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Persistence of fixed points under rigid perturbations of maps

by

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Abstract. Let $f : S^1 \times [0,1] \to S^1 \times [0,1]$ be a real-analytic diffeomorphism which is homotopic to the identity map and preserves an area form. Assume that for some lift $\tilde{f} : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ we have $\mathrm{Fix}(\tilde{f}) = \mathbb{R} \times \{0\}$ and that $\tilde{f}$ positively translates points in $\mathbb{R} \times \{1\}$. Let $\tilde{f}_\epsilon$ be the perturbation of $\tilde{f}$ by the rigid horizontal translation $(x,y) \mapsto (x+\epsilon, y)$. We show that $\mathrm{Fix}(\tilde{f}_\epsilon) = \emptyset$ for all $\epsilon > 0$ sufficiently small. The proof follows from Kerékjártó's construction of Brouwer lines for orientation preserving homeomorphisms of the plane with no fixed points. This result turns out to be sharp with respect to the regularity assumption: there exists a diffeomorphism f with all the properties above, except that f is not real-analytic but only smooth, such that the above conclusion is false. Such a map is constructed via generating functions.

1. Introduction. For $k \geq 1$, let us denote by $\mathrm{Diff}^k(\mathbb{D})$ the set of orientation and area preserving C^k-diffeomorphisms $\hat{h} : \mathbb{D} \to \mathbb{D}$, defined in the closed disk $\mathbb{D} := \{z \in \mathbb{R}^2 : |z| \leq 1\}$, which fix the origin $0 \in \mathbb{D}$. We denote by $\mathrm{Diff}^k_0(\mathbb{D}) \subset \mathrm{Diff}^k(\mathbb{D})$ the subset of diffeomorphisms satisfying

$$\mathrm{Fix}(\hat{h}) := \{\hat{h}(z) = z\} = \{0\} \quad \text{and} \quad D\hat{h}(0) = \mathrm{Id}.$$

Here we are considering the usual area form $dz_1 \wedge dz_2$ on $\mathbb{R}^2$ with coordinates (z_1, z_2).

In this paper we address the following question:

(Q1) Under what conditions can we find $\hat{g} \in \mathrm{Diff}^k(\mathbb{D})$ arbitrarily C^k-close to $\hat{h}$ such that $\mathrm{Fix}(\hat{g}) = \{0\}$ and $D\hat{g}(0) = e^{2\pi\epsilon i}$ for some $\epsilon \in \mathbb{R} \setminus \mathbb{Q}$?

Before stating the main results we need some definitions.

Definition 1.1. (a) Let $A := S^1 \times [0,1]$ be the closed annulus, where S^1 is identified with $\mathbb{R}/\mathbb{Z}$. Let $\tilde{A} := \mathbb{R} \times [0,1]$ be the infinite strip and $p : \tilde{A} \to A$ be the covering map $(x,y) \mapsto (x \bmod 1, y)$.

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(b) Let $p_1 : \tilde{A} \to \mathbb{R}$ and $p_2 : \tilde{A} \to \mathbb{R}$ be the projections of $\tilde{A}$ into the first and second factors, respectively. We also denote by p_1 and p_2 the respective projections defined on A.

(c) Let $\mathrm{Diff}^k(A)$ be the space of area preserving C^k-diffeomorphisms $f : A \to A$, where $k \in \mathbb{N} \cup \{\infty, \omega\}$, which are homotopic to the identity map. Let $\mathrm{Diff}^k_0(A) \subset \mathrm{Diff}^k(A)$ denote the diffeomorphisms which satisfy the following conditions: $f(x,0) = (x,0)$ for all $x \in S^1$ and if $\tilde{f} : \tilde{A} \to \tilde{A}$ is the lift of f such that $\tilde{f}(x,0) = (x,0)$ for all $x \in \mathbb{R}$, then $\mathrm{Fix}(\tilde{f}) = \mathbb{R} \times \{0\}$. Moreover, we require that

$$p_1 \circ \tilde{f}(x,1) > x, \qquad \forall x \in \mathbb{R}. \tag{1}$$

(d) Let $\mathrm{Diff}^k_0(\tilde{A})$ be the lifts of maps in $\mathrm{Diff}^k_0(A)$ which fix all points in $\mathbb{R} \times \{0\}$.

Now if $\hat{h} \in \mathrm{Diff}^k_0(\mathbb{D})$, we obtain a map $f := b^{-1} \circ \hat{h} \circ b$ induced by $b : A \to \mathbb{D}$ defined by

$$b(x,y) := (\sqrt{y}\cos 2\pi x, -\sqrt{y}\sin 2\pi x),$$

where (x,y) are coordinates in A. Notice that f preserves the area form $dx \wedge dy$. We assume that f extends to a map in $\mathrm{Diff}^k(A)$. Clearly, $S^1 \times \{0\}$ corresponds to the blow up of $0 \in \mathbb{D}$, and $S^1 \times \{1\}$ corresponds to $\partial\mathbb{D}$. Also, since $\hat{h} \in \mathrm{Diff}^k_0(\mathbb{D})$, it follows that either f or f^{-1} admits a lift $\tilde{f} \in \mathrm{Diff}^k_0(\tilde{A})$. In fact, either $p_1 \circ \tilde{f}(x,1) > x$ for all $x \in \mathbb{R}$, or $p_1 \circ \tilde{f}(x,1) < x$ for all $x \in \mathbb{R}$. After possibly interchanging f with f^{-1} we may assume without loss of generality that (1) is satisfied.

Given $\epsilon \in \mathbb{R}$ we consider the diffeomorphism

$$\tilde{f}_\epsilon : \tilde{A} \to \tilde{A} : (x,y) \mapsto \tilde{f}(x,y) + (\epsilon, 0). \tag{2}$$

The map $\tilde{f}_\epsilon$ naturally induces a diffeomorphism $f_\epsilon : A \to A$ given by

$$f_\epsilon = p \circ \tilde{f}_\epsilon \circ p^{-1}. \tag{3}$$

Notice that the translated map f_ϵ corresponds to blowing up the map $\hat{h} \in \mathrm{Diff}^k_0(\mathbb{D})$ after composing it with the rigid rotation $z \mapsto e^{2\pi\epsilon i}z$.

Our first result is the following theorem.

Theorem 1.2. *Let $f \in \mathrm{Diff}^\omega_0(A)$ and $\tilde{f} \in \mathrm{Diff}^\omega_0(\tilde{A})$ be a lift of f. Then there exists $\epsilon_0 > 0$ such that $\mathrm{Fix}(\tilde{f}_\epsilon) = \emptyset$ for all $0 < \epsilon < \epsilon_0$.*

Remark 1.3. The hypothesis $\tilde{f}(x,0) = (x,0)$ for all $x \in \mathbb{R}$ can be weakened to $p_1 \circ \tilde{f}(x,0) \geq x$ for all $x \in \mathbb{R}$, as is easily seen from the proof.

Remark 1.4. From the classical Poincaré–Birkhoff theorem, $\tilde{f}_\epsilon$ has fixed points in interior$(\tilde{A})$ for all $\epsilon < 0$ sufficiently small.

Our next result proves sharpness of the real-analyticity assumption in Theorem 1.2, i.e. this phenomenon does not occur assuming only smoothness.

THEOREM 1.5. *There exist $f \in \mathrm{Diff}_0^\infty(A)$ and a sequence of positive real numbers $\epsilon_n \to 0^+$ as $n \to \infty$ such that $\mathrm{Fix}(\tilde{f}_{\epsilon_n}) \neq \emptyset$, where $\tilde{f} \in \mathrm{Diff}_0^\infty(\tilde{A})$ is the special lift of f and $\tilde{f}_{\epsilon_n}$ is defined as in* (2)*, for all $n \in \mathbb{N}$.*

The proof of Theorem 1.2 strongly relies on a construction due to B. de Kerékjártó [3] of Brouwer lines for orientation preserving homeomorphisms of the plane which have no fixed point. Here, the hypothesis of real-analyticity of f plays an important role. We argue indirectly assuming the existence of a sequence $\epsilon_n \to 0^+$ such that $\tilde{f}_{\epsilon_n}$ admits a fixed point z_n. We can assume that z_n converges to a point $\bar{z}$ at the lower boundary component of $\tilde{A}$. The real-analyticity hypothesis then allows one to deduce the existence of a small real-analytic curve γ_0 starting at $\bar{z}$, which is a graph in the vertical direction, so that $\tilde{f}$ moves its points horizontally to the left. Since $\tilde{f}$ has no fixed point in interior$(\tilde{A})$, the curve γ_0 is then prolonged to a Brouwer line $L \subset \tilde{A}$, following Kerékjártó's construction. We analyse all possibilities for the behaviour of L and each of them yields a contradiction. Here, we strongly use the fact that $\tilde{f}$ moves points in the upper boundary of $\tilde{A}$ to the right.

