

Julia and John revisited

by

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Abstract. We show that the Fatou components of a semi-hyperbolic rational map are John domains. The converse does not hold. This compares to a famous result of Carleson, Jones and Yoccoz for polynomials, in which case the two conditions are equivalent.

We show that a connected Julia set is locally connected for a large class of non-uniformly hyperbolic rational maps. This class is more general than semi-hyperbolicity and includes Collet–Eckmann maps, topological Collet–Eckmann maps and maps satisfying a summability condition (as considered by Graczyk and Smirnov).

1. Introduction. Hyperbolic rational dynamics is very well understood and the Julia sets of hyperbolic maps have good geometric and statistical properties. Allow critical points in the Julia set and one may lose these good properties. During the last two decades various classes of rational maps have been considered which display some form of non-uniform hyperbolicity. Such classes include sub-hyperbolic maps ([5], [11], [21]), semi-hyperbolic maps ([6], [23]), Collet–Eckmann maps ([15], [14], [9]), topological Collet–Eckmann maps ([15], [17]), recurrent Collet–Eckmann maps ([12]) and maps satisfying a summability condition ([8]). Maps from these classes retain some of the good geometric and statistical properties of the hyperbolic setting. The main result of Carleson, Jones and Yoccoz [6] states equivalence for *polynomial* maps between a geometric condition, John regularity of the basin of infinity, and a topological condition on critical orbits, semi-hyperbolicity.

In Theorem 1, we extend the result of Carleson et al. to the rational setting: semi-hyperbolicity implies John regularity for all components of the Fatou set. This result has been obtained previously by Yin [23] for rational maps with *connected* Julia set. In contrast to what was shown by Carleson et al. in the case of polynomials, there are several known counterexamples to the converse in the rational setting.

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In Theorem 2, we prove local connectivity of connected Julia sets for all of the classes of rational maps mentioned in the first paragraph.

Carleson et al. use a criterion for John regularity of a simply connected domain in terms of the hyperbolic metric, applied for the basin of attraction of infinity A_∞ . For the general case when the Julia set is not connected, thus A_∞ is not simply connected, a more involved proof is needed.

We propose a new approach, using Herron's characterization of John regularity [10] in terms of the quasihyperbolic metric. His criterion works for all domains. We thus provide a proof that covers both cases, of a connected and disconnected Julia set.

Let f be a rational map of degree at least 2. We say that f is *semi-hyperbolic* if it has no parabolic cycles and all critical points in its Julia set J are non-recurrent. We say that x is *non-recurrent* if $x \notin \omega(x)$ where $\omega(x)$ is the accumulation set of the orbit of x ,

$$\omega(x) = \bigcap_{N \geq 0} \overline{\{f^n(x) : n \geq N\}}.$$

A domain $\Omega \subseteq \bar{\mathbb{C}}$ is an ε -*John domain* if there is $z_0 \in \Omega$ such that for all $z_1 \in \Omega$ there exists an arc $\gamma \subseteq \Omega$ connecting z_1 to z_0 and for all $z \in \gamma$,

$$\delta(z) \geq \varepsilon \delta(z, z_1),$$

where δ denotes the distance with respect to the spherical metric σ , and by $\delta(z)$ we mean $\delta(z, \partial\Omega)$.

A closed set $A \subseteq \bar{\mathbb{C}}$ is called *locally connected* if for every $\tau > 0$ there is $\theta > 0$ such that, for any two points $a, b \in A$ with $\delta(a, b) < \theta$, we can find a continuum $B \subseteq A$ (i.e. a compact connected set containing at least two points) such that

$$a, b \in B \quad \text{and} \quad \text{diam } B < \tau.$$

As a consequence of the main result of [12] (see also Proposition 3.1 in [23]), semi-hyperbolic rational maps satisfy the *Exponential Shrinking of components* condition (denoted *ExpShrink*). This property was proved in [6] to hold for semi-hyperbolic polynomials. Przytycki [15, Section 4] showed that *ExpShrink* is equivalent to the *Topological Collet–Eckmann* condition (*TCE*) and other conditions, for example the *Uniform Hyperbolicity on Periodic Orbits*. These results were later extended by Przytycki, Rivera-Letelier and Smirnov [17]. As *TCE* is a topological invariant, all the conditions above have the same property.

A rational map f satisfies the *Exponential Shrinking of components* condition if there are $\lambda > 1$ and $r > 0$ such that for all $z \in J$, $n > 0$ and every connected component W of $f^{-n}(B(z, r))$,

$$\text{diam } W < \lambda^{-n}.$$

We introduce a weaker version of *ExpShrink*. We say that a rational map f satisfies the *Summable Shrinking of components* condition (*SumShrink*) if there are $r > 0$ and a sequence $(\omega_n)_{n \geq 1}$ of positive numbers such that

$$\sum_{n > 0} \omega_n < \infty,$$

and for all $z \in J$, $n > 0$ and every connected component W of $f^{-n}(B(z, r))$,

$$\text{diam } W < \omega_n.$$

This property rules out the existence of rotation domains, Cremer points and parabolic cycles. Therefore, by the classification of periodic Fatou components (see Theorem IV.2.1 in [5]), the Julia set of such a map without attracting cycles is the Riemann sphere.

In this paper we prove the following facts.

THEOREM 1. *The Fatou components of a rational semi-hyperbolic map are John domains. The converse does not hold.*

THEOREM 2. *If the Julia set of a SumShrink rational map is connected then it is locally connected.*

In [19], Rivera-Letelier shows that if a rational map has a fully invariant attracting John domain Ω , then it is semi-hyperbolic. The idea of the proof is the following. The map satisfies *ExpShrink* (see Corollary 4) and *TCE* (see Section 4 in [15]). The Julia set is the boundary of Ω . By the John property of Ω , the Julia set is porous. He shows that the Julia set of a *TCE* map is not porous in a neighborhood of a recurrent critical point. So there are no recurrent critical points in J and therefore the map is semi-hyperbolic.

