Expanding repellers in limit sets
for iterations of holomorphic functions

by

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Abstract. We prove that for $\Omega$ being an immediate basin of attraction to an attracting fixed point for a rational mapping of the Riemann sphere, and for an ergodic invariant measure $\mu$ on the boundary $\text{Fr}\Omega$, with positive Lyapunov exponent, there is an invariant subset of $\text{Fr}\Omega$ which is an expanding repeller of Hausdorff dimension arbitrarily close to the Hausdorff dimension of $\mu$. We also prove generalizations and a geometric coding tree abstract version. The paper is a continuation of a paper in Fund. Math. 145 (1994) by the author and Anna Zdunik, where the density of periodic orbits in $\text{Fr}\Omega$ was proved.

1. Introduction. Let $\Omega$ be a simply connected domain in $\overline{\mathbb{C}}$ and $f$ be a holomorphic map defined on a neighbourhood $W$ of $\text{Fr}\Omega$ to $\overline{\mathbb{C}}$. Assume $f(W \cap \Omega) \subset \Omega$, $f(\text{Fr}\Omega) \subset \text{Fr}\Omega$ and $\text{Fr}\Omega$ repells to the side of $\Omega$, that is, $\bigcap_{n=0}^{\infty} f^{-n}(W \cap \Omega) = \text{Fr}\Omega$. An important special case is where $\Omega$ is an immediate basin of attraction of an attracting fixed point for a rational function. This covers also the case of a component of the immediate basin of attraction to a periodic attracting orbit, as one can consider an iterate of $f$ mapping the component to itself. Distances and derivatives are considered in the Riemann spherical metric on $\mathbb{C}$.

Let $R : \mathbb{D} \to \Omega$ be a Riemann mapping from the unit disc onto $\Omega$ and let $g$ be a holomorphic extension of $R^{-1} \circ f \circ R$ to a neighbourhood of the unit circle $\partial\mathbb{D}$. It exists and it is expanding on $\partial\mathbb{D}$ (see [P2, Section 7]). We prove the following

**Theorem A.** Let $\nu$ be an ergodic $g$-invariant probability measure on $\partial\mathbb{D}$ such that for $\nu$-a.e. $\zeta \in \partial\mathbb{D}$ the radial limit $\hat{R}(\zeta) := \lim_{r \to 1} R(r\zeta)$ exists. Assume that the measure $\mu := \hat{R}_{*}(\nu)$ has positive Lyapunov exponent $\chi_\mu(f)$. 

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Let \( \varphi : \partial \mathbb{D} \to \mathbb{R} \) be a continuous real-valued function. Then for every \( \varepsilon > 0 \) there exist a \( g \)-invariant expanding repeller \( Y \subset \partial \mathbb{D} \) and \( C > 0 \) such that for all positive integers \( n \) and all \( \zeta \in Y \),

(i) \(- \ln C + n(\int \varphi \, d\nu - \varepsilon) \leq \sum_{j=0}^{n-1} \varphi(g^j(\zeta)) \leq \ln C + n(\int \varphi \, d\nu + \varepsilon)\).

(ii) \( \hat{R} \) is defined on all of \( Y \) and finite-to-one on \( Y \). Moreover \( R(r\zeta) \to \hat{R}(\zeta) \) uniformly as \( r \not\to 1 \) for \( \zeta \in Y \). The set \( X := \hat{R}(Y) \) is an expanding repeller for \( f \) contained in \( \text{Fr} \Omega \). Both \( Y \) and \( X \) are Cantor sets.

(iii) \( C^{-1} \exp n(x_\mu(f) - \varepsilon) \leq |(f^n)'(\hat{R}(\zeta))| \leq C \exp n(x_\mu(f) + \varepsilon)\).

(iv) \( \text{HD}(X) \geq \text{HD}(\mu) - \varepsilon \).

The existence of an expanding repeller \( X \subset \text{Fr} \Omega \) satisfying (iii) for all \( x \in X \) (in place of \( \hat{R}(\zeta) \)) and (iv) holds without the assumption that \( \Omega \) is simply connected.

Above, \( X \) being an expanding repeller for \( f \) means that \( X \) is compact, \( f(X) \subset X \) and the map \( f \) restricted to \( X \) is open, topologically mixing and expanding, that is, there exist \( C > 0 \) and \( \lambda > 1 \), called an expanding constant, such that \( |(f^n)'(x)| \geq C\lambda^n \) for every \( x \in X \). The property that \( f|_X \) is open is equivalent to the existence of a neighbourhood \( U \) of \( X \) in \( \mathbb{C} \), called a repelling neighbourhood, such that every forward \( f \)-trajectory \( x, f(x), \ldots, f^n(x), \ldots \) staying in \( U \) must be contained in \( X \). The definition of an expanding repeller \( Y \subset \partial \mathbb{D} \) for \( g \) is similar. \( \text{HD}(X) \) denotes the Hausdorff dimension of the set \( X \), and \( \text{HD}(\mu) \) the Hausdorff dimension of the measure \( \mu \) which is defined as the infimum of the Hausdorff dimensions of sets of full measure \( \mu \).

