes the standard Riemann measure on $\overline{\mathbb{C}}$.

For the proof see [PZ, Lemma 3], where $\mathrm{cl}\, \Lambda$ is replaced by a formally larger set $\widehat{\Lambda} := \{$all limit points of the sequences $z_n(\alpha^n)$, $\alpha^n \in \Sigma^d, n \to \infty\}$. It is easy to see that the above conditions about the tree $\mathcal{T}$ hold if $\mathcal{T}$ is in $W \cap \Omega$, close enough to $\mathrm{Fr}\, \Omega$, as in the situation of Theorem A (see Section 5).

4. Proof of Theorem B

STEP 1: *Good backward branches and their number.* Denote the natural extension of the one-sided shift $\sigma : \Sigma^d \to \Sigma^d$ preserving a Borel probability measure ν, i.e. the corresponding two-sided shift, by $(\widetilde{\Sigma}^d, \widetilde{\nu}, \widetilde{\sigma})$. Denote the projection $\widetilde{\Sigma}^d \to \Sigma^d$ mapping α to $(\alpha_0, \alpha_1, \ldots)$ by π_+. For each $\alpha \in \widetilde{\Sigma}^d$ denote $\pi_+(\alpha)$ by α^+.

By Pesin theory (see [PZ, Lemma 1] for the version we apply) and by the Birkhoff Ergodic Theorem applied to φ, for every $\varepsilon > 0$ we can find a set $K \subset \widetilde{\Sigma}^d$, constants $C, \delta > 0$ and a positive integer M such that $\widetilde{\nu}(K) > 1-\varepsilon$ and for all $\alpha \in K$ and $n \geq 0$,

B(i) $-\ln C + n(\int \varphi\, d\nu - \varepsilon/2) \leq \sum_{j=0}^{n-1} \varphi(\sigma^j(\alpha^+)) \leq \ln C + n(\int \varphi\, d\nu + \varepsilon/2)$.

B(ii) $b_M(\alpha^+) \subset B(z_\infty(\alpha^+), \delta/3)$.

B(iii) There exist univalent branches f_α^{-n} of f^{-n} on $B(z_\infty(\alpha^+), \delta)$ for all $n = 1, 2, \ldots$ mapping $z_\infty(\alpha^+)$ to $z_\infty(\widetilde{\sigma}^{-n}(\alpha)^+)$.

In the notation accompanying property (1) these branches are the continuations along $b(\alpha^+)$ of $f^{-n}_{(\alpha_{-n}, \ldots, \alpha_{-1})}$, i.e. the branches $f^{-n}_{(\alpha_{-n}, \ldots, \alpha_{-1}), \alpha^+}$.

Moreover

B(iv) $$C^{-1} \exp n(\chi_\mu(f) - \varepsilon/2) \leq |(f^n)'(z_\infty(\widetilde{\sigma}^{-n}(\alpha)^+))| \leq C \exp n(\chi_\mu(f) + \varepsilon/2).$$

B(v) $|(f_\alpha^{-n})'(x)|/|(f_\alpha^{-n})'(y)| < C$ for all $x, y \in B(z_\infty(\alpha^+), \delta)$.

For $-\infty \leq r \leq s \leq \infty$ and $\alpha \in \widetilde{\Sigma}^d$ or $\alpha \in \Sigma_{r,s} = \{1, \ldots, d\}^{\{r, r+1, \ldots, s\}}$, we denote by $C_{r,s}(\alpha)$ the cylinder $\{w \in \widetilde{\Sigma}^d : w_j = \alpha_j \text{ for all } j : r \leq j \leq s\}$. The projection $\widetilde{\Sigma}^d \ni (\ldots, \alpha_j, \ldots) \mapsto (\alpha_r, \ldots, \alpha_s) \in \Sigma_{r,s}$ will be denoted by $\pi_{r,s}$. Note that $C_{r,s}(\alpha) = \pi_{r,s}^{-1} \pi_{r,s}(\alpha)$.

Choose an arbitrary cylinder $C_M := C_{0,M}(\beta)$, for a fixed sequence $\beta = (\beta_0, \ldots, \beta_M) \in \Sigma_M := \Sigma_{0,M}$, such that $\widetilde{\nu}(C_M \cap K) \geq \widetilde{\nu}(C_M)/2$, which is possible provided $\varepsilon \leq 1/2$.