Several counterexamples to this implication are known in the general case. Probably the most interesting is an example of a rational map without neutral cycles, with all Fatou components being John domains with uniform constant but which has a recurrent critical orbit in the Julia set. This example is given by Rivera-Letelier [19], using a construction of Roesch [20]. In [23], Yin presents an example given by Carsten Petersen of a rational map whose Julia set is the unit circle but which has a parabolic fixed point (therefore it is not semi-hyperbolic). We use a construction of Yampolsky and Zakeri [22] of a rational map of degree two having two Siegel disks which are John domains, each having a recurrent critical point on the boundary.

We prove in Proposition 10 that if the Julia set of a semi-hyperbolic rational map is connected then there exists $\varepsilon > 0$ such that each Fatou component is an ε -John domain. This implies a stronger version of local connectivity, namely θ depends linearly on τ , in the notation of the definition. This is related to the Julia set being *fractal* as defined in [6]: small balls centered on J are pushed forward to the large scale with bounded degree.

This gives control on the geometric distortion so J resembles itself at any scale. Using the fact that semi-hyperbolic rational maps satisfy *ExpShrink*, it is fairly easy to check that semi-hyperbolicity is equivalent to the Julia set being fractal. See Theorem 2.1 in [6] for the complete proof in the polynomial case. In the rational case, this has been proven by Yin [23].

TCE is defined (see Section 4 in [15], for example) by the existence of a radius $r > 0$, a bound for criticality $M \geq 1$ and a lower bound $P > 0$ for the density of times $n > 0$ such that the degree of the n th pullback of $B(f^n(x), r)$ is bounded by M , independently of $x \in J$. Semi-hyperbolicity is therefore equivalent to TCE with $P = 1$, that is, the degree is uniformly bounded for all times. We may also remark that hyperbolicity is equivalent to TCE with $P = 1$ and $M = 1$.

The assumption that J is connected in Proposition 10 can be replaced by the condition that there are only finitely many multivalent Fatou components. If this condition fails then it is not hard to show that there are two critical points that are separated by infinitely many Fatou components. *A priori*, this situation cannot be excluded. Similar phenomena may occur even for hyperbolic dynamics see examples of dynamics in the last chapter of [2].

In [6], the existence of the basin of attraction of infinity, which is super-attracting in the polynomial case, and properties of the hyperbolic metric are used to prove relations between the geometry of the Fatou set and the dynamics. John regularity can be better understood in full generality (for domains which are not simply connected) using the quasi-hyperbolic metric, as demonstrated in [10]. In our construction we emulate features like equipotential curves and geodesic rays in an arbitrary attracting cycle of a rational map.

Let $\gamma \subseteq \Omega$ be an arc. We define its *quasi-hyperbolic length* by

$$l_{\text{qh}}(\gamma) = \int_{\gamma} \frac{|d\sigma(z)|}{\delta(z)}.$$

This induces the quasi-hyperbolic distance $\text{dist}_{\text{qh}}(\cdot, \cdot)$ on Ω by the standard construction. Let also $l(\gamma)$ define the length of γ with respect to the spherical metric.

The quasi-hyperbolic distance has been used in [10] to give an alternative definition of John domains. It has also been extensively employed in [9] and [8] to study Hölder regularity (defined in Section 3) and more general integrable domains (defined on page 83).

In the polynomial case, local connectivity of connected Julia sets is easier to check. Assume J is connected and denote by A_{∞} the basin of attraction of infinity. Then A_{∞} is simply connected, so if it is a John or even

a Hölder domain, the Riemann mapping extends to a Hölder continuous map on \bar{D} . Therefore, by Carathéodory's theorem, $J = \partial A_\infty$ is locally connected.

Every John domain is a Hölder domain and every Hölder domain is an integrable domain. Graczyk and Smirnov [8] show that every connected component of the boundary of an integrable domain is locally connected. Suppose that all Fatou components of a rational map are integrable domains. *A priori*, this does not imply that J is locally connected even if it is connected, since in general there are infinitely many Fatou components.

2. Further remarks. A stronger version of John regularity, *uniformly John* property, is considered in [1]. In the case of simply connected domains it is equivalent to John regularity. Polynomials whose basin of infinity has this property are characterized in terms of topological properties of critical orbits in the Julia set.

Graczyk and Smirnov [9] proved that Fatou components of a *Collet–Eckmann* map (see definition below) are Hölder domains. The converse problem was considered by Przytycki [15]. Hölder regularity implies the Collet–Eckmann property provided the orbit of each critical point in the Julia set of a polynomial does not accumulate at other critical points. The existence of a fully invariant Fatou component is essential, as is the case in Corollary 4.

Relations between derivative growth and the geometry of Fatou components have also been studied in [4]. All aforementioned regularity conditions are discussed there in a systematic way.

In [18] it is proven that polynomial derivative growth on repelling periodic orbits of a polynomial implies that the basin of attraction of infinity is an integrable domain. More precisely, it is required that the derivative on repelling periodic orbits of period n is of order at least $n^{5+\varepsilon}$. As a consequence, if the Julia set is connected then it is locally connected. This result has been improved in [16]: only growth of order $n^{3+\varepsilon}$ is required and the result holds for attracting cycles of rational maps.

Let us define the summability condition as considered by Graczyk and Smirnov [8]. Let f be a rational map of degree at least 2, J its Julia set and Crit its critical set. For technical reasons we assume that critical orbits in the Julia set do not contain critical points, but an additional construction overcomes this obstacle. Let

$$\sigma_n := \min\{|(f^n)'(f(c))| : c \in \text{Crit} \cap J\}.$$

Suppose also that f has no parabolic periodic points. We say that f satisfies

the *summability condition* with exponent α if

$$\sum_{n=1}^{\infty} (\sigma_n)^{-\alpha} < \infty.$$

This condition generalizes the *Collet–Eckmann condition* which requires exponential growth of $(\sigma_n)_{n \geq 1}$. Let also μ_{\max} be the maximal multiplicity of critical points in J . Proposition 7.2 in [8] shows that if f satisfies the summability condition with exponent

$$\alpha = \frac{1}{1 + \mu_{\max}},$$

then f satisfies *SumShrink*, so Theorem 2 applies.