Property (iv) is a version of the fact that the hyperbolic Hausdorff dimension of the Julia set \( J(f) \) for a rational mapping (= supremum of the Hausdorff dimensions of expanding repellers contained in \( J(f) \)) is equal to the hyperbolic dynamical dimension (= supremum of the Hausdorff dimensions of invariant probability measures on \( J(f) \) of positive Lyapunov exponents); see for example [PU].

Theorem A, with property \((v')\) below added to the conclusions, extends the main theorem from the paper [PZ], where the density of periodic orbits in \( \text{Fr} \Omega \) was proved. The idea of the proof, as in [PZ], is to apply Pesin and Katok theories; see [HK, Suplement] for a general theory and [PU, Ch. 9] for its adaptation in holomorphic iteration. The problem is, as in [PZ], that the standard Katok method to produce a large hyperbolic (here expanding) set does not guarantee that the set is in \( \text{Fr} \Omega \). It does not give the set \( Y \) either.

We needed this theorem in [P3], applied to \( \varphi = \ln |g'| \) and \( \mu \) in the harmonic measure class, but it is of independent interest, so we have decided to put it in a separate paper.
2. Additional properties. The following additional properties of suitably constructed $X$ in Theorem A will be proved:

(v) $X$ can be arbitrarily close to the topological support $\text{supp } \mu$ in the Hausdorff metric in the space of compact subsets of $\text{Fr } \Omega$.

(vi) For any finite families of real-valued continuous functions $\varphi_1, \ldots, \varphi_k$ on $\partial \mathbb{D}$, $\psi_1, \ldots, \psi_{k'}$ on $\text{Fr } \Omega$, for every $i = 1, \ldots, k$ and $i = 1, \ldots, k'$ respectively, for all $\zeta \in Y$, $x \in X$ and positive integers $n$,

$$-\ln C + n \left( \int_{\partial \mathbb{D}} \varphi_i \, d\nu - \varepsilon \right) \leq \sum_{j=0}^{n-1} \varphi_i(g^j(\zeta)) \leq \ln C + n \left( \int_{\partial \mathbb{D}} \varphi_i \, d\nu + \varepsilon \right),$$

$$-\ln C + n \left( \int_{\text{Fr } \Omega} \psi_i \, d\mu - \varepsilon \right) \leq \sum_{j=0}^{n-1} \psi_i(f^j(x)) \leq \ln C + n \left( \int_{\text{Fr } \Omega} \psi_i \, d\mu + \varepsilon \right).$$

(vii) For $P$ denoting the topological pressure and $h_{\text{top}}$ the topological entropy,

$$P(f | X, \psi_i) \geq h_\mu(f) + \int_{\text{Fr } \Omega} \psi_i \, d\mu - \varepsilon,$$

$$P(g | Y, \varphi_i) \geq h_\nu(g) + \int_{\partial \mathbb{D}} \varphi_i \, d\nu - \varepsilon,$$

in particular

(viii) $h_{\text{top}}(f | X) \geq h_\mu(f) - \varepsilon$ and $h_{\text{top}}(g | Y) \geq h_\nu(g) - \varepsilon$.

(xi) $\text{HD}(Y) \geq \text{HD}(\nu) - \varepsilon$.

Remark 1. Property (v) implies

(v') If $\text{supp } \mu = \text{Fr } \Omega$ then $X$ is arbitrarily close to $\text{Fr } \Omega$ in the Hausdorff metric.

The assumption $\text{supp } \mu = \text{Fr } \Omega$ holds for every $\mu = \hat{R}_*(\nu)$ for $\nu$ being a $g$-invariant Gibbs state (measure) for a Hölder continuous potential function on $\partial \mathbb{D}$ (see [PZ]). In this case $\nu$ has positive entropy, hence the existence of the radial limit $\nu$-a.e. holds automatically (see [PZ] and references there, in particular [P1]). This automatically implies $\chi_\mu(f) > 0$, since $0 < h_\nu(g) = h_\mu(f) \leq 2\chi_\mu(f)$ (Ruelle inequality).

Remark 2. The radial convergence in (ii) automatically implies the nontangential convergence. This means the following: For every $\zeta \in \partial \mathbb{D}$, $0 < \vartheta < \pi/2$ and $t > 0$ define

$$S_{\vartheta, t}(\zeta) = \zeta \cdot (1 + \{x \in \mathbb{C} \setminus \{0\} : \pi - \vartheta \leq \text{Arg}(x) \leq \pi + \vartheta, |x| < t\}).$$

Such a set is called a Stolz angle. If $t$ is irrelevant we skip it and write $S_\vartheta$.

