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Univoque sets for real numbers

by

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Abstract. For $x \in (0,1)$, the univoque set for x, denoted $\mathcal{U}(x)$, is defined to be the set of $\beta \in (1,2)$ such that x has only one representation of the form $x = x_1/\beta + x_2/\beta^2 + \cdots$ with $x_i \in \{0,1\}$. We prove that for any $x \in (0,1)$, $\mathcal{U}(x)$ contains a sequence $\{\beta_k\}_{k\geq 1}$ increasing to 2. Moreover, $\mathcal{U}(x)$ is a Lebesgue null set of Hausdorff dimension 1; both $\mathcal{U}(x)$ and its closure $\overline{\mathcal{U}(x)}$ are nowhere dense.

1. Introduction. Given $\beta > 1$ and $x \in \mathbb{R}$, an infinite sequence $(\varepsilon_i) = \varepsilon_1\varepsilon_2\cdots$ of integers with $0 \leq \varepsilon_i < \beta$ for all i is called an *expansion of x in base β* if

$$x = \frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \cdots.$$

It is clear that x has such an expansion only if x lies in the interval $J_\beta = [0, (\lceil\beta\rceil - 1)/(\beta - 1)]$, where $\lceil\beta\rceil$ denotes the smallest integer larger than or equal to β. Conversely, for any $x \in J_\beta$, the expansions always exist, for instance, the following so-called *quasi-greedy algorithm* [DK95] gives an expansion of x in base $\beta > 1$ as follows: If $x = 0$, then $x_i = 0$ for all i. Otherwise, x_1 is defined to be the *largest integer in $\Omega_\beta = \{0, 1, \ldots, \lceil\beta\rceil - 1\}$* such that $x_1/\beta < x$. Inductively, if $x_1, \ldots, x_{i-1}$ have already been defined, then x_i is chosen to be the largest integer in Ω_β such that

$$\frac{x_1}{\beta} + \cdots + \frac{x_i}{\beta^i} < x.$$

It is easy to show that (x_i) is an expansion of x, called the *quasi-greedy expansion of x in base β*.

For $\beta > 1$, let Ω_β^∞ be the collection of infinite sequences over Ω_β, i.e., $\Omega_\beta^\infty = \{(\varepsilon_i) = \varepsilon_1\varepsilon_2\cdots : \varepsilon_i \in \Omega_\beta \text{ for all } i\}$. We define the projection map

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$\pi_\beta : \Omega_\beta^\infty \to \mathbb{R}$ as

$$\pi_\beta\big((\varepsilon_i)\big) = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{\beta^i} \quad \text{for } (\varepsilon_i) \in \Omega_\beta^\infty.$$

The map π_β is not injective; there is an easy example: taking $\beta = G$ to be the golden ratio $(1+\sqrt{5})/2 \approx 1.618$, we have

$$1 = \frac{1}{G} + \frac{1}{G^2} = \frac{1}{G} + \frac{1}{G^3} + \frac{1}{G^5} + \cdots .$$

It was generally believed that for any given $\beta \in (1,2)$, the number 1 has infinitely many expansions [Ko]. After Erdős, Horváth and Joó's startling discovery [EHJ] that $\#\pi_\beta^{-1}(1) = 1$ (i.e., the quasi-greedy expansion is the unique expansion of 1 in base β) for a continuum of $\beta \in (1,2)$, many works [DK95, dVK09, EJK, KoL98, KoL07] have been devoted to the study of the set

$$\mathcal{U}(1) = \{\beta \in (1,2) : \#\pi_\beta^{-1}(1) = 1\}.$$

In particular, it was proved that both $\mathcal{U}(1)$ and its closure $\overline{\mathcal{U}(1)}$ are of Lebesgue measure 0 and of Hausdorff dimension 1 [DK95, KoL07], and thus they are nowhere dense.

In the same way, for any $x \in [0,1]$, we define

$$\mathcal{U}(x) = \{\beta \in (1,2) : \#\pi_\beta^{-1}(x) = 1\}.$$

Note that 0 has the unique expansion $00\cdots$ in any base $\beta \in (1,2)$.

Recently, de Vries and Komornik [dVK11] proved that $\overline{\mathcal{U}(x)} \setminus \mathcal{U}(x)$ is (at most) countable for any $x \in [0,1]$. Moreover, they studied the set

$$\mathbf{U} = \{(x,\beta) \in \mathbb{R} \times (1,\infty) : x \in \mathcal{U}_\beta\},$$

where $\mathcal{U}_\beta = \{x \in J_\beta : \#\pi_\beta^{-1}(x) = 1\}$, proving that:

(1) $\mathbf{U}$ is not closed and $\overline{\mathbf{U}}$ is a Cantor set;
(2) $\mathbf{U}$ and $\overline{\mathbf{U}}$ are two-dimensional Lebesgue null sets;
(3) $\mathbf{U}$ and $\overline{\mathbf{U}}$ have Hausdorff dimension 2.

When β is an integer, all but countably many $x \in [0,1]$ have a unique expansion, i.e., $\mathcal{U}_\beta = [0,1]$ up to a countable set; when β is not an integer, Sidorov [S03] showed that $\mathcal{U}_\beta$ is a Lebesgue null set (see also [DdV, Theorem 7]). Daróczy, Kátai and Kallós [DK93, KK, Ka99, Ka01] developed a strategy for the computation of the Hausdorff dimension of $\mathcal{U}_\beta$ for any given $\beta > 1$. Glendinning and Sidorov [GS], Sidorov [S07], Kong, Li and Dekking [KLD], and de Vries and Komornik [dVK11] proved that:

(1) $\dim_{\mathrm{H}} \mathcal{U}_\beta = 1$ if $\beta \in \mathbb{N}$, and $\dim_{\mathrm{H}} \mathcal{U}_\beta < 1$ if $\beta \notin \mathbb{N}$;
(2) $\dim_{\mathrm{H}} \mathcal{U}_\beta \to 1$ as $\beta \nearrow 2$.