Denote $C_M \cap K$ by K'. For all $n \geq 0$ consider $K_n := \widetilde{\sigma}^{-n}(K')$. By the invariance of $\widetilde{\nu}$ we have $\widetilde{\nu}(K_n) \geq \widetilde{\nu}(C_M)/2 =: \xi$.

By the Birkhoff Ergodic Theorem there exists $N \geq 0$ such that

$$\nu(\{\alpha \in K_n : \exists i : 0 \leq i \leq N,\ \widetilde{\sigma}^{-i}(\alpha) \in K'\} \geq \xi/2.$$

Therefore for every $n \geq 0$ there exists N' with $0 \leq N' \leq N$ such that, setting $n' := n + N'$, for $A(n') := \{\alpha \in K' : \widetilde{\sigma}^{-n'}(\alpha) \in K'\}$ we have

$$\widetilde{\nu}(A(n')) \geq \xi/2N. \tag{2}$$

For every $\alpha \in A(n')$ we obtain $b_M(\widetilde{\sigma}^{-n'}(\alpha)^+) \subset B(z_M(\alpha^+), \delta/3)$. Indeed, for $\alpha' = \sigma^{-n'}(\alpha)$ we have $\pi_{0,M}(\alpha') = \beta$, as we have landed with α' in C_M. The length of $b_M(\alpha'^+)$ is at most $\delta/3$ as $\alpha' \in K$.

Hence

$$f_\alpha^{-n'}(\mathrm{cl}(B(z_M(\beta), 2\delta/3))) \subset B(z_M(\beta), 2\delta/3) \tag{3}$$

for all n large enough, more precisely for n such that

(4) $$|(f_\alpha^{-n'})'(x)| < 1/2 \quad \text{for all } x \in B(z_M(\beta), 2\delta/3).$$

By B(ii)–B(iv) this holds for $n \geq (2\ln C + \ln 2)/(\chi_\mu(f) - \varepsilon)$.

CLAIM. *The branches $f_\alpha^{-n'}$ on $B(z_\infty(\alpha^+), \delta)$ depend only on $\pi_{-n',M}(\alpha)$, more precisely on $\pi_{-n',-1}(\alpha)$ as $\pi_{0,M}(\alpha) = \beta$ has been fixed, on the common domain $B := B(z_M(\beta), 2\delta/3)$.*

This is so since if two α's in $A(n')$, say α and α', have the same block $(\alpha_{-n'}, \ldots, \alpha_{-1})$, then the branches $f_\alpha^{-n'}$ and $f_{\alpha'}^{-n'}$ are continuations of the same branch at z along curves coinciding till $z_M(\beta)$ and next contained in the common domain B (see Figure 1).

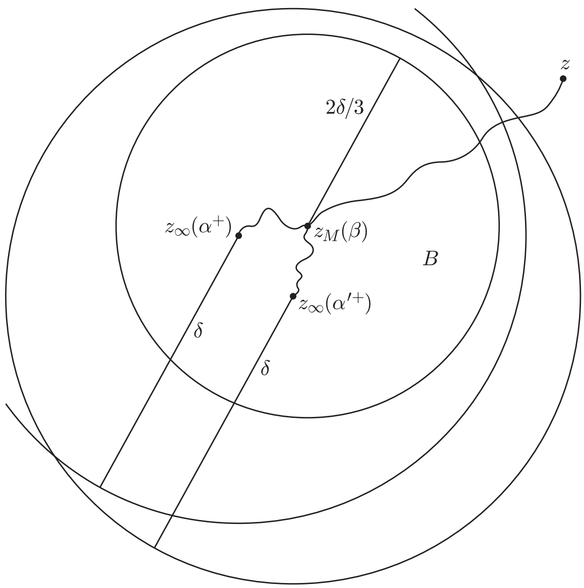

Fig. 1

We shall not use this claim directly, but we put it to help the reader understand the proof and to simplify notation later on.