The smooth map f in Theorem 1.5 is obtained from a special generating function on $\tilde{A}$. More precisely, first we define a diffeomorphism $\psi : \tilde{A} \to \tilde{A}$ supported in the sequence of balls $B_k \subset \tilde{A}$ centred at $(0, 3/2^{k+2})$ and of radius $1/2^{k+3}$, converging to the origin. Using the function $h(t) = e^{-1/t}$, which extends smoothly at $t = 0$ as a flat point, we define the generating function by $g(p) = h \circ p_2 \circ \psi(p)$, where p_2 is the projection in the vertical direction. The diffeomorphism associated to g, which is a priori defined only in a small neighbourhood of the origin, is then suitably rescaled in order to get a diffeomorphism f of the annulus satisfying all the requirements.

As one can see, f satisfies all hypotheses of Theorem 1.2 except that it is not real-analytic at a unique point in the lower boundary. This follows from the flatness of h at $t = 0$, and therefore the example in Theorem 1.5 shows the sharpness of the regularity assumption in Theorem 1.2.

2. Kerékjártó's construction of Brouwer lines. In this section we denote by $h : \mathbb{R}^2 \to \mathbb{R}^2$ an orientation preserving homeomorphism of the plane satisfying

$$\mathrm{Fix}(h) = \emptyset. \tag{4}$$

The following periodicity in x is assumed:

$$h(x+1, y) = h(x, y) + (1, 0), \qquad \forall (x, y) \in \mathbb{R}^2. \tag{5}$$

Definition 2.1. (a) We call $\alpha \subset \mathbb{R}^2$ a *simple arc* if α is the image of a topological embedding $\psi : [0,1] \to \mathbb{R}^2$. We may consider the parametrization ψ of the arc α, which will also be denoted by α. We also identify all the parametrizations of α which are induced by orientation preserving homeomorphisms of the respective domains. The *internal points* of the simple arc α are defined by $\alpha \setminus \{\alpha(0), \alpha(1)\}$ and denoted $\dot{\alpha}$. Given distinct points $B_1, B_2, \dots$ in $\mathbb{R}^2$, we denote by $B_1B_2\dots$ the polygonal arc connecting them by straight line segments following that order. We may also denote by AB a simple arc with end points $A \neq B$, which is not necessarily a line segment.

(b) Given any two simple arcs η_0 and η_1 with a unique common end point, we denote by $\eta_0 \cup \eta_1$ the simple arc obtained by concatenating η_0 and η_1 in the usual way and respecting the orientation from η_0 to η_1.

(c) We say that the simple arc $\alpha \subset \mathbb{R}^2$ is a *translation arc* if $\alpha(0) = z$, $\alpha(1) = h(z) \neq z$ and

$$\alpha \cap h(\alpha) = \{h(z)\}.$$

(d) Let α be a simple arc with end points b and c. We say that α *abuts on its inverse or direct image*, respectively, if $b \notin h^{-1}(\alpha) \cup h(\alpha) = \emptyset$ and one of the following conditions holds:

(i) $\dot{\alpha} \cap h^{-1}(\alpha) = \emptyset$ and $c \in h^{-1}(\alpha)$.
(ii) $\dot{\alpha} \cap h(\alpha) = \emptyset$ and $c \in h(\alpha)$.

(e) We say that $L \subset \mathbb{R}^2$ is a *Brouwer line* for h if L is the image of a proper topological embedding $\psi : \mathbb{R} \to \mathbb{R}^2$ so that $h(L)$ and $h^{-1}(L)$ lie in different components of $\mathbb{R}^2 \setminus L$.

Let AB be a translation arc with end points A and $B := h(A)$. Let $C = h(B)$ and denote by BC the simple arc given by $h(AB)$. Assume without loss of generality that the vertical line passing through B intersects the arcs AB and BC only at B. Otherwise, we can perform a topological change of coordinates in order to achieve this property.

We will construct two half-lines L_1 and L_2 issuing from B, with L_1 starting upwards and L_2 starting downwards, so that $L = L_1 \cup L_2$ is a Brouwer line for h. L_1 and L_2 will be referred to as *Brouwer half-lines* since both are topological embeddings of $[0, \infty)$ into $\mathbb{R}^2$ and $h(L_i) \cap L_i = \emptyset$, $i = 1, 2$.

Let us start with L_1. Consider the vertical arc γ_1 starting upwards from B which is defined by $\gamma_1(t) = B + (0, t)$, where $t \in [0, t^*]$ (with t^* to be defined below), or $t \in [0, \infty)$. One of the following conditions is met:

(i) There exists $t^* > 0$ such that γ_1 abuts on its inverse image and $P := \gamma_1(t^*)$ is such that $h(P) =: P'$ is an internal point of γ_1.
(ii) There exists $t^* > 0$ such that γ_1 abuts on its image and $P := \gamma_1(t^*)$ is an internal point of $h(\gamma_1)$. In this case we set $P' := h^{-1}(P)$, which is an internal point of γ_1.

(iii) γ_1 is defined for all $t \geq 0$, $(h^{-1}(\dot{\gamma}_1) \cup \dot{\gamma}_1 \cup h(\dot{\gamma}_1)) \cap (AB \cup BC) = \emptyset$ and $h(\gamma_1) \cap \gamma_1 = \emptyset$.

In case (iii) our construction of L_1 ends and we define $L_1 = \gamma_1$. In cases (i) and (ii), we define PP' to be the simple arc in γ_1 from P' to P. Notice that by construction, PP' is a translation arc. Kerékjártó proves the following theorem.

THEOREM 2.2 (see [3, Theorems II, III and IV]). *In cases* (i) *and* (ii) *above, we have*

$$h(\gamma_1) \cap AB = h^{-1}(\gamma_1) \cap BC = h(\gamma_1) \cap h^{-1}(\gamma_1) = \emptyset.$$

Moreover, in case (i) *there exists a subarc ν_1 of $h^{-1}(\gamma_1)$ from A to P such that $\nu_1 \cup PP' \cup h(\gamma_1) \cup BC \cup AB$ is a simple closed curve which bounds an open domain $U_1 \subset \mathbb{R}^2$. In case* (ii) *there exists a subarc ν_1 of $h(\gamma_1)$ from C to P such that $\nu_1 \cup PP' \cup h^{-1}(\gamma_1) \cup AB \cup BC$ is a simple closed curve which bounds an open domain $U_1 \subset \mathbb{R}^2$.*

DEFINITION 2.3. The *free side* of PP' is defined to be the side of PP' outside U_1 as in Theorem 2.2. See Figure 1.

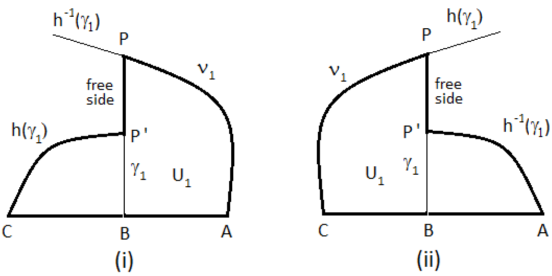

Fig. 1. In this picture, γ_1 abuts on its inverse and direct image as in cases (i) and (ii), respectively.

The free side of the translation arc $PP' \subset \gamma_1$ only depends on which side γ_1 lies on with respect to the oriented arc $AB \cup BC$ and how γ_1 abuts on its image according to cases (i) or (ii). This dependence strongly follows from the assumption that h has no fixed points and is exemplified in Figure 1.

