

# Metric spaces nonembeddable into Banach spaces with the Radon–Nikodým property and thick families of geodesics

by

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**Abstract.** We show that a geodesic metric space which does not admit bilipschitz embeddings into Banach spaces with the Radon–Nikodým property does not necessarily contain a bilipschitz image of a thick family of geodesics. This is done by showing that no thick family of geodesics is Markov convex, and comparing this result with results of Cheeger–Kleiner, Lee–Naor, and Li. The result contrasts with the earlier result of the author that any Banach space without the Radon–Nikodým property contains a bilipschitz image of a thick family of geodesics.

**1. Introduction.** The Radon–Nikodým property (RNP) is one of the most important isomorphic invariants of Banach spaces. We refer to [1, 2, 3, 8, 9, 20] for systematic presentations of results on the RNP.

In the recent work on metric embeddings a substantial role is played by existence and nonexistence of bilipschitz embeddings of metric spaces into Banach spaces with the RNP (see [6, 7, 12]). At the seminar “Nonlinear geometry of Banach spaces” (Texas A & M University, August 2009) Bill Johnson suggested the problem of metric characterization of reflexivity and the RNP [22, Problem 1.1]; see also [17, p. 307]. In [18] the RNP was characterized in terms of thick families of geodesics defined in the following way:

**DEFINITION 1.1.** Let  $u$  and  $v$  be two elements in a metric space  $(M, d_M)$ . A  $uv$ -geodesic is a distance-preserving map  $g : [0, d_M(u, v)] \rightarrow M$  such that  $g(0) = u$  and  $g(d_M(u, v)) = v$  (where  $[0, d_M(u, v)]$  is an interval of the real line with the distance inherited from  $\mathbb{R}$ ). A family  $T$  of  $uv$ -geodesics is called *thick* if there is  $\alpha > 0$  such that for every  $g \in T$  and for any finite collection of points  $r_1, \dots, r_n \in [0, d_M(u, v)]$ , to be called *control points*, there is another  $uv$ -geodesic  $\tilde{g} \in T$  and a sequence  $0 < s_1 < q_1 < s_2 < q_2 < \dots < s_m < q_m <$

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$s_{m+1} < d_M(u, v)$  satisfying the conditions:

- The set  $\{0, q_1, \dots, q_m, d_M(u, v)\}$  contains  $r_1, \dots, r_n$ .
- $g(q_i) = \tilde{g}(q_i)$ .
- We have

$$(1) \quad \sum_{i=1}^{m+1} d_M(g(s_i), \tilde{g}(s_i)) \geq \alpha.$$

The following result gives a metric characterization of the RNP.

**THEOREM 1.2** ([18]). *A Banach space  $X$  does not have the RNP if and only if there exists a metric space  $M_X$  containing a thick family  $T_X$  of geodesics which admits a bilipschitz embedding into  $X$ .*

Studying metric characterizations of the RNP, it would be much more useful and interesting to get a characterization of all metric spaces which do not admit bilipschitz embeddings into Banach spaces with the RNP. In view of Theorem 1.2 it is natural to ask whether the presence of bilipschitz images of thick families of geodesics characterizes metric spaces which do not admit bilipschitz embeddings into spaces with the RNP. It is clear that the answer to this question in full generality is negative: we may just consider a dense subset of a Banach space without the RNP which does not contain any continuous curves (e.g. the subset of all vectors with rational coordinates in  $c_0$ ). So we need to restrict our attention to spaces containing sufficiently large collections of continuous curves. Our main result is a negative answer even in the case of geodesic metric spaces (we use the terminology of [4] on metric spaces and of [17] on metric embeddings):

**THEOREM 1.3.** *There exist geodesic metric spaces which*

- *do not contain bilipschitz images of thick families of geodesics,*
- *do not admit bilipschitz embeddings into Banach spaces with the Radon–Nikodým property.*

More precisely, we prove that the Heisenberg group with its subriemannian (Carnot–Carathéodory) metric (see [5, 10, 14, 16]) does not admit a bilipschitz embedding of a thick family of geodesics. This result proves Theorem 1.3 since it is known that the Heisenberg group does not admit a bilipschitz embedding into a Banach space with the RNP—see [6] and [12], where the observation made in [21] on the consequences of the differentiability result of [19] was generalized to RNP targets.