3. John regularity. In this section we prove the aforementioned results. The first tool relates the quasi-hyperbolic metric to John regularity. In [10], the hypothesis of Lemma 3 is shown to be an equivalent definition of John regularity. We only need one implication. As its proof is reasonably short, we include it for completeness. As a general remark, all derivatives are spherical derivatives unless specified otherwise.

LEMMA 3. *Let $\Omega \subseteq \bar{\mathbb{C}}$ be a domain, $z_0 \in \Omega$ and $M > 0$. Suppose that for all $z_1 \in \Omega$ there exists an arc $\gamma \subseteq \Omega$ connecting z_1 to z_0 such that for each (orientation preserving) arc $\gamma' \subseteq \gamma$ connecting w_1 to w_0 with*

$$l_{\text{qh}}(\gamma') \geq M,$$

one has

$$\delta(w_1) \leq \frac{1}{2} \delta(w_0).$$

Then Ω is an $\varepsilon(M)$ -John domain.

For simply connected domains, quasi-hyperbolic and hyperbolic metrics are comparable. In this case the previous lemma has been used in [6] (see also [11]). In [7] it is proved that quasi-hyperbolic geodesics can replace arbitrary paths in the definition of John regularity only in the simply connected case.

Proof of Lemma 3. Let γ be a concatenation $\gamma_0 \cdot \gamma_1 \cdot \dots \cdot \gamma_m$ of arcs with $l_{\text{qh}}(\gamma_i) \leq M$ for $i = 0, \dots, m$. Let $w_0 = z_0, w_1, \dots, w_m = z_1$ be their endpoints. By hypothesis we may assume that for all $i = 0, \dots, m - 1$,

$$\delta(w_i) = 2^{-i} \delta(w_0).$$

Set $\delta_i^+ = \max\{\delta(z) : z \in \gamma_i\}$ and $\delta_i^- = \min\{\delta(z) : z \in \gamma_i\}$. Then one may observe that

$$M \geq l_{\text{qh}}(\gamma_i) \geq \int_{\delta_i^-}^{\delta_i^+} \frac{dx}{x}$$

and therefore

$$(1) \quad \delta_i^+ \leq e^M \delta_i^-.$$

As a consequence, for all $i = 0, \dots, m$,

$$l(\gamma_i) \leq l_{\text{qh}}(\gamma_i) \delta_i^+ \leq M e^M \delta(w_i) \leq M e^M 2^{-i} \delta(w_0),$$

so for all $z \in \gamma_i$,

$$(2) \quad \delta(z, z_1) \leq 2^{-i} (2M e^M \delta(w_0)).$$

Using inequality (1), for all $z \in \gamma_i$ and $i = 0, \dots, m$,

$$\delta(z) \geq e^{-M} \delta(w_i) = e^{-M} 2^{-i} \delta(w_0),$$

which combined with (2) shows that for all $z \in \gamma$,

$$\delta(z) \geq \frac{e^{-2M}}{2M} \delta(z, z_1). \quad \blacksquare$$

Hölder regularity is more general than John regularity. In the particular case when the domain Ω is simply connected, it is equivalent to the condition that the Riemann mapping $\varphi : \mathbb{D} \rightarrow \Omega$ can be extended to a Hölder continuous mapping on the closed unit disk (see Lemma 6 in [9]). In this case $\partial\Omega$ is locally connected by Carathéodory's theorem.

Let us write $A(\cdot) \lesssim B(\cdot)$ whenever A has order at most $O(B)$, that is, there are constants $C_0 > 0$ and $C_1 > 0$ such that

$$A(\cdot) \leq C_0 B(\cdot) + C_1.$$

We also write $A(\cdot) \approx B(\cdot)$ when $A(\cdot) \lesssim B(\cdot)$ and $B(\cdot) \lesssim A(\cdot)$.

A domain $\Omega \subseteq \bar{\mathbb{C}}$ is a *Hölder domain* if there is $z_0 \in \Omega$ such that for all $z \in \Omega$,

$$\text{dist}_{\text{qh}}(z, z_0) \lesssim -\log \delta(z).$$

As a consequence of Proposition 3 in [9], the Main Theorem and the Complement to the Main Theorem (page 49) in [17] we obtain the following fact. See also Proposition 5.2 in [15].

COROLLARY 4. *Let f be a rational map of degree at least 2. If f satisfies *ExpShrink* then all connected components of the Fatou set are Hölder domains. If f has a fully invariant attractive Fatou component that is a Hölder domain, then f satisfies *ExpShrink*.*

Proof. Using the aforementioned results, if f satisfies *ExpShrink*, then almost every point z_0 in $\bar{\mathbb{C}}$ satisfies the *backward Collet–Eckmann* condition (denoted by $CE2(z_0)$). A point z_0 satisfies $CE2(z_0)$ if there are constants $C > 0$ and $\lambda > 1$ such that for all $n > 0$ and $y \in f^{-n}(z_0)$,

$$|(f^n)'(y)| > C \lambda^n.$$

We have seen that periodic Fatou components of f are attractive. As they contain points z_0 with $CE2(z_0)$, they are Hölder domains. With a bit of

work, one can see that a pullback of a Hölder domain by a rational map is still a Hölder domain.

Conversely, if f has a fully invariant attractive Fatou component Ω that is a Hölder domain, then there is $z_0 \in \Omega$ satisfying the previous inequality for all $n > 0$ and $y \in f^{-n}(z_0) \cap \Omega$. As Ω is fully invariant, z_0 satisfies $CE2(z_0)$, which in turn implies *ExpShrink*. ■

For any set $A \subseteq \bar{\mathbb{C}}$ and $\varepsilon > 0$ we denote by $B(A, \varepsilon)$ the ε -neighborhood of A ,

$$B(A, \varepsilon) = \{z \in \bar{\mathbb{C}} : \text{dist}(z, A) < \varepsilon\}.$$

Let us describe a classical construction inside a fixed (or periodic) attracting Fatou component.