Now we need a couple of definitions in order to start the construction of L_1.

DEFINITION 2.4. (a) Let $R_n = [-n, n] \times [-n, n]$ for all $n \in \mathbb{N}^*$ and

$$\epsilon_n = \inf\{|h(x) - x|, |h^{-1}(x) - x| : x \in R_n\} > 0.$$

Define $\eta_n > 0$ to be the largest $t \in (0, \epsilon_n/2]$ such that $|h(x) - h(y)| \leq \epsilon_n/2$ and $|h^{-1}(x) - h^{-1}(y)| \leq \epsilon_n/2$ whenever $x, y \in R_n$ and $|x - y| \leq t$.

(b) Let $n \in \mathbb{N}^*$ and assume $PP' \subset R_n$. By a *mid-segment* of PP' we mean a segment $M \subset PP'$ such that the distances of its points to P and to P' are at least η_n. Notice that $M \neq \emptyset$.

(c) A *base point* associated to the vertical translation arc PP' and to a given free side of PP' is a point B_1 in a mid-segment $M \subset PP'$ such that either the half-line l_{B_1} starting from B towards the free side of PP' is such that $l_{B_1} \cap (h(l_{B_1}) \cup h(PP') \cup h^{-1}(PP')) = \emptyset$, or there exists a simple arc β starting from B_1, perpendicular to PP' and going towards the free side of PP' such that β abuts on its image and $\beta \cap (h(PP') \cup h^{-1}(PP')) = \emptyset$. In the former case, we say that the base point B_1 with that given free side is *unbounded*, and in the latter case B_1 with the given free side is *bounded*. One of the end points of β is B_1 and the other is denoted by P_1.

The proof of the existence of at least one base point associated to a translation arc PP' and to any given free side of PP' is found in [3, Section 2.2].

Remark 2.5. If the translation arc PP' is horizontal, then the definitions above are the same and analogous results hold.

Continuing our construction, we pick a base point B_1 associated to the vertical translation arc PP'. The *initial part* of L_1 is then defined to be the segment BB_1. If B_1 is unbounded then we are finished and $L_1 = BB_1 \cup l_{B_1}$ is the desired half-line. If B_1 is bounded then the horizontal segment $\beta = B_1P_1$ abuts on its image and we find an internal point $P_1' = h(P_1)$ or $P_1' = h^{-1}(P_1)$ as before such that the horizontal arc $P_1P_1' \subset \beta$ is a translation arc. The translation arc P_1P_1' admits a free side according to the description above. Observe that now the free side of P_1P_1' is either the upper or the lower side. Again we find a base point $B_2 \in P_1P_1'$ associated to the free side of P_1P_1' and add the simple arc B_1B_2 to L_1, now given by $L_1 = BB_1B_2$. Repeating this procedure indefinitely we arrive at one of the following cases:

(i) After a finite number of steps we find an unbounded base point $B_j \in P_{j-1}P_{j-1}'$ and our broken half-line is given by $L_1 = BB_1B_2 \ldots B_j l_{B_j}$.
(ii) All base points B_j found in the construction are bounded and we define $L_1 = BB_1B_2B_3 \ldots$. Then the following holds: given $n \in \mathbb{N}^*$, there exists $k_0 \in \mathbb{N}^*$ such that $B_k \notin R_n$ for all $k \geq k_0$. This follows from the definition of base points and is proved in [3].

Notice that the construction of L depends on the choices of the internal base points $B_k \in P_{k-1}P_{k-1}'$. Also, the half-line L_1 goes to infinity and

$$h(L_1) \cap L_1 = (h(\dot{L}_1) \cup h^{-1}(\dot{L}_1) \cup \dot{L}_1) \cap (AB \cup BC) = \emptyset. \tag{6}$$

We still need a modification trick from [3] in the construction of L_1. It is called the *deviation* of the path. Let $V_k = \{(x, y) \in \mathbb{R}^2 : x = k\}$, $k \in \mathbb{Z}$, be the vertical lines at integer values and assume that

$$0 < l := \# V_0 \cap h^{-1}(V_0) < \infty. \tag{7}$$

Notice that from (5), hypothesis (7) must hold as well for each V_k, $k \in \mathbb{Z}$, and the respective intersections are shifted by $(k, 0)$.

Let $V_0 \cap h^{-1}(V_0) = \{w_1, \dots, w_l\}$ and $w_i' := h(w_i)$, $i = 1, \dots, l$. Consider the vertical arcs $\gamma_i = w_i w_i' \subset V_0, i = 1, \dots, l$. If γ_j does not properly contain any γ_i with $i \neq j$, then γ_j is a translation arc. We consider only such translation arcs on V_0 and keep denoting them by γ_j, now with $j = 1, \dots, l_0$, $l_0 \leq l$. Given j, assume a free side of γ_j is given and is to the left. Then there exists a base point $u_{j,l} \in \dot{\gamma}_j$ associated to γ_j and to that free side. Accordingly, if the given free side of γ_j is to the right, we can also find a base point $u_{j,r} \in \dot{\gamma}_j$ associated to γ_j and to that free side. Let $\gamma_j^i = \gamma_j + (i, 0)$ be the respective translation arcs on V_i for all $i \in \mathbb{Z}$ and let $u_{j,l}^i := u_{j,l} + (i, 0), u_{j,r}^i := u_{j,r} + (i, 0), i \in \mathbb{Z}$, be their respective base points. In the following we fix these base points $u_{j,l}^i$ and $u_{j,r}^i$ in each γ_j^i.

In the construction of L_1 above suppose that at some point we find a vertical translation arc $P_{k-1}P_{k-1}'$ with a given free side and the horizontal arc issuing from a bounded base point $B_k \subset P_{k-1}P_{k-1}'$ towards the free side intersects some V_j at an internal point $z \in B_k B_{k+1}$ so that the arc $B_k z$ intersects no other vertical $V_i, i \neq j$. Instead of adding the segment $B_k B_{k+1}$ to L_1 we add only the segment $B_k z$ and the new B_{k+1} is determined according to one of the alternatives found in the following theorem.

Theorem 2.6 ([3], [2]). *We have:*

- (i) *There exists a vertical half-line $l_z \subset V_j$ through z so that the broken half-line $\alpha := B_k z \cup l_z$ satisfies $\alpha \cap h(\alpha) = \emptyset$ and $(h(P_{k-1}P_{k-1}') \cup h^{-1}(P_{k-1}P_{k-1}')) \cap \alpha = \emptyset$. In this case we have $L_1 = BB_1 \dots B_k z l_z$ and the construction of L_1 is finished.*
- (ii) *There exists $c \in V_j$, $c \neq z$, such that the broken arc $\alpha := B_k z \cup zc$ abuts on its image, satisfies $(h(P_{k-1}P_{k-1}') \cup h^{-1}(P_{k-1}P_{k-1}')) \cap \alpha = \emptyset$, and contains a translation arc $\gamma_m^j \subset V_j$ for some $m \in \{1, \dots, l_0\}$. Let $B_{k+1} \in \{u_{m,l}^j, u_{m,r}^j\}$ be the base point in γ_m^j associated to the free side of α. In this case we have $L_1 = BB_1 \dots B_k z B_{k+1} \dots$ and we keep constructing L_1 through the horizontal arc issuing from $B_{k+1} \in V_j$ towards the free side of α as before.*
- (iii) *The point z is an internal point of a translation arc γ_m^j for some m so that $\gamma_m^j \cap (h(P_{k-1}P_{k-1}') \cup P_{k-1}P_{k-1}' \cup h^{-1}(P_{k-1}P_{k-1}')) = \emptyset$ and $\bigcup_{n \in \mathbb{Z}} h^n(\gamma_m^j) \cap B_k z = \{z\}$. In this case the free side of γ_m^j is the side opposite to $B_k z$ and we find a base point $B_{k+1} \in \{u_{m,l}^j, u_{m,r}^j\}$ associated to γ_m^j and to that free side. We have $L_1 = BB_1 \dots B_k z B_{k+1} \dots$*

and we keep constructing L_1 through the horizontal arc issuing from B_{k+1} towards that free side.