In this paper, we deal with the structure and properties of the set $\mathcal{U}(x)$ for any $x \in (0,1)$. It is well known, in the study of the set $\mathcal{U}(1)$, that the uniqueness of the expansion of 1 depends only on the comparison of the expansion with its shifts. But, in the criterion of the uniqueness of the expansion of any $x \in (0,1)$, the quasi-greedy expansion of 1 and the expansion of x itself are involved, while usually, one does not know both expansions simultaneously; this causes the main difficulty in the construction.

The following are our main results:

THEOREM 1.1. *For any* $x \in (0,1)$, *there exists a sequence* $\{\beta_k\}_{k\ge 1}$ *in* $\mathcal{U}(x)$ *satisfying* $\beta_k \nearrow 2$ *as* $k \to \infty$.

THEOREM 1.2. *We have*

$$\mathcal{L}(\mathcal{U}(x)) = 0 \quad \text{for all } x \in (0,1),$$

where $\mathcal{L}$ *denotes the one-dimensional Lebesgue measure.*

THEOREM 1.3. *We have*

$$\dim_{\mathrm{H}} \mathcal{U}(x) = 1 \quad \text{for all } x \in (0,1),$$

where $\dim_{\mathrm{H}}$ *means the Hausdorff dimension.*

We make several remarks here:

1. By Theorem 1.1, for any $x \in (0,1)$, we know that $\mathcal{U}(x)$ is nonempty, and the projection of $\mathbf{U}$ onto the x-axis contains the interval $[0,1]$.

2. Together with the result of de Vries and Komornik that $\overline{\mathcal{U}(x)} \setminus \mathcal{U}(x)$ is (at most) countable, Theorem 1.2 implies that for any $x \in (0,1)$, we have $\mathcal{L}(\overline{\mathcal{U}(x)}) = 0$. Therefore $\mathcal{U}(x)$ and $\overline{\mathcal{U}(x)}$ are nowhere dense in $(1,2)$.

3. In light of [F, Corollary 7.10], Theorem 1.3 provides an alternative proof of the fact that both $\mathbf{U}$ and $\overline{\mathbf{U}}$ are of Hausdorff dimension 2.

The rest of this paper is organized as follows: In the next section, some basic facts about quasi-greedy expansion and the set $\mathcal{U}(x)$ are presented. Section 3 is devoted to the proof of Theorem 1.2. Theorem 1.1 is proved in Section 4. We will prove Theorem 1.3 in the last section, based on the construction in the proof of Theorem 1.1.

2. Preliminaries. More than fifty years ago, Rényi [R] introduced the well-known greedy expansions. Parry [P] then gave a lexicographic characterization of greedy expansions, which became an excellent tool in investigating the combinatorial and topological nature of such expansions. In Parry's lexicographic characterization, the quasi-greedy expansion of 1 plays a very important role. Given $\beta > 1$ and $x \in \mathbb{R}$, note that x has a unique expansion in base β if and only if the lexicographically largest and smallest expansions of x in base β coincide.

From now on, for simplicity, we always consider β in the interval $(1,2]$, whence $\Omega_\beta = \{0,1\}$. It is readily checked that the quasi-greedy expansion of 1 for $\beta \in (1,2]$ should contain infinitely many 1's. Moveover, a sequence $(\delta_i) \in \{0,1\}^\infty$ of infinitely many 1's is the quasi-greedy expansion of 1 for some $\beta \in (1,2]$ if and only if it is self-admissible [BK], i.e.,

$$(\delta_{j+i})_i \le (\delta_i) \quad \text{for all } j \ge 1,$$

here and below, the notation $<$ or $\le$ between sequences means the lexicographic order: $(\varepsilon_i) < (\eta_i)$ if there exists $i_0 \ge 1$ such that $\varepsilon_{i_0} < \eta_{i_0}$ and $\varepsilon_i = \eta_i$ for all $i < i_0$; $(\varepsilon_i) \le (\eta_i)$ means either $(\varepsilon_i) < (\eta_i)$, or $(\varepsilon_i) = (\eta_i)$. The symbol $(\delta_{j+i})_i$ denotes the sequence $\delta_{j+1}\delta_{j+2}\cdots$; we will write (δ_{j+i}) instead if it causes no confusion.

Given $\beta \in (1,2]$ and $x \in [0,1]$, a *criterion* for x to have a unique expansion (in base β) is (see [Ko]): there exists an expansion (ε_i) of x satisfying

$$\begin{cases} (\varepsilon_{j+i}) < (\delta_i) & \text{if } \varepsilon_j = 0, \\ (\overline{\varepsilon_{j+i}}) < (\delta_i) & \text{if } \varepsilon_j = 1, \end{cases} \tag{2.1}$$

where (δ_i) is the quasi-greedy expansion of 1, $\overline{\varepsilon_i} = 1-\varepsilon_i$ and $(\overline{\varepsilon_i}) = \overline{\varepsilon_1}\,\overline{\varepsilon_2}\cdots$.

A sequence $(\delta_i) \in \{0,1\}^\infty$ is the unique expansion of 1 in some base $\beta \in (1,2]$ if and only if it is admissible, that is,

$$\begin{cases} (\delta_{j+i}) < (\delta_i) & \text{if } \delta_j = 0, \\ (\overline{\delta_{j+i}}) < (\delta_i) & \text{if } \delta_j = 1. \end{cases}$$

Moreover, the quasi-greedy algorithm provides a strictly increasing bijection between the set $\mathcal{U}(1)$ of bases β in which the expansion of 1 is unique and the collection $\mathcal{A}$ of admissible sequences (equipped with the lexicographic order defined as above); see [EJK] for more information.

