FUNDAMENTA
MATHEMATICAE
222 (2013)

On Levi subgroups and the Levi decomposition for groups definable in o-minimal structures

by

Annalisa Conversano (Auckland) and **Anand Pillay** (Leeds)

Abstract. We study analogues of the notions from Lie theory of Levi subgroup and Levi decomposition, in the case of groups G definable in an o-minimal expansion of a real closed field. With a rather strong definition of *ind-definable semisimple subgroup*, we prove that G has a unique maximal ind-definable semisimple subgroup S, up to conjugacy, and that $G = R \cdot S$ where R is the solvable radical of G. We also prove that any semisimple subalgebra of the Lie algebra of G corresponds to a unique ind-definable semisimple subgroup of G.

1. Introduction and preliminaries. The "Levi–Mal'tsev" theorem sometimes refers to Lie algebras (over any field of characteristic 0) and sometimes to Lie groups. For Lie algebras L it says that L is the semidirect product of a solvable ideal $\mathfrak{r}$ and a semisimple subalgebra $\mathfrak{s}$ (with certain uniqueness properties) and, as such, is valid for Lie algebras over real closed fields. $\mathfrak{s}$ is sometimes called a *Levi factor* of L. For connected Lie groups G it says that G has a unique, up to conjugacy, maximal connected semisimple Lie subgroup S, and for any such S, $G = R \cdot S$ where R is the solvable radical (maximal connected solvable normal subgroup). And of course $R \cap S$ is 0-dimensional. S need not be closed, but when G is simply connected, S *is* closed and $R \cap S = \{1\}$ (so G is the semidirect product of R and S). S is sometimes called a *Levi subgroup* of G. See Theorems 3.14.1, 3.14.2 and 3.18.13 of [17] for example. We also refer to the latter book for basic facts and definitions concerning Lie algebras and Lie groups.

In this paper we are concerned with a (definably connected) group G definable in an o-minimal expansion M of a real closed field K, so this goes outside the Lie group context unless $K = \mathbb{R}$. We are interested not only in the existence of a "Levi" subgroup and decomposition of the group G but also in definability properties. Even in the case where $M = (\mathbb{R}, +, \cdot)$ and

2010 *Mathematics Subject Classification*: 03C64, 22E15.
Key words and phrases: Levi decomposition, ind-definable semisimple, o-minimality.

DOI: 10.4064/fm222-1-3

so G is a *Nash group* (semialgebraic Lie group), this is a nontrivial issue and S need not be semialgebraic, as pointed out in the first author's thesis (see also Example 2.10 of [4]). In the general situation G will have a Lie algebra $L(G)$ (over the relevant real closed field K) which has its own Levi decomposition as a sum of a solvable ideal and a semisimple algebra, so the issue is what kind of subgroup of G, if any, corresponds to the semisimple subalgebra, and also to what extent it is unique.

We will be forced into the category of "ind-definable" subgroups, i.e. defined by a possibly infinite, but countable, disjunction of formulas. In Definition 2.5 below we give a rather strong definition of an *ind-definable semisimple* group S: S should be ind-definable, (locally) definably connected, with "discrete" centre $Z(G)$ and $G/Z(G)$ should be a *definable* semisimple group. This is equivalent to saying that S is a quotient of the o-minimal universal cover (in the sense of [8]) of a definable semisimple group. We should say that the existence of a natural candidate for a maximal "ind-definable semisimple subgroup" S of a definable group G is straightforward: $G/Z(G)^0$ has a maximal definable (definably connected) semisimple subgroup H, and if H_1 is the preimage of H in G then take S to be the commutator subgroup of H_1. However to prove that S has the required properties (ind-definable semisimple), and is unique up to conjugacy modulo those properties, requires some additional work. (If we use a weaker notion of ind-definable semisimple such as ind-definable, (locally) definably connected, and with semisimple Lie algebra, the uniqueness up to conjugacy of S will fail.) Definability of S in this general context corresponds more or less to S being a closed subgroup of G in the classical context. We will also give a number of situations where S is definable (for example when the "semisimple" quotient of G has finite o-minimal fundamental group).