Our proof is based on the notion of Markov convexity which was introduced in [13], with further important progress achieved in [15].

**DEFINITION 1.4** ([13]; we use a slightly modified version of [15]). Let  $\{X_t\}_{t \in \mathbb{Z}}$  be a Markov chain on a state space  $\Omega$ . Given an integer  $k \geq 0$ , we denote by  $\{\tilde{X}_t(k)\}_{t \in \mathbb{Z}}$  the process which equals  $X_t$  for time  $t \leq k$ , and

evolves independently (with respect to the same transition probabilities) for time  $t > k$ . Fix  $p > 0$ . A metric space  $(X, d_X)$  is called *Markov  $p$ -convex with constant  $\Pi$*  if for every Markov chain  $\{X_t\}_{t \in \mathbb{Z}}$  on a state space  $\Omega$ , and every  $f : \Omega \rightarrow X$ ,

$$(2) \quad \sum_{k=0}^{\infty} \sum_{t \in \mathbb{Z}} \frac{\mathbb{E}[d_X(f(X_t), f(\tilde{X}_t(t-2^k)))^p]}{2^{kp}} \leq \Pi^p \cdot \sum_{t \in \mathbb{Z}} \mathbb{E}[d_X(f(X_t), f(X_{t-1}))^p].$$

The least constant  $\Pi$  for which (2) holds for all Markov chains is called the *Markov  $p$ -convexity constant* of  $X$ , and is denoted  $\Pi_p(X)$ . We say that  $(X, d_X)$  is *Markov  $p$ -convex* if  $\Pi_p(X) < \infty$ .

Our proof of Theorem 1.3 is based on the following result:

**THEOREM 1.5.** *A metric space with a thick family of geodesics is not Markov  $p$ -convex for any  $p \in (0, \infty)$ .*

Theorem 1.5 implies that thick families of geodesics do not admit bilipschitz embeddings into the Heisenberg group because it is known [14, Theorem 7.4] that the Heisenberg group is Markov convex for some  $p \in (0, \infty)$ .

Theorem 1.5 is a generalization of the result of [15, Section 3] stating that the Laakso space (we mean the one defined in [11, p. 290]) is not Markov  $p$ -convex for any  $p \in (0, \infty)$ . It is easy to see that the Laakso space has a thick family of geodesics.

**REMARK 1.6.** The Heisenberg group can be identified with  $\mathbb{R}^3$  in such a way that all geodesics of the Heisenberg group with its subriemannian metric are spirals, projecting down to circles in two dimensions (see [16, Section 1.3]). With this representation it is easy to verify that the family of  $uv$ -geodesics, where  $u$  and  $v$  are elements of the Heisenberg group, is never thick. It is natural to expect that one can prove Theorem 1.3 by combining this description with some differentiability theory. I preferred to use Markov convexity because I think that Theorem 1.5 is of independent interest.

**2. Proof of Theorem 1.5.** Let  $(M, d_M)$  be a metric space containing a thick family of geodesics. We assume that each of the geodesics in the family has length 1 and is parameterized by the interval  $[0, 1]$ .

The general idea of the proof is the same as the idea of the proof in [15] of the fact that the Laakso space is not Markov convex. Namely, given  $h \in \mathbb{N}$  we find in the thick family of geodesics in  $M$  (a thick family is necessarily infinite) a finite collection  $\mathbb{G}_h$  consisting of  $2^h$  geodesics, and a collection of finitely many points on each of them, such that there is a Markov chain on

this collection of points with the the left-hand side of (2) greater than or equal to  $C(p)h^{1/p}$  times the right-hand side of (2) without  $II^p$ .