By (2) we have $\widetilde{\nu}(\pi_{-n',M}^{-1}\pi_{-n',M}(A(n'))) \geq \xi/2N$. By the Shannon–McMillan–Breiman theorem for every $\eta > 0$ and all integers k large enough,

$$\nu\Big(\bigcup\{C_{0,k}(w) : w \in \Sigma_{0,k},\ \nu(C_{0,k}(w)) \leq \exp k(\mathrm{h}_\nu(\sigma) + \varepsilon/3)\}\Big) \geq 1 - \eta.$$

Setting $\eta = \xi/4N$ and $k = M + n'$ we get

$$\widetilde{\nu}\Big(\bigcup\{C_{-n',M}(w) : w \in A(n')),\ \widetilde{\nu}(C_{-n',M}(w)) \leq \exp\,(n'+M)(\mathrm{h}_\nu(\sigma)+\varepsilon/3)\}\Big) \geq \xi/2N - \xi/4N = \xi/2N.$$

Therefore for n large enough, the number of "good backward trajectories" of length n' can be estimated as follows:

$$\#(\pi_{-n',-1}(A(n'))) \geq \exp n'(\mathrm{h}_\nu(\sigma) - \varepsilon/2). \tag{5}$$

STEP 2: *The sets X, Y and IFS.* Now define $Y' \subset \Sigma^d$ as the set of one-sided sequences which are concatenations of blocks v^k belonging to $G_{n'} := \pi_{0,n'-1}\widetilde{\sigma}^{-n'}(A(n'))$, that is,

$$Y' = \{\alpha = v^0 v^1 \ldots \in \Sigma^d : v^k = \pi_{0,n'-1}\sigma^{kn'}(\alpha) \in G_{n'} \ \forall k = 0, 1, \ldots\},$$

and set

$$X' = z_\infty(Y').$$

Finally, define

$$Y = \bigcup\{\sigma^j(Y') : j = 0, \ldots, n'-1\},$$
$$X = \bigcup\{f^j(X') : j = 0, \ldots, n'-1\} = z_\infty(Y).$$

For each $\alpha \in \Sigma^d$ and $r \leq s$ denote by $b_{r,s}$ the part of the branch $b(\alpha)$ starting from $z_{r-1}(\alpha)$ and ending at $z_s(\alpha)$.

Now, to put it briefly, by (3) and (4) for every $\alpha \in Y'$ the length of $b_{kn',(k+1)n'-1}(\alpha)$ is less than $C2^{-k}$ for a constant $C > 0$. Hence $z_n(\alpha) \to z_\infty(\alpha)$ uniformly (even exponentially fast), which proves (ii) on Y', hence on Y by the uniform continuity of f. By (3) and (4), X', and hence X, are expanding repellers for $f^{n'}$ and f respectively.

Let us now be more precise. Let $\alpha \in Y'$ be a concatenation of $v^k = \pi_{0,n'-1}\widetilde{\sigma}^{-n'}(w^k)$, for $w^k \in A(n')$, for $k = 0, 1, \ldots$. We want to analyse $b(\alpha)$. Note that by (1),

$$b_{(k-1)n',kn'-1}(\alpha) = f^{-n'}_{v^0,v^1v^2\ldots v^k}(\ldots(f^{-n'}_{v^{k-1},v^k}(b_{0,n'-1}(v^k)))). \tag{6}$$

Assume that $n' > M$. Then all $b(\alpha)$ for $\alpha = v^0v^1\ldots \in Y'$ pass through $z_M(\beta)$ since $v^0 \in G_{n'}$ implies that $b_{0,n'-1}(\alpha)$ depending only on v^0 passes through $z_M(\beta)$. (There is no reason for α to belong to $A(n')$, which would imply passing through $z_M(\beta)$ by definition, as in the Claim. So for the first time in the proof we need to use $n' > M$.)

Now we apply induction on k. Suppose that for every $\alpha \in \Sigma_{0,kn'-1}$ which is a concatenation $v^0v^1\ldots v^{k-1}$ of blocks v^j in $G_{n'}$ we have $b_{M+1,kn'-1}(\alpha) \subset B$ (see Figure 2)). Take an arbitrary $v \in G_{n'}$ which is the truncation of $w \in A(n')$, more precisely $v = \pi_{0,n'-1}\widetilde{\sigma}^{-n'}(w)$. Then $f^{-n'}_{v,\alpha}$ and $f^{-n'}_w$ coincide on B, in particular on $b_{M,kn'-1}(\alpha)$, since also $b_M(w^+)$ is contained in B, as $w \in K$, yielding a path in $\mathcal{T}$ joining $z_{kn'-1}(\alpha)$ to $z_\infty(w^+)$ and entirely contained in B (compare the proof of Claim). Hence, by (3) applied to $f^{-n'}_w$ we get $b_{M+1,(k+1)n'-1}(v\alpha) \subset B$, which finishes the induction.