Now (ii) can be written as
(ii') For every $0 < \vartheta < \pi/2$ the convergence $R(z) \to \tilde{R}(\zeta)$ is uniform for $\zeta \in X$ as $z \to \zeta$ and $z \in S_\vartheta$. The rate of convergence is exponential, more precisely, there exists $C > 0$ such that for $z \in S_{\vartheta,r}(\zeta)$,

$$C^{-1}(1 - r)\chi_\alpha(f) / (\chi_\nu(g) - \varepsilon) \leq \text{dist}(R(z), \tilde{R}(\zeta)) \leq C(1 - r)\chi_\nu(g) / (\chi_\mu(f) + \varepsilon).$$

3. Geometric coding tree version. As in [PZ], we prove a more general, abstract version of these results, in the language of a geometric coding tree. We recall the definitions and notation:

Let $U$ be an open connected subset of the Riemann sphere $\mathbb{C}$. Consider any holomorphic mapping $f : U \to \mathbb{C}$ such that $f(U) \supset U$ and $f : U \to f(U)$ is a proper map. Define $\text{Crit}(f) = \{z : f'(z) = 0\}$, the set of critical points for $f$. Suppose that $\text{Crit}(f)$ is finite. Consider any $z \in f(U)$. Let $z^1, \ldots, z^d$ be some of the $f$-preimages of $z$ in $U$ where $d \geq 2$. Consider continuous curves $\gamma^j : [0, 1] \to f(U)$, $j = 1, \ldots, d$, joining $z$ to $z^j$ respectively (i.e. $\gamma^j(0) = z$, $\gamma^j(1) = z^j$) such that there are no critical values for the iterates of $f$ in $\bigcup_{j=1}^d \gamma^j$, i.e. $\gamma^j \cap f^n(\text{Crit}(f)) = \emptyset$ for every $j$ and $n > 0$.

Let $\Sigma^d := \{1, \ldots, d\}^\mathbb{Z}_+$ denote the one-sided shift space and $\sigma$ the shift to the left, i.e. $\sigma((\alpha_n)) = (\alpha_{n+1})$. For every sequence $\alpha = (\alpha_n)_{n=0}^\infty \in \Sigma^d$ we define $\gamma_0(\alpha) := \gamma^\alpha_0$. Suppose that for some $n \geq 0$, every $0 \leq m \leq n$, and all $\alpha \in \Sigma^d$, the curves $\gamma_m(\alpha)$ are already defined. Suppose that for $1 \leq m \leq n$ we have $f \circ \gamma_m(\alpha) = \gamma_{m-1}(\sigma(\alpha))$, and $\gamma_m(\alpha)(0) = \gamma_{m-1}(\alpha)(1)$.

Define the curves $\gamma_{n+1}(\alpha)$ so that the previous equalities hold by taking suitable $f$-preimages of $\gamma_n$. For every $\alpha \in \Sigma^d$ and $n \geq 0$ set $z_n(\alpha) := \gamma_n(\alpha)(1)$. Note that $z_n(\alpha)$ and $\gamma_n(\alpha)$ depend only on $(\alpha_0, \ldots, \alpha_n)$ so sometimes we consider $z_n$ and $\gamma_n$ as functions on blocks of symbols of length $n + 1$. Sometimes it is convenient to denote $z$ by $z_{-1}$.

The graph $T(z, \gamma^1, \ldots, \gamma^d)$ with vertices $z$ and $z_n(\alpha)$ and edges $\gamma_n(\alpha)$ is called a geometric coding tree with root at $z$. For every $\alpha \in \Sigma^d$ the subgraph composed of $z, z_n(\alpha)$ and $\gamma_n(\alpha)$ for all $n \geq 0$ is called a geometric branch and denoted by $b(\alpha)$.

For each $j = 1, \ldots, d$ we define $f^{-1}_j$ on a small neighbourhood of $z$ as the branch of $f^{-1}$ mapping $z$ to $z^j$. For each $\alpha \in \Sigma^d$ the branch $f^{-1}_j$ has an analytic continuation $f^{-1}_{j,\alpha}$ along the curve $b(\alpha)$. Note that by construction $f^{-1}_{j,\alpha}(b(\alpha)) = b(j\alpha)$, where $j\alpha$ is the concatenation of the symbol $j$ and the sequence $\alpha$. By induction, for any block $w$ of $k$ symbols in $\{1, \ldots, d\}$, for $f^{-k}_w$ being the branch of $f^{-k}$ mapping $z$ to $z_{k-1}(w)$ and for $f^{-k}_{w,\alpha}$ being the analytic continuation along $b(\alpha)$, we get

$$f^{-k}_{w,\alpha}(b(\alpha)) = b(w\alpha).$$
Expanding repellers in limit sets

Similar notation is used and properties hold for finite sequences $\alpha$, where for $\alpha = (\alpha_0, \ldots, \alpha_n)$, $b(\alpha)$ is the path in $T$ from $z$ to $z_n(\alpha)$.

For infinite $\alpha$ the branch $b(\alpha)$ is called convergent if the sequence $\gamma_n(\alpha)$ is convergent to a point in $\text{cl}U$ in the Hausdorff metric. We define the coding map $z_\infty : D(z_\infty) \rightarrow \text{cl}U$ by $z_\infty(\alpha) := \lim_{n \rightarrow \infty} z_n(\alpha)$ on the domain $D = D(z_\infty)$ of all $\alpha$’s for which $b(\alpha)$ is convergent.