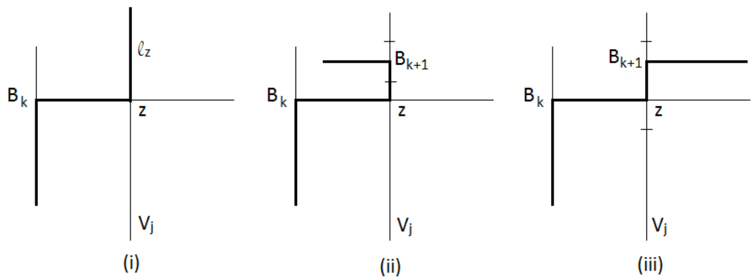

Fig. 2. The deviation of the path according to cases (i)–(iii) above

Proceeding as above indefinitely and using the deviation of the path whenever its conditions are met, we find the desired half-line L_1. As explained before, L_1 goes to infinity and satisfies (6).

Remark 2.7. Given $a \in \mathbb{R}$, let $G_a = \{(x, y) \in \mathbb{R}^2 : y \geq a\}$. We can extract some more information on how L_1 goes to infinity if we assume the following twist condition on h:

$$(8) \qquad \exists y_0 \in \mathbb{R}, \quad p_1 \circ h(x, y) > x \text{ and } p_1 \circ h^{-1}(x, y) < x, \ \forall (x, y) \in G_{y_0}.$$

Suppose that at some point in the construction of L_1 we find a horizontal translation arc $P_{k-1}P'_{k-1}$ with the free side coinciding with its upside and an associated base point $B_k \in P_{k-1}P'_{k-1}$. Assume that there exists a vertical segment V starting from B_k towards the free side so that its other extremity lies inside G_{y_0} and that $h(V) \cap V = \emptyset$. We claim that B_k is an unbounded base point and the construction of L_1 ends by adding to it the vertical half-line l_{B_k}, i.e., $L_1 = BB_1 \ldots B_k l_{B_k}$. To see this we argue indirectly and assume the existence of a vertical segment W starting from B_k and containing V such that W abuts on its image. Let $w \neq B_k$ be the other extremity of W. Then either $h(w) \in W$ or $h^{-1}(w) \in W$. However, this contradicts (8) and proves our claim.

Remark 2.8. Using the deviation of the path explained above we know that if L_1 is horizontally unbounded then L_1 is eventually periodic. This follows from the finiteness of the points $u^i_{j,l}, u^i_{j,r} \in V_i$ for each $i \in \mathbb{Z}$. For instance, suppose that L_1 is deviated at V_i for some $i \in \mathbb{Z}$, and leaves it to the right at $u^i_{j,r} \in V_i$ for some $j \in \{1, \ldots, l_0\}$. Suppose that after this deviation, L_1 is now deviated at V_{i+N}, for some $N \in \mathbb{Z}^*$, leaving it to the right at $u^{i+N}_{j,r} \in V_{i+N}$. We continue the construction of L_1 from $u^{i+N}_{j,r}$, proceeding in exactly the same way as we did from $u^i_{j,r}$. This implies that, except perhaps for its initial segments, L_1 is periodic, i.e., there exists a connected subset $W_0 \subset L_1$ from $u^i_{j,r}$ to $u^{i+N}_{j,r}$ such that $W_0 + (kN, 0) \subset L$ for all $k \in \mathbb{N}$. Hence we find a new periodic Brouwer line L_{per} given by $W_0 + (kN, 0)$, $k \in \mathbb{Z}$. By construction we must have $h(L_{\text{per}}) \cap L_{\text{per}} = \emptyset$.

REMARK 2.9. The construction of the other half-line L_2 from B (now starting downwards) can be made in exactly the same way as we did for L_1 so that by construction $L = L_1 \cup L_2$ is a Brouwer line. An alternative construction for L_2, which will be used in the proof of Theorem 1.2, is the following: let $\psi : [0, \infty) \to \mathbb{R}^2$ be a proper topological embedding with $\psi(0) = B$ and let $L_2 = \psi([0, \infty))$. Assume that $h(L_2) \cap L_2 = \emptyset$ and that $\dot{L}_1$ and $\dot{L}_2$ lie in different components of $\mathbb{R}^2 \setminus (h^{-1}(L_2) \cup AB \cup BC \cup h(L_2))$. Then one easily checks that $L = L_1 \cup L_2$ is a Brouwer line for h.

We end this section with a proposition that will be useful in the proof of Theorem 1.2 in the next section. Its proof is entirely contained in Kerékjártó's construction of Brouwer lines explained above.

PROPOSITION 2.10. *Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving homeomorphism of the plane which has no fixed points and satisfies the following assumptions:*

(i) *There exists $y_0 \in \mathbb{R}$ such that*

$$p_1 \circ h(x, y) > x \text{ and } p_1 \circ h^{-1}(x, y) < x, \qquad \forall (x, y) \in \mathbb{R}^2,\ y \geq y_0.$$

(ii) *$h(x + 1, y) = h(x, y) + (1, 0)$ for all $(x, y) \in \mathbb{R}^2$.*

(iii) *There exists a vertical line V_0 such that*

$$0 < l := \#V_0 \cap h^{-1}(V_0) < \infty.$$

Then through any point $B \in \mathbb{R}^2$ as above, there exists a Brouwer half-line L_1 issuing from B upwards such that:

- *L_1 contains only horizontal and vertical segments.*
- *If L_1 contains a point $q \in \{y \geq y_0\}$ then it contains the vertical upper half-line through q.*
- *If L_1 is horizontally unbounded then L_1 is eventually periodic, i.e., there exists a simple arc $W_0 \subset L_1$ and an integer $N \neq 0$ such that $W_0 + (kN, 0) \subset L_1$ for all $k \in \mathbb{N}$; and $|N|$ is the least positive integer with this property. In particular, this implies that $W_0 \cap W_0 + (N, 0) = \{\text{point}\}$.*
- *If L_2 is a given Brouwer half-line issuing from B downwards and $\dot{L}_1$ and $\dot{L}_2$ lie in different components of $\mathbb{R}^2 \setminus (h^{-1}(L_2) \cup AB \cup BC \cup h(L_2))$, then $L_1 \cup L_2$ is a Brouwer line.*

Here, as above, $B = h(A)$, $C = h(B)$, AB is a translation arc and $BC = h(AB)$ is horizontal.

3. Proof of Theorem 1.2. We start with the following lemma.

Lemma 3.1. *Let $\tilde{g}:\tilde{U}\subset\tilde{A}\to\tilde{A}$ be a real-analytic area preserving diffeomorphism onto its image, defined in an open neighbourhood $\tilde{U}$ of $\mathbb{R}\times\{0\}\subset\tilde{A}$ such that $\mathrm{Fix}(\tilde{g})=\mathbb{R}\times\{0\}$. Assume that there exists a sequence of positive real numbers $\epsilon_n\to 0^+$ such that each $\tilde{g}_{\epsilon_n}$, defined as in (2), admits a fixed point $p_n\in\tilde{U}$, with $p_n\to\bar{p}=(\bar{x},0)\in\mathbb{R}\times\{0\}$ as $n\to\infty$. Then there exists a real-analytic curve $\gamma_0:[0,1]\to\tilde{U}$, $\gamma_0(t)=(x(t),y(t))$, such that $\tilde{g}\circ\gamma_0(t)=(w(t),y(t))$ and*

$$w(0)=\bar{x},\qquad w(t)<x(t),\qquad y'(t)>0,\qquad \forall t\in(0,1]. \tag{9}$$

Proof. Write $\tilde{g}(x,y)=(g_1(x,y),g_2(x,y))$ and let $G_2(x,y):=g_2(x,y)-y$. We may express G_2 as a power series in $x-\bar{x}$ and y near $\bar{p}$ which converges in $B_\epsilon:=\{(x,y)\in\mathbb{R}^2:(x-\bar{x})^2+y^2\le\epsilon^2\}$ with $\epsilon>0$ small.