Let us now state formally the main result. We assume M to be an o-minimal expansion of a real closed field, and G to be a definably connected definable group in M. Then G has a unique maximal definably connected solvable subgroup which we call R. We denote the quotient G/R by P, a definable (equivalently, interpretable) group which is definably connected and *semisimple* in the sense that it has no infinite normal solvable (equivalently, abelian) subgroups, or equivalently the quotient of P by its finite centre is a direct product of finitely many definably simple (noncommutative) definable groups. We sometimes call P the *semisimple part* of G.

THEOREM 1.1. *G has a maximal ind-definable semisimple subgroup S, unique up to conjugacy in G. Moreover:*

(i) $G = R \cdot S$.
(ii) *The centre of S, denoted $Z(S)$, is finitely generated and contains $R \cap S$.*

As mentioned above, it will also follow from material discussed in Section 2 that if $\pi : G \to P$ is the canonical surjective homomorphism, then the (surjective) homomorphism from S to P induced by π is a quotient of the o-minimal universal cover of P. We will call S as in Theorem 1.1 an *ind-definable Levi subgroup* of G, and the decomposition of G given by Theorem 1.1 the *ind-definable Levi decomposition* of G. When G is a definable real Lie group this decomposition coincides with the usual Levi decomposition of G referred to earlier. Note that by uniqueness of the (ind-definable) Levi subgroup up to conjugacy, some Levi subgroup will be definable iff all are. When $K = \mathbb{R}$ the examples of nondefinability of the Levi subgroup, given in [4], [5] and [3], come from encoding the universal cover of P as an ind-definable but nondefinable subgroup of G, and for this to be possible P has to have infinite "fundamental group".

Our methods will also yield:

THEOREM 1.2. *Let $\mathfrak{s}$ be a semisimple Lie subalgebra of $L(G)$. Then there is a unique ind-definable semisimple subgroup S of G such that $\mathfrak{s} = L(S)$.*

In Section 2 we discuss ind-definable groups, semisimplicity, and universal covers. In Sections 3 and 4 we will prove Theorem 1.1 (and Theorem 1.2) in some special cases, and then combine these in Section 5 to give the proofs in general. At the end of Section 5 we will list a number of hypotheses which imply definability of the Levi subgroups. In the remainder of this introduction we recall some basic facts and notions.

Usually M denotes an o-minimal expansion of a real closed field K, and G a group definable in M. For various reasons, especially when dealing with ind-definable objects, we should bear in mind a saturated elementary extension $\bar{M}$ of M. We refer to earlier papers such as [14] for an account of the general theory of definable sets and definable groups in M, as well as the existence and properties of tangent spaces and Lie algebras of definable groups. But we repeat that, for any k, a definable group can be equipped with an (essentially unique) definable C^k-manifold structure over K with respect to which the group operation is C^k. Likewise for definable homogeneous spaces. Definable connectedness of a definable group has two equivalent descriptions: no proper definable subgroup of finite index, and no proper open definable subgroup with respect to the topological structure referred to above. Definability means with parameters unless we said otherwise.

DEFINITION 1.3. G is *semisimple* if G is definably connected and has no infinite normal abelian (definable) subgroup.

REMARK 1.4. Assume G definably connected. Then G is semisimple if and only if $Z(G)$ is finite and $G/Z(G)$ is a direct product of finitely many definably simple, noncommutative, definable groups.

We now list some basic facts, from [11], [13], [14], [15], which we will use:

Fact 1.5.