LEMMA 5. *Let f be a rational map of degree at least 2 and Ω a fixed attracting Fatou component. Then for all $\varepsilon > 0$ there exists a connected domain V with $\bar{V} \subseteq \Omega$ satisfying*

- $\overline{f(V)} \subseteq V$;
- $\Omega \cap f^{-1}(f(V)) = V$;
- $\Omega \setminus f(V) \subseteq B(\partial\Omega, \varepsilon)$.

Proof. Let $p \in \Omega$ with $f(p) = p$ and $|f'(p)| < 1$. Then all orbits in Ω are attracted by p , that is, for all $z \in \Omega$,

$$\lim_{n \rightarrow \infty} f^n(z) = p.$$

For any open $W \subseteq \Omega$ that contains p , we define $n_W : \Omega \rightarrow \mathbb{N}$ such that $n_W(z)$ is the smallest iterate of z that enters W . As Ω is the immediate basin of attraction of p , n_W is well defined on Ω .

Let $W = B(p, r_0)$ for some $r_0 > 0$ such that ∂W does not intersect critical orbits and $\overline{f(W)} \subseteq W$. Therefore $\partial f^{-k}(W)$ is smooth and $\overline{f^{-k}(W)} \subseteq f^{-(k+1)}(W)$ for all $k \geq 0$. For all $k \geq 0$, let $W_k = \text{Comp}_p f^{-k}(W)$ be the connected component of $f^{-k}(W)$ that contains p . Note that $\bar{W}_k \subseteq \Omega$ for all $k \geq 0$. Let also

$$\{p, p_1, \dots, p_m\} = f^{-1}(p) \cap \Omega.$$

Then there are arcs $\gamma_1, \dots, \gamma_m \subseteq \Omega$ connecting p to p_1, \dots, p_m . By compactness, there is $k_0 \geq 0$ such that for all $i = 1, \dots, m$,

$$\gamma_i \subseteq W_{k_0}.$$

Then for all $k \geq k_0$, W_k has the following properties:

- $\overline{f(W_k)} \subseteq W_k$;
- $f^{-1}(W_k) \cap \Omega = W_{k+1}$.

Indeed, $f^{-1}(W_k) \cap \Omega$ is connected for all $k \geq k_0$. Otherwise it would contain a preimage of p in Ω outside $W_{k_0} \subseteq W_{k+1}$.

By compactness there is $k_1 \geq k_0$ such that $\Omega \subseteq W_{k_1}$ is contained in an ε -neighborhood of $\partial\Omega \subseteq J$. We set $V = W_{k_1+1}$. ■

Let us state a simplified version of a classical distortion control tool, the Koebe Theorem. As derivatives and distances are expressed with respect to the spherical metric we add a condition on the diameter of the image of the unit disk.

KOEBE THEOREM. *There exists $\kappa > 0$ and for all $D > 1$ there is $\rho > 0$ such that if $g : \mathbb{D} \rightarrow \mathbb{D}$ is univalent then*

$$B(g(0), \kappa|g'(0)|) \subseteq g(\mathbb{D}),$$

and for all $z \in B(0, \rho)$,

$$D^{-1} \leq \left| \frac{g'(z)}{g'(0)} \right| \leq D.$$

For more general statements of this theorem, see Theorem 1.3 and Corollary 1.4 in [13] or Lemma 2.5 in [3].

The following lemma will be used together with Koebe's Theorem and is a direct consequence of the Monodromy Theorem.

LEMMA 6. *Let U be a simply connected open set, g a rational map and U' a connected component of $g^{-1}(U)$. If g has no critical points in U' then it is univalent on U' , and U' is simply connected.*

Unless otherwise explicitly stated, f is assumed to be a rational map of degree at least 2; J denotes its Julia set, F its Fatou set and Crit the set of critical points of f .

Let us prove a bound for the quasi-hyperbolic length of pullbacks of arcs in the Fatou set.

LEMMA 7. *Let $d_c > 0$ be the distance from J to the critical orbits in F . There exists a universal constant $K > 1$ such that if $\gamma \subseteq F \cap B(J, d_c/2)$ is an arc and γ' is a connected component of $f^{-k}(\gamma)$ for some $k > 0$, then*

$$l_{\text{qh}}(\gamma') \leq K l_{\text{qh}}(\gamma).$$

Proof. By hypothesis, for any $z \in \gamma$ and $k > 0$, all branches of f^{-k} are univalent on $B(z, \delta(z))$. Indeed, $B(z, \delta(z)) \subseteq F \cap B(x, d_c)$ for some $x \in J$ so all preimages of $B(z, \delta(z))$ are simply connected by Lemma 6, as they do not contain critical points. Koebe's Theorem shows that the statement holds locally. The lemma follows by compactness of γ' . ■

The following statements are Lemmas 3 and 5 in [12]. See also Lemma 1.4 in [14].

LEMMA 8. *Let g be a rational map, $z \in \bar{\mathbb{C}}$ and $0 < r < R < 1$. Let W and W' be connected components of $g^{-1}(B(z, R))$ and $g^{-1}(B(z, r))$ respectively,*

with $W' \subseteq W$ and $\text{diam } W < 1$. If $\deg_W(g) \leq \mu$ then

$$\frac{\text{diam } W'}{\text{diam } W} < 64 \left(\frac{r}{R} \right)^{1/\mu}.$$

If A is an annulus and C_1, C_2 are the connected components of $\bar{C} \setminus \bar{A}$ then we denote

$$\text{dist}(\bar{C} \setminus A) = \text{dist}(C_1, C_2).$$

We also write

$$\text{dist}(\partial A) = \inf\{r > 0 : \partial C_1 \subseteq \partial C_2 + B(0, r) \text{ and } \partial C_2 \subseteq \partial C_1 + B(0, r)\}$$

for the Hausdorff distance between the two components of the boundary of A . Let us remark that

$$(3) \quad \text{dist}(\bar{C} \setminus A) \leq \text{dist}(\partial A),$$

with equality only when A is a round annulus.