For each geometric branch $b(\alpha)$ denote by $b_m(\alpha)$ the part of $b(\alpha)$ starting from $z_m(\alpha)$, i.e. consisting of the vertices $z_k(\alpha)$, $k \geq m$, and of the edges $\gamma_k(\alpha)$, $k > m$.

If the map $f$ extends holomorphically to a neighbourhood of the closure of the limit set $\Lambda$ of a geometric coding tree, $\Lambda = z_\infty(D(z_\infty))$, then $\Lambda$ is called a quasi-repeller (see [PUZ]). Note that $f(\Lambda) \subset \Lambda$ and $fz_\infty = z_\infty\sigma$.

**Theorem B.** Let $\Lambda$ be a quasi-repeller for a geometric coding tree $T(z, \gamma_1, \ldots, \gamma^d)$ for a holomorphic map $f : U \rightarrow \mathbb{C}$. Let $\nu$ be an ergodic $\sigma$-invariant probability measure on $\Sigma^d$ such that for $\nu$-a.e. $\alpha \in \Sigma^d$ the limit $z_\infty(\alpha)$ exists. Assume that the measure $\mu := z_\infty(\nu)$ has positive Lyapunov exponent $\chi_\mu(f)$. Let $\varphi, \varphi_j, \psi_j$ be continuous real-valued functions on $\Sigma^d$ or $\text{cl}\Lambda$ respectively. Then all the properties (i)–(ix) hold, with $\widehat{R} : \partial D \rightarrow \text{Fr} \Omega$ replaced by $z_\infty : \Sigma^d \rightarrow \text{cl} \Lambda$ defined $\nu$-a.e. and $\text{R}(r\zeta) \rightarrow \widehat{R}(\zeta)$ replaced by $\gamma_n(\alpha) \rightarrow z_\infty(\alpha)$ as $n \rightarrow \infty$.

The assumption that $z_\infty(\alpha)$ exists for $\nu$-a.e. $\alpha \in \Sigma^d$, i.e. $\nu(D) = 1$, holds for every $\nu$ of positive entropy (compare Remark 1; see [PZ, Convergence Theorem], where further references are given). As in the Riemann mapping case, $\chi_\mu(f) > 0$ then holds automatically.

In the setting of Theorem B property (v’) also holds, with $\text{Fr} \Omega$ replaced by $\text{cl} \Lambda$, which immediately follows from (v).

The assumption supp $\mu = \text{cl} \Lambda$ holds whenever $\nu$ is a $\sigma$-invariant Gibbs state for a Hölder continuous function on $\Sigma^d$ (cf. Remark 1), and if additionally the tree $T$ satisfies $\gamma^j \cap \text{cl}(\bigcup_{n \geq 0} f^n(\text{Crit} f)) = \emptyset$ for all $j = 1, \ldots, d$ and there exists a neighbourhood $U^j \subset f(U)$ of $\gamma^j$ such that area$(f^{-n}(U^j)) \rightarrow 0$, where area denotes the standard Riemann measure on $\mathbb{C}$.

For the proof see [PZ, Lemma 3], where $\text{cl} \Lambda$ is replaced by a formally larger set $\hat{\Lambda} := \{\text{all limit points of the sequences } z_n(\alpha^n), \alpha^n \in \Sigma^d, n \rightarrow \infty\}$. It is easy to see that the above conditions about the tree $T$ hold if $T$ is in $W \cap \Omega$, close enough to Fr $\Omega$, as in the situation of Theorem A (see Section 5).

4. Proof of Theorem B

**Step 1:** Good backward branches and their number. Denote the natural extension of the one-sided shift $\sigma : \Sigma^d \rightarrow \Sigma^d$ preserving a Borel probability measure $\nu$, i.e. the corresponding two-sided shift, by $(\hat{\Sigma}^d, \hat{\nu}, \hat{\sigma})$. Denote the
projection \( \tilde{\Sigma}^d \to \Sigma^d \) mapping \( \alpha \) to \( (\alpha_0, \alpha_1, \ldots) \) by \( \pi_+ \). For each \( \alpha \in \tilde{\Sigma}^d \) denote \( \pi_+(\alpha) \) by \( \alpha^+ \).

By Pesin theory (see [PZ, Lemma 1] for the version we apply) and by the Birkhoff Ergodic Theorem applied to \( \varphi \), for every \( \varepsilon > 0 \) we can find a set \( K \subset \tilde{\Sigma}^d \), constants \( C, \delta > 0 \) and a positive integer \( M \) such that \( \bar{\nu}(K) > 1 - \varepsilon \) and for all \( \alpha \in K \) and \( n \geq 0 \),

\[
\begin{align*}
B(i) & \quad \ln C + n(\int_{\varphi} d\nu - \varepsilon/2) \leq \sum_{j=0}^{n-1} \varphi(\sigma^j(\alpha^+)) \leq \ln C + n(\int_{\varphi} d\nu + \varepsilon/2). \\
B(ii) & \quad b_M(\alpha^+) \subset B(z_\infty(\alpha^+), \delta/3). \\
B(iii) & \quad \text{There exist univalent branches } f_{\alpha}^{-n} \text{ of } f^{-n} \text{ on } B(z_\infty(\alpha^+), \delta) \text{ for all } n = 1, 2, \ldots \text{ mapping } z_\infty(\alpha^+) \text{ to } z_\infty(\tilde{\sigma}^{-n}(\alpha^+)).
\end{align*}
\]