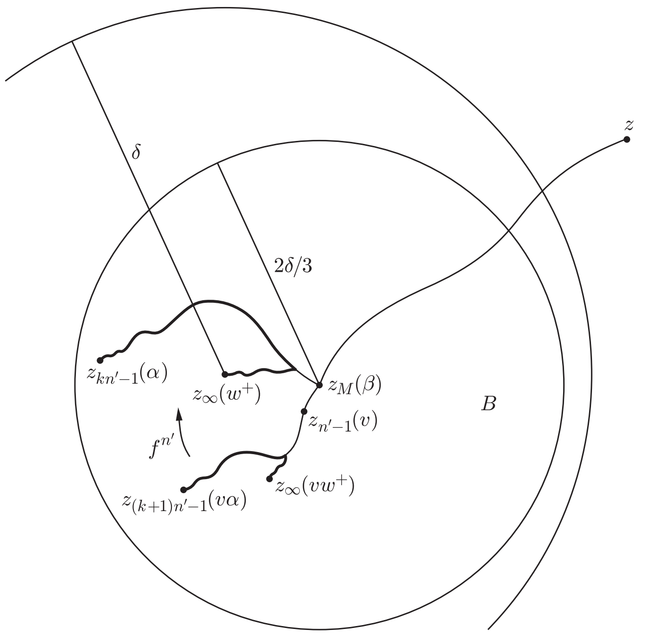

Fig. 2

Therefore in (6) we can replace $f^{-n'}_{v^j,v^{j+1}v^{j+2}\dots v^k}$ by $f^{-n'}_{w^j}$ for all $j = 0,1,\dots,k-1$, in particular these branches of $f^{-n'}$ act on branches of the tree $\mathcal{T}$ in the common domain B (except $b_{0,n'-1}(v^k)$).

One can view the family of branches $F_v := f_v^{-n'}$ for $v \in G_{n'}$ as an iterated function system (IFS) on B. It satisfies the so-called Strong Open Set Condition, i.e. all $F_v(B)$ have pairwise disjoint closures. The Claim allows us to write v in place of w, where v is the truncation of w. These branches also act on (extend to) $b_{0,M}(\beta)$, the line in the tree joining z to $z_M(v)$ which need not be contained in B. So F_v need not contract it. But further iteration contracts them exponentially since $F_v(b_{0,M}(\beta))$ lies already in B.

The limit set is contained in $\operatorname{cl} z_\infty(\Sigma^d)$, since the F_v preserve the tree $\mathcal{T}$.

STEP 3: *Proving properties* (i)–(ix) *in Theorem B.* To prove (i) consider an arbitrary $\alpha = v^0v^1\ldots \in Y'$ for $v^k = \pi_{0,n'-1}\widetilde{\sigma}^{-n'}(w^k)$, where $w^k \in A(n')$. Then, for each $k = 1,2,\dots,$

$$(7)\qquad \sum_{j=0}^{kn'-1} \varphi(\sigma^j(\alpha^+)) - \sum_{i=0}^{k-1}\sum_{j=0}^{n'-1} \varphi(\sigma^j((\widetilde{\sigma}^{-n'}(w^i))^+)) \le kn'\varepsilon/2$$

for n large enough. This follows from the continuity of φ since $\sigma^{in'+j}(\alpha)$ and $\sigma^j((\widetilde{\sigma}^{-n'}(w^i))^+)$ are very close to each other for all i and $0 \le j \ll n'$. This is so because both one-sided sequences have the same beginning of length

$n'-j$. Now (i) follows from the estimate B(i) on $\sum_{j=0}^{n'-1}\varphi(\sigma^j((\widetilde{\sigma}^{n'}(w^i))^+)$. Passing from Y' to Y changes only the constant C in (i).

These considerations also prove (vi). Indeed, in the case of ψ one ensures the property of K analogous to B(i), namely

$$\text{B(vi)}\qquad -\ln C + n\Big(\int\psi\,d\mu - \varepsilon/2\Big) \le \sum_{j=0}^{n-1}\psi(f^j(z_\infty(\alpha^+))) \le \ln C + n\Big(\int\psi\,d\mu + \varepsilon/2\Big),$$

following from the ν-integrability of $\psi\circ z_\infty$ and the Birkhoff Ergodic Theorem. Use also the property analogous to (7), for ψ and f in place of φ and σ, which follows from the continuity of ψ and the fact that the preimages of points in B under the same branch $f^j f_{v^i}^{-n'}$ of $f^{-(n'-j)}$ are very close to each other for $0\le j\ll n'$.

