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Remainders of metrizable spaces and a generalization of Lindelöf Σ-spaces

by

A. V. Arhangel'skii (Moscow)

Abstract. We establish some new properties of remainders of metrizable spaces. In particular, we show that if the weight of a metrizable space X does not exceed 2^ω, then any remainder of X in a Hausdorff compactification is a Lindelöf Σ-space. An example of a metrizable space whose remainder in some compactification is not a Lindelöf Σ-space is given. A new class of topological spaces naturally extending the class of Lindelöf Σ-spaces is introduced and studied. This leads to the following theorem: if a metrizable space X has a remainder Y with a G_δ-diagonal, then both X and Y are separable and metrizable. Some new results on remainders of topological groups are also established.

1. Introduction. A *compactification* of a space X is any compact space bX containing X as a subspace such that X is dense in bX.

Especially important are Hausdorff compactifications of a space X, that is, those compactifications bX of X which satisfy the Hausdorff separation axiom. It is well-known [10] that a space X has a Hausdorff compactification if and only if X is Tikhonov.

By a *remainder* of a Tikhonov space X we understand the subspace $bX \setminus X$ of a Hausdorff compactification bX of X. We study how properties of a Tikhonov space X are related to properties of some or all remainders of X.

A famous classical result in this direction is the following theorem of M. Henriksen and J. Isbell [12]:

Theorem 1.1. *A Tikhonov space X is of countable type if and only if the remainder in any* (*or some*) *Hausdorff compactification of X is Lindelöf.*

Recall that a space X is of *countable type* if every compact subspace P of X is contained in a compact subspace $F \subset X$ which has a countable

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base of open neighbourhoods in X [1]. All metrizable spaces and all locally compact Hausdorff spaces, as well as all Čech-complete spaces, are of countable type [1]. It follows from the theorem of Henriksen and Isbell that every remainder of a metrizable space is Lindelöf, and hence paracompact.

Very few results on remainders can compare in their importance with Henriksen–Isbell's Theorem.

In this paper we establish some new properties of remainders of metrizable spaces and of paracompact p-spaces. In this connection, a new class of topological spaces naturally extending the class of Lindelöf Σ-spaces is introduced and studied. This leads to some new results on remainders of topological groups.

The next statement is easily obtained with the help of Henriksen and Isbell's Theorem:

Corollary 1.2. *If a nowhere locally compact metrizable space X has a metrizable remainder Y in some Hausdorff compactification bX of X, then both X and Y are separable and metrizable.*

Proof. Since X and Y are metrizable spaces, they are both of countable type [1]. Therefore, by Theorem 1.1, X and Y are Lindelöf spaces, since each of them is the remainder of the other one in bX. Hence, X and Y are separable. ∎

If γ is a family of subsets of a space X, and $x \in X$, then $\mathrm{St}_\gamma(x) = \bigcup\{U \in \gamma : x \in U\}$.

Recall that *paracompact p-spaces* are preimages of metrizable spaces under perfect mappings. A mapping is said to be *perfect* if it is continuous, closed, and all fibers are compact. A *Lindelöf p-space* is the preimage of a separable metrizable space under a perfect mapping.

A Tikhonov space X is a *p-space* [1] if for any (or some) compactification bX of X there exists a countable family $\xi = \{\gamma_n : n \in \omega\}$ of families γ_n of open subsets of bX such that $x \in \bigcap\{\mathrm{St}_{\gamma_n}(x) : n \in \omega\} \subset X$ for each $x \in X$. Under these restrictions we automatically have $X \subset \bigcup \gamma_n$, for each $n \in \omega$. It was shown in [1] that every p-space is of countable type, and that every metrizable space is a p-space.

Clearly, every separable metrizable space has a separable metrizable remainder. Here is a parallel result from [3]. We sketch a proof of it below for the sake of completeness.