In the notation accompanying property (1) these branches are the continuations along \( b(\alpha^+) \) of \( f_{(\alpha_n, \ldots, \alpha_{-1})}^{-n} \), i.e. the branches \( f_{(\alpha_n, \ldots, \alpha_{-1}), \alpha^+}^{-n} \).

Moreover

\[
\text{B(iv) } C^{-1} \exp n(\chi_\mu(f) - \varepsilon/2) \leq |(f^n)'(z_\infty(\tilde{\sigma}^{-n}(\alpha^+)))| \leq C \exp n(\chi_\mu(f) + \varepsilon/2).
\]

\[
\text{B(v) } |(f_{\alpha}^{-n})'(x)| |(f_{\alpha}^{-n})'(y)| < C \text{ for all } x, y \in B(z_\infty(\alpha^+), \delta).
\]

For \( -\infty \leq r \leq s \leq \infty \) and \( \alpha \in \tilde{\Sigma}^d \) or \( \alpha \in \Sigma_{r,s} = \{1, \ldots, d\}^{\{r, r+1, \ldots, s\}} \), we denote by \( C_{r,s}(\alpha) \) the cylinder \( \{w \in \tilde{\Sigma}^d : w_j = \alpha_j \text{ for all } j : r \leq j \leq s\} \).

The projection \( \tilde{\Sigma}^d \ni (\alpha_j, \ldots) \mapsto (\alpha_r, \alpha_s) \in \Sigma_{r,s} \) will be denoted by \( \pi_{r,s} \). Note that \( C_{r,s}(\alpha) = \pi_{r,s}^{-1} \pi_{r,s}(\alpha) \).

Choose an arbitrary cylinder \( C_M := C_{0,M}(\beta) \), for a fixed sequence \( \beta = (\beta_0, \ldots, \beta_M) \in \Sigma_M := \Sigma_{0,M} \), such that \( \bar{\nu}(C_M \cap K) \geq \bar{\nu}(C_M)/2 \), which is possible provided \( \varepsilon \leq 1/2 \).

Denote \( C_M \cap K \) by \( K' \). For all \( n \geq 0 \) consider \( K_n := \tilde{\sigma}^{-n}(K') \). By the invariance of \( \bar{\nu} \) we have \( \bar{\nu}(K_n) \geq \bar{\nu}(C_M)/2 =: \xi \).

By the Birkhoff Ergodic Theorem there exists \( N \geq 0 \) such that

\[
\nu(\{\alpha \in K_n : \exists i : 0 \leq i \leq N, \tilde{\sigma}^{-i}(\alpha) \in K'\} \geq \xi/2.
\]

Therefore for every \( n \geq 0 \) there exists \( N' \) with \( 0 \leq N' \leq N \) such that, setting \( n' := n + N' \), for \( A(n') := \{\alpha \in K' : \tilde{\sigma}^{-n'}(\alpha) \in K'\} \) we have

\[
\bar{\nu}(A(n')) \geq \xi/2N.
\]

For every \( \alpha \in A(n') \) we obtain \( b_M(\tilde{\sigma}^{-n'}(\alpha^+)) \subset B(z_M(\alpha^+), \delta/3) \). Indeed, for \( \alpha' = \sigma^{-n'}(\alpha) \) we have \( \pi_{0,M}(\alpha') = \beta \), as we have landed with \( \alpha' \) in \( C_M \).

The length of \( b_M(\alpha'^+) \) is at most \( \delta/3 \) as \( \alpha' \in K \).

Hence

\[
f_{\alpha}^{-n'}(\text{cl}(B(z_M(\beta), 2\delta/3))) \subset B(z_M(\beta), 2\delta/3)
\]
for all $n$ large enough, more precisely for $n$ such that
\begin{equation}
|\left((f^{-n'}_\alpha) '\right|(x)| < 1/2 \quad \text{for all } x \in B(z_M(\beta), 2\delta/3).
\end{equation}
By B(ii)–B(iv) this holds for $n \geq (2\ln C + \ln 2)/(\chi_M(f) - \varepsilon)$.

**Claim.** The branches $f^{-n'}_\alpha$ on $B(z_\infty(\alpha^+), \delta)$ depend only on $\pi_{-n',M}(\alpha)$, more precisely on $\pi_{-n',-1}(\alpha)$ as $\pi_{0,M}(\alpha) = \beta$ has been fixed, on the common domain $B := B(z_M(\beta), 2\delta/3)$.