The uniform (exponential) convergence in (ii) has already been proven. The injectivity and the property of X' of being a Cantor set follow from the Strong Open Set Condition of the IFS $\{F_v\}$. This implies that z_∞ is finite-to-one on Y and X is also a Cantor set.

By (5) and (i) and by the definition of pressure,

$$\mathrm{P}\Big(\sigma^{n'}|_{Y'}, \sum_{j=0}^{n'-1}\varphi\circ\sigma^j\Big) \ge \mathrm{h}_\nu(\sigma^{n'}) + n'\Big(\int\varphi\,d\nu - \varepsilon\Big),$$

hence easily $\mathrm{P}(\sigma|_Y,\varphi)\ge \mathrm{h}_\nu(f)+\int_Y\varphi\,d\nu-\varepsilon$, proving (vii) for $\mathrm{P}(\sigma|_Y,\varphi)$. The argument for $\mathrm{P}(f|_X,\psi)$ is similar, using (vi) for ψ.

Note that one cannot pull back to Σ^d to refer to (vii) for $\mathrm{P}(\sigma,\psi\circ z_\infty)$ on Y since $\psi\circ z_\infty$ need not be continuous on $\partial\mathbb{D}$, even not defined, so we might not have (7).

By [M], or [P1, Sec. 3] where further references are provided, we have $\mathrm{HD}(\mu)=\mathrm{h}_\mu(f)/\chi_\mu(f)$. Consider an arbitrary $\varepsilon'>0$ and set $t':=\mathrm{HD}(\mu)-\varepsilon'$. Then $t'=\mathrm{h}_\mu(f)/\chi_\mu(f)-\varepsilon'$. By (iii) and (5),

$$\begin{aligned}\mathrm{P}(f|_X, -t'\ln|f'|_X|) &\ge \mathrm{h}_{\mathrm{top}}(f|_X) - t'(\chi_\mu(f)+\varepsilon)\\ &\ge \mathrm{h}_\mu(f)-\varepsilon-(\mathrm{h}_\mu(f)/\chi_\mu(f)-\varepsilon')(\chi_\mu(f)+\varepsilon)\\ &\ge -\varepsilon-\varepsilon\mathrm{h}_\mu(f)/\chi_\mu(f)+\varepsilon'\chi_\mu(f)+\varepsilon\varepsilon',\end{aligned}$$

which is positive if

$$\varepsilon' > \frac{\varepsilon(1+\mathrm{h}_\mu(f))/\chi_\mu(f)}{\chi_\mu(f)+\varepsilon}.$$

Hence $\mathrm{HD}(X)>t'$ as $\mathrm{HD}(X)$ is not smaller than the first zero of the pressure function $t\mapsto\mathrm{P}(f|_X,-t\ln|f'|_X|)$, by the Bowen theorem (see for example [PU]). If we choose ε small we obtain ε' small, hence $\mathrm{HD}(X)$ arbitrarily close to $\mathrm{HD}(\mu)$, which proves (iv).

We prove (ix) similarly.

To prove (v) consider the cylinder $C_M = C_{0,M}(\beta)$ for β being the truncation of a sequence α dense in $\operatorname{supp}\nu$ and M large. The proof of Theorem B is finished. ∎

5. Conclusions. Theorem B easily implies Theorem A. One builds the tree $\mathcal{T}$ in the basin of attraction. It is only sufficient to note that the branches of the tree $R^{-1}(\mathcal{T})$ converge to $\partial\mathbb{D}$ nontangentially, so the convergence of each branch $b(\alpha)$ in $\mathcal{T}$ implies the nontangential, in particular radial, convergence of R at $\lim R^{-1}(b(\alpha)) \in \partial\mathbb{D}$, with the same limit. One considers the pull-back $\varphi\circ(R^{-1}(z))_\infty : \Sigma^d \to \mathbb{R}$, finds Y in Σ^d, maps it by $(R^{-1}(z))_\infty$ with the use of $R^{-1}(\mathcal{T})$ into $\partial\mathbb{D}$ and with the use of $\mathcal{T}$ to $X \subset \operatorname{Fr}\Omega$ as in Theorem B. The map $\widehat{R}$ is finite-to-one on Y since z_∞ is. The rate of the exponential convergence in (ii) and more precisely in (ii′) in Remark 2 follows easily from (iii), (i) applied to $\varphi = \ln|g'|$, and the chain rule $R'(z) = (f^{-n})'(R(g^n(z)))\cdot R'(g^n(z))\cdot(g^n)'(z)$ for $z = r\zeta$, the integer n such that for the first time $g^n(z)$ is far from $\partial\mathbb{D}$, and the appropriate branch of f^{-n}; for details see [P2]. See also [P3].