From now on, we put $\Omega = \{0,1\}$. For $x \in [0,1]$, define $\Phi_x : (1,2] \to \Omega^\infty$ as follows: for $\beta \in (1,2]$,

$$\Phi_x(\beta) = (x_i),$$

where (x_i) is the quasi-greedy expansion of x in base β.

We list some basic facts about the quasi-greedy expansion here, and the readers are referred to [dVK09, dVK11] for more details.

PROPOSITION 2.1. *For any $x \in (0,1]$, Φ_x is strictly increasing.*

PROPOSITION 2.2. *Given $\beta \in (1,2]$ and $x \in (0,1]$, let (x_i) and (δ_i) be the quasi-greedy expansions of x and 1 (in base β) respectively. Then*

(1) *The sequence (x_i) is infinite, i.e., there are infinitely many 1's in (x_i);*
(2) *For any infinite expansion $(\varepsilon_i) \in \Omega^\infty$ of x in base β, $(\varepsilon_i) \le (x_i)$;*
(3) *$(x_i) \le (\delta_i)$ with equality if and only if $x = 1$;*

(4) *If* $x_j = 0$ *for some* $j \geq 1$, *then* $\frac{x_1}{\beta} + \cdots + \frac{x_{j-1}}{\beta^{j-1}} + \frac{1}{\beta^j} \geq x$;
(5) *If* $\frac{1}{\beta} + \cdots + \frac{1}{\beta^j} < x$ *for some* $j \geq 1$, *then* $x_1 \cdots x_j = 1^j$.

PROPOSITION 2.3 ([Ko]). *If* $\beta \in (1, G)$ *with* G *the golden ratio, then any* $x \in (0, 1]$ *has a continuum of expansions.*

As a result, for any $x \in (0, 1]$, the elements of $\mathcal{U}(x)$ must be larger than or equal to G. In fact, they are strictly larger than G (see [dVK09]) and G is sharp in this respect.

3. The Lebesgue measure of $\mathcal{U}(x)$. In this section, we study the Lebesgue measure of $\mathcal{U}(x)$ and prove Theorem 1.2.

Given $x \in (0, 1)$, for $j \geq 1$ and $\omega \in \Omega^j$, we define p_ω and q_ω as follows:

(1) Put $p_\omega = 1$ if $\omega = 0^j$; otherwise, define p_ω to be the unique solution larger than 1 of the equation
$$x = \frac{\omega_1}{p} + \cdots + \frac{\omega_j}{p^j}.$$
(2) Define q_ω to be the unique solution larger than 1 of the equation
$$x = \frac{\omega_1}{q} + \cdots + \frac{\omega_j}{q^j} + \frac{1}{q^{j+1}} + \frac{1}{q^{j+2}} + \cdots.$$

Denoting $\omega\tau = \omega_1 \cdots \omega_j \tau$ for $\tau = 0$ or 1, we have

$$p_\omega = p_{\omega 0} < \min\{p_{\omega 1}, q_{\omega 0}\} \leq \max\{p_{\omega 1}, q_{\omega 0}\} < q_{\omega 1} = q_\omega. \tag{3.1}$$

When $p_{\omega 1} < 2$ or $q_{\omega 0} < 2$, we have $p_{\omega 1} < q_{\omega 0}$, and thus

$$p_{\omega 0} = p_\omega < p_{\omega 1} < q_{\omega 0} < q_\omega = q_{\omega 1}. \tag{3.2}$$

LEMMA 3.1. *For* $j \geq 1$ *and* $\omega \in \Omega^j$, *we have:*

(1) *If* $p_\omega \in (1, 2)$, *then* x *has at least two expansions in base* p_ω; *if* $q_\omega \in (1, 2)$, *then* x *has at least two expansions in base* q_ω.
(2) *For any* $\beta \in (1, 2)$, *we have* $\beta \in [p_\omega, q_\omega]$ *if and only if there exists a sequence* $(\varepsilon_i) \in \pi_\beta^{-1}(x)$ *satisfying* $\varepsilon_1 \cdots \varepsilon_j = \omega$.

Proof. (1) can be deduced directly from the criterion (2.1) for $x \in (0, 1)$ to have a unique expansion in base $\beta \in (1, 2)$.

(2) "$\Rightarrow$": This is true for $\beta = p_\omega$ or q_ω. Assume that $\beta \in (p_\omega, q_\omega)$. Then
$$x - \sum_{i=1}^{\infty} \frac{1}{q_\omega^{j+i}} = \sum_{i=1}^{j} \frac{\omega_i}{q_\omega^i} < \sum_{i=1}^{j} \frac{\omega_i}{\beta^i} < \sum_{i=1}^{j} \frac{\omega_i}{p_\omega^i} \leq x,$$
and thus
$$0 < x - \sum_{i=1}^{j} \frac{\omega_i}{\beta^i} < \sum_{i=1}^{\infty} \frac{1}{q_\omega^{j+i}} < \sum_{i=1}^{\infty} \frac{1}{\beta^{j+i}} = \frac{1}{\beta^j(\beta - 1)},$$

i.e.,

$$y = \beta^j \left(x - \sum_{i=1}^{j} \frac{\omega_i}{\beta^i} \right) \in J_\beta.$$

Let (y_i) be the quasi-greedy expansion of y in base β. Then

$$x = \sum_{i=1}^{j} \frac{\omega_i}{\beta^i} + \sum_{i=1}^{\infty} \frac{y_i}{\beta^{j+i}},$$

that is, the sequence $(\varepsilon_i) = \omega_1 \cdots \omega_j y_1 y_2 \cdots$ is an expansion of x.