This is so since if two $\alpha$’s in $A(n')$, say $\alpha$ and $\alpha'$, have the same block $(\alpha_{-n'}, \ldots, \alpha_{-1})$, then the branches $f^{-n'}_{\alpha}$ and $f^{-n'}_{\alpha'}$ are continuations of the same branch at $z$ along curves coinciding till $z_M(\beta)$ and next contained in the common domain $B$ (see Figure 1).

![Fig. 1](image)

We shall not use this claim directly, but we put it to help the reader understand the proof and to simplify notation later on.

By (2) we have $\nu(\pi_{-n',M}^{-1} \pi_{-n',M}(\alpha), \bar{A}(n')) \geq \xi/2N$. By the Shannon–McMillan–Breiman theorem for every $\eta > 0$ and all integers $k$ large enough,

$$\nu\left(\bigcup\{C_{0,k}(w) : w \in \Sigma_{0,k}, \nu(C_{0,k}(w)) \leq \exp k(h_\nu(\sigma) + \varepsilon/3)\}\right) \geq 1 - \eta.$$  

Setting $\eta = \xi/4N$ and $k = M + n'$ we get

$$\nu\left(\bigcup\{C_{-n',M}(w) : w \in \bar{A}(n')\}, \nu(C_{-n',M}(w)) \leq \exp (n' + M)(h_\nu(\sigma) + \varepsilon/3)\right) \geq \xi/2N - \xi/4N = \xi/2N.$$
Therefore for \( n \) large enough, the number of “good backward trajectories” of length \( n' \) can be estimated as follows:

\[
\#(\pi_{-n',-1}(A(n'))) \geq \exp n' (h_\nu(\sigma) - \varepsilon/2).
\]

**Step 2:** The sets \( X, Y \) and IFS. Now define \( Y' \subset \Sigma^d \) as the set of one-sided sequences which are concatenations of blocks \( v^k \) belonging to \( G_{n'} := \pi_{0,n'-1}\hat{\sigma}^{-n'}(A(n')) \), that is,

\[
Y' = \{ \alpha = v^0v^1\ldots \in \Sigma^d : v^k = \pi_{0,n'-1}\sigma^{kn'}(\alpha) \in G_{n'} \ \forall k = 0,1,\ldots \},
\]

and set

\[
X' = z_\infty(Y').
\]

Finally, define

\[
Y = \bigcup \{ \sigma^j(Y') : j = 0,\ldots,n'-1 \},
\]

\[
X = \bigcup \{ f^j(X') : j = 0,\ldots,n'-1 \} = z_\infty(Y).
\]

For each \( \alpha \in \Sigma^d \) and \( r \leq s \) denote by \( b_{r,s} \) the part of the branch \( b(\alpha) \) starting from \( z_{r-1}(\alpha) \) and ending at \( z_s(\alpha) \).

Now, to put it briefly, by (3) and (4) for every \( \alpha \in Y' \) the length of \( b_{kn',(k+1)n'-1}(\alpha) \) is less than \( C2^{-k} \) for a constant \( C > 0 \). Hence \( z_n(\alpha) \to z_\infty(\alpha) \) uniformly (even exponentially fast), which proves (ii) on \( Y' \), hence on \( Y \) by the uniform continuity of \( f \). By (3) and (4), \( X' \), and hence \( X \), are expanding repellers for \( f^{n'} \) and \( f \) respectively.

Let us now be more precise. Let \( \alpha \in Y' \) be a concatenation of \( v^k = \pi_{0,n'-1}\hat{\sigma}^{-n'}(w^k) \), for \( w^k \in A(n') \), for \( k = 0,1,\ldots \). We want to analyse \( b(\alpha) \). Note that by (1),

\[
(b_{(k-1)n',kn'-1}(\alpha)) = f_{-n'}^{-n'}(f_{-n',v^1,v^2,\ldots,v^k}(\ldots(f_{-n',v^{k-1},v^k}(b_{0,n'-1}(v^k))))).
\]

Assume that \( n' > M \). Then all \( b(\alpha) \) for \( \alpha = v^0v^1\ldots \in Y' \) pass through \( z_M(\beta) \) since \( v^0 \in G_{n'} \) implies that \( b_{0,n'-1}(\alpha) \) depending only on \( v^0 \) passes through \( z_M(\beta) \). (There is no reason for \( \alpha \) to belong to \( A(n') \), which would imply passing through \( z_M(\beta) \) by definition, as in the Claim. So for the first time in the proof we need to use \( n' > M \).)