Remark 3. If ν is mixing, which is the case for Gibbs ν as in Remark 1, then one can ensure that f on X is topologically mixing, that is, for any open subsets U, V of X there exists n_0 such that $f^n(U)\cap V \neq \emptyset$ for all $n \geq n_0$.

Indeed, for n large we have by mixing $\widetilde{\nu}(\widetilde{\sigma}^{-n}(C_M)\cap C_M) \sim \nu(C_M)^2$. Hence, if $\nu(K)\approx 1$, then $\widetilde{\nu}(A(n)) \geq \mathrm{const} > 0$ for all n large (compare (2)). We can repeat the previous construction by taking instead of one n' two different mutually prime integers.

Remark 4. Theorem A holds in the case Ω is an immediate connected simply connected basin of attraction to a parabolic fixed point p, i.e. $p \in \operatorname{Fr}\Omega$ such that $f(p) = p$ and $f'(p)$ is a root of unity.

Indeed, in this case $R^{-1}\circ f\circ R$ extends to $\overline{\mathbb{C}}$ to yield g which is a Blaschke product such that $\mathbb{D}$ (and $\overline{\mathbb{C}}\setminus \operatorname{cl}\mathbb{D}$) is a basin of a parabolic fixed point for g in $\partial\mathbb{D}$. As in the conclusion that Theorem B implies Theorem A, we consider the trees $\mathcal{T}$ and $R^{-1}(\mathcal{T})$. All the branches of $R^{-1}(\mathcal{T})$ converge (polynomially fast, but not necessarily nontangentially), and at each limit point $\zeta = (R^{-1}(z))_\infty(\alpha)$ for $\alpha \in \mathcal{D}(z_\infty)$, in particular in Y, the radial limit $\widehat{R}(\zeta)$ coincides with $z_\infty(\alpha)$ by Lindelöf's theorem. Hence $z_\infty = \widehat{R}\circ(R^{-1}(z))_\infty$ on Y and all the maps involved are finite-to-one since z_∞ is finite-to-one on Y.

References

[HK] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, with supplement by A. Katok and L. Mendoza, Cambridge Univ. Press, 1995.

[M] R. Mañé, *The Hausdorff dimension of invariant probabilities of rational maps*, in: Dynamical Systems (Valparaiso, 1986), Lecture Notes in Math. 1331, Springer, 1988, 86–117.

[P1] F. Przytycki, *Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map*, Invent. Math. 80 (1985), 161–179.

[P2] —, *Riemann map and holomorphic dynamics*, ibid. 85 (1986), 439–455.

[P3] —, *An improvement of J. Rivera-Letelier result on weak hyperbolicity on periodic orbits for polynomials*, submitted; http://www.impan.gov.pl/~feliksp.

[PS] F. Przytycki and J. Skrzypczak, *Convergence and pre-images of limit points for coding trees for iterations of holomorphic maps*, Math. Ann. 290 (1991), 425–440.

[PU] F. Przytycki and M. Urbański, *Fractals in the Plane, Ergodic Theory Methods*, Cambridge Univ. Press, to appear; http://www.math.unt.edu/~urbanski and http://www.impan.gov.pl/~feliksp.

[PUZ] F. Przytycki, M. Urbański and A. Zdunik, *Harmonic, Gibbs and Hausdorff measures for holomorphic maps*, Part 1, Ann. of Math. 130 (1989), 1-40; Part 2, Studia Math. 97 (1991), 189–225.

[PZ] F. Przytycki and A. Zdunik, *Density of periodic sources in the boundary of a basin of attraction for iteration of holomorphic maps*: *geometric coding trees technique*, Fund. Math. 145 (1994), 65–77.

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