"$\Leftarrow$": Trivial. ∎

For $j \geq 1$ and $\omega \in \Omega^j$, set $I_\omega = (p_\omega, q_\omega) \cap (1,2)$ and

$$I^*_\omega = I_\omega \setminus \bigcup_{\zeta \in \Omega^j \setminus \{\omega\}} \overline{I_\zeta}.$$

It is readily checked that both I_ω and I^*_ω are (possibly empty) intervals, and for any pair of distinct $\omega, \zeta \in \Omega^j$, we have $I^*_\omega \cap I^*_\zeta = \emptyset$. By (3.1),

$$(p_{\omega 0}, q_{\omega 0}) \cup (p_{\omega 1}, q_{\omega 1}) \subseteq (p_\omega, q_\omega).$$

Then with the help of (3.2) it is not difficult to see that $I_{\omega 0} \cup I_{\omega 1} = I_\omega$, and thus we infer that $I^*_{\omega 0} \cup I^*_{\omega 1} \subseteq I^*_\omega$, in other words, the intervals $\{I^*_\omega\}_w$ have a nested structure.

Lemma 3.2. *With the notations above, we have* $\mathcal{U}(x) = \bigcap_{j=1}^\infty \bigcup_{\omega \in \Omega^j} I^*_\omega$.

Proof. First we have

$$\bigcap_{j=1}^{\infty} \bigcup_{\omega \in \Omega^j} I^*_\omega = \bigcup_{\eta \in \Omega^\infty} \bigcap_{j=1}^{\infty} I^*_{\eta|_j},$$

where $\eta|_j = \eta_1 \cdots \eta_j$ for $\eta = (\eta_i) \in \Omega^\infty$. We divide the proof into two steps:

"$\subseteq$": For $\beta \in \mathcal{U}(x)$, we assume that η is the unique expansion of x in base β. Then by Lemma 3.1, we have $\beta \in I_{\eta|_j}$ for all $j \geq 1$. Also by Lemma 3.1, for any $\omega \in \Omega^j$ with $\omega \neq \eta|_j$, we have $\beta \notin \overline{I_\omega}$. We infer that $\beta \in I^*_{\eta|_j}$ for all $j \geq 1$, and hence

$$\beta \in \bigcap_{j=1}^{\infty} I^*_{\eta|_j} \subseteq \bigcap_{j=1}^{\infty} \bigcup_{\omega \in \Omega^j} I^*_\omega.$$

"$\supseteq$": Suppose that $\beta \in \bigcup_{\eta \in \Omega^\infty} \bigcap_{j=1}^\infty I^*_{\eta|_j}$. Then $\beta \in \bigcap_{j=1}^\infty I^*_{\eta|_j}$ for some $\eta \in \Omega^\infty$, and thus $\beta \in I^*_{\eta|_j} \subseteq I_{\eta|_j}$ for all $j \geq 1$. If $\beta \notin \mathcal{U}(x)$, then by Lemma 3.1 we can find some $\omega \in \Omega^j$ with $\omega \neq \eta|_j$ such that $\beta \in [p_\omega, q_\omega]$. Then $\beta \in I_{\eta|_j} \cap [p_\omega, q_\omega] \subseteq I_{\eta|_j} \cap \overline{I_\omega}$, and thus $\beta \notin I^*_{\eta|_j}$, a contradiction. ∎

Proof of Theorem 1.2. For $\beta \in \mathcal{U}(x)$, let η be the unique expansion of x in base β. Then by Lemma 3.1, we have $\beta \in I^*_{\eta|_j} \subseteq I_{\eta|_j}$ for all $j \geq 1$. Let i_0

be the first place where 1 occurs in the sequence η (recall that $x \neq 0$ and thus $\eta \neq 0^\infty := 000\cdots$), i.e., $\eta_{i_0} = 1$ and $\eta_i = 0$ for $i < i_0$.

For $j \geq i_0$, we write $p = p_{\eta|_j}$ and $q = q_{\eta|_j}$. Since $p < \beta < q$ and

$$\sum_{i=1}^{j} \frac{\eta_i}{p^i} = x = \sum_{i=1}^{j} \frac{\eta_i}{q^i} + \frac{1}{q^j(q-1)},$$

we have

$$\frac{1}{q^j(q-1)} = \sum_{i=1}^{j} \frac{\eta_i}{p^i} - \sum_{i=1}^{j} \frac{\eta_i}{q^i} \geq \frac{1}{p^{i_0}} - \frac{1}{q^{i_0}} > \frac{q-p}{pq^{i_0}},$$

then

$$q - p < \frac{pq^{i_0}}{q^j(q-1)} < \frac{\beta}{\beta^{j-i_0}(\beta-1)}; \tag{3.3}$$

therefore,

$$|I_{\eta|_j}| = q - p \to 0 \quad \text{as } j \to \infty, \tag{3.4}$$

where $|\cdot|$ denotes the length of an interval. Then there exists a constant $\gamma > 0$ (depending on β) such that $I_{\eta|_j} \subseteq (1+\gamma, 2-\gamma)$ for large j, say $j \geq j_0$. Writing $p^* = p_{(\eta|_j)1}$ and $q^* = q_{(\eta|_j)0}$, we have

$$\sum_{i=1}^{j} \frac{\eta_i}{(p^*)^i} + \frac{1}{(p^*)^{j+1}} = x = \sum_{i=1}^{j} \frac{\eta_i}{(q^*)^i} + \frac{1}{(q^*)^{j+1}(q^*-1)},$$

and thus

$$\frac{1}{(q^*)^{j+1}(q^*-1)} - \frac{1}{(q^*)^{j+1}} = \sum_{i=1}^{j} \frac{\eta_i}{(p^*)^i} - \sum_{i=1}^{j} \frac{\eta_i}{(q^*)^i} + \frac{1}{(p^*)^{j+1}} - \frac{1}{(q^*)^{j+1}}. \tag{3.5}$$