**Short description of the Markov chain.** We introduce the state space  $\Omega$  as  $\Omega = \mathbb{Z} \times \mathbb{G}_h$ . Let  $\varphi$  be a positive integer (to be specified later) and let  $f : \Omega \rightarrow M$  be given by

$$f(t, g) = \begin{cases} g(0) & \text{if } t < 0, \\ g(t2^{-\varphi}) & \text{if } t \in \{0, 1, 2, 3, \dots, 2^\varphi\}, \\ g(1) & \text{if } t > 2^\varphi. \end{cases}$$

The Markov chain  $\{X_t\}_{t \in \mathbb{Z}}$  is defined as follows:

- $X_t = (t, g)$  for some  $g \in \mathbb{G}_h$  (so the chain  $X_t$  *remembers* the geodesic which *it is on*).
- If  $X_t = (t, g)$  and  $t < 0$  or  $t \geq 2^\varphi$ , then  $X_{t+1} = (t + 1, g)$  with probability 1.
- If  $X_t = (t, g)$  and  $t \in \{0, 1, 2, 3, \dots, 2^\varphi - 1\}$ , then  $X_{t+1} = (t + 1, \widehat{g})$ , where  $\widehat{g} \in \mathbb{G}_h$  and either  $\widehat{g} = g$  or  $\widehat{g} = \widetilde{g}$ , where  $\widetilde{g}$  is any geodesic of the family  $\mathbb{G}_h$  which has what we call a *crossing with  $g$*  in the interval  $[t2^{-\varphi}, (t + 1)2^{-\varphi}]$ . The probabilities of all permissible choices of  $\widehat{g}$  are the same. Crossings,  $\varphi$ , and the family  $\mathbb{G}_h$  of geodesics are defined in such a way that a geodesic cannot have two crossings in one interval of the form  $[t2^{-\varphi}, (t + 1)2^{-\varphi}]$ .

We describe the needed notion of crossing below. At this point we just mention that each crossing of geodesics corresponds to their intersection, but not all of the intersections of geodesics are crossings.

The description of the allowed moves from one geodesic to another in Theorem 1.5 is substantially more complicated than in the case of the Laakso space in [15], because the geodesics can have infinitely many points of intersection. Therefore to get the desired estimate we need the Markov chain to move from one geodesic to another in a well-organized manner, because we have lower estimates for distances between geodesics only for small sets of pairs of points (the only available estimate of this type is (1)).

We label geodesics of  $\mathbb{G}_h$  by *binary strings of length  $h$*  and sets of crossings by vertices of a *binary tree of depth  $h - 1$* .

Recall that a *binary tree*  $B_h$  of depth  $h$  is a finite graph whose vertices are finite sequences of 0s and 1s of length at most  $h$ , including the empty sequence denoted  $\emptyset$ ; two vertices are joined by an edge if the corresponding sequences are  $(\theta_1, \dots, \theta_{n-1})$  and  $(\theta_1, \dots, \theta_{n-1}, \theta_n)$  for some  $\theta_n \in \{0, 1\}$  ( $(\theta_1, \dots, \theta_{n-1})$  can be empty).

We pick one element in the thick family of geodesics and label it by the sequence consisting of  $h$  zeros, so we denote it  $g_{(0, \dots, 0)}$ . We apply the

condition of Definition 1.1 to  $g_{(0,\dots,0)}$  with control points 0 and 1, and get a geodesic which we label  $g_{(1,0,\dots,0)}$  and points which we denote  $q_1^\emptyset, \dots, q_m^\emptyset \in [0, 1]$  and  $s_1^\emptyset, \dots, s_{m+1}^\emptyset \in [0, 1]$  so that the conditions of Definition 1.1 are satisfied. We introduce the set

$$R_\emptyset = \{0, q_1^\emptyset, \dots, q_m^\emptyset, 1\}.$$

This is the set of common crossings of all geodesics of  $\mathbb{G}_h$ .

In the next step we pick two geodesics  $g_{(0,1,0,\dots,0)}$  and  $g_{(1,1,0,\dots,0)}$  and find two subsets  $R_{(0)}$  and  $R_{(1)}$  of  $[0, 1]$ . The sets  $R_{(0)}$  and  $R_{(1)}$  will be the sets of common crossings of all geodesics whose labels start with 0 and with 1, respectively.