(i) *Assume G is definably connected. Then G has a unique maximal definable definably connected normal solvable subgroup R and G/R is semisimple.*
(ii) *If G is semisimple then G is perfect* (*i.e.* *G equals its commutator subgroup $[G, G]$*), *and moreover for some r every element of G is a product of at most r commutators.*
(iii) *If G is definably connected, then $G/Z(G)$ is* linear, *that is, definably embeds in some $\mathrm{GL}_n(K)$.*
(iv) *Let G be definably connected. Then G is semisimple iff $L(G)$ is semisimple.*
(v) *If $\mathfrak{s}$ is a semisimple Lie subalgebra of $\mathfrak{gl}_n(K)$, then there is a (unique) definably connected definable subgroup S of $\mathrm{GL}_n(K)$ such that $\mathfrak{s} = L(S)$. Moreover S is semialgebraic* (*and semisimple by* (iv)).
(vi) *If G is definable, semisimple and centreless, then G is definably isomorphic to a semialgebraic subgroup of some $\mathrm{GL}_n(K)$ which is defined over $\mathbb{R}$* (*in fact over $\mathbb{Z}$*).

2. Ind-definability, semisimplicity, and universal covers. The expressions *ind-definable*, $\vee$*-definable*, and *locally definable* are more or less synonymous, and refer to definability by a possibly infinite disjunction of first order formulas. There is a considerable literature on ind-definability and the "category" of ind-definable sets. See for example the detailed treatment in Section 2.2 of [10]. Likewise there is a lot written on ind-definable spaces and groups in the *o*-minimal setting, especially in the context of universal covers and fundamental groups. See for example [1], [6] and [7]. So we refer to these other sources for more details and restrict ourselves here to fixing notation suitable for the purposes of this paper.

We start with T an arbitrary complete theory in a countable language L, say and $\bar{M}$ a saturated model of T. A definable set means a definable set in $\bar{M}$ unless we say otherwise. M denotes a small elementary substructure.

Definition 2.1.

(i) By an (*abstract*) *ind-definable set* X we mean a countable collection $(X_i : i < \omega)$ of definable sets together with definable injections $f_i : X_i \to X_{i+1}$ for each i, where we identify X with the directed union (via the f_i) of the X_i. By a *definable subset* of X we mean a definable subset of some X_i (in the obvious sense). We say X is *defined over* M if the X_i and f_i are, in which case we also have $X(M)$.

(ii) An *ind-definable group* is an ind-definable set as in (i) such that X has a group operation which is definable, that is, the restriction to each $X_i \times X_j$ is definable, hence with image in some X_k.
(iii) An ind-definable set X as in (i) is called a *concrete* ind-definable set if for some definable set Y, all the X_i are subsets of Y and each f_i is the identity map restricted to X_i, that is, $X_i \subseteq X_{i+1}$ for all i, so that X is simply $\bigcup_i X_i$, an ind-definable subset of Y.

REMARK 2.2.

(i) If X is an ind-definable subset of the definable set Y as in (iii) above, and Z is a definable subset of Y contained in X, then by compactness Z is contained in some X_i. Hence the notion of a *definable subset* of the abstract ind-definable set X is consistent with the natural notion when X is concrete.
(ii) There are obvious notions of a function between (abstract) ind-definable sets being definable (or we should say ind-definable). Note in particular that if X is ind-definable, Y is definable and $f : X \to Y$ is definable and surjective, then already the restriction of f to some X_i is surjective (by compactness).

We can formulate some basic notions such as definable connectedness for groups at this level of generality.

DEFINITION 2.3.

(i) Let X be an ind-definable set and Y a subset of X. We will say that Y is *discrete* if for any definable subset Z of X, $Z \cap Y$ is finite.
(ii) Let G be an ind-definable group. We will call G *definably connected* if G has *no* proper subgroup H with the properties: for each definable subset Z of G, $Z \cap H$ is definable, and Z meets only finitely many distinct cosets of H in G.