By (3.2), the left hand side of (3.5) is

$$\frac{1}{(q^*)^{j+1}(q^*-1)} - \frac{1}{(q^*)^{j+1}} = \frac{1}{(q^*)^{j+1}} \left(\frac{1}{q^*-1} - 1 \right) \geq \frac{\gamma}{q^{j+1}(1-\gamma)},$$

and the right hand side is

$$\begin{aligned} &\sum_{i=1}^{j} \frac{\eta_i}{(p^*)^i} - \sum_{i=1}^{j} \frac{\eta_i}{(q^*)^i} + \frac{1}{(p^*)^{j+1}} - \frac{1}{(q^*)^{j+1}} \\ &\qquad \leq \sum_{i=1}^{\infty} \left[\frac{1}{(p^*)^i} - \frac{1}{(q^*)^i} \right] \leq \frac{1}{p^*-1} - \frac{1}{q^*-1} \leq \frac{q^*-p^*}{\gamma^2}. \end{aligned}$$

Comparing the two inequalities above, we infer that

$$|I_{(\eta|_j)0} \cap I_{(\eta|_j)1}| = q^* - p^* \geq \frac{\gamma^3}{q^{j+1}(1-\gamma)} \geq \frac{\gamma^3}{q^j(1-\gamma)(2-\gamma)}.$$

On the other hand, by (3.3),

$$|I_{\eta|_j}| = q - p < \frac{pq^{i_0}}{q^j(q-1)} < \frac{(2-\gamma)^{i_0+1}}{q^j\gamma}.$$

Therefore, when $j \geq j_0$,

$$\frac{|I_{(\eta|_j)0} \cap I_{(\eta|_j)1}|}{|I_{\eta|_j}|} = \frac{q^* - p^*}{q - p} > C \tag{3.6}$$

for some constant $C > 0$ which only depends on x and β.

By (3.4), for any $r > 0$ small enough, we can find $j \geq j_0$ such that

$$I_{\eta|_{(j+1)}} \subseteq (\beta - r, \beta + r) \quad \text{but} \quad I_{\eta|_j} \nsubseteq (\beta - r, \beta + r).$$

This means that $|I_{\eta|_j}| \geq r$, and thus by (3.6), we have

$$|I_{\eta|_{(j+1)}}| \geq |I_{\eta|_j 0} \cap I_{\eta|_j 1}| \geq Cr.$$

Now by Lemma 3.2, we have $\mathcal{U}(x) \subseteq \bigcup_{\omega \in \Omega^{j+2}} I_\omega^*$, so

$$I_{\eta|_{(j+1)}0} \cap I_{\eta|_{(j+1)}1} \cap \mathcal{U}(x) = \emptyset.$$

Notice that $I_{\eta|_{(j+1)}0} \cap I_{\eta|_{(j+1)}1} \subseteq I_{\eta|_{(j+1)}} \subseteq (\beta - r, \beta + r)$, and thus by (3.6),

$$\mathcal{L}(\mathcal{U}(x) \cap (\beta - r, \beta + r)) \leq 2r - |I_{\eta|_{(j+1)}0} \cap I_{\eta|_{(j+1)}1}| \leq 2r - C^2 r.$$

Therefore,

$$\limsup_{r \to 0} \frac{\mathcal{L}(\mathcal{U}(x) \cap (\beta - r, \beta + r))}{2r} \leq 1 - \frac{C^2}{2} < 1,$$

which implies that β is not a density point of $\mathcal{U}(x)$. Since $\beta \in \mathcal{U}(x)$ is arbitrary, the Lebesgue density theorem yields $\mathcal{L}(\mathcal{U}(x)) = 0$. ■

4. Sequence increasing to 2 in $\mathcal{U}(x)$. In this section we prove Theorem 1.1. To this end, we construct a sequence of β's increasing to 2 in $\mathcal{U}(x)$.

For $x \in (0,1)$, let (x_i) be the quasi-greedy expansion of x in base 2. Recalling that (x_i) cannot end with 0^∞ by Proposition 2.2(1), we consider the following two cases according to the tails of (x_i).

CASE I: (x_i) *ends with* 1^∞. In this case, we write

$$(x_i) = \alpha_1 \cdots \alpha_m 0 1^\infty,$$

where $m \geq 0$ and $\alpha_i \in \Omega$ for all $1 \leq i \leq m$.

For $k \geq 2$, we take two sequences of integers (a_i), (b_i) with $1 \leq a_i, b_i \leq k-1$, and put

$$(x_i^{(k)}) = \alpha_1 \cdots \alpha_m 0 1^k 0^{a_1} 1^{b_1} 0^{a_2} 1^{b_2} \cdots,$$
$$(\varepsilon_i^{(k)}) = 1^k 0^{a_1} 1^{b_1} 0^{a_2} 1^{b_2} \cdots,$$

where, e.g., 1^k denotes the block of 1's of length k. We remark that when $k \geq m+1$, we have

$$\text{(4.1)} \qquad \begin{cases} (x_{j+i}^{(k)}) \leq (\varepsilon_i^{(k)}) \\ \overline{(x_{j+i}^{(k)})} < (\varepsilon_i^{(k)}) \end{cases} \quad \text{for all } j \geq 0.$$