LEMMA 9. *Let $A \subseteq \bar{C}$ be an annulus and C_1, C_2 the components of $\bar{C} \setminus \bar{A}$. For each $\alpha > 0$ there exists $\delta_\alpha > 0$ that depends only on α such that if $\text{mod } A \geq \alpha$ then*

$$\text{dist}(\bar{C} \setminus A) \geq \delta_\alpha \min(\text{diam } C_1, \text{diam } C_2).$$

Proof of Theorem 1. Let f be semi-hyperbolic. Then by the aforementioned results all periodic components of its Fatou set are attracting, as f satisfies *ExpShrink*. If $J = \bar{C}$ then there is nothing to prove.

Let us first show that an attracting periodic Fatou component Ω is a John domain. Without loss of generality we may assume that $f(\Omega) = \Omega$. Let $\lambda > 1$ and $r > 0$ be provided by the *ExpShrink* property of f . As there are no parabolic cycles, critical orbits in the Fatou set do not accumulate on the Julia set. By possibly decreasing r we may assume that for all $z \in J$, $n \geq 0$ and each component U of $f^{-n}(B(z, r))$,

$$U \cap \text{Crit} \subseteq J,$$

where Crit is the set of critical points of f . As the critical orbits in the Julia set are not recurrent, we may also assume that there exists $\mu \geq 1$ such that

$$(4) \quad \deg_U f^n \leq \mu.$$

As f is locally holomorphic, we may assume that the diameter of any such pullback U is sufficiently small, so that, by induction, it is simply connected.

Let V be given by Lemma 5 with $\varepsilon = r/100$. We continue to use the notations W_k , $n_W(z)$ and p introduced during its proof. Let also $V_n = f^{-n}(V) \cap \Omega = W_{k_1+n+1}$ and $n(z) = n_V(z)$ for all $z \in \Omega$. If $n(z) > 0$ for some $z \in \Omega$ then

$$f^{n(z)} \in V \setminus f(V),$$

thus for all $k > 0$,

$$n^{-1}(k) = V_k \setminus V_{k-1}.$$

For all $z \in \Omega$ we construct an arc $\gamma_z \subseteq \Omega$ without self-intersections that connects z to p and avoids critical orbits with the exception of p if it is critical. By compactness, there exists $L > 0$ such that for all $z \in \bar{V}$ there is an arc $\gamma_z \subseteq \bar{V}$ that connects z to p with

$$l_{\text{qh}}(\gamma_z) \leq L,$$

and such that $\gamma'_z = \gamma_z \setminus f(V)$ has at most one connected component. Let $z \in \Omega \setminus \bar{V}$ and $m = n(z)$ except if $z \in \partial V_{n(z)-1}$ when $m = n(z) - 1$. Let $y = f^m(z) \in \bar{V} \setminus f(\bar{V})$ and let $\gamma'_y = f^{-m}(\gamma'_y)$ connect z to $z' \in \partial V_{m-1}$. We define inductively γ_z as the concatenation

$$\gamma_z = \gamma'_z \cdot \gamma_{z'}.$$

Figure 1 illustrates this inductive procedure for $m = 1$.

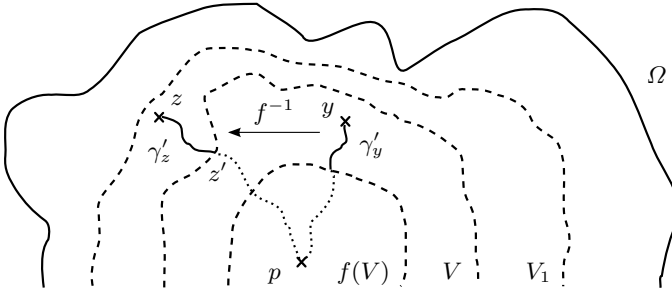


Fig. 1. Construction of γ_z

By construction, this family of arcs is invariant under f outside $f(V)$. That is, for all $z \in \Omega \setminus \bar{V}$, we have

$$(5) \quad f(\gamma_z) \setminus f(V) = \gamma_{f(z)} \setminus f(V).$$

Using Lemma 7 and the inclusion $\bar{V} \subseteq V_1$ we conclude that for all $z \in \Omega$,

$$(6) \quad l_{\text{qh}}(\gamma_z) \lesssim n(z).$$

Let $z \in \Omega \setminus V$. Then $y = f^{n(z)}(z) \in V \setminus f(V)$, therefore $\delta(y) < r/4$. Using *ExpShrink* we obtain

$$\delta(z) \leq \lambda^{-n(z)}.$$

One may also remark that

$$\delta(\partial V) \|f'\|_{\infty}^{-n(z)} \leq \delta(z).$$

As a consequence of these inequalities we conclude that for all $z \in \Omega$,

$$(7) \quad -\log \delta(z) \approx n(z).$$

REMARK. Relations (6) and (7) show that Ω is a Hölder domain. This is an alternative proof of the direct implication of Corollary 4, as we do not use (4). With a similar construction, a stronger version of (6) and an estimate of $\delta(z)$ that implies (7) have been proved in Lemma 7 of [9].

Set $\gamma_z(k) = \gamma_z \cap \bar{V}_k \setminus V_{k-1}$ and $\gamma_z^k = \gamma_z \setminus V_{k-1}$ for all $k = 1, \dots, n(z)$. By the previous relation and Lemma 7 there exists $A > 1$ such that for all $z \in \Omega \setminus V$ and $0 < k < n(z)$,

$$A^k \cdot l(\gamma_z(k)) \lesssim \int_{\gamma_z(k)} \frac{|d\xi|}{\delta(\xi)} = l_{\text{qh}}(\gamma_z(k)) \leq KL,$$

therefore by summation

$$(8) \quad l(\gamma_z^k) \lesssim A^{-k}.$$

We may therefore find $n_0 > 0$ such that for all $z \in \Omega \setminus \overline{n^{-1}(n_0)}$,

$$l(\gamma_z^{n_0}) \leq \frac{r}{100}.$$

For $z \in \Omega$ and $z' \in \gamma_z$ we denote by $\gamma_z^{z'}$ the arc $\gamma' \subseteq \gamma$ that connects (or lifts) z to z' . By compactness and using relations (7) and (6), for all $\eta > 0$ there exists $M > 0$ such that if $n(z') \leq n_0$ then

$$(9) \quad l_{\text{qh}}(\gamma_z^{z'}) \geq M \Rightarrow \delta(z) \leq \eta \delta(z').$$

Let $\gamma_w^{w'} \subseteq \Omega \setminus V_{n_0}$ with $l_{\text{qh}}(\gamma_w^{w'}) \geq KM$ where K is provided by Lemma 7. We show that if η is sufficiently small then

$$(10) \quad \delta(w) \leq \frac{1}{2} \delta(w').$$

By Lemma 3 this means that Ω is a John domain.