Maybe we should rather use the expression "locally definably connected" in (ii) above, but we leave it as is. In any case, when X is a definable set (G a definable group) the above notions reduce to Y being finite (G has no proper definable subgroup of finite index). Let us state for the record:

LEMMA 2.4. *Let G be a definably connected ind-definable group. Then any discrete normal subgroup of G is central.*

Proof. Let N be a discrete normal ind-definable subgroup of G. Then G acts on N by conjugation. Let $n \in N$ and let H be $C_G(n)$, which clearly meets each definable subset of G in a definable set. Let Z be a definable subset of G. Then $\{gng^{-1} : g \in Z\}$ is a definable subset of N, hence finite as N is discrete. So only finitely many distinct cosets of H in G meet Z.

As this is true for all definable Z and G is definably connected, we see that $H = G$, i.e. n is central in G. ■

We now specialize to the o-minimal case, i.e. T is an o-minimal expansion of RCF. We will only work with concrete ind-definable sets. When $X = G$ is a (concrete) ind-definable group, then by [6], X can be definably equipped with a topology such that the group operation is continuous (as in the case for definable groups), and in fact C^k for arbitrarily large k. Definable connectedness as defined above has a "topological" interpretation. Also G has a well-defined Lie algebra (over the ambient real closed field). Here is our main definition (which agrees with the usual one when G is definable).

Definition 2.5. We will call G *ind-definable semisimple* if G is ind-definable and definably connected, $Z(G)$ is discrete, and $G/Z(G)$ is definable and semisimple (that is, there is a definable semisimple group D and a definable surjective homomorphism from G to D with kernel $Z(G)$, and note that D will be centreless).

Remark 2.6. An equivalent definition is: G is ind-definable, definably connected, and there is a definable surjective homomorphism π from G to a definable (not necessarily centreless) semisimple group D such that $\ker(\pi)$ is discrete.

Lemma 2.7. *An ind-definable semisimple group is perfect.*

Proof. Let G be our ind-definable semisimple group, and $\pi : G \to D$ definable with D definable semisimple. Let $G_1 = [G, G]$. We want to argue that (i) the intersection of G_1 with any definable subset Z of G is definable, and moreover (ii) Z intersects only finitely many distinct cosets of G_1 in G. Definable connectedness of G will then imply that $G_1 = G$.

We first prove (i). It suffices to show that for arbitrarily large definable subsets Y of G, $G_1 \cap Y$ is definable. As $D = [D, D]_r$ (the collection of products of r commutators), we may assume, by enlarging Y, that $Y \cap [G, G]$ maps onto D under π. Now clearly $Y \cap [G, G]$ is ind-definable.

Claim. *$Y \setminus [G, G]$ is ind-definable.*

Proof of Claim. Let $y \in Y \setminus [G, G]$, and let $\pi(y) = d \in D$. By our assumption above there is $x \in Y \cap [G, G]$ such that $\pi(x) = d$. Hence $x^{-1}y \in \ker(\pi) = Z(G)$. Note that $x^{-1}y \in Y^{-1} \cdot Y$, a definable subset of G. By definition of G being ind-definable semisimple, $Z(G) \cap (Y \cdot Y^{-1})$ is finite. Hence $Y \setminus [G, G]$ equals the union of translates $c \cdot (Y \cap [G, G])$ for c ranging over the (finite) set of elements of $Z(G) \cap (Y \cdot Y^{-1})$ which are not in $[G, G]$. This proves the Claim.

By the Claim and compactness, $Y \cap [G, G]$ is definable. This proves (i). The proof of the Claim shows (ii). So the lemma is proved. ■

LEMMA 2.8. *Let G be an ind-definable semisimple group. Then $L(G)$ is semisimple.*

Proof. Let $\pi : G \to D$ be the canonical surjective homomorphism to a definable semisimple group. As $\ker(\pi)$ is discrete, π induces an isomorphism between $L(G)$ and $L(D)$ and the latter is semisimple by 1.5(iv). ∎

As remarked earlier, there is a body of work on o-minimal universal covers, which it will be convenient to refer to (although we could use other methods, such as in Section 5 of [5]). The content of Theorem 1.4 of [8] is:

FACT 2.9. *Let G be a definable, definably connected group. Then:*

(i) *The family* $\mathrm{Cov}(G) = \{f : H \to G$*: H is ind-definable, definably connected, f is surjective and definable with discrete kernel*$\}$ *has a universal object, that is, some $\pi : \tilde{G} \to G$ in* $\mathrm{Cov}(G)$ *such that for any $f : H \to G$ in* $\mathrm{Cov}(G)$ *there is a (unique) surjective definable homomorphism $h : \tilde{G} \to H$ such that $h \circ f = \pi$.*
(ii) *The kernel of $\pi : \tilde{G} \to G$ is finitely generated.*

We remark that $\pi : \tilde{G} \to G$ is what is known as the "o-minimal universal cover of G" and it is proved in [8] that $\ker(\pi)$ coincides with the "o-minimal fundamental group" $\pi_1(G)$ of G given in terms of definable paths and homotopies in [2]. Also if G is definable over M so is $\pi : \tilde{G} \to G$.

Although not required for the purposes of this paper, one would also expect $\pi : \tilde{G} \to G$ to have the additional property: For any (locally) definable central extension $f : H \to G$ of G there is a (unique) definable (but not necessarily surjective) homomorphism $h : \tilde{G} \to H$ such that $\pi = h \circ f$.

REMARK 2.10. If G is an ind-definable semisimple group and $f : G \to G/Z(G) = H$ is the canonical surjective homomorphism from G to a definable semisimple group, then by Fact 2.9(i), f is a quotient of the o-minimal universal cover $\pi : \tilde{H} \to H$, and by Fact 2.9(ii), $Z(G)$ is finitely generated.

We now briefly recall the relation of the o-minimal universal covers to the classical universal covers of connected Lie groups. Let us suppose that G is a definably connected definable group (identified with its group of $\bar{M}$-points). Let L^- be a sublanguage of the language $L(T)$ of T including the language of ordered fields, such that for some copy $\mathbb{R}$ of the reals living inside K, $\mathbb{R}$ with its induced L^--structure is an elementary substructure of $\bar{M}|L^-$. Let us also suppose that G is definable in L^- with parameters from $\mathbb{R}$. Then $G(\mathbb{R})$ is a connected Lie group. Moreover the o-minimal universal cover $\tilde{G}$ of G is (ind-)definable in L^- over $\mathbb{R}$, so that $\tilde{G}(\mathbb{R})$ makes sense as a topological (in fact Lie) group. Then Theorem 8.5 from [11] and its proof say that $\tilde{G}(\mathbb{R})$ is the classical universal cover of $G(\mathbb{R})$. Moreover the kernel of $\tilde{G} \to G$

coincides with the kernel of $\tilde{G}(\mathbb{R}) \to G(\mathbb{R})$, which is the fundamental group of the Lie group $G(\mathbb{R})$ (see also Theorem 3.3 in [9]).

This applies in particular to the case when G is semisimple: By [11], G is definably isomorphic in $\bar{M}$ to a group definable in the ordered field language over $\mathbb{R}$. So we may assume G to be already definable in the ordered field language over $\mathbb{R}$. Hence the o-minimal fundamental group of G coincides with the fundamental group of the semisimple real Lie group $G(\mathbb{R})$ (see also Corollary 1.3 in [9]).

In the rest of the paper, the model M will be an arbitrary model of an o-minimal expansion of RCF. When we speak of an ind-definable set (resp. group) in M we mean $X(M)$ (resp. $G(M)$) for X (resp. G) an ind-definable set (resp. group) in $\bar{M}$ which is defined over M.

3. Central extensions of definable semisimple groups. Here we prove Theorem 1.1 when R is central in G, hence $R = Z(G)^0$. In fact S will turn out to be the commutator subgroup $[G, G]$, but one has to check the various properties claimed of S.