Let β_k be the unique solution between 1 and 2 of the equation

$$x = \sum_{i=1}^{\infty} \frac{x_i^{(k)}}{\beta^i}.$$

We claim that when $k \geq \max\{2, m+1\}$, the number x has a unique expansion in base β_k. To prove the claim, by the criterion for the uniqueness of the expansion (2.1) and the inequalities (4.1), we only need to show that $(\varepsilon_i^{(k)}) < (\delta_i^{(k)})$ with $(\delta_i^{(k)})$ the quasi-greedy expansion of 1 in base β_k, or equivalently, $\sum_{i=1}^{\infty} \varepsilon_i^{(k)}/\beta_k^i < 1$ (see (2) and (3) in Proposition 2.2). To this end, we note that

$$\sum_{i=1}^{\infty} \frac{x_i}{2^i} = \sum_{i=1}^{m} \frac{\alpha_i}{2^i} + \frac{1}{2^{m+1}} = x = \sum_{i=1}^{\infty} \frac{x_i^{(k)}}{\beta_k^i} = \sum_{i=1}^{m} \frac{\alpha_i}{\beta_k^i} + \frac{1}{\beta_k^{m+1}} \sum_{i=1}^{\infty} \frac{\varepsilon_i^{(k)}}{\beta_k^i},$$

hence

$$\sum_{i=1}^{\infty} \frac{\varepsilon_i^{(k)}}{\beta_k^i} = \beta_k^{m+1} \left(\sum_{i=1}^{m} \frac{\alpha_i}{2^i} + \frac{1}{2^{m+1}} - \sum_{i=1}^{m} \frac{\alpha_i}{\beta_k^i} \right) \leq \frac{\beta_k^{m+1}}{2^{m+1}} < 1,$$

and the claim follows.

To see that $\beta_k \to 2$ as $k \to \infty$, we remark that

$$p_{\alpha_1 \cdots \alpha_m 01^k} < \beta_k < q_{\alpha_1 \cdots \alpha_m 01^k} = 2,$$

and by a similar estimation to (3.4), that $q_{\alpha_1 \cdots \alpha_m 01^k} - p_{\alpha_1 \cdots \alpha_m 01^k} \to 0$ as $k \to \infty$.

Case II: *(x_i) does not end with 1^∞.* Then there are infinite 0's and 1's in (x_i). This case can be divided into the following two subcases according to the first digit of (x_i).

Subcase II.1: *(x_i) begins with* 1. Then

$$(x_i) = 1^{r_1} 0^{s_1} 1^{r_2} 0^{s_2} \cdots,$$

where r_j, s_j are positive integers for all j.

When $k \geq 2$, we take two sequences of integers $(a_i), (b_i)$ satisfying $1 \leq a_i, b_i \leq \nu_k - 3$, where $\nu_k = \sum_{j=1}^{k} (r_j + s_j)$, and put

$$(x_i^{(k)}) = 1^{r_1} 0^{s_1} \cdots 1^{r_k} 0^{s_k} 0 1^{a_1} 0^{b_1} 1^{a_2} 0^{b_2} \cdots.$$

It is readily checked that when $k \geq 3$,

$$(4.2)\qquad \begin{cases} (x_{j+i}^{(k)}) < 1^{\nu_k - 2}0^\infty \\ \overline{(x_{j+i}^{(k)})} < 1^{\nu_k - 2}0^\infty \end{cases} \quad \text{for all } j \geq 0.$$

Let β_k be the unique solution in the interval $(1, 2)$ of the equation

$$x = \sum_{i=1}^{\infty} \frac{x_i^{(k)}}{\beta^i}.$$

We claim that when $k \geq 3$, the expansion of x in base β_k is unique. To see this, by (4.2) and Proposition 2.2(5), we only need to prove that

$$(4.3)\qquad \sum_{i=1}^{\nu_k - 2} \frac{1}{\beta_k^i} < 1.$$

Note that

$$\sum_{i=1}^{\nu_k} \frac{x_i}{\beta_k^i} = \sum_{i=1}^{\nu_k} \frac{x_i^{(k)}}{\beta_k^i} < \sum_{i=1}^{\infty} \frac{x_i^{(k)}}{\beta_k^i} = x = \sum_{i=1}^{\infty} \frac{x_i}{2^i} < \sum_{i=1}^{\nu_k} \frac{x_i}{2^i} + \frac{1}{2^{\nu_k}}.$$

Then

$$\frac{1}{\beta_k} - \frac{1}{2} \leq \sum_{i=1}^{\nu_k} \frac{x_i}{\beta_k^i} - \sum_{i=1}^{\nu_k} \frac{x_i}{2^i} < \frac{1}{2^{\nu_k}},$$

where the first inequality is due to the facts that $x_1 = 1$ and $\beta_k < 2$. Hence

$$(4.4)\qquad 2 - \beta_k < \frac{\beta_k}{2^{\nu_k - 1}} < \frac{1}{\beta_k^{\nu_k - 2}},$$

and thus

$$\sum_{i=1}^{\nu_k - 2} \frac{1}{\beta_k^i} = \frac{1 - \beta_k^{2 - \nu_k}}{\beta_k - 1} < 1,$$

and this completes the proof of the claim.

At last, the fact that $\beta_k \to 2$ as $k \to \infty$ is due to (4.3).