If G_2 vanishes identically then $g_2(x,y)=y$ near $\bar{p}$. By preservation of area and the fact that $g_1(x,0)=x$ for all x, we have $g_1(x,y)=x+yR(y)$ for a real-analytic function R defined near $y=0$. Since $\mathrm{Fix}(\tilde{g})=\mathbb{R}\times\{0\}$, R does not vanish identically. The existence of p_n as in the hypothesis implies $R(y)<0$ for y small. In this case we can define the curve γ_0 by $\gamma_0(t)=(\bar{x},t)$, with $t\ge 0$ small.

Now assume that G_2 does not vanish identically. We investigate the zeros of G_2 near $\bar{p}$ in B_ϵ for $\epsilon>0$ small. Notice that $\bar{p}\in\mathbb{R}\times\{0\}\cap B_\epsilon\subset\{G_2=0\}$ and thus $\bar{p}$ is not an isolated point of $\{G_2=0\}$. Since G_2 is real-analytic, we take $\epsilon>0$ small and find an even number of real-analytic embedded curves $\eta_i:[0,1]\to B_\epsilon$, $i=1,\dots,2m$, with $\eta_i(0)=\bar{p}$, such that $\{G_2=0\}\cap B_\epsilon=\bigcup_{i=1}^{2m}\mathrm{Image}(\eta_i)$ (see [4, Lemmas 3.1 and 3.3]). Taking $\epsilon>0$ even smaller, we may assume that the images of any two of these curves intersect each other only at $\bar{p}$. Also, we may choose $\eta_1(t)=(\bar{x}+\epsilon t,0)$ and $\eta_2(t)=(\bar{x}-\epsilon t,0)$ for $t\in[0,1]$, since $\mathbb{R}\times\{0\}\subset\{G_2=0\}$. The existence of a sequence $p_n\to\bar{p}$ as in the hypothesis implies that $m\ge 2$ and therefore we find $j_0\in\{3,\dots,2m\}$ and a subsequence of p_n, still denoted by p_n, such that $p_n\in\mathrm{Image}(\eta_{j_0})$. Moreover, since $\eta_{j_0}(t)=(x_{j_0}(t),y_{j_0}(t))$ is real-analytic, we have $y'_{j_0}(t)>0$ for all $t\in(0,\mu]$, for some $\mu>0$ small, and therefore $\mathrm{Image}(\eta_{j_0}|_{[0,\mu]})$ projects injectively into the y-axis. Finally, we define $\gamma_0(t)=\eta_{i_0}(\mu t)$, $t\in[0,1]$. By the properties of p_n and the fact that $\mathrm{Fix}(\tilde{g})=\mathbb{R}\times\{0\}$, we conclude that γ_0 has the properties as in the statement. ∎

To prove Theorem 1.2 we argue indirectly. Assume that there exists a sequence $\epsilon_n\to 0^+$ such that $\tilde{f}_{\epsilon_n}$ defined as in (2) admits a fixed point p_n. By the periodicity of $p_1\circ\tilde{f}(x,y)-x$ in x we can assume that $p_n\to\bar{p}=(\bar{x},0)\in\mathbb{R}\times\{0\}$. This implies that $\tilde{f}$ restricted to a neighbourhood $\tilde{U}$ of $\mathbb{R}\times\{0\}$ satisfies the conditions of Lemma 3.1. So we can find a real-analytic curve $\gamma_0:[0,1]\to\tilde{A}$, $\gamma_0(t)=(x(t),y(t))$, such that $\tilde{f}\circ\gamma_0(t)=(w(t),y(t))$

satisfies (9). In what follows, the curve γ_0 will be prolonged to a Brouwer line $\tilde{L}$ in $\tilde{A}$ satisfying one of the possibilities:

- $\tilde{L}$ hits $\mathbb{R}\times\{1\}$. Since $\tilde{f}$ moves points in $\mathbb{R}\times\{1\}$ to the right and moves γ_0 to the left, $\tilde{L}$ must intersect its image, a contradiction.
- $\tilde{L}$ is eventually periodic. In this case we obtain an area preserving diffeomorphism of the closed annulus with a homotopically non-trivial simple closed curve which is disjoint from its image, again a contradiction.
- $\tilde{L}$ is bounded and accumulates at $\mathbb{R}\times\{0\}$. In this case we obtain an area preserving homeomorphism of the 2-sphere admitting a simple closed curve bounding a topological disk whose image is properly contained inside itself, a contradiction.

Given $t\in(0,1]$ let $B_t := \gamma_0(t)$, $C_t := \tilde{f}(B_t)$ and $A_t := \tilde{f}^{-1}(B_t)$. Let B_tC_t be the horizontal segment connecting B_t and C_t, and let $A_tB_t := \tilde{f}^{-1}(B_tC_t)$ be its inverse image. We claim that A_tB_t is a translation arc for $\tilde{f}$ if t is small enough. To see this, assume this is not the case, so that we can find a point $B_t \neq z \in B_tC_t$ which is also contained in A_tB_t. It follows that $p_1(z) > p_1(B_t)$. Since $\frac{\partial(p_1\circ\tilde{f})}{\partial x}(x,y)\to 1$ as $(x,y)\to\bar{p}$, we have

$$p_1\circ\tilde{f}(z) - p_1\circ\tilde{f}(B_t) = \frac{\partial(p_1\circ\tilde{f})}{\partial x}(\xi)(p_1(z)-p_1(B_t)) > 0$$

for some $\xi\in B_tC_t$ and $t>0$ small. As $\tilde{f}(z)\in B_tC_t$, we also have $p_1\circ\tilde{f}(z) \le p_1(C_t) = p_1\circ\tilde{f}(B_t)$. This contradiction proves that indeed A_tB_t is a translation arc for $\tilde{f}$ and

$$p_1(z) > p_1(B_t) \tag{10}$$

for all internal points $z\in A_tB_t$ where $t\in(0,1]$ is fixed and small.

Let us fix a sufficiently small $t_0>0$ such that for some $c_0\in\mathbb{R}$, the set $\tilde{f}^{-1}(\gamma_0([0,t_0]))\cup\gamma_0([0,t_0])\cup\tilde{f}(\gamma_0([0,t_0]))\cup B_{t_0}C_{t_0}\cup A_{t_0}B_{t_0}$ is disjoint from all the verticals $V_{k+c_0} = \{(x,y)\in\tilde{A} : x = k+c_0\}$, $k\in\mathbb{Z}$. We may assume without loss of generality that $c_0=0$.

In order to directly apply elements of Kerékjártó's construction of Brouwer lines in the plane as stated in Proposition 2.10, we consider the homeomorphism $d : \mathrm{interior}(\tilde{A})\to\mathbb{R}^2$ given by

$$d(x,y) = \left(x, \frac{y-1/2}{y(1-y)}\right)$$

and the induced orientation preserving homeomorphism $h:\mathbb{R}^2\to\mathbb{R}^2$ given by $h = d\circ\tilde{f}\circ d^{-1}$. From the hypothesis $\mathrm{Fix}(\tilde{f}) = \mathbb{R}\times\{0\}$, we get $\mathrm{Fix}(h)=\emptyset$.