To pick  $g_{(0,1,0,\dots,0)}$  and  $g_{(1,1,0,\dots,0)}$  we apply the condition of Definition 1.1 to  $g_{(0,\dots,0)}$  and  $g_{(1,0,\dots,0)}$ , respectively, with the collection of control points defined as the union of two subsets:

- The points  $q_1^\emptyset, \dots, q_m^\emptyset$  and  $s_1^\emptyset, \dots, s_{m+1}^\emptyset \in [0, 1]$ . Observe that  $g(0)$  is the same for all geodesics of the family, so we do not have to list 0 among control points. The same applies to 1.
- The points  $k2^{-\gamma(1)}$ ,  $k = 1, \dots, 2^{\gamma(1)}$ , where  $\gamma(1) \in \mathbb{N}$  is sufficiently large. The conditions on  $\gamma(1)$  are the following:
  1.  $4m \cdot 2^{-\gamma(1)} \leq \alpha/10$ , where  $m$  is the cardinality of  $\{q_i^\emptyset\}$ .
  2.  $2^{-\gamma(1)} \leq \frac{1}{4} \min_i (s_i^\emptyset - q_{i-1}^\emptyset)$ .

We denote the sequences obtained by applying the condition of Definition 1.1 to  $g_{(0,\dots,0)}$  by  $q_1^{(0)}, \dots, q_{m(0)}^{(0)} \in [0, 1]$  and  $s_1^{(0)}, \dots, s_{m(0)+1}^{(0)} \in [0, 1]$ , and the sequences obtained by applying the condition of Definition 1.1 to  $g_{(1,0,\dots,0)}$  by  $q_1^{(1)}, \dots, q_{m(1)}^{(1)} \in [0, 1]$  and  $s_1^{(1)}, \dots, s_{m(1)+1}^{(1)} \in [0, 1]$ . We set  $R_{(0)} = \{q_1^{(0)}, \dots, q_{m(0)}^{(0)}\} \setminus R_\emptyset$  and  $R_{(1)} = \{q_1^{(1)}, \dots, q_{m(1)}^{(1)}\} \setminus R_\emptyset$ . The set  $R_{(0)}$  is the set of crossings of all geodesics whose label starts with 0. The set  $R_{(1)}$  is the set of crossings of all geodesics whose label starts with 1.

At this point we give a generic description which will be used for all further selections of geodesics and sets of crossings.

Suppose that we have already picked  $g_{(\theta_1, \dots, \theta_n, 0, \dots, 0)}$  and constructed all crossing sets  $R_{(\theta_1, \dots, \theta_k)}$ , where  $\theta_1, \dots, \theta_k$  is an initial segment of  $\theta_1, \dots, \theta_{n-1}$ , as well as the sequences

$$(3) \quad q_1^{(\theta_1, \dots, \theta_{n-1})}, \dots, q_{m(\theta_1, \dots, \theta_{n-1})}^{(\theta_1, \dots, \theta_{n-1})} \in [0, 1],$$

$$(4) \quad s_1^{(\theta_1, \dots, \theta_{n-1})}, \dots, s_{m(\theta_1, \dots, \theta_{n-1})+1}^{(\theta_1, \dots, \theta_{n-1})} \in [0, 1].$$

To pick the geodesic  $g_{(\theta_1, \dots, \theta_n, 1, 0, \dots, 0)}$  we apply the conditions of Definition 1.1 to  $g_{(\theta_1, \dots, \theta_n, 0, \dots, 0)}$  and the collection of control points defined as the

union of two subsets:

- All points listed in (3) and (4).
- The points  $k2^{-\gamma(n)}$ ,  $k = 1, \dots, 2^{\gamma(n)}$ , where  $\gamma(n)$  is a sufficiently large number. The conditions on  $\gamma(n)$  are

$$(5) \quad 4m(\theta_1, \dots, \theta_{n-1}) \cdot 2^{-\gamma(n)} \leq \frac{\alpha}{10}.$$

$$(6) \quad 2^{-\gamma(n)} \leq \frac{1}{4} \min_{(\theta_1, \dots, \theta_{n-1})} \min_i (s_i^{(\theta_1, \dots, \theta_{n-1})} - q_{i-1}^{(\theta_1, \dots, \theta_{n-1})}).$$

We denote the resulting sequences

$$(7) \quad q_1^{(\theta_1, \dots, \theta_n)}, \dots, q_{m(\theta_1, \dots, \theta_n)}^{(\theta_1, \dots, \theta_n)} \in [0, 1],$$

$$(8) \quad s_1^{(\theta_1, \dots, \theta_n)}, \dots, s_{m(\theta_1, \dots, \theta_n)+1}^{(\theta_1, \dots, \theta_n)} \in [0, 1].$$