Subcase II.2: (x_i) *begins with* 0. In this case, we write

$$(x_i) = 0^{r_1}1^{s_1}0^{r_2}1^{s_2}\cdots.$$

For $k \geq 3$ and $1 \leq a_i, b_i \leq \nu_{k-1} + r_k - r_1 - 2$, where $\nu_k = \sum_{j=1}^{k}(r_j + s_j)$, we put

$$(x_i^{(k)}) = 0^{r_1}1^{s_1}\cdots 0^{r_k}01^{a_1}0^{b_1}1^{a_2}0^{b_2}\cdots.$$

Then if $k \geq r_1 + 2$,

$$\begin{cases} (x_{j+i}^{(k)}) < 1^{\nu_{k-1} + r_k - r_1 - 1}0^\infty \\ \overline{(x_{j+i}^{(k)})} < 1^{\nu_{k-1} + r_k - r_1 - 1}0^\infty \end{cases} \quad \text{for all } j \geq 1.$$

Let $\beta_k \in (1,2)$ be the unique solution of the equation

$$x = \sum_{i=1}^{\infty} \frac{x_i^{(k)}}{\beta^i}.$$

We claim that when $k \geq r_1 + 2$, the expansion of x in base β_k is unique, and that $\beta_k \to 2$ as $k \to \infty$. To see this, as in Subcase II.1, we only need to show that

$$\sum_{i=1}^{\nu_{k-1}+r_k-r_1-1} \frac{1}{\beta_k^i} < 1.$$

In fact,

$$\sum_{i=1}^{\nu_{k-1}+r_k} \frac{x_i}{\beta_k^i} = \sum_{i=1}^{\nu_{k-1}+r_k} \frac{x_i^{(k)}}{\beta_k^i} < \sum_{i=1}^{\infty} \frac{x_i^{(k)}}{\beta_k^i} = x = \sum_{i=1}^{\infty} \frac{x_i}{2^i} < \sum_{i=1}^{\nu_{k-1}+r_k} \frac{x_i}{2^i} + \frac{1}{2^{\nu_{k-1}+r_k}}.$$

Then

$$\frac{1}{2^{\nu_{k-1}+r_k}} > \sum_{i=1}^{\nu_{k-1}+r_k} \frac{x_i}{\beta_k^i} - \sum_{i=1}^{\nu_{k-1}+r_k} \frac{x_i}{2^i} \geq \frac{1}{\beta_k^{r_1+1}} - \frac{1}{2^{r_1+1}} \geq \frac{2-\beta_k}{2^{r_1}\beta_k},$$

and thus

$$2 - \beta_k < \frac{\beta_k}{2^{\nu_{k-1}+r_k-r_1}} < \frac{1}{\beta_k^{\nu_{k-1}+r_k-r_1-1}},$$

therefore,

$$\sum_{i=1}^{\nu_{k-1}+r_k-r_1-1} \frac{1}{\beta_k^i} = \frac{1-\beta_k^{1+r_1-r_k-\nu_{k-1}}}{\beta_k - 1} < 1.$$

5. The Hausdorff dimension of $\mathcal{U}(x)$. In this section, we prove Theorem 1.3, based on the construction of $(x_i^{(k)})$ in the preceding section.

As usual, we equip the space $\Omega^\infty = \{0,1\}^\infty$ with the metric D defined as follows: For $(\varepsilon_i), (\eta_i) \in \Omega^\infty$, if $(\varepsilon_i) = (\eta_i)$, then $D((\varepsilon_i),(\eta_i)) = 0$; otherwise,

$$D((\varepsilon_i),(\eta_i)) = 2^{-\min\{i\,:\,\varepsilon_i \neq \eta_i\}}.$$

For $x \in (0,1)$, let (x_i) be the quasi-greedy expansion of x in base 2. As in the preceding section, the proof of Theorem 1.3 can be divided into several cases according to the expansion (x_i). We will only prove the theorem for Case I; the other cases can be treated in the same way. From now on, we assume that

$$(x_i) = \alpha_1 \cdots \alpha_m 0 1^\infty$$

with $m \geq 0$ and $\alpha_i \in \Omega$ for $1 \leq i \leq m$.

For $k \geq 2$, let $\Sigma_{k-1} = \{1, \ldots, k-1\}$ and let $\Sigma_{k-1}^* = \bigcup_{j=1}^{\infty} \Sigma_{k-1}^j$ be the set of finite blocks over Σ_{k-1}. For $w = w_1 \cdots w_j \in \Sigma_{k-1}^j$, define

$$y(w) = \epsilon_1^{w_1} \cdots \epsilon_j^{w_j} \epsilon_{j+1},$$

where $\epsilon_i = 0$ if i is odd and $\epsilon = 1$ otherwise. Thus $y(w)$ is a block of length $w_1 + \cdots + w_j + 1$ over Ω.

For $k \geq \max\{2, m+1\}$, put $x^{(k)} = \alpha_1 \cdots \alpha_m 01^k$. For $\omega \in \Sigma_{k-1}^*$, define

$$E_\omega^{(k)} = \{(\varepsilon_i) \in \Omega^\infty : (\varepsilon_i) \text{ begins with the block } x^{(k)} y(w)\}.$$

Then $E_\omega^{(k)}$ is a compact set and $E_{\omega\tau}^{(k)} \subseteq E_\omega^{(k)}$ for any $\tau \in \Sigma_{k-1}$, and for two distinct blocks $\omega, \zeta \in \Sigma_{k-1}^j$, we have $E_\omega^{(k)} \cap E_\zeta^{(k)} = \emptyset$. At last, we define

$$E^{(k)} = \bigcap_{j=1}^{\infty} \bigcup_{\omega \in \Sigma_{k-1}^j} E_\omega^{(k)} = \bigcup_{\eta \in \Sigma_{k-1}^\infty} \bigcap_{j=1}^{\infty} E_{\eta|_j}^{(k)}. \tag{5.1}$$

Then from the proof of Theorem 1.1 and Proposition 2.1, we know that for any $k \geq \max\{2, m+1\}$, $E^{(k)}$ is a nonempty compact set and

$$\Phi_x^{-1}(E^{(k)}) \subseteq \mathcal{U}(x).$$

So, to prove Theorem 1.3, we only need to show that for any $s \in (0,1)$,

$$\dim_{\mathrm{H}} \Phi_x^{-1}(E^{(k)}) \geq s \tag{5.2}$$

when k is large enough. This fact will be proved via the following two lemmas.

