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A theorem on generic intersections in an o-minimal structure

by

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Abstract. Consider a transitive definable action of a Lie group G on a definable manifold M. Given two (locally) definable subsets A and B of M, we prove that the dimension of the intersection $\sigma(A) \cap B$ is not greater than the expected one for a generic $\sigma \in G$.

For a fixed o-minimal expansion $\mathcal{R}$ of the real field $\mathbb{R}$, consider a transitive definable (left) action α of a definable group G on a definable manifold M of dimension m:

$$\alpha : G \times M \to M, \qquad \alpha(\sigma, x) = \sigma \cdot x.$$

Here, "definable" means "definable with parameters from $\mathcal{R}$". When $M = \mathbb{R}^m$ is an affine space, the most natural examples of groups which can occur in what follows are perhaps the group of affine automorphisms, the group of isometries and the group of translations of $\mathbb{R}^m$. One can also consider the transitive action of the general linear group on the punctured affine space $\mathbb{R}^m \setminus \{0\}$.

The main purpose of this paper is to establish the theorem below, which was inspired by a question of Jan Mycielski concerning the intersections of translates of analytic sets in $\mathbb{R}^2$. Jan Mycielski and Grzegorz Tomkowicz apply our theorem to prove that a bounded subset A of the real plane $\mathbb{R}^2$, which is a countable union of analytic sets of dimension ≤ 1, can be packed by a finite decomposition and isometries into an arbitrarily small disk D. Moreover, the image A' of A in D can be constructed so that $D \setminus A'$ is equivalent to D by a finite decomposition and isometries (equivalence in the sense of the Banach–Tarski paradox, investigated by Mycielski and Tomkowicz in [4, 5]). In their recent manuscript [6], they establish (using only the principle of dependent choices and our theorem) that bounded subsets of the

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Euclidean space $\mathbb{R}^n$, the sphere $\mathbb{S}^n$ and the hyperbolic space $\mathbb{H}^n$, included in countable unions of proper analytic subsets of these spaces, are of measure zero in the sense of Tarski.

Theorem on Generic Intersections. *Assume that A and B are two definable subsets of M of dimension k and l, respectively, and put $d := \max\{k+l-m, -1\}$. Then there is a nowhere dense definable subset Z of G such that*

$$\dim(\sigma(A) \cap B) \leq d \quad \textit{for all } \sigma \in G \setminus Z;$$

here $\dim \emptyset = -1$. In particular, the intersection $\sigma(A) \cap B$ is finite or empty for every $\sigma \in G \setminus Z$ according as $k+l = m$ or $k+l < m$, respectively.

Remark. When the structure under study is the expansion of the real field by restricted analytic functions or, more generally, by restricted quasi-analytic functions, the family of definable sets coincides with the family of globally subanalytic or quasi-subanalytic sets (i.e. sets which are subanalytic or quasi-subanalytic in a semialgebraic compactification; see e.g. [11, 7]). In turn, the locally definable sets are then precisely the subanalytic or quasi-subanalytic ones.

We immediately obtain the following

Corollary. *Under the above notation, let A and B be two locally definable subsets of M, i.e. each point $a \in M$ has a neighbourhood U such that the sets $A \cap U$ and $B \cap U$ are definable. Then there is a meagre* (*in the sense of Baire*) *subset $Z \subset G$ of zero Haar measure such that*

$$\dim(\sigma(A) \cap B) \leq d \quad \textit{for all } \sigma \in G \setminus Z.$$

In particular, the intersection $\sigma(A) \cap B$ is discrete (*whence at most countable*) *or empty for every $\sigma \in G \setminus Z$ according as $k+l = m$ or $k+l < m$, respectively.*

Before proceeding with the proof, we make some remarks about definable groups in o-minimal structures. It is well known that, for any non-negative integer n, every definable group G can be equipped with a definable C^n-manifold structure which makes G a definable Lie group of differentiability class C^n. The case $n = 0$ was proven by Pillay [10] for arbitrary o-minimal structures, but his proof can be repeated verbatim for a positive integer n whenever the structure under study is an o-minimal expansion of a real closed field R. When a given o-minimal structure is on the real field $\mathbb{R}$, G is a real analytic (not necessarily definable) Lie group (which follows from the Baker–Campbell–Hausdorff formula; cf. [3, Remark 4.30]).