Now we apply induction on \( k \). Suppose that for every \( \alpha \in \Sigma_{0, kn'-1} \) which is a concatenation \( v^0v^1\ldots v^{k-1} \) of blocks \( v^j \) in \( G_{n'} \) we have \( b_{M+1, kn'-1}(\alpha) \subset B \) (see Figure 2)). Take an arbitrary \( v \in G_{n'} \) which is the truncation of \( w \in A(n') \), more precisely \( v = \pi_{0,n'-1}\hat{\sigma}^{-n'}(w) \). Then \( f^{-n'}_{v,\alpha} \) and \( f^{-n'}_{w} \) coincide on \( B \), in particular on \( b_{M, kn'-1}(\alpha) \), since also \( b_{M}(w^+) \) is contained in \( B \), as \( w \in K \), yielding a path in \( T \) joining \( z_{kn'-1}(\alpha) \) to \( z_\infty(w^+) \) and entirely contained in \( B \) (compare the proof of Claim). Hence, by (3) applied to \( f^{-n'}_{w} \) we get \( b_{M+1,(k+1)n'-1}(w\alpha) \subset B \), which finishes the induction.
Therefore in (6) we can replace $f^{-n'} v \cdot v_{j+1} \cdot v_{j+2} \cdot \ldots \cdot v_k$ by $f^{-n'} w \cdot w_{j+1} \cdot w_{j+2} \cdot \ldots \cdot w_k$ for all $j = 0, 1, \ldots, k - 1$, in particular these branches of $f^{-n'}$ act on branches of the tree $T$ in the common domain $B$ (except $b_{0, n'-1}(w_k)$).

One can view the family of branches $F_v := f^{-n'} v$ for $v \in G_{n'}$ as an iterated function system (IFS) on $B$. It satisfies the so-called Strong Open Set Condition, i.e. all $F_v(B)$ have pairwise disjoint closures. The Claim allows us to write $v$ in place of $w$, where $v$ is the truncation of $w$. These branches also act on (extend to) $b_{0, M} (\beta)$, the line in the tree joining $z$ to $z_M(v)$ which need not be contained in $B$. So $F_v$ need not contract it. But further iteration contracts them exponentially since $F_v(b_{0, M} (\beta))$ lies already in $B$.

The limit set is contained in $\text{cl} z_\infty(\Sigma^d)$, since the $F_v$ preserve the tree $T$.

**Step 3: Proving properties (i)–(ix) in Theorem B.** To prove (i) consider an arbitrary $\alpha = v^0 v^1 \ldots \in Y'$ for $v^k = \pi_{n'-1}(w^k)$, where $w^k \in A(n')$. Then, for each $k = 1, 2, \ldots$,

$$
\sum_{j=0}^{kn'-1} \varphi(\sigma^j(\alpha^+)) - \sum_{i=0}^{k-1} \sum_{j=0}^{n'-1} \varphi(\sigma^j((\widetilde{\sigma}^{-n'}(w^i))^+)) \leq kn' \varepsilon/2
$$

for $n$ large enough. This follows from the continuity of $\varphi$ since $\sigma^{in'+j}(\alpha)$ and $\sigma^j((\widetilde{\sigma}^{-n'}(w^i))^+)$ are very close to each other for all $i$ and $0 \leq j \ll n'$. This is so because both one-sided sequences have the same beginning of length.
$n' - j$. Now (i) follows from the estimate $B(i)$ on $\sum_{j=0}^{n' - 1} \varphi(\sigma^j((\tilde{\sigma}^n(w))^{+}))$.

Passing from $Y'$ to $Y$ changes only the constant $C$ in (i).

These considerations also prove (vi). Indeed, in the case of $\psi$ one ensures the property of $K$ analogous to $B(i)$, namely

$$B(vi) \quad -\ln C + n\left(\int \psi \, d\mu - \varepsilon / 2\right) \leq \sum_{j=0}^{n' - 1} \psi(f^j(z_{\infty}(\alpha^+)))$$

following from the $\nu$-integrability of $\psi \circ z_{\infty}$ and the Birkhoff Ergodic Theorem. Use also the property analogous to (7), for $\psi$ and $f$ in place of $\phi$ and $\sigma$, which follows from the continuity of $\psi$ and the fact that the preimages of points in $B$ under the same branch $f^j f_{n' - j}^{-1}$ of $f^{-(n' - j)}$ are very close to each other for $0 \leq j \ll n'$.

The uniform (exponential) convergence in (ii) has already been proven. The injectivity and the property of $X$' of being a Cantor set follow from the Strong Open Set Condition of the IFS $\{F_v\}$. This implies that $z_{\infty}$ is finite-to-one on $Y$ and $X$ is also a Cantor set.

By (5) and (i) and by the definition of pressure,

$$P\left(\sigma^{n'}|_{Y'}, \sum_{j=0}^{n' - 1} \varphi \circ \sigma^j\right) \geq h_\nu(\sigma^{n'}) + n'\left(\int \varphi \, d\nu - \varepsilon\right),$$

hence easily $P(\sigma|_{Y'}, \varphi) \geq h_\nu(f) + \int_Y \varphi \, d\nu - \varepsilon$, proving (vii) for $P(\sigma|_{Y'}, \varphi)$.

The argument for $P(f|_X, \psi)$ is similar, using (vi) for $\psi$.

Note that one cannot pull back to $\Sigma^d$ to refer to (vii) for $P(\sigma, \psi \circ z_{\infty})$ on $Y$ since $\psi \circ z_{\infty}$ need not be continuous on $\partial \mathbb{D}$, even not defined, so we might not have (7).