Theorem 1.3. *If X is a Lindelöf p-space, then any remainder of X is a Lindelöf p-space.*

Proof. Fix a perfect mapping of X onto a separable metrizable space M, and let Z be some separable metrizable remainder of M. The Stone–Čech remainder Y of X is mapped by a perfect mapping onto Z. Therefore, Y is a Lindelöf p-space. An arbitrary remainder Y_1 of X is the image of Y under a perfect mapping. Hence, Y_1 is also a Lindelöf p-space, by a theorem of V. V. Filippov [11]. ∎

The following partial converse to Theorem 1.3 holds:

COROLLARY 1.4. *If X is a nowhere locally compact space with a remainder which is a Lindelöf p-space, then X is also a Lindelöf p-space.*

Unfortunately, Theorem 1.3 cannot be generalized to paracompact p-spaces, and it follows easily from Corollary 1.2 that not every metrizable space has a metrizable remainder [3].

Some of the main results in this article concern topological groups. These objects are much more sensitive to properties of their remainders, and vice versa. See in this connection the papers [4], where a Dichotomy Theorem for remainders of topological groups was proved, and [5], where in particular some metrizability criteria for topological groups in terms of their remainders are discussed, and some further references are given.

However, in this article we are mostly concerned with remainders of spaces which are not necessarily topological groups, especially in Sections 2 and 3.

A "space" below stands for a Tikhonov topological space.

K. Nagami has defined the important class of Σ-spaces [13] which contains the class of metrizable spaces and the class of spaces with a σ-discrete network. Lindelöf Σ-spaces can be characterized as continuous images of Lindelöf p-spaces. In particular, every Lindelöf p-space is in this class. Every space with a countable network, and every σ-compact space is in this class as well (recall that a space is σ-compact if it is the union of some countable family of compacta).

2. Some basic facts on remainders of metrizable spaces. In this section, we establish some basic properties of remainders of metrizable spaces and of spaces closely related to metrizable spaces. It is well-known that every metrizable space has a point-countable base.

The following general statement makes use of Mischenko's Theorem on metrizability of an arbitrary compact space with a point-countable base (see [10]).

THEOREM 2.1. *If a nowhere locally compact space X with a point-countable base has a metrizable remainder Y, then X and Y are separable and metrizable.*

Proof. Every compact subspace F of X is separable and metrizable, by Mischenko's Theorem. Since X has a point-countable base, it follows that every compact subspace F of X has a countable base of open neighbourhoods in X. Thus, X is of countable type. Therefore, by Henriksen–Isbell's Theorem, the remainder Y is Lindelöf. Since Y is metrizable, it follows that Y is a Lindelöf p-space. Since X is nowhere locally compact, Corollary 1.4 now implies that X is also a Lindelöf p-space. However, every Lindelöf p-space with a point-countable base is separable and metrizable [14]. Hence, X is separable and metrizable. ■

Let us fix some terminology. Let X and Y be subspaces of a space Z, and γ be a family of subsets of Z. We consider the following two conditions:

(1) For any $x \in X$ and any $y \in Y$ where $x \neq y$, there is $P \in \gamma$ such that $x \in P$ and $y \notin P$.
(2) For any $x \in X$ and any $y \in Y$ where $x \neq y$, there are $P_1, P_2 \in \gamma$ such that $x \in P_1$, $y \in P_2$, and $P_1 \cap P_2 = \emptyset$.

If condition (1) is satisfied, we say that γ is a T_0-*separator* for the pair (X, Y).

If condition (2) is satisfied, we say that γ is a *Hausdorff separator* for (X, Y).

A T_0-separator (respectively a Hausdorff separator) will be called *open* (or *closed*) if all elements of it are open (respectively, closed) subsets of Z.

It is well-known that a Tikhonov space X is a Lindelöf Σ-space if and only if, for any compactification bX of X, there is a countable T_0-separator γ for the pair $(X, bX \setminus X)$ such that every $P \in \gamma$ is compact [13].

Recall that the *i-weight* $iw(X)$ of a space X is the smallest infinite cardinal τ such that there exists a one-to-one continuous mapping of X onto a space of weight $\leq \tau$.