Lemma 5.1. *For $s \in (0,1)$, there exists $k_0 \geq \max\{2, m+1\}$ such that $\dim_{\mathrm{H}} E^{(k)} > s$ whenever $k \geq k_0$.*

Lemma 5.2. *Given $\lambda > 0$ and $k \geq \max\{2, m+1\}$, there exists a constant $C > 0$ such that for any $\beta_1, \beta_2 \in \Phi_x^{-1}(E^{(k)})$, we have*

$$D(\Phi_x(\beta_1), \Phi_x(\beta_2)) \leq C|\beta_1 - \beta_2|^{1-\lambda}.$$

In fact, if the two lemmas are proven, then by the following proposition,

$$\dim_{\mathrm{H}} \Phi_x^{-1}(E^{(k)}) \geq s(1-\lambda),$$

and thus (5.2) follows.

Proposition 5.3 ([F]). *Suppose that (X, d_1), (Y, d_2) are metric spaces and $f : X \to Y$ is a map. If there exist constants $C, \alpha > 0$ such that*

$$d_2(f(a), f(b)) \leq C(d_1(a,b))^\alpha \quad \text{for all } a, b \in X,$$

then $\dim_{\mathrm{H}} X \geq \alpha \dim_{\mathrm{H}} f(X)$.

Now we prove the lemmas.

Proof of Lemma 5.1. Let $\sigma : \Omega^\infty \to \Omega^\infty$ be the shift operator, i.e., $\sigma(\varepsilon_i) = (\varepsilon_{i+1})$ for $(\varepsilon_i) \in \Omega^\infty$. Set $F^{(k)} = \sigma^{m+1+k} E^{(k)}$, here σ^{m+1+k} is a similarity between $E^{(k)}$ and $F^{(k)}$. As a result, the set $F^{(k)}$ is compact, and $\dim_H E^{(k)} = \dim_H F^{(k)}$.

For any $\omega \in \Sigma_{k-1}^2$, we define $\varphi(\omega) : \Omega^\infty \to \Omega^\infty$ as follows:

$$\varphi(\omega)\big((\varepsilon_i)\big) = 0^{\omega_1}1^{\omega_2}(\varepsilon_i) \quad \text{for } (\varepsilon_i) \in \Omega^\infty.$$

Then $\varphi(\omega)$ is a contractive similarity for any $\omega \in \Sigma_{k-1}^2$. Moreover, by (5.1), we infer that

$$F^{(k)} = \bigcup_{\omega \in \Sigma_{k-1}^2} \varphi(\omega)(F^{(k)})$$

with the union pairwise disjoint. In other words, the set $F^{(k)}$ is a self-similar set satisfying the open set condition. Therefore, denoting $s_k = \dim_H F^{(k)}$, we have

$$(2^{-s_k} + 2^{-2s_k} + \cdots + 2^{-(k-1)s_k})^2 = 1,$$

i.e., $2^{-s_k} + 2^{-2s_k} + \cdots + 2^{-(k-1)s_k} = 1$. The lemma then follows from the fact that $s_k \to 1$ as $k \to \infty$. ∎

Proof of Lemma 5.2. Fix $\beta_1, \beta_2 \in \Phi_x^{-1}(E^{(k)})$ with $\beta_2 > \beta_1$. By Proposition 2.3, we have $\beta_2 > \beta_1 \geq G$, where G is the golden ratio.

Let (ε_i) and (η_i) be the unique expansions of x in base β_1 and β_2 respectively. Then by Proposition 2.1, we have $(\varepsilon_i) < (\eta_i)$. Hence, there exists $i_0 \geq 1$ such that

$$\varepsilon_{i_0} = 0,\ \eta_{i_0} = 1, \quad \text{and} \quad \varepsilon_i = \eta_i \quad \text{for all } i < i_0.$$

Therefore $D(\Phi_x(\beta_1), \Phi_x(\beta_2)) = D((\varepsilon_i), (\eta_i)) = 2^{-i_0}$.

By Proposition 2.2(4) and the definition of $E^{(k)}$, we have

$$\sum_{i=1}^{i_0-1} \frac{\eta_i}{\beta_2^i} + \frac{1}{\beta_2^{i_0}} + \frac{1}{\beta_2^{i_0+k}} < x \leq \sum_{i=1}^{i_0-1} \frac{\varepsilon_i}{\beta_1^i} + \frac{1}{\beta_1^{i_0}},$$

and thus

$$\frac{1}{\beta_2^{i_0+k}} < \sum_{i=1}^{i_0} \left(\frac{1}{\beta_1^{i_0}} - \frac{1}{\beta_2^{i_0}} \right) \leq \frac{\beta_2 - \beta_1}{(G-1)^2};$$

therefore,

$$\beta_2 - \beta_1 \geq \beta_2^{-(i_0+k)}(G-1)^2 > 2^{-(i_0+k)}(G-1)^2. \tag{5.3}$$

Fix $\lambda > 0$. Assume that $\beta_2 - \beta_1 < 2^{-k/\lambda}(G-1)^2$. Then $i_0 > k(1-\lambda)/\lambda$ by (5.3). So

$$D(\Phi_x(\beta_1), \Phi_x(\beta_2)) = 2^{-i_0} < \left(\frac{\beta_2 - \beta_1}{(G-1)^2} \right)^{\frac{i_0}{i_0+k}} \leq \frac{|\beta_1 - \beta_2|^{1-\lambda}}{(G-1)^2},$$

and the lemma follows. ∎

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