Let $A := d(A_{t_0})$, $B := d(B_{t_0})$ and $C := d(C_{t_0})$. Denote by AB the simple arc $d(A_{t_0}B_{t_0})$ and by BC its image under h. Notice that AB is a translation arc and BC is a horizontal simple arc. From (10), the vertical line through

B intersects AB and BC only at B. Hence we can start the construction of a Brouwer line for h with the vertical line starting from B towards the upside and proceeding as in Section 2, thus obtaining the half-line L_1. To obtain L_2 we simply define $L_2 = d(\gamma_0|_{(0,t_0]})$. It follows that $L = L_1 \cup L_2$ is a Brouwer line (see Remark 2.9). Let $\tilde{L} := d^{-1}(L) = \tilde{L}_1 \cup \tilde{L}_2$ and observe that

$$\tilde{f}(\tilde{L}) \cap \tilde{L} = \emptyset. \tag{11}$$

Now we prove that the existence of the Brouwer line $\tilde{L}$ leads to a contradiction. First, from the twist condition (1), we can find $0 < \delta < 1$ such that for all $(x, y) \in S_\delta := \{(x, y) \in \tilde{A} : \delta \leq y \leq 1\}$, we have $p_1 \circ \tilde{f}(x, y) > x$ and $p_1 \circ \tilde{f}^{-1}(x, y) < x$. This implies that h satisfies condition (8) for $y_0 = \frac{\delta - 1/2}{\delta(1-\delta)}$ (see Remark 2.7). It follows that if $\tilde{L}$ hits S_δ then $\overline{\tilde{L}}$ contains a vertical segment with end point $z_0 \in \mathbb{R} \times \{1\}$. By construction, points of $\overline{\tilde{L}}$ near but different from $\bar{p}$ are mapped under $\tilde{f}$ to the left, while points near z_0 are mapped to the right. This implies that $\tilde{f}(\tilde{L}) \cap \tilde{L} \neq \emptyset$, which contradicts (11). Hence we can assume that $\tilde{L}$ does not accumulate at $\mathbb{R} \times \{1\}$.

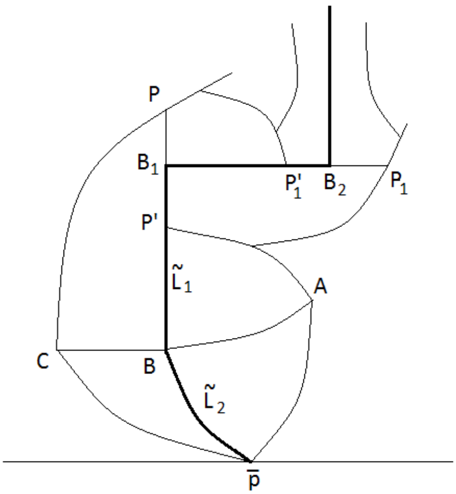

Fig. 3. The Brower line $\tilde{L} = \tilde{L}_1 \cup \tilde{L}_2 \subset \tilde{A}$

Since $\tilde{f}$ is analytic and $p_1 \circ \tilde{f}(x, 1) > x$ for all $x \in \mathbb{R}$, we deduce that $\tilde{f}^{-1}(V_k) \cap V_k = (\tilde{f}^{-1}(V_0) \cap V_0) + (k, 0)$ is a finite set for all $k \in \mathbb{Z}$, and therefore condition (7) holds for h. This implies that if $\widetilde{L}$ is horizontally unbounded then, as explained in Remark 2.8, we can find $N \in \mathbb{Z}$ and another Brouwer line $\widetilde{L}_{\text{per}} = d^{-1}(L_{\text{per}}) \subset \text{interior}(\tilde{A})$ which is N-periodic in x, i.e., $\widetilde{L}_{\text{per}} + (kN, 0) = \widetilde{L}_{\text{per}}$ for all $k \in \mathbb{Z}$. Let $f_N : A_N \to A_N$ be the map induced by $\tilde{f}$ on the annulus $A_N := \tilde{A}/T_N$, where $T_N : \tilde{A} \to \tilde{A}$ is the horizontal translation $T_N(x,y) = (x+N,y)$. Let $p_N : \tilde{A} \to A_N$ be the associated covering map and let $L_N := p_N(\tilde{L}_{\text{per}})$. From the properties of $\tilde{L}_{\text{per}}$ and of the map $\tilde{f}$ we see that L_N and $f_N(L_N)$ are disjoint simple closed curves which are homotopically non-trivial. Let C_- be the topological closed annulus bounded by L_N and $p_N(\mathbb{R} \times \{0\})$. Then either f_N or f_N^{-1} maps C_- properly into itself. Since f_N preserves a finite area form, we get a contradiction. Hence we can assume that $\tilde{L}$ is horizontally bounded and accumulates only at $\mathbb{R} \times \{0\}$.

Now we find $N_0 \in \mathbb{N}$ large enough that $\tilde{L} \cap (\tilde{L} + (N_0, 0)) = \emptyset$, which implies by Brouwer's lemma (see for instance [1]) that

$$\tilde{L} \cap (\tilde{L} + (kN_0, 0)) = \emptyset, \qquad \forall k \in \mathbb{Z}^*. \tag{12}$$

As before we consider the annulus $A_{N_0} := \tilde{A}/T_{N_0}$ and identify the points in each component of ∂A_{N_0} to obtain a topological sphere S^2. We end up with a map $\widehat{f}_{N_0} : S^2 \to S^2$ induced by f_{N_0} which preserves orientation and a finite area form. The closure of $p_{N_0}(\tilde{L})$ corresponds to a simple closed curve $\gamma_0 \subset S^2$ passing through the pole p_0, which corresponds to the lower component of ∂A_{N_0}. This last assertion follows from (12). Since $\tilde{L}$ is a Brouwer line we see that $\widehat{f}_{N_0}(\gamma_0) \cap \gamma_0 = \{p_0\}$ and that $\widehat{f}_{N_0}$ maps properly one component of $S^2 \setminus \gamma_0$ into itself. This contradicts the preservation of a finite area form and shows that $\tilde{L}$ cannot exist. The proof of Theorem 1.2 is complete.

4. Proof of Theorem 1.5. Our aim in this section is to construct a diffeomorphism $f : S^1 \times [0,1] \to S^1 \times [0,1]$, homotopic to the identity map, which satisfies:

- (i) f is smooth and area preserving.
- (ii) $\text{Fix}(f) = S^1 \times \{0\}$.
- (iii) If $\tilde{f} : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ is the lift of f satisfying $\tilde{f}(x,0) = (x,0)$ for all $x \in \mathbb{R}$, then $p_1 \circ \tilde{f}(x,1) > x$ for all $x \in \mathbb{R}$.
- (iv) Given $\epsilon > 0$, if $f_\epsilon : S^1 \times [0,1] \to S^1 \times [0,1]$ is induced by the lift $\tilde{f}_\epsilon := \tilde{f} + (\epsilon, 0)$ as before, then there exists a positive sequence $(\epsilon_n)_{n\in\mathbb{N}}$ with $\epsilon_n \to 0^+$ as $n \to \infty$ such that $\text{Fix}(\tilde{f}_{\epsilon_n}) \neq \emptyset$ for all n.

As proved in Theorem 1.2, such a diffeomorphism cannot exist if smoothness is replaced by real-analyticity in (i).

4.1. Area preserving maps and generating functions. We start by recalling basic facts on area preserving maps associated to generating functions. Let $U := \{(X,y) \in \mathbb{R} \times [0,1] : X^2 + y^2 < \varepsilon\}$, $\varepsilon > 0$, and $g : U \to \mathbb{R}$ be a smooth function such that

$$D^\nu g|_{\{y=0\}\cap U} \equiv 0, \qquad \forall 0 \le |\nu| \le 2. \tag{13}$$

We denote by $G : U \to \mathbb{R}$ the function given by

$$G(X, y) := Xy - g(X, y). \tag{14}$$

Let $(x, Y) \in \mathbb{R} \times [0, 1]$ be given by

$$x := G_y = X - g_y(X, y), \qquad Y := G_X = y - g_X(X, y). \tag{15}$$

We see from the first equation of (15) and the hypothesis (13) on g that we can use the implicit function theorem to write $X = \alpha(x, y)$ for $|(x, y)|$ small, where α is a smooth map satisfying $\alpha(x, 0) = x$ for all x. In this case $Y = y - g_X(\alpha(x, y), y) = \beta(x, y) \geq 0$, where β is smooth and satisfies $\beta(x, 0) = 0$ for all x. Let $\bar{f} : V \subset \mathbb{R} \times [0, 1] \to \mathbb{R} \times [0, 1]$ be given by

$$(X, Y) = \bar{f}(x, y) := (\alpha(x, y), \beta(x, y)),$$

where V is a small neighbourhood of $(0, 0) \in \mathbb{R} \times [0, 1]$. We say that $\bar{f}$ is a *local map* associated to the *generating function* G. Moreover, $\bar{f}|_{\mathbb{R}\times\{0\}}$ is the identity map.