Let $m = n(w') - n_0$ so $n(z') = n(f^m(w')) = n_0$. Let also $z = f^m(w)$ and $x, x' \in \partial\Omega \subseteq J$ with $\delta(x, z) = \delta(z)$ and $\delta(x', z') = \delta(z')$. By property (5),

$$\gamma_z^{z'} = f^m(\gamma_w^{w'}),$$

and $\delta(z') < r/100$, $l(\gamma_z^{z'}) < r/100$. Figure 2 illustrates this construction. By the choice of $\gamma_w^{w'}$, Lemma 7 and inequality (9),

$$\delta(z) < \eta \delta(z').$$

Let U be the connected component of $f^{-m}(B(x, r))$ that contains w and w' . Let also $y, y' \in U$ be preimages of x and x' respectively, under the same branch of f^{-m} (i.e. connected by a homeomorphic pullback of the path $[x, z] \cdot \gamma_z^{z'} \cdot [z', x']$ that contains w and w'). Let $B_0 = B(z, \delta(z))$, $B_1 = B(z, r/8)$, $B_2 = B(z', r/4)$, $B_3 = B(z', r/2)$ and U_0, U_1, U_2, U_3 their

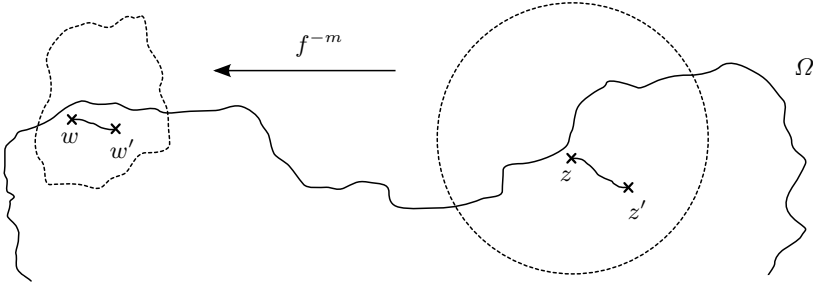


Fig. 2. Pullback of $\gamma_z^{z'}$

respective pullbacks by f^{-m} such that $w \in U_0 \subset U_1 \subset U_2 \subset U_3 \subset U$. By Lemma 8,

$$\begin{aligned} \delta(w) &\leq \text{diam } U_0 \leq 64 \text{diam } U_1 \left(\frac{8\delta(z)}{r} \right)^{1/\mu} \\ &\leq 64\eta^{1/\mu} \text{diam } U_2 \left(\frac{8\delta(z')}{r} \right)^{1/\mu}. \end{aligned}$$

Therefore

$$(11) \quad \delta(w) < 64\eta^{1/\mu} \text{diam } U_2,$$

as $8\delta(z') < r$.

As $\text{mod}(B_3 \setminus \bar{B}_2) > C_0$, a universal constant (the modulus is a constant in the Euclidean metric), an application of the Grötzsch inequality on conformal pullbacks of subannuli of $B_3 \setminus \bar{B}_2$ that separate U_2 from the complement of U_3 shows that

$$\text{mod}(U_3 \setminus \bar{U}_2) > C_0/\mu.$$

For an explicit construction one may check the proof of Lemma 8 in [12]. By Lemma 9 there exists $d > 0$ that depends only on μ such that

$$B(w', d \text{diam } U_2) \subseteq U_3.$$

Let $D = B(0, r')$ for some $0 < r' < 1$. The spherical, Euclidean and hyperbolic metric ρ_D on D are (uniformly in r') comparable on $B(0, r'/2)$. Therefore there exists $\beta \in (0, 1)$ that does not depend on r' such that for all $0 < \theta \leq \beta/2$,

$$B(0, \beta\theta r') \subseteq \{\zeta \in D : \rho_D(0, \zeta) < \theta\} \subseteq B(0, \beta^{-1}\theta r').$$

Let $D' = B(w', d \cdot \text{diam } U_2)$ and

$$\theta = 2\beta\delta(z')/r,$$

which is bounded from below as $n(z') = n_0$. Then

$$B(w', \beta\theta d \cdot \text{diam } U_2) \subseteq \{\zeta \in D' : \rho_{D'}(w', \zeta) < \theta\} \subseteq \{\zeta \in U_3 : \rho_{U_3}(w', \zeta) < \theta\}$$

and by the Schwarz Lemma

$$\begin{aligned} f^m(B(w', \beta\theta d \cdot \text{diam } U_2)) &\subseteq \{\zeta \in B_3 : \rho_{B_3}(z', \zeta) < \theta\} \\ &\subseteq B(z', \beta^{-1}\theta r/2) = B(z', \delta(z')). \end{aligned}$$

Therefore $\beta\theta d \cdot \text{diam } U_2 \leq \delta(w')$, which combined with inequality (11) and the lower bound for θ shows inequality (10), provided η is sufficiently small.

We have shown that each periodic Fatou component is a John domain. There are only finitely many such components. As any other component is a pullback of a periodic one, it is enough to show that pullbacks of Ω are John domains. Let Ω' be such a component with $f^p(\Omega') = \Omega$ and $V' = f^{-p}(V_{n_0}) \subseteq \Omega'$. We may recall that for all $z \in \Omega$, γ_z avoids critical orbits. For $w \in \Omega'$ let $z = f^p(w)$ and γ_w be the component of $f^{-p}(\gamma_z)$ that contains w . It connects w to a preimage of p in Ω' . Paths $\gamma_w^{w'} \subseteq \Omega' \setminus V'$ are lifted to $\gamma_z^{z'} \subseteq \Omega$ with $n(z') = n_0$. Lemma 7 applies to $\gamma_z^{z'}$ and its pullback $\gamma_w^{w'}$, and the previous argument shows inequality (10). As there are only finitely many preimages of p in Ω' , it follows that Ω' is a John domain.