In particular, the i-weight of X is countable if and only if X condenses onto a separable metrizable space.

The next two statements look as technical results; we need them to prove Theorem 2.4.

LEMMA 2.2. *Suppose that B is a compact space, and that a Lindelöf Σ-space L is a subspace of B. Suppose further that $iw(M)$ is countable for the subspace $M = B \setminus L$. Then any subspace X of B with $L \subset X$ is a Lindelöf Σ-space.*

Proof. There is a countable Hausdorff separator γ for the pair (M, M) such that each $P \in \gamma$ is open in M; this is so, since M condenses onto a separable metrizable space. Let η be the family of closures of elements of γ in B. Then η is a countable family of compact subsets of B.

Put $Y = X \setminus L$. Since γ is a Hausdorff separator for (M, M), and each $P \in \gamma$ is open in M, the family η is a T_0-separator for $(Y, M \setminus Y)$.

Denote by F the closure of L in B. Since L is a Lindelöf Σ-space, there is a countable T_0-separator ξ for $(L, B \setminus L)$ such that each $P \in \xi$ is compact (it may be necessary to include F in ξ).

Clearly, $M \setminus X = M \setminus Y$ and $L \cup Y = X$. It follows that $\eta \cup \xi$ is a countable T_0-separator for $(X, B \setminus X)$, and that all elements of this separator are compact.

Indeed, consider any $x \in X$ and $z \in B \setminus X$. Then $z \in M \setminus X \subset M \setminus Y$. If $x \in Y$, then we can use the fact that η is a T_0-separator for $(Y, M \setminus Y)$. If $x \notin Y$, then $x \in L$, and we can also refer to the fact that ξ is a T_0-separator for $(L, B \setminus L)$ (notice that $z \in B \setminus X \subset B \setminus L$). It follows that X is a Lindelöf Σ-space. ∎

PROPOSITION 2.3. *Suppose that B is a compact space, and that a Lindelöf Σ-space L is a subspace of B. Suppose further that $M = B \setminus L$ is a metrizable space of weight $\leq 2^\omega$. Then any subspace X of B with $L \subset X$ is a Lindelöf Σ-space.*

Proof. This follows from Lemma 2.2, since every metrizable space X of cardinality $\leq 2^\omega$ admits a one-to-one continuous mapping onto a separable metrizable space (see, for example, Lemma 2.20 in [7]). In fact, it is easy to construct directly a countable open Hausdorff separator in a space X with these properties. ∎

Proposition 2.3 is crucial for the proof of the next statement which is one of our main results on remainders of metrizable spaces.

THEOREM 2.4. *For every metrizable space X of weight $\leq 2^\omega$ and every compactification bX, the remainder $bX \setminus X$ is a Lindelöf Σ-space.*

Proof. Let M be the completion of X with respect to a metric on X generating the topology of X. Then M is a Čech-complete metrizable space [10].

Now let $B = bM$ be any compactification of M, and put $L = B \setminus M$. Then L is σ-compact, since M is Čech-complete. It follows that L is a Lindelöf Σ-space. Put $Z = B \setminus X$. Observe that $B = L \cup M$ and $L \subset Z \subset B$.

By Proposition 2.3 (with Z in the role of X), the subspace Z is a Lindelöf Σ-space. However, B is also a compactification of X, and $Z = B \setminus X$ is a remainder of X. Since the class of Lindelöf Σ-spaces is invariant under continuous mappings and is preserved when taking preimages under perfect mappings [13], it follows that the remainder $bX \setminus X$ is a Lindelöf Σ-space for every compactification bX of X. ∎

The domain covered by Theorem 2.4 can be extended in the following way:

COROLLARY 2.5. *For every paracompact p-space X with weight $\leq 2^\omega$ and every compactification bX, the remainder $bX \setminus X$ is a Lindelöf Σ-space.*

Proof. The space X can be mapped onto a metrizable space Y by a perfect mapping [1]. Under the continuous extension of this mapping to the Čech–Stone compactifications βX and βY, $\beta X \setminus X$ is mapped onto $\beta Y \setminus Y$

by a perfect mapping. The weight of Y is not greater than the weight of X, since the weight cannot increase under onto perfect mappings.