Let us mention that Pillay's proof was an adaptation to the o-minimal setting of Hrushovski's proof of the Weil theorem that an algebraic group over an algebraically closed field can be built from birational data. Pillay's approach was later adapted by Peterzil–Pillay–Starchenko [8] to strengthen the result as follows. Consider a definable transitive action α of a definable group G on a definable set A. Then, for any non-negative integer n, A can be equipped with a definable C^n-manifold structure which makes α a definable action of differentiability class C^n.

For a point $x \in A$, denote by G_x the isotropy subgroup of x. Then the map

$$\alpha^x : G \ni \sigma \mapsto \sigma \cdot x \in A$$

factors through the canonical map $\pi : G \to G/G_x$ to a G-equivariant diffeomorphism $G/G_x \to A$ (cf. [3, Theorem 6.4]). In other words, A is diffeomorphic to the homogeneous space of G with respect to G_x. Obviously, we get

$$\dim G = \dim A + \dim G_x.$$

While many semialgebraic groups are listed in [12], some examples of definable linear groups which are not definably isomorphic to semialgebraic groups are given in [9].

We shall still need an elementary proposition relying on definable cell decomposition, which can be found e.g. in [2, Chap. 4, Proposition 1.5].

Proposition. *Let $f : V \to W$ be a definable map between non-empty definable sets. Then the following three implications hold:*

$$\begin{aligned}
&\dim f^{-1}f(v) \le k \text{ for all } v \in V \ \Rightarrow\ \dim V \le k + \dim f(V);\\
&\dim f^{-1}f(v) \ge k \text{ for all } v \in V \ \Rightarrow\ \dim V \ge k + \dim f(V);\\
&\dim f^{-1}f(v) = k \text{ for all } v \in V \ \Rightarrow\ \dim V = k + \dim f(V). \ \blacksquare
\end{aligned}$$

Proof of the Theorem. It is convenient to regard elements $\sigma \in G$ as definable diffeomorphisms of M, and so we shall write $\sigma(x) = \sigma \cdot x$ for $x \in M$. Let

$$\Delta = \Delta_M := \{(x,x) : x \in M\} \quad \text{and} \quad \pi : \Delta \to M$$

be the diagonal and the projection onto the first factor. Then

$$\begin{aligned}
\sigma(A) \cap B &= \pi((\sigma(A) \times B) \cap \Delta)\\
&= \pi \circ (\sigma \times \mathrm{Id}_M)((A \times B) \cap \{(x, \sigma(x)) : x \in M\}).
\end{aligned}$$

Therefore the sets $\sigma(A) \cap B$ and $(A \times B) \cap \{(x,\sigma(x)) : x \in M\}$ are diffeomorphic, and thus we must find a nowhere dense definable subset Z of G such that

$$\dim\,(A \times B) \cap \{(x,\sigma(x)) : x \in M\} \le d \quad \text{for all } \sigma \in G \setminus Z.$$

It is thus sufficient to prove the following

Lemma. *Let E be a definable subset of M^2 of dimension s and $d := \max\{s - m, -1\}$. Then the subset Z of those $\sigma \in G$ such that*

$$\dim E \cap \{(x,\sigma(x)) : x \in M\} > d$$

is definable and nowhere dense in G.