Let γ be a Jordan curve and $D > 1$. We say that γ is a D -quasicircle if for all $x, y \in \gamma$, the subarc γ' of γ of smaller diameter that joins x and y satisfies

$$\text{diam } \gamma' \leq D \text{ dist}(x, y).$$

Both components of the complement of a quasicircle on $\bar{\mathbb{C}}$ are John domains.

Let us show that there exists a rational map whose Fatou components are John domains but which is not semi-hyperbolic. Corollary 4.4 in [22] provides a degree 2 rational map g which has two fixed Siegel disks Δ^0 and Δ^∞ with the following properties: $\partial\Delta^0$ and $\partial\Delta^\infty$ are disjoint quasicircles, each containing a critical point c_0 and c_∞ respectively.

$\partial\Delta^0$ and $\partial\Delta^\infty$ are forward invariant sets and by Theorem V.1.1 in [5], the orbits of c_0 and c_∞ are dense in $\partial\Delta^0$ and $\partial\Delta^\infty$ respectively, as g has no other critical points. Therefore both critical points are recurrent. By Theorems III.2.2, III.2.3, IV.2.1 and V.1.1 of [5], all Fatou components are preimages of Δ^0 or Δ^∞ . By Lemma 6, all Fatou components are simply connected and univalent. It is not hard to check that a preimage of a quasidisk (component of the complement of a quasicircle) under a rational map is a John domain. Therefore all Fatou components of g are John domains but both critical points are recurrent, thus g is not semi-hyperbolic. ■

An example of Carsten Lunde Petersen of a Blaschke product with a parabolic fixed point, thus not semi-hyperbolic, is given in [23]. As its Julia set is the unit circle, all Fatou components are John domains.

Let us show that if the Julia set of a semi-hyperbolic map is connected then all Fatou components are John with a uniform constant. In the follow-

ing section we use this result to show a stronger version of local connectivity of the Julia set.

PROPOSITION 10. *Let f be a semi-hyperbolic rational map with connected Julia set. There exists $\varepsilon > 0$ such that any Fatou component of f is an ε -John domain.*

Proof. Let U be a Fatou domain of f . We call U *multivalent* if f is not univalent on U . As the Julia set is connected, all Fatou components are simply connected, and therefore by Lemma 6 there are only finitely many multivalent Fatou components.

We show that univalent pullbacks of Ω are uniformly John domains. The general case can be treated with minor modifications. Let us use the notation introduced in the previous proof and assume that $f^p : \Omega' \rightarrow \Omega$ is univalent. By the proof of Theorem 1 there exists $M > 0$ that does not depend on the choice of Ω' such that if $l_{\text{qh}}(\gamma_w^{w'}) \geq M$ and $\gamma_z^{z'} = f^p(\gamma_w^{w'}) \subseteq \Omega \setminus V_{n_0}$ then

$$\delta(w) \leq \frac{1}{2}\delta(w').$$

Therefore the only obstacle to uniformity is related to $l_{\text{qh}}(\gamma_w)$ and $\delta(w)$ when $w \in V' \Leftrightarrow z \in V_{n_0}$. As f^p is univalent on Ω' , Lemma 7 applies to γ_z for all $z \in \Omega$. Therefore $l_{\text{qh}}(\gamma_w)$ is uniformly bounded (independently of the choice of Ω'). To complete the proof we show that there is a bound $R > 0$ that depends only on Ω and V_{n_0} such that for all $w, w' \in V'$,

$$(12) \quad \frac{\delta(w)}{\delta(w')} \leq R.$$

Let $g : \Omega \rightarrow \Omega'$ be a univalent branch of f^{-p} . Let $\rho = \rho(2)$ be provided by Koebe's Theorem. Let us cover \bar{V}_{n_0} with m balls $B(x_i, r_i)$ such that for all $i = 1, \dots, m$,

$$r_i \leq \rho\delta(x_i).$$

Then for all $z, z' \in V_{n_0}$,

$$\left| \frac{g'(z)}{g'(z')} \right| \leq 4^m.$$

If

$$S = \sup_{z, z' \in V_{n_0}} \frac{\delta(z)}{\delta(z')},$$

then again by the Koebe Theorem applied to g and g^{-1} , we may define R in inequality (12) by

$$R = \kappa^{-2}4^m S. \blacksquare$$

4. Local connectivity. Let us show that connected Julia sets of semi-hyperbolic maps satisfy a slightly stronger version of local connectivity. The construction developed for this purpose is extended to prove Theorem 2.

We use an alternative definition of simply connected John domains given by Theorem 4.4 in [7]. As we only need the easy part of this theorem, we include a proof for completeness.

If U is a simply connected domain, we say that the segment $[a, b]$ is a *crosscut* of U if $[a, b] \cap \partial U = \{a, b\}$ and $[a, b] \subseteq \bar{U}$.