Therefore, $w(Y) \leq 2^\omega$. Now Theorem 2.4 implies that $\beta Y \setminus Y$ is a Lindelöf Σ-space. Since $\beta X \setminus X$ is the preimage of $\beta Y \setminus Y$ under a perfect mapping, it follows that $\beta X \setminus X$ is a Lindelöf Σ-space.

Every remainder of X is a continuous image of $\beta X \setminus X$. Hence, every remainder of X is a Lindelöf Σ-space. ∎

It would be very nice if Theorem 2.4 could be extended to all metrizable spaces: this result would become a basic tool in the further study of remainders of metrizable spaces. However, this is impossible. Let us show this.

Recall that if τ is a cardinal number, then τ^+ is the smallest cardinal number greater than τ.

Theorem 2.6. *There is a locally separable metrizable nowhere locally compact space X such that $|X| \leq (2^\omega)^+$ and no remainder of X is a Lindelöf Σ-space.*

Proof. Let M be a discrete space of cardinality $\tau = (2^\omega)^+$, and consider $I = [0,1] \times [0,1]$ with the usual topology. The product space $I \times M$ is metrizable, locally separable, and locally compact. For $a \in M$, put $I_a = I \times \{a\}$.

Clearly, the cardinality of the set of all nowhere locally compact subspaces of I which are dense in I is greater than 2^ω. Therefore, there exists a family $\mathcal{P} = \{A_m : m \in M\}$ of subsets of I such that $A_k \neq A_m$ whenever $k \neq m$.

Put $B_m = A_m \times \{m\}$ for $m \in M$, and let $X = \bigcup\{B_m : m \in M\}$, a subspace of $Z = I \times M$. Clearly, X is dense in Z. We also put $Y = Z \setminus X$.

Claim 1. *Let $\gamma = \{P(i) : i \in \omega\}$ be an arbitrary sequence of closed subsets of Z. Then γ is not a T_0-separator for the pair (Y, X).*

To prove this, for an arbitrary subset P of $I \times M$ and for any $m \in M$, we denote by $(P)_m$ the subset of I such that $(P)_m \times \{m\} = P \cap (I \times \{m\})$. Put $(\gamma)_m = \{(P(i))_m : i \in \omega\}$ for $m \in M$. Then $(\gamma)_m$ is a sequence of closed subsets of I.

Since the cardinality of the set of all sequences of closed subsets of I does not exceed 2^ω, and the cardinality of M is greater than 2^ω, there are distinct $m, k \in M$ such that $(\gamma)_m = (\gamma)_k$, that is, $(P(i))_m = (P(i))_k$ for every $i \in \omega$.

Observe that $A_m \neq A_k$, since $m \neq k$. We may assume that $A_m \setminus A_k \neq \emptyset$; fix $c \in A_m \setminus A_k$. Put $c_m = (c, m)$ and $c_k = (c, k)$. Clearly, $c_m \in B_m \subset X$ and $c_k \notin B_k$, $c_k \notin X$, that is, $c_k \in Y$.

Take any $P(i) \in \gamma$ such that $c_k \in P(i)$. Then $c_k \in (P(i))_k \times \{k\}$, which implies that $c \in (P(i))_k$. Hence, $c \in (P(i))_m$, and therefore $c_m \in (P(i))_m \times \{m\}$, which implies that $c_m \in P(i)$. Thus, γ is not a T_0-separator for (Y, X).

CLAIM 2. *No remainder of X in a compactification of X is a Lindelöf Σ-space.*

It is enough to show that some remainder of X in a compactification is not a Lindelöf Σ-space. Let B be any compactification of $Z = I \times M$. Since X is dense in Z, we can consider B as a compactification bX of X. Put $H = bX \setminus X$. Then H is a remainder of X in bX, and $Y = Z \setminus X \subset H$.