The set Z is definable, because the dimension of fibers from a definable family depends definably on the parameters (*loc. cit.*). Suppose, on the contrary, that Z is not nowhere dense. Then it would contain an open cell $C \subset G$. Further, put

$$\mathcal{E} := \{(\sigma, x, y) \in G \times E : y = \sigma(x)\},$$

and let $p : \mathcal{E} \to G$ and $q : \mathcal{E} \to E$ be the canonical projections. Obviously, for $(x, y) \in E$ the fibre

$$q^{-1}(x, y) = \{\sigma \in G : \sigma(x) = y\} \times \{(x, y)\}$$

is diffeomorphic to the isotropy subgroup of x, and thus is of dimension $\dim G - m$; notice that $\dim G \geq m$. Hence and by the foregoing proposition,

$$\dim \mathcal{E} = \dim G + s - m.$$

Now, observe that

$$p^{-1}(\sigma) = \{\sigma\} \times (E \cap \{(x, \sigma(x)) : x \in M\}).$$

Since $\dim p^{-1}(\sigma) > d$ for every $\sigma \in C$, it follows again from the Proposition that

$$\dim \mathcal{E} > \dim G + d.$$

Hence we get a contradiction $\dim G + s - m > \dim G + d$, which completes the proof. ■

Proof of the Corollary. By the second countability axiom, every locally definable set is a countable, locally finite union of definable sets. We can thus write

$$A = \bigcup_{i=1}^{\infty} A_i \quad \text{and} \quad B = \bigcup_{j=1}^{\infty} B_j,$$

where A_i and B_j, $i, j = 1, 2, \ldots$, are locally finite families of definable subsets of M. It follows from the Theorem that, for each $i, j = 1, 2, \ldots$, there is a nowhere dense definable subset Z_{ij} of G such that

$$\dim(\sigma(A_i) \cap B_j) \leq d \quad \text{for all } \sigma \in G \setminus Z_{ij}.$$

Then the countable union $Z := \bigcup_{i,j=1}^{\infty} Z_{ij}$ is a set we are looking for. ■

We conclude this paper with some comments. Sometimes definable groups have better properties than Lie groups. For instance, every definable subgroup is closed and every definable group has the descending chain condition for definable subgroups. On the other hand, unlike Lie groups, definable groups do not in general enjoy the passage from Lie algebras to Lie groups (described by theorems on existence of subgroups and homomorphisms; cf. e.g. [1, Chap. IV] or [3, Chap. II, Section 5]).

Finally, observe that the results of this paper remain valid with the same proofs for o-minimal expansions $\mathcal{R}$ of arbitrary real closed fields R.

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References

[1] C. Chevalley, *Theory of Lie Groups*, Princeton Univ. Press, 1946.
[2] L. van den Dries, *Tame Topology and O-minimal Structures*, Cambridge Univ. Press, 1998.
[3] P. W. Michor, *Topics in Differential Geometry*, Grad. Stud. Math. 93, Amer. Math. Soc., 2008.
[4] J. Mycielski, *The Banach–Tarski paradox for the hyperbolic plane*, Fund. Math. 132 (1989), 143–149.
[5] J. Mycielski and G. Tomkowicz, *The Banach–Tarski paradox for the hyperbolic plane (II)*, Fund. Math. 222 (2013), 289–290.
[6] J. Mycielski and G. Tomkowicz, *On small subsets in Euclidean spaces and the universe $L(\mathbb{R})$*, manuscript (yet unpublished).
[7] K. J. Nowak, *Decomposition into special cubes and its application to quasi-subanalytic geometry*, Ann. Polon. Math. 96 (2009), 65–74.
[8] Y. Peterzil, A. Pillay and S. Starchenko, *Definable simple groups in o-minimal structures*, Trans. Amer. Math. Soc. 352 (2000), 4421–4450.
[9] Y. Peterzil, A. Pillay and S. Starchenko, *Linear groups definable in o-minimal structures*, J. Algebra 247 (2002), 1–23.
[10] A. Pillay, *On groups and fields definable in o-minimal structures*, J. Pure Appl. Algebra 53 (1988), 239–255.
[11] J.-P. Rolin, P. Speissegger and A. J. Wilkie, *Quasianalytic Denjoy–Carleman classes and o-minimality*, J. Amer. Math. Soc. 16 (2003), 751–777.
[12] A. Strzeboński, *Euler characteristic in semialgebraic and other o-minimal structures*, J. Pure Appl. Algebra 96 (1994), 173–204.

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