LEMMA 11. *Let U be an ε -John simply connected domain, $[a, b]$ a crosscut of U , and U_1, U_2 the connected components of $U \setminus [a, b]$. Then*

$$\min(\text{diam } U_1, \text{diam } U_2) \leq \varepsilon^{-1} \delta(a, b).$$

Proof. Let z_0 be the base point of U with respect to which it is an ε -John domain. Let also U' be the component of $U \setminus [a, b]$ that does not contain z_0 . Let $x, y \in U'$ and γ_x, γ_y the paths that connect z_0 to x and y respectively, provided by the definition of John domains. Let $x' \in [a, b] \cap \gamma_x$ and $y' \in [a, b] \cap \gamma_y$. We may choose the order of $x', y' \in [a, b]$ such that

$$\delta(a, b) = \delta(a, x') + \delta(x', y') + \delta(y', b).$$

As $\varepsilon \delta(x, x') \leq \delta(x') \leq \delta(a, x')$ and $\varepsilon \delta(y', y) \leq \delta(y') \leq \delta(y', b)$, the triangle inequality completes the proof. ■

PROPOSITION 12. *If the Julia set of a semi-hyperbolic rational map is connected then it is locally connected. Moreover, there is $\varepsilon > 0$ such that the Julia set satisfies the condition defining local connectivity with $\tau = \varepsilon^{-1} \theta$.*

Proof. By Proposition 10 there is $\varepsilon > 0$ such that any Fatou component U is a simply connected ε -John domain. Let $\tau > 0$ and $a, b \in J$ with $\delta(a, b) < \varepsilon \tau$. It is sufficient to find a continuum $C \subseteq J$ that contains a and b with

$$(13) \quad \text{diam } C < \tau$$

to show that J is locally connected and complete the proof.

Let us consider the segment $[a, b]$. We replace each connected component (c, d) of $[a, b] \cap F$ with a piece of the boundary of the Fatou component U that contains (c, d) . Using the notation of Lemma 11 and assuming $\text{diam } U_1 \leq \text{diam } U_2$, we replace (c, d) by $\partial U_1 \setminus (c, d)$. Letting C be the closure of the union of all these components and of the points that were not replaced in $[a, b]$, it is not hard to see that C is a continuum that satisfies (13). ■

The following classical topology result has a straightforward proof, along the lines of that of Proposition 12, which we omit.

PROPOSITION 13. *Let $K \subseteq \bar{\mathbb{C}}$ be a continuum and $(U_n)_{n \geq 0}$ the sequence of connected components of its complement $\bar{\mathbb{C}} \setminus K$. If all ∂U_n are locally connected and*

$$\lim_{n \rightarrow \infty} \text{diam } U_n = 0,$$

then K is locally connected.

A domain regularity that is more general than Hölder regularity is considered in [8]. A domain Ω is called *integrable* if there exists $z_0 \in \Omega$ and an integrable function $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$,

$$\int_0^\infty H(r) dr < \infty,$$

such that Ω satisfies the following *quasi-hyperbolic boundary condition*: for all $z \in \Omega$,

$$\delta(z) \leq H(\text{dist}_{\text{qh}}(z, z_0)).$$

Hölder domains correspond to *exponentially fast integrable* domains, that is, with $H(r) = \exp(C - \varepsilon r)$. However, John domains and Hölder domains cannot be distinguished by their integrability function H .

An immediate consequence of Lemma 11.5 and Fact 11.1 in [8] is that all connected components of the boundary of an integrable domain are locally connected. For any attracting periodic Fatou component of a rational map, integrability is characterized in terms of derivative growth on backward orbits inside the domain (see Lemma 11.1 in [8]). By Koebe’s Theorem, *SumShrink* implies this condition, which yields the following fact.

COROLLARY 14. *Periodic simply connected Fatou components of a SumShrink rational map have locally connected boundary.*

Theorem 11 in [8] shows that *SumShrink* holds for rational maps that satisfy a certain *summability condition*, a generalization of the Collet–Eckmann condition. This condition does not imply *ExpShrink*, nor is it a consequence of it.

Suppose the Julia set is not connected, the components of the Fatou set are integrable domains, and their diameter tends to 0. Then one may show that the connected components of the Julia set are locally connected; only minor modifications in the proof of Proposition 13 are needed.

Proof of Theorem 2. If $J = \bar{\mathbb{C}}$ there is nothing to prove, therefore we assume that the Fatou set is non-empty. As discussed in the introduction, by *SumShrink*, the Fatou set consists of finitely many attracting periodic components and their preimages. The Julia set is connected, therefore Fatou components are simply connected. Thus the boundaries of periodic Fatou components are locally connected by Corollary 14. Being pullbacks of locally connected compact sets by rational maps (iterates of f), the boundaries of all Fatou components are locally connected.

By Lemma 6, there are only finitely many multivalent Fatou components. By Proposition 13, it is enough to show that the diameters of univalent pullbacks of some Fatou component U tend to 0. Let

$$\varphi : \mathbb{D} \rightarrow U$$

be the Riemann mapping which extends continuously to \bar{D} by Carathéodory's Theorem. Let $A = U \setminus \varphi(\overline{B(0, R)})$ be an annulus with $0 < R < 1$ such that

$$\text{dist}(\partial A) < r/2,$$

where r is given by *SumShrink* and $\text{dist}(\partial A)$ denotes the Hausdorff distance between the components of ∂A (see definition on page 76).

Let $(U_n)_{n \geq 0}$ be a sequence of univalent pullbacks of $U = U_0$ such that $f(U_{n+1}) = U_n$ for all $n \geq 0$. Let also $(A_n)_{n \geq 1}$ be the corresponding pullbacks of A . Then for all $n > 0$,

$$\text{mod } A = \text{mod } A_n,$$

and using a cover of A with balls of radius r centered on $\partial U \subseteq J$, by *SumShrink*,

$$\lim_{n \rightarrow \infty} \text{dist}(\partial A_n) = 0.$$

Let C_n and K_n be the connected components of $\bar{\mathbb{C}} \setminus A_n$ with $\text{diam } C_n \leq \text{diam } K_n$ for all $n > 0$. Note that

$$(14) \quad \text{diam}(C_n \cup A_n) \leq \text{diam } C_n + 2 \text{dist}(\partial A_n).$$

If n is sufficiently large then $\text{dist}(\partial A_n) < 1/4$ and by Lemma 9, $\text{diam } C_n < 1/2$. Then K_n contains half the Riemann sphere. Therefore there is at most one (sufficiently large) n such that $K_n \subseteq U_n$. Therefore, for all but finitely many $n > 0$,

$$U_n = A_n \cup C_n.$$

By Lemma 9 and inequalities (3) and (14),

$$\lim_{n \rightarrow \infty} \text{diam } U_n = 0,$$

which completes the proof. ■

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