CLAIM 3. *H is not a Lindelöf Σ-space.*

Assume the contrary. Then there exists a countable closed T_0-separator η in bX for the pair (H, X). Put $\gamma = \{P \cap Z : P \in \eta\}$. Then, clearly, γ is a countable closed T_0-separator in Z for (Y, X). However, this contradicts Claim 1. Claims 3 and 2 are established. ∎

The proof of the last theorem uses an idea in the proof of Lemma 4.3 in [7].

3. Charming spaces. Since Theorem 2.4 does not extend to all metrizable spaces, we will try to identify weaker topological properties which are common to all remainders of metrizable spaces. This leads us to a curious class of spaces.

A space X will be called *charming* if there is a Lindelöf Σ-subspace Y of X (called *a Lindelöf Σ-kernel* of X) such that, for each open neighbourhood U of Y in X, the subspace $X \setminus U$ is a Lindelöf Σ-space.

This notion can be used to construct various new classes of spaces. A similar idea was introduced in [8], where it had been applied to define some natural extensions of certain classes of spaces.

Notice that every Lindelöf Σ-space is charming. Hence, all separable metrizable spaces and all Lindelöf p-spaces are charming.

The class of charming spaces has some nice stability properties. Since the proofs of the next three statements are almost obvious and quite standard, we omit them.

PROPOSITION 3.1. *Any image of a charming space under a continuous mapping is a charming space.*

PROPOSITION 3.2. *Any preimage of a charming space under a perfect mapping is a charming space.*

PROPOSITION 3.3. *Every charming space is Lindelöf.*

Since every compact space is charming, we see that not every subspace of a charming space is charming.

A motivation for the study of the class of charming spaces comes from the following statement:

THEOREM 3.4. *For every metrizable space X and every compactification bX of X, the remainder $bX \setminus X$ is a charming space.*

Proof. Due to Propositions 3.2 and 3.1, it is enough to show that the remainder $bX \setminus X$ in some compactification bX of X is a charming space.

Fix a metric on X generating the topology of X, and take the completion M of X with respect to this metric. Then M is a Čech-complete space containing X as a dense subspace. Let B be any compactification of M. Clearly, B is a compactification of X as well. We denote by Z the remainder $B \setminus X$ of X in B.

The subspace $L = B \setminus M$ of Z is σ-compact, since M is Čech-complete.

CLAIM. *L is a Lindelöf Σ-kernel of the space Z.*

Indeed, take any open neighbourhood U of L in Z. Then, clearly, $Z \setminus U$ is a subspace of the metrizable space M. Therefore, $Z \setminus U$ is metrizable.

On the other hand, $Z \setminus U$ is a closed subspace of Z. Since X is metrizable (and therefore of countable type), the remainder Z of X is Lindelöf [12]. Hence, $Z \setminus U$ is also Lindelöf, which implies that $Z \setminus U$ is a separable metrizable space.

Thus, $Z \setminus U$ is a Lindelöf Σ-space, and L is a Lindelöf Σ-kernel of Z. It follows that Z is charming. ∎

Theorem 3.4 can be considerably extended.

COROLLARY 3.5. *For every paracompact p-space X and every compactification bX of X, the remainder $bX \setminus X$ is a charming space.*

Proof. This obviously follows from Theorem 3.4 and Proposition 3.2. ∎

The reader could have noticed that remainders of metrizable spaces and, more generally, of paracompact p-spaces are charming spaces of a rather special kind. To sharpen conclusions in some theorems above, we introduce the following definitions.

Let $\mathcal{P}$ and $\mathcal{Q}$ be some classes of topological spaces. A space X will be called $(\mathcal{P}, \mathcal{Q})$-*structured* if there exists a subspace Y of X such that $Y \in \mathcal{P}$ and, for every open neighbourhood U of Y in X, the subspace $X \setminus U$ of X belongs to $\mathcal{Q}$. In this situation, we call Y a $(\mathcal{P}, \mathcal{Q})$-*shell* of the space X.