We start with a trivial fact about *abstract groups*, which we give a proof of, for completeness. Recall that an (abstract) group G is said to be *perfect* if G coincides with its commutator subgroup $[G, G]$.

FACT 3.1. *If G is a central extension (as an abstract group) of a perfect group P, then $[G, G]$ is perfect.*

Proof. Let N be the kernel of the surjective homomorphism $\pi : G \to P$. By assumption, N is central in G. Let $H = [G, G]$. As P is perfect, $\pi(H) = P$, and for the same reason $\pi([H, H]) = P$. If by way of contradiction $H' = [H, H]$ is a proper subgroup of H, then as $G = N \cdot H'$ we see that $[G, G] = H$ is contained in H', impossible. So H is perfect. ∎

We now return to the o-minimal context.

LEMMA 3.2. *Suppose $R = Z(G)^0$. Let $S = [G, G]$. Then:*

(i) *S is the unique maximal ind-definable semisimple subgroup of G.*
(ii) *$G = R \cdot S$.*
(iii) *$R \cap S$ is contained in $Z(S)$ and the latter is finitely generated.*

Proof. Let P be the semisimple part of G, that is, P is definable semisimple and $\pi : G \to P$ is surjective with kernel $R = Z(G)^0$.

(i) We first prove that S is "ind-definable semisimple". Clearly S is ind-definable. By Fact 1.5(ii), P is perfect, hence π induces a surjective homomorphism $\pi|S : S \to P$. By Remark 2.6 it suffices to prove that S is definably connected and $\ker(\pi|S)$ is discrete. By Lemma 1.1 of [5], for each n, $[G, G]_n \cap Z(G)^0$ is finite, clearly showing that $\ker(\pi|S)$ is discrete. If S were not definably connected, let this be witnessed by the subgroup S_1

of S. As P is definably connected, $\pi(S_1) = P$, hence $G = R \cdot S_1$, but then $S = [G, G]$ is contained in S_1, so $S = S_1$, a contradiction.

We will now show maximality and uniqueness simultaneously by showing that *any* ind-definable semisimple subgroup S_1 of G is contained in S. So let S_1 be such. By Lemma 2.7, S_1 is perfect, hence $S_1 = [S_1, S_1] \leq [G, G] = S$. So (i) is proved.

We now look at (ii) and (iii). As P is perfect (1.5(ii)), S maps onto P, so $G = R \cdot S$. As remarked earlier, $[G, G]_n \cap R$ is finite for all n, so that $R \cap S$ is discrete, hence by Lemma 2.4 it is central in S. Also, by Fact 2.9, $Z(S)$ is finitely generated. ∎

4. The almost linear case. Assume first that G is *linear*, that is, a definable subgroup of $\mathrm{GL}_n(K)$. Let $\mathfrak{g}$ be its Lie algebra, a subalgebra of $\mathfrak{gl}_n(K)$. Let $\mathfrak{r}$ be $L(R)$ where remember that R is the solvable radical of G.

Lemma 4.1. (G *linear.*) *Let S be a maximal ind-definable definably connected semisimple subgroup of G. Then S is definable* (*in fact semialgebraic*). *Moreover $G = R \cdot S$ and $R \cap S$ is contained in the* (*finite*) *centre of S.*

Proof. Let $\mathfrak{s}$ be the Lie algebra of S; it is semisimple by Lemma 2.8. By [17], $\mathfrak{s}$ extends to a Levi factor $\mathfrak{s}_1$ of $\mathfrak{g}$ (i.e. a semisimple subalgebra such that $\mathfrak{g}$ is the semidirect product of $\mathfrak{r}$ and $\mathfrak{s}_1$). By Fact 1.5(v) there is a definable semisimple subgroup S_1 of G such that $L(S_1) = \mathfrak{s}_1$. We will prove that $S \leq S_1$, so by maximality $S = S_1$ and is definable.