By [M], or [P1, Sec. 3] where further references are provided, we have $\text{HD}(\mu) = h_\mu(f)/\chi_\mu(f)$. Consider an arbitrary $\varepsilon' > 0$ and set $t' := \text{HD}(\mu) - \varepsilon'$. Then $t' = h_\mu(f)/\chi_\mu(f) - \varepsilon'$. By (iii) and (5),

$$P(f|_X, -t' \ln |f'|_X|) \geq h_{\text{top}}(f|_X) - t'(\chi_\mu(f) + \varepsilon)$$

$$\geq h_\mu(f) - \varepsilon - (h_\mu(f)/\chi_\mu(f) - \varepsilon')(\chi_\mu(f) + \varepsilon)$$

$$\geq -\varepsilon - \varepsilon h_\mu(f)/\chi_\mu(f) + \varepsilon' \chi_\mu(f) + \varepsilon \varepsilon',$$

which is positive if

$$\varepsilon' > \frac{\varepsilon (1 + h_\mu(f))/\chi_\mu(f)}{\chi_\mu(f) + \varepsilon}.$$

Hence $\text{HD}(X) > t'$ as $\text{HD}(X)$ is not smaller than the first zero of the pressure function $t \mapsto P(f|_X, -t \ln |f'|_X|)$, by the Bowen theorem (see for
example [PU]). If we choose \( \varepsilon \) small we obtain \( \varepsilon' \) small, hence HD(\( X \)) arbitrarily close to HD(\( \mu \)), which proves (iv).

We prove (ix) similarly.

To prove (v) consider the cylinder \( C_M = C_{0,M}(\beta) \) for \( \beta \) being the truncation of a sequence \( \alpha \) dense in supp \( \nu \) and \( M \) large. The proof of Theorem B is finished. \( \blacksquare \)

5. Conclusions. Theorem B easily implies Theorem A. One builds the tree \( T \) in the basin of attraction. It is only sufficient to note that the branches of the tree \( R^{-1}(T) \) converge to \( \partial \mathbb{D} \) nontangentially, so the convergence of each branch \( b(\alpha) \) in \( T \) implies the nontangential, in particular radial, convergence of \( R \) at \( \lim R^{-1}(b(\alpha)) \in \partial \mathbb{D} \), with the same limit. One considers the pull-back \( \varphi \circ (R^{-1}(z))_\infty : \Sigma^d \to \mathbb{R} \), finds \( Y \) in \( \Sigma^d \), maps it by \( (R^{-1}(z))_\infty \) with the use of \( R^{-1}(T) \) into \( \partial \mathbb{D} \) and with the use of \( T \) to \( X \subset Fr \Omega \) as in Theorem B. The map \( \hat{R} \) is finite-to-one on \( Y \) since \( z_\infty \) is.

The rate of the exponential convergence in (ii) and more precisely in (ii') follows easily from (iii), (i) applied to \( R \) rule.

Remark 2 follows easily from (iii), (i) applied to \( R \) rule.

Remark 3. If \( \nu \) is mixing, which is the case for Gibbs \( \nu \) as in Remark 1, then one can ensure that \( f \) on \( X \) is topologically mixing, that is, for any open subsets \( U, V \) of \( X \) there exists \( n_0 \) such that \( f^n(U) \cap V \neq \emptyset \) for all \( n \geq n_0 \).

Indeed, for \( n \) large we have by mixing \( \tilde{\nu}(\tilde{\sigma}^{-n}(C_M) \cap C_M) \sim \nu(C_M)^2 \).

Hence, if \( \nu(K) \approx 1 \), then \( \tilde{\nu}(A(n)) \geq \text{const} > 0 \) for all \( n \) large (compare (2)). We can repeat the previous construction by taking instead of one \( n' \) two different mutually prime integers.

Remark 4. Theorem A holds in the case \( \Omega \) is an immediate connected simply connected basin of attraction to a parabolic fixed point \( p \), i.e. \( p \in Fr \Omega \) such that \( f(p) = p \) and \( f'(p) \) is a root of unity.

Indeed, in this case \( R^{-1} \circ f \circ R \) extends to \( \mathbb{C} \) to yield \( g \) which is a Blaschke product such that \( \mathbb{D} \) (and \( \mathbb{C} \setminus \text{cl} \mathbb{D} \)) is a basin of a parabolic fixed point for \( g \) in \( \partial \mathbb{D} \). As in the conclusion that Theorem B implies Theorem A, we consider the trees \( T \) and \( R^{-1}(T) \). All the branches of \( R^{-1}(T) \) converge (polynomially fast, but not necessarily nottangentially), and at each limit point \( \zeta = (R^{-1}(z))_\infty(\alpha) \) for \( \alpha \in D(z_\infty) \), in particular in \( Y \), the radial limit \( \hat{R}(\zeta) \) coincides with \( z_\infty(\alpha) \) by Lindelöf’s theorem. Hence \( z_\infty = \hat{R} \circ (R^{-1}(z))_\infty \) on \( Y \) and all the maps involved are finite-to-one since \( z_\infty \) is finite-to-one on \( Y \).
References


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