We introduce the set

$$R_{(\theta_1, \dots, \theta_n)} = \{q_1^{(\theta_1, \dots, \theta_n)}, \dots, q_{m(\theta_1, \dots, \theta_n)}^{(\theta_1, \dots, \theta_n)}\} \setminus \bigcup_{k=0}^{n-1} R_{(\theta_1, \dots, \theta_k)},$$

where by the set corresponding to  $k = 0$  we mean  $R_\emptyset$ . The set  $R_{(\theta_1, \dots, \theta_n)}$  is the set of common crossings of all geodesics whose label starts with  $(\theta_1, \dots, \theta_n)$ .

After we pick all geodesics for  $\mathbb{G}_h$  and construct all the sets of crossings, we pick the number  $\varphi \in \mathbb{N}$ . The choice of  $\varphi$  should satisfy two conditions:

- $2^{-\varphi}$  should be strictly less than the distance between any two crossings for the same geodesic.
- Moreover

$$(9) \quad 2^{-\varphi} \leq \frac{1}{16} d_M(g_{(\theta_1, \dots, \theta_n, 0, \dots, 0)}(s_i^{(\theta_1, \dots, \theta_n)}), g_{(\theta_1, \dots, \theta_n, 1, 0, \dots, 0)}(s_i^{(\theta_1, \dots, \theta_n)}))$$

for all  $(\theta_1, \dots, \theta_n)$  and  $1 \leq i \leq m(\theta_1, \dots, \theta_n) + 1$  satisfying

$$(10) \quad g_{(\theta_1, \dots, \theta_n, 0, \dots, 0)}(s_i^{(\theta_1, \dots, \theta_n)}) \neq g_{(\theta_1, \dots, \theta_n, 1, 0, \dots, 0)}(s_i^{(\theta_1, \dots, \theta_n)}).$$

Now we are ready to complete the short description of the Markov chain given at the beginning of the proof. Namely we provide more details on the way in which the Markov chain can move from one geodesic to another. If  $X_t = (t, g_{(\theta_1, \dots, \theta_h)})$ , and the interval  $[t2^{-\varphi}, (t+1)2^{-\varphi}]$  contains a crossing labeled by some initial segment  $(\theta_1, \dots, \theta_d)$  of  $(\theta_1, \dots, \theta_h)$ , then  $X_{t+1} = (t+1, \tilde{g})$ , where  $\tilde{g}$  is any of the  $2^{h-d}$  geodesics whose labels have  $(\theta_1, \dots, \theta_d)$  as their initial segment, and each of these  $2^{h-d}$  choices has the same probability. Observe that the choice of  $\varphi$  is such that a segment of the form  $[t2^{-\varphi}, (t+1)2^{-\varphi}]$  cannot contain more than one crossing.

For each collection  $(\theta_1, \dots, \theta_n)$ ,  $n < h$ , we find a subset

$$\{s_i^{(\theta_1, \dots, \theta_n)}\}_{i \in A(\theta_1, \dots, \theta_n)}$$

in the set  $\{s_i^{(\theta_1, \dots, \theta_n)}\}_{i=1}^{m(\theta_1, \dots, \theta_n)+1}$  which is sufficiently large in the sense that

$$\sum_{i \in A(\theta_1, \dots, \theta_n)} d_M(g_{(\theta_1, \dots, \theta_n, 0, \dots, 0)}(s_i^{(\theta_1, \dots, \theta_n)}), g_{(\theta_1, \dots, \theta_n, 1, 0, \dots, 0)}(s_i^{(\theta_1, \dots, \theta_n)})) \geq \frac{\alpha}{4}.$$

We require that each  $i \in A(\theta_1, \dots, \theta_n)$  satisfies (10) and two additional conditions needed for our estimates—see conditions **(a)** and **(b)** below.