Clearly, the class of charming spaces is the same as the class of $(\mathcal{P}, \mathcal{Q})$-structured spaces, where $\mathcal{P} = \mathcal{Q}$ is the class of all Lindelöf Σ-spaces.

Let $\mathcal{P}_0$ be the class of σ-compact spaces, $\mathcal{P}_1$ be the class of separable metrizable spaces, $\mathcal{P}_2$ be the class of spaces with a countable network, $\mathcal{P}_3$ be the class of Lindelöf p-spaces, and $\mathcal{P}_4$ be the class of Lindelöf Σ-spaces.

Take some $i, j \in \{0, 1, 2, 3, 4\}$. A space X will be called *(i, j)-structured* if it is $(\mathcal{P}_i, \mathcal{P}_j)$-structured.

I believe that each of the classes of spaces so defined is worth studying. However, in this paper we are primarily interested in charming spaces, that is, in spaces which are $(4, 4)$-structured.

Observe that our arguments and results in this section show that *every remainder of a metrizable space in a compactification is $(0, 1)$-structured.* The next theorem is obtained with the help of this result.

THEOREM 3.6. *Suppose that X is a paracompact p-space, and that the remainder $Y = bX \setminus X$ of X in some compactification bX of X has a G_δ-diagonal. Then the space Y is $(2, 1)$-structured and $(0, 1)$-structured.*

Proof. Indeed, Y is $(0, 3)$-structured—this is clear from the proof of Theorem 3.4. Since every compact space with a G_δ-diagonal has a countable base (see [10]), it follows that every σ-compact space with a G_δ-diagonal has a countable network. Therefore, Y is $(2, 3)$-structured.

It remains to take into account that every Lindelöf p-space with a G_δ-diagonal has a countable base [6]. Thus, Y is $(2, 1)$-structured. ∎

4. Some further results involving charming spaces. We show below that some very general restrictions on cardinal invariants of remainders of metrizable spaces guarantee that the remainder is a Lindelöf Σ-space. The main result in this section is the next theorem. It is well-known that determining the cardinality of Lindelöf spaces with countable pseudocharacter is a very delicate question. Below we will see that for charming spaces this problem gets a complete solution.

THEOREM 4.1. *The cardinality of every charming space X of countable pseudocharacter does not exceed 2^ω.*

Proof. Case 1. Let us first assume that X is a Lindelöf Σ-space. Fix a compactification B of X and a countable T_0-separator S for the pair $(X, B \setminus X)$ in B such that all members of S are compacta.

Put $\eta = \{\bigcap \lambda : \lambda \subset S,\ \bigcap \lambda \subset X\}$. Then, clearly, $|\eta| \le 2^\omega$, and $\bigcup \eta = X$. For each $P \in \eta$ we have $|P| \le 2^\omega$, since P is a first-countable compactum. Therefore, $|X| \le 2^\omega$. Thus, if X is a Lindelöf Σ-space, then the conclusion holds.

Let us now consider the general case. Take a Lindelöf Σ-kernel Y of X. Then, by Case 1, $|Y| \le 2^\omega$. Since the pseudocharacter of X at every point of Y is countable, it follows that there is a family γ of open subsets of X which T_0-separates Y from $X \setminus Y$ and satisfies $|\gamma| \le 2^\omega$.

Consider the family $E = \{\bigcup \lambda : \lambda \subset \gamma,\ |\lambda| \leq \omega,\ Y \subset \bigcup \lambda\}$. Clearly, the cardinality of E is not greater than 2^ω, and $\bigcup\{X \setminus U : U \in E\} = X \setminus Y$ (we recall that every charming space is Lindelöf).