PROPOSITION 4.1. *The map $\bar{f}$ preserves the area form $dx \wedge dy$ on the strip $\mathbb{R} \times [0, 1]$, i.e.,*

$$dX \wedge dY = dx \wedge dy.$$

Proof. From (15) we have

$$\begin{aligned} dx &= (1 - g_{yX})dX - g_{yy}dy, \quad &\text{so} \quad dx \wedge dy &= (1 - g_{yX})dX \wedge dy, \\ dY &= (1 - g_{Xy})dy - g_{XX}dX, \quad &\text{so} \quad dX \wedge dY &= (1 - g_{Xy})dX \wedge dy. \end{aligned} \tag{16}$$

Since g is smooth, the proposition follows. ■

4.2. A special generating function. Let $\rho : [0, \infty) \to [0, 1]$ be a smooth function satisfying $\rho \equiv 1$ in $[0, 1/16]$, $\rho \equiv 0$ in $[1/4, \infty)$ and $\rho' < 0$ in $(1/16, 1/4)$. We define the vector field X on the strip $W_1 := \mathbb{R} \times [-1, 1]$ by

$$X(x, y) = \rho(x^2 + y^2) \cdot (-y, x).$$

It is clear that X is smooth, $X \equiv 0$ outside $B(1/2) := \{(x, y) \in W_1 : x^2 + y^2 \leq 1/4\}$ and $X(x, y) = (-y, x)$ in $B(1/4)$.

The flow $\{\varphi_t\}$ of X on W_1 is defined for all $t \in \mathbb{R}$ and satisfies

$$\begin{aligned} \varphi_t(x, y) &= (x, y), \qquad &\forall (x, y) \in W_1 \setminus B(1/2), \forall t, \\ \varphi_\pi(x, y) &= -(x, y), \qquad &\forall (x, y) \in B(1/4). \end{aligned} \tag{17}$$

Now let $W_0 := \mathbb{R} \times [0, 1]$. For each $k \in \mathbb{N}$, let

$$F_k := \mathbb{R} \times (1/2^{k+1}, 1/2^k] \subset W_0, \qquad F_\infty := \mathbb{R} \times \{0\} \subset W_0,$$

and consider the diffeomorphism

$$\begin{aligned} t_k : F_k &\to W_1 \setminus \mathbb{R} \times \{-1\}, \qquad k \in \mathbb{N}, \\ (x, y) &\mapsto (2^{k+2}x, 2^{k+2}y - 3). \end{aligned} \tag{18}$$

Letting $\partial_k^+ := \mathbb{R} \times \{1/2^k\} \subset F_k$, we observe that $t_k(\partial_k^+) = \mathbb{R} \times \{1\}$.

Next define a map $\psi : W_0 \to W_0$ by

$$\psi|_{F_k} := t_k^{-1} \circ \varphi_\pi \circ t_k, \qquad \forall k, \tag{19}$$
$$\psi|_{F_\infty} := \mathrm{Id}|_{F_\infty}.$$

Let $p_k := (0, 3/2^{k+2}) \in F_k \subset W_0$ be the 'midpoint' of F_k. From (17)–(19) we note that

$$\operatorname{supp}\psi = \overline{\bigcup_{k\geq 0} B_{p_k}(1/2^{k+3})}, \tag{20}$$

where $B_p(r)$ denotes the closed ball centred at p with radius r. Moreover, ψ is the identity map when restricted to a small neighbourhood of each ∂_k^+. These observations together with the second equality in (19) imply that ψ is smooth in $W_0 \setminus \{(0,0)\}$. We also see that ψ is a diffeomorphism when restricted to $W_0 \setminus \{(0,0)\}$. From the second equality in (17) we have

$$\psi(x,y) = 2p_k - (x,y), \qquad \forall (x,y) \in B_{p_k}(1/2^{k+4}),\ \forall k. \tag{21}$$

Let $h : [0,1] \to [0,\infty)$ be the smooth function given by

$$h(t) = e^{-1/t}, \qquad \forall t > 0, \tag{22}$$
$$h(0) = 0.$$

Note that h is flat at $t = 0$, i.e.,

$$h^{(n)}(0) = 0, \qquad \forall n. \tag{23}$$

Observe also that given $l \in \mathbb{N}$, we find polynomial functions P_l, Q_l such that

$$h^{(l)}(t) = e^{-1/t}\frac{P_l(t)}{Q_l(t)}, \qquad \forall t > 0. \tag{24}$$

This can easily be proved by induction.

Now let $g : W_0 \to [0,\infty)$ be defined by

$$g := h \circ p_2 \circ \psi. \tag{25}$$

Proposition 4.2. *We have the following:*

(i) *The function g is smooth and $D^\nu g|_{\mathbb{R}\times\{0\}} \equiv 0$ for all $\nu \geq 0$.*
(ii) *The set* Crit(g) *of critical points of g coincides with* $\mathbb{R} \times \{0\}$.
(iii) *There exists a positive sequence $(s_k)_{k\in\mathbb{N}}$ with $s_k \to 0^+$ as $k \to \infty$ such that $\nabla g|_{\partial_k^+} = (0, s_k)$ for all k.*
(iv) *There exists a positive sequence $(m_k)_{k\in\mathbb{N}}$ with $m_k \to 0^+$ as $k \to \infty$ such that $\nabla g(p_k) = (0, -m_k)$ for all k.*

Postponing its proof to Section 4.3, we use Proposition 4.2 to show that g induces a diffeomorphism $f : S^1 \times [0,1] \to S^1 \times [0,1]$ satisfying conditions (i)–(iv) as given at the beginning of this section.

Let $G(X,y) = Xy - g(X,y)$ be the function defined in (14). Then, as explained before, we find a small neighbourhood V of $(0,0)$ in $\mathbb{R} \times [0,1]$ and a smooth area preserving map $\bar{f} : V \to \mathbb{R} \times [0,1]$, $\bar{f}(x,y) = (X,Y)$, such that $\bar{f}|_{V \cap F_\infty}$ is the identity map, i.e., $\bar{f}$ is the local map associated to the generating function G. From (15) we see that the fixed points of $\bar{f}$ correspond to the critical points of g. This implies that $\mathrm{Fix}(\bar{f}) = \mathrm{Crit}(g) = V \cap F_\infty$. Now observe that since $\nabla g|_{\partial_k^+} = (0, s_k)$ with $s_k \to 0^+$ as $k \to \infty$, we see from (15) that $\bar{f}(x,y) = (x + s_k, y)$ for all $(x,y) \in \partial_k^+$. In the same way, since $\nabla g(p_k) = (0, -m_k)$, we have $\bar{f}(m_k, 3/2^{k+2}) = p_k = (0, 3/2^{k+2})$ for all k, with $m_k \to 0^+$ as $k \to \infty$. This implies that for all $k \geq 0$, the map $\bar{f} + (m_k, 0)$ has $(m_k, 3/2^{k+2})$ as a fixed point. From (20) and the definition of g, we see that given any $x_1 > 0$ small we have $\bar{f}(x,y) = (x + h'(y), y)$ for all $(x,y) \in \{|x| = x_1, 0 \leq y \leq 2x_1\}$. Given $\lambda > 0$, let $T_\lambda : \mathbb{R} \times [0,\infty) \to \mathbb{R} \times [0,\infty)$ be the map $T_\lambda = (\lambda x, \lambda y)$. If necessary we replace $\bar{f}$ by $(T_{1/2^{k_0}})^{-1} \circ \bar{f} \circ T_{1/2^{k_0}}$, for a fixed k_0 sufficiently large, in order to find a map defined in $[-1/2, 1/2] \times [0,1]$ with the same properties above. Identifying $(-1/2, y) \in \{-1/2\} \times [0,1]$ with $(1/2, y) \in \{1/2\} \times [0,1]$ we finally find a map $f : S^1 \times [0,1] \to S^1 \times [0,1]$ with all the desired properties.

Notice that the diffeomorphism induced by the generating function g is defined in the open neighbourhood $V \subset \mathbb{R} \times [0,1]$ which might be very small. This explains why property (iii) is necessary in Proposition 4.2. Its proof is not straightforward and is left to the next section.