We estimate the sum on the left-hand side of (2) from below as follows. We assign to each point  $s_i^{(\theta_1, \dots, \theta_n)}$  with  $i \in A(\theta_1, \dots, \theta_n)$  a scale  $2^k$  ( $k = k(i, (\theta_1, \dots, \theta_n)) \in \mathbb{N}$ ), an interval of integers

$$(11) \quad I_{i, (\theta_1, \dots, \theta_n)} = [t, t + 1, \dots, T],$$

and a subset  $\mathbb{G} = \mathbb{G}_{(\theta_1, \dots, \theta_n)}$  in  $\mathbb{G}_h$ , in such a way that no triple (scale, integer, geodesic) is ever repeated. In the remainder of the argument we use the following notation and terminology. If  $\mathbb{G}$  is a subset of  $\mathbb{G}_h$  we write  $X_t \in \mathbb{G}$  as a shorthand for the condition  $X_t = (t, g)$  with  $g \in \mathbb{G}$ . If  $X_t = (t, g)$ , we say that  $X_t$  is *on g*.

Then the left-hand side of (2) can be estimated from below by

$$(12) \quad \sum_{\substack{(\theta_1, \dots, \theta_n) \\ i \in A(\theta_1, \dots, \theta_n)}} \sum_{t \in I_{i, (\theta_1, \dots, \theta_n)}} \frac{\mathbb{E}[d_X(f(X_t), f(\tilde{X}_t(t - 2^k)))^p | X_{t-2^k} \in \mathbb{G}] \mathbb{P}(X_{t-2^k} \in \mathbb{G})}{2^{kp}},$$

where the conditional probability is with respect to the event  $X_{t-2^k} \in \mathbb{G}_{(\theta_1, \dots, \theta_n)}$ . Note that although this is not reflected in our notation,  $k$  also depends on  $i$  and  $(\theta_1, \dots, \theta_n)$ .

Now we describe how we pick the scale  $2^k$ , the interval in (11), and the set of geodesics  $\mathbb{G}_{(\theta_1, \dots, \theta_n)}$ .

**(1)** We pick  $k$  to be the smallest positive integer such that  $2^k 2^{-\varphi}$  exceeds  $s_i^{(\theta_1, \dots, \theta_n)} - q_{i-1}^{(\theta_1, \dots, \theta_n)}$  (we use 0 instead of  $q_{i-1}^{(\theta_1, \dots, \theta_n)}$  if  $i = 1$ ).

**(2)** We let

$$(13) \quad \begin{aligned} L &= L_{i, (\theta_1, \dots, \theta_n)} \\ &= d_M(g_{(\theta_1, \dots, \theta_n, 0, \dots, 0)}(s_i^{(\theta_1, \dots, \theta_n)}), g_{(\theta_1, \dots, \theta_n, 1, 0, \dots, 0)}(s_i^{(\theta_1, \dots, \theta_n)})) \end{aligned}$$

and introduce the interval (11) as the set of  $\tau \in \mathbb{Z}$  for which  $\tau 2^{-\varphi}$  is in the interval of length  $\frac{1}{4}L$  which ends at the point  $s_i^{(\theta_1, \dots, \theta_n)}$ .

The set of such  $\tau$  is nonempty because, by (9),  $2^{-\varphi} < \frac{1}{16}L$ . Furthermore, (9) implies that  $2^{-\varphi}|I| \geq \frac{1}{8}L$ .

**(3)** We define  $\mathbb{G}_{(\theta_1, \dots, \theta_n)}$  as the set of geodesics whose labels start with  $(\theta_1, \dots, \theta_n)$ .

Now we impose the second of the conditions under which a number  $i \in \{1, \dots, m(\theta_1, \dots, \theta_n) + 1\}$  is included into  $A(\theta_1, \dots, \theta_n)$ . (Below we introduce the third condition which we label **(b)**.)

- (a)** The interval of length  $\frac{1}{4}L + 2^k 2^{-\varphi}$  which ends at the point  $s_i^{(\theta_1, \dots, \theta_n)}$  does not contain any crossings belonging to  $\bigcup_{k=0}^{n-1} R_{(\theta_1, \dots, \theta_k)}$ .