For each $U \in E$, $X \setminus U$ is a Lindelöf Σ-space of countable pseudocharacter; therefore, by Case 1, $|X \setminus U| \leq 2^\omega$. It follows that $|X \setminus Y| \leq 2^\omega$, and finally $|X| \leq 2^\omega$. ∎

Theorem 4.2. *Suppose that X is a nowhere locally compact metrizable space, and that bX is a compactification of X such that the remainder $Y = bX \setminus X$ has locally a G_δ-diagonal. Then both X and Y are separable metrizable spaces.*

Proof. It follows from Theorem 3.4 that Y is a charming space. Observe that every point of Y is a G_δ-point in Y, since Y has locally a G_δ-diagonal. Therefore, by Theorem 4.1, the cardinality of Y does not exceed 2^ω. Hence, the Suslin number of Y does not exceed 2^ω as well. The subspace Y is dense in bX, since X is nowhere locally compact. It follows that the Suslin number of bX does not exceed 2^ω. Since X is dense in bX, we conclude that the Suslin number of X does not exceed 2^ω. Since X is metrizable, the cardinality and the weight of X also do not exceed 2^ω.

Now we can apply the basic Theorem 2.4 which implies that Y is a Lindelöf Σ-space.

It is well-known that every Lindelöf Σ-space with a G_δ-diagonal has a countable network ([13], see also [6]). Therefore, Y locally has a countable network, since Y is regular and every closed subspace of Y is a Lindelöf Σ-space. Since Y is Lindelöf, it follows that Y has a countable network.

Since Y is dense in bX, it follows that the Suslin number of bX is countable. Hence, the Suslin number of X is countable. Since X is metrizable, it follows that it is separable. Thus, X is a separable metrizable space, and hence a Lindelöf p-space. Therefore, by Theorem 1.3, Y is also a Lindelöf p-space.

Since Y is a Lindelöf p-space with a G_δ-diagonal locally, it follows that Y is separable and metrizable as well ([13], see also [6]). ∎

Here is another result of similar kind. It has a similar proof and weaker assumptions and conclusion.

Theorem 4.3. *Suppose that X is a nowhere locally compact metrizable space, and that bX is a compactification of X such that every point in the remainder $Y = bX \setminus X$ is a G_δ-point. Then Y is a Lindelöf Σ-space, and the cardinality of X does not exceed 2^ω.*

Proof. By Theorem 3.4, Y is a charming space. Therefore, by Theorem 4.1, the cardinality of Y does not exceed 2^ω. Hence, the Suslin number of Y does not exceed 2^ω as well. The subspace Y is dense in bX, since X is nowhere locally compact. It follows that the Suslin number of bX does not exceed 2^ω. Since X is dense in bX, we conclude that the Suslin number of X does not exceed 2^ω. Since X is metrizable, it follows that the cardinality and the weight of X also do not exceed 2^ω. Now we can apply Theorem 2.4 which implies that Y is a Lindelöf Σ-space. ∎

5. Remainders of topological groups and charming spaces. Using results in preceding sections, we characterize below topological groups G such that some remainder of G is a charming space. To do that, we need the Dichotomy Theorem proved in [4] which says that *for every topological group G, either every remainder of G in a compactification is Lindelöf, or every remainder of G in a compactification is pseudocompact.* It follows from the Dichotomy Theorem that *a remainder Y of a non-locally compact topological group G is paracompact if and only if this remainder is Lindelöf, which is the case if and only if the topological group G is a paracompact p-space.* Combining this result with Corollary 3.5, we come to the following conclusion:

Corollary 5.1. *For any topological group G, the following conditions are equivalent:*

- *Some remainder of G is paracompact.*
- *Some remainder of G is Lindelöf.*
- *All remainders of G are charming spaces.*

Corollary 3.5 also allows us to characterize topological groups G with a paracompact p-remainder. For that, we need two more results on charming spaces.

Theorem 5.2. *Every charming topological group G has a dense subgroup that is a Lindelöf Σ-space.*

Proof. Let L be a Lindelöf Σ-kernel of G. If L is dense in G, it remains to take the smallest subgroup H of G such that $L \subset H$. It is well-known that H is also a Lindelöf Σ-space (see, for example, [2]).