4.3. Proof of Proposition 4.2. As observed before, ψ is smooth on $W_0 \setminus \{(0,0)\}$. Hence g is smooth in this set as well. Moreover, since ψ is the identity map near $(\bar{x}, 0)$ for each $\bar{x} \neq 0$, we see that g is given by $g(x,y) = h(y)$ near $(\bar{x}, 0)$. It follows from (23) that

$$D^\nu g(\bar{x}, 0) = 0, \qquad \forall \bar{x} \neq 0, \forall |\nu| \geq 0. \tag{26}$$

It remains to prove that g is smooth at $(0,0)$ and $D^\nu g(0,0) = 0$ for all $|\nu| \geq 0$. Let $p = p_2 \circ \psi$. From the definition of ψ we see that

$$p(x,y) = \frac{1}{2^n} p(2^n x, 2^n y), \qquad \forall (x,y) \in W_0 \setminus F_\infty. \tag{27}$$

For any given smooth function $a : U \subset \mathbb{R}^2 \to \mathbb{R}$, we write $D^\alpha a = \frac{\partial^{|\alpha|} a}{\partial x^i \partial y^j}$ where $\alpha = (i,j) \in \mathbb{N}^2$ and $|\alpha| = i + j$.

Lemma 4.3. *In $W_0 \setminus F_\infty$, we have*

$$D^\alpha g = \sum_{l=1}^{|\alpha|} h^{(l)}(p) T_{\alpha,l}(\{D^\beta p\}_{1 \leq |\beta| \leq |\alpha| - l + 1}), \tag{28}$$

where $T_{\alpha,l}$ is a multi-variable polynomial function of the $D^\beta p$.

Proof. Observe that $D^{(1,0)}g = h'(p)p_x$ and $D^{(0,1)}g = h'(p)p_y$ have the form above with

$$T_{(1,0),1}(p_x, p_y) = p_x, \quad T_{(0,1),1}(p_x, p_y) = p_y.$$

In the same fashion we have $D^{(2,0)}g = h'(p)p_{xx} + h''(p)p_x^2$, $D^{(1,1)}g = h'(p)p_{xy} + h''(p)p_x p_y$, $D^{(0,2)}g = h'(p)p_{yy} + h''(p)p_y^2$ and

$$\begin{aligned} T_{(2,0),1}(p_x, p_y, p_{xx}, p_{xy}, p_{yy}) &= p_{xx}, \quad T_{(2,0),2}(p_x, p_y) = p_x^2, \\ T_{(1,1),1}(p_x, p_y, p_{xx}, p_{xy}, p_{yy}) &= p_{xy}, \quad T_{(1,1),2}(p_x, p_y) = p_x p_y, \end{aligned}$$

and so on. Let $\tilde{\alpha} = \alpha + (1,0)$ and observe that $D^{\tilde{\alpha}}g = D^{(1,0)}D^{\alpha}g$. The case $\tilde{\alpha} = \alpha + (0,1)$ is similar. Now an easy induction argument establishes the claim. ∎

It follows from (24) and (28) that

$$D^{\alpha}g = e^{-1/p} \sum_{l=1}^{|\alpha|} \frac{P_l(p)}{Q_l(p)} T_{\alpha,l}(\{D^{\beta}p\}), \tag{29}$$

where P_l, Q_l are polynomial functions in p.

LEMMA 4.4. *There are constants $C_{\beta} > 0$ depending on $(0,0) \neq \beta \in \mathbb{N} \times \mathbb{N}$ such that*

$$|D^{\beta}p(x,y)| \leq \frac{C_{\beta}}{y^{|\beta|}}, \quad \forall (x,y) \in W_0 \setminus F_{\infty}.$$

Proof. Given $(x,y) \in W_0 \setminus F_{\infty}$, let $n(x,y) \in \mathbb{N}$ be the unique positive integer such that $2^{n(x,y)}(x,y) \in F_0 = \mathbb{R} \times (1/2, 1]$. From (27), we have

$$D^{\beta}p(x,y) = 2^{n(x,y)|\beta|-1} D^{\beta}p(2^{n(x,y)}x, 2^{n(x,y)}y).$$

Let

$$0 < C_{\beta} := \sup_{(x,y) \in F_0} D^{\beta}p(x,y) < \infty.$$

It follows from the definition of $n(x,y)$ that

$$|D^{\beta}p(x,y)| \leq 2^{n(x,y)|\beta|-1} C_{\beta}.$$

Now since $2^{n(x,y)} \leq 1/y \Rightarrow 2^{n(x,y)|\beta|-1} \leq 1/y^{|\beta|}$, the claim follows. ∎

LEMMA 4.5. $|D^{\beta}g(x,y)| \to 0$ *as* $(x,y) \to (0,0)$ *for all* β.

Proof. From (26) it suffices to consider $(x,y) \in W_0 \setminus F_{\infty}$. From Lemmas 4.3 and 4.4 we find constants $C_{\alpha,l}, n_{\alpha,l} > 0$ such that

$$|T_{\alpha,l}(\{D^{\beta}p\})| \leq \frac{C_{\alpha,l}}{y^{n_{\alpha,l}}}. \tag{30}$$

We can also find constants $K_l, m_l > 0$ such that

$$\left| \frac{P_l(p)}{Q_l(p)} \right| \leq \frac{K_l}{p^{m_l}}. \tag{31}$$

Now since $0 < y/2 \le p(x,y) \le 2y$, from (29)–(31) we get

$$(32)\qquad |D^\beta g(x,y)| \le e^{-1/(2y)} \sum_{l=1}^{|\beta|} \frac{2^{m_l} K_l C_{\alpha,l}}{y^{m_l+n_{\alpha,l}}} \le e^{-1/(2y)} \frac{M_\beta}{y^{m_\beta}} \to 0$$

as $y \to 0$, where $M_\beta, m_\beta > 0$ are suitable constants. ∎

Let $\beta = (b_1, 0)$, where $b_1 \in \mathbb{N}$. Since $g|_{F_\infty \setminus \{(0,0)\}} = 0$, we have $D^\beta g(0,0) = 0$. From (32), $D^\beta g$ is continuous at $(0,0)$.

Now assume $b_2 > 0$ and let $\beta = (b_1, b_2)$. Then

$$D^\beta g(0,0) = \lim_{y\to 0^+} \frac{D^{\beta-(0,1)} g(0,y) - D^{\beta-(0,1)} g(0,0)}{y}.$$

Using induction on b_2 and inequality (32) again, we find

$$(33)\qquad D^\beta g(0,0) = 0, \quad \forall \beta.$$

Finally, from (32) and (33) we see that $D^\beta g$ is continuous at $(0,0)$. The proof of (i) is finished.

It is clear from the considerations above that $\mathrm{Crit}(g) \supseteq F_\infty$. Since ψ is a local diffeomorphism in $W_0 \setminus F_\infty$, p_2 is a submersion and $h'(y) > 0$ for all $y > 0$, we find that also g is a submersion when restricted to $W_0 \setminus F_\infty$. This implies that $\mathrm{Crit}(g) \subseteq F_\infty$, and therefore $\mathrm{Crit}(g) = F_\infty = \mathbb{R} \times \{0\}$. This proves (ii).

Since ψ is the identity map near each ∂_k^+, we have $g(x,y) = h(y)$ for all (x,y) near ∂_k^+. This implies that

$$\nabla g(x,y) = (0, h'(1/2^k)), \quad \forall (x,y) \in \partial_k^+.$$

Since $h'(1/2^k) > 0$ for all k and $\lim_{k\to\infty} h'(1/2^k) = 0$, (iii) follows.

To prove (iv), observe from (21) that $g(x,y) = h(p_2(2p_k - (x,y))) = h(3/2^{k+1} - y)$ for all $(x,y) \in B_{p_k}(1/2^{k+4})$. This implies in particular that

$$\nabla g(p_k) = (0, -h'(3/2^{k+2})).$$

Since $h'(3/2^{k+2}) > 0$ for all k and $\lim_{k\to\infty} h'(3/2^{k+2}) = 0$, (iv) follows. The proof of Proposition 4.2 is now complete.

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