Observe that under the condition **(a)** the conditional expectation in (12) is at least  $\frac{1}{2}(L - 2(\frac{1}{4}L))^p = \frac{1}{2^{p+1}}L^p$ . The reason is that **(a)** implies that all of the geodesics in  $\mathbb{G}_{(\theta_1, \dots, \theta_n)}$  have  $q_{i-1}^{(\theta_1, \dots, \theta_n)}$  as their common crossing, the crossing occurs “after” time  $t - 2^k$  if  $t \in I_{i, (\theta_1, \dots, \theta_n)}$ , and there are no crossings which could lead outside  $\mathbb{G}_{(\theta_1, \dots, \theta_n)}$  in the interval between  $t - 2^k$  and  $t$  if  $t \in I_{i, (\theta_1, \dots, \theta_n)}$ . Therefore with probability 1/2 at the crossing  $q_{i-1}^{(\theta_1, \dots, \theta_n)}$  the Markov chains  $X_t$  and  $\tilde{X}_t$  will “go” in different “directions”, one of them to  $g_{(\theta_1, \dots, \theta_n, 0, \dots, 0)}(s_i^{(\theta_1, \dots, \theta_n)})$ , and the other to  $g_{(\theta_1, \dots, \theta_n, 1, 0, \dots, 0)}(s_i^{(\theta_1, \dots, \theta_n)})$ . It remains to use the triangle inequality.

Next, it is easy to verify that the probability that  $X_t$  ( $t = 0, 1, \dots, 2^\varphi$ ) is on a geodesic  $g$  is  $2^{-h}$  if  $t2^{-\varphi}$  is not a crossing involving  $g$ . If  $t2^{-\varphi}$  is a crossing of  $2^{h-n}$  geodesics, the probability that  $X_t$  is on one of them is  $2^{-n}$ . The verification of this statement can be done by moving from 0 to 1. Therefore the probability in (12) is  $2^{-n}$ .

The third condition on  $i \in A(\theta_1, \dots, \theta_n)$  is

- (b)**  $L \geq \frac{\alpha}{2}(q_i^{(\theta_1, \dots, \theta_n)} - q_{i-1}^{(\theta_1, \dots, \theta_n)})$ , where  $L$  is defined in (13).

Under the condition **(b)** we can estimate each of the summands in (12). In fact, by the choice of  $k$  we have  $2^k 2^{-\varphi} < 2(q_i^{(\theta_1, \dots, \theta_n)} - q_{i-1}^{(\theta_1, \dots, \theta_n)})$ . Therefore  $L > \frac{\alpha}{4} \cdot 2^{k-\varphi}$  and

$$\frac{\frac{1}{2^{p+1}}L^p}{2^{kp}} > \frac{\alpha^p}{2^{3p+1}} 2^{-p\varphi}.$$

Hence, for  $i$  satisfying **(a)** and **(b)** each term in the sum (12) is  $\geq C2^{-n}2^{-p\varphi}$ , where  $C$  is a constant which depends only on  $\alpha$  and  $p$ .

Now we fix  $(\theta_1, \dots, \theta_n)$  and consider the sum

$$(14) \quad \sum_{i \in A(\theta_1, \dots, \theta_n)} \sum_{t \in I_{i, (\theta_1, \dots, \theta_n)}} \frac{\mathbb{E}[d_X(f(X_t), f(\tilde{X}_t(t - 2^k)))^p | X_{t-2^k} \in \mathbb{G}]}{2^{kp}} \mathbb{P}(X_{t-2^k} \in \mathbb{G}).$$

As we observed above, the number of terms in the sum  $\sum_{t \in I_{i, (\theta_1, \dots, \theta_n)}}$  is at least  $2^\varphi \cdot \frac{1}{8}L_{i, (\theta_1, \dots, \theta_n)}$ . We shall show that this implies that the sum in (14)

is at least

$$(15) \quad \frac{2^\varphi}{8} \sum_{i \in A(\theta_1, \dots, \theta_n)} L_{i, (\theta_1, \dots, \theta_n)} C 2^{-n} 2^{-p\varphi} \geq \frac{C 2^{(1-p)\varphi} 2^{-n}}{8} \left( \frac{\alpha}{2} - \frac{\alpha}{10} \right).$$

To get this we use the inequality

$$\sum_{i \text{ satisfies (b)}} L_{i, (\theta_1, \dots, \theta_n)} \geq \frac{\alpha}{2},$$

which follows from (1) by the Markov-type inequality. The condition (10) is included in **(b)**, but we also have to exclude  $i$  which fail to satisfy **(a)**.