Now assume that L is not dense in G. Then there is a non-empty open subset V of G such that $\overline{V} \cap L = \emptyset$. Since $U = G \setminus \overline{V}$ is an open neighbourhood of L, it follows that $\overline{V}$ is a Lindelöf Σ-space.

According to Proposition 3.3, the space G is Lindelöf. Since V is open in G and G is a topological group, the space G can be covered by a countable family of subspaces homeomorphic to $\overline{V}$. It follows that G is a Lindelöf Σ-space. ∎

Theorem 5.3. *The Suslin number of an arbitrary charming topological group G is countable.*

Proof. By Theorem 5.2, G has a dense subgroup H which is a Lindelöf Σ-space. According to a theorem of V. V. Uspenskiĭ [16], the Suslin number of H is countable. Since H is dense in G, it follows that $c(G) \leq \omega$. ∎

The next result was obtained in [3]:

THEOREM 5.4. *Suppose that G is a non-locally compact topological group, and that bG is a compactification of G. Then the remainder $bG \setminus G$ is a p-space if and only if at least one of the following conditions holds:*

(a) *G is a Lindelöf p-space;*
(b) *G is σ-compact.*

We now present an independent proof of a closely related result which can be used to give an alternative proof of Theorem 5.4.

THEOREM 5.5. *A topological group G has a paracompact p-remainder if and only if at least one of the following conditions holds:*

(a) *G is locally compact;*
(b) *G is a Lindelöf p-space.*

Proof. If (a) holds, then G has a compact remainder. If (b) holds, then G has a Lindelöf p-remainder. This takes care of the sufficiency.

Assume now that G is not locally compact, and fix a compactification bG of G such that the remainder $Y = bG \setminus G$ is a paracompact p-space. Then Y is dense in bG, that is, bG is a compactification of Y as well, and G is the remainder of Y in this compactification. By Corollary 3.5, G is a charming space. By Theorem 5.3, the Suslin number of G is countable. Then the Suslin number of bG is countable, and since Y is dense in bG, the Suslin number of Y is countable as well. Since Y is paracompact, it follows that Y is a Lindelöf p-space. Hence, G is also a Lindelöf p-space. ∎

Now we are ready to prove the next theorem.

THEOREM 5.6. *If a Lindelöf topological group G has a compactification B such that the remainder $H = B \setminus G$ is also homeomorphic to a topological group, then both G and H are Lindelöf p-spaces.*

Proof. Since G is Lindelöf, the remainder H is a space of countable type. Since H is homeomorphic to a topological group, it follows that H is a paracompact p-space. If H is not dense in B, then, obviously, G is locally compact and $H = B \setminus G$ is compact. Thus, in this case, both G and H are Lindelöf p-spaces.

Assume now that H is dense in B. Then G is the remainder of H in the compactification B of H. By Corollary 3.5, the space G is charming. Since G is a topological group, it follows that the Suslin number of G is countable, by Theorem 5.3. Since G is dense in B, this implies that $c(B) \leq \omega$. Since

$\overline{H} = B$, it follows that $c(H) \leq \omega$. Since the space H is paracompact, we conclude that it is Lindelöf. Thus, H is a Lindelöf p-space, which implies that G is a Lindelöf p-space as well. Hence, in any case, G and H are Lindelöf p-spaces. ∎

The last theorem shows that if a compact space B is decomposed into two dense disjoint subspaces homeomorphic to topological groups at least one of which is Lindelöf, then these subspaces have to be of a very special kind, namely Lindelöf p-spaces.

Clearly, the space Q of rational numbers has a separable metrizable compactification B such that the complement $B \setminus Q$ is not homeomorphic to a topological group. Thus, there is no way to reverse the above theorem.

THEOREM 5.7. *If a Lindelöf nowhere locally compact space X has a remainder homeomorphic to a topological group, then X is charming.*

Proof. This is so, since every Lindelöf remainder of any topological group is a charming space. ∎

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A. V. Arhangel'skii
Moscow 121165, Russia
E-mail: arhangel.alex@gmail.com

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