Comparing condition **(a)** with our definitions we see that it suffices to require that  $s_i^{(\theta_1, \dots, \theta_n)}$  is not in the interval of length  $2(s_i^{(\theta_1, \dots, \theta_n)} - q_{i-1}^{(\theta_1, \dots, \theta_n)}) + \frac{1}{4}L \leq 3(s_i^{(\theta_1, \dots, \theta_n)} - q_{i-1}^{(\theta_1, \dots, \theta_n)}) \leq 3 \cdot 2^{-\gamma(n)}$  (see (5)) following one of the elements of  $\bigcup_{k=0}^{n-1} R_{(\theta_1, \dots, \theta_k)}$ . Hence the total length of the intervals  $[q_{i-1}^{(\theta_1, \dots, \theta_n)}, q_i^{(\theta_1, \dots, \theta_n)}]$  which have to be excluded does not exceed  $4m(\theta_1, \dots, \theta_{n-1}) \cdot 2^{-\gamma(n)} \leq \alpha/10$  (see (5)). It is clear that the sum of  $L_{i, (\theta_1, \dots, \theta_n)}$  over all excluded in this way  $i$  also does not exceed  $\alpha/10$ . The inequality (15) follows.

Therefore the sum (14) is  $\geq 2^{(1-p)\varphi} 2^{-n} C(\alpha, p)$ . Adding over  $(\theta_1, \dots, \theta_n)$  for fixed  $n$  we get  $\geq 2^{(1-p)\varphi} C(\alpha, p)$ . Adding over  $n = 0, 1, \dots, h-1$  we get  $\geq 2^{(1-p)\varphi} C(\alpha, p)h$ . On the other hand, the sum on the right-hand side of (2) is  $2^{-p\varphi} \cdot 2^\varphi = 2^{(1-p)\varphi}$ . We get  $II^p \geq C(\alpha, p)h$ , which is the desired inequality.

To complete the proof we need to explain why for different choices of  $i$  and  $(\theta_1, \dots, \theta_n)$  the sets of triples (scale, integer, geodesic) are disjoint. First let us consider the case where  $(\theta_1, \dots, \theta_{n_1})$  and  $(\tilde{\theta}_1, \dots, \tilde{\theta}_{n_2})$  are such that  $n_1 \neq n_2$ . Observe that  $2^{k(i, (\theta_1, \dots, \theta_{n_1}))}$  is 2-equivalent to

$$(16) \quad 2^\varphi (s_i^{(\theta_1, \dots, \theta_{n_1})} - q_{i-1}^{(\theta_1, \dots, \theta_{n_1})}),$$

and  $2^{k(j, (\tilde{\theta}_1, \dots, \tilde{\theta}_{n_2}))}$  is 2-equivalent to

$$(17) \quad 2^\varphi (s_j^{(\tilde{\theta}_1, \dots, \tilde{\theta}_{n_2})} - q_{j-1}^{(\tilde{\theta}_1, \dots, \tilde{\theta}_{n_2})}),$$

and for  $n_1 \neq n_2$  the numbers (16) and (17) cannot be 4-equivalent—see (6) and the description of the choice of geodesics  $g_{(\theta_1, \dots, \theta_h)}$ .

If  $n_1 = n_2$ , but  $(\theta_1, \dots, \theta_{n_1})$  is not the same as  $(\tilde{\theta}_1, \dots, \tilde{\theta}_{n_2})$ , then the families  $\mathbb{G}_{i, (\theta_1, \dots, \theta_{n_1})}$  and  $\mathbb{G}_{j, (\tilde{\theta}_1, \dots, \tilde{\theta}_{n_2})}$  do not contain common geodesics.

Finally, if we consider labels  $i, (\theta_1, \dots, \theta_n)$  and  $j, (\theta_1, \dots, \theta_n)$ , then either

$$k(i, (\theta_1, \dots, \theta_n)) \neq k(j, (\theta_1, \dots, \theta_n))$$

(and we are done) or

$$k(i, (\theta_1, \dots, \theta_n)) = k(j, (\theta_1, \dots, \theta_n)).$$

In the latter case, as is easy to check, the intervals  $I_{i,(\theta_1,\dots,\theta_n)}$  and  $I_{j,(\theta_1,\dots,\theta_n)}$  (see (2) for the definition) are disjoint.

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