This is a slight adaptation of material from [14] to the present context. Consider the definable homogeneous space $X = G/S_1$. We have the natural action $\alpha : G \times X \to X$ of G on X by multiplication on the left, which is differentiable (when X is definably equipped with suitable differentiable structure). Let $a \in X$ and let $f : G \to X$ be $f(g) = \alpha(g, a)$. By Theorem 2.19(ii) of [14], $L(S_1)$ is precisely the kernel of the differential df_e of f at the identity e of G. Consider the restriction f_1 of f to S. As $L(S) = \mathfrak{s}$ is contained in $\mathfrak{s}_1 = L(S_1)$, we see that $(df_1)_e$ is 0. By Theorem 2.19(i) of [14], $(df_1)_h = 0$ for all $h \in S$, so f_1 is "locally constant" on S. It follows that $\mathrm{Fix}(a) = \{h \in S : f_1(h) = a\}$ is a subgroup of S which is "locally" of finite index (as in Definition 2.3(ii)), hence $\mathrm{Fix}(a) = S$, which means that $S \leq S_1$, as desired.

For dimension reasons $G = R \cdot S$. Clearly $R \cap S$ is finite (for dimension reasons again, or because it is solvable and normal in S), hence central in S, as S is definably connected. ∎

Now we want to prove conjugacy.

Lemma 4.2. (G *linear.*) *Any two maximal ind-definable definably connected semisimple subgroups of G are conjugate.*

Proof. Let S, S_1 be such. By Lemma 4.1 both S, S_1 are semialgebraic subgroups of $G \leq \mathrm{GL}_n(K)$. We may assume that G is already semialgebraic, by using 4.1 of [15] which says that there are semialgebraic $G_1 < G < G_2 \leq \mathrm{GL}_n(K)$ with G_1 normal in G normal in G_2 and G_1 normal in G_2 such that G_2/G_1 is abelian. (One could also get to this conclusion by using Lemma 3.1 in [16] and deduce that the (abstract) subgroup H of $\mathrm{GL}_n(K)$ generated by S and S_1 is contained in some algebraic subgroup H_1 of $\mathrm{GL}_n(K)$ such that moreover H contains an open semialgebraic subset of H_1. Therefore $\dim H = \dim H_1$ and the definably (or equivalently, semialgebraically) connected component of H_1 is contained in G.)

We now make use of transfer to the reals together with the classical Levi theorem to conclude the proof. Without loss of generality G, S, S_1 are defined by formulas $\phi(x, b)$, $\psi(x, b)$, $\psi_1(x, b)$ where these are formulas in the language of ordered fields with parameters witnessed by b. We may assume that these formulas include conditions on the parameters b expressing that the group defined by $\phi(x, b)$ is definably connected and of the given dimension, also that the subgroups defined by $\psi(x, b)$, $\psi_1(x, b)$ are *maximal* (*semialgebraic*) *semisimple*. For example the family of definable abelian subgroups of a definable group is uniformly definable in terms of centralizers, so we can express that the subgroup defined by $\psi(x, b)$ is semisimple (definably connected with no infinite normal abelian subgroup). We can also express maximality, by witnessing a solvable definable normal subgroup R such that G is $R{\cdot}\psi(x, b)^K$.

Let σ be the sentence in the language of ordered fields expressing that for any choice c of parameters, the subgroups of $\phi(x, c)^K$ defined by $\psi(x, c)$ and $\psi_1(x, c)$ are conjugate.

Claim. *The sentence σ is true in the model* $(\mathbb{R}, +, \cdot, <)$.

Proof of Claim. Choose parameters c from $\mathbb{R}$. Let H, W, W_1 be the groups (subgroups of $\mathrm{GL}_n(\mathbb{R})$) defined by the formulas $\phi(x, c)$, $\psi(x, c)$, $\psi_1(x, c)$. As $H = RW = RW_1$ (R being solvable definable normal, as explained before), it follows that W, W_1 are maximal semisimple Lie subgroups of H (which also happen to be closed) hence are conjugate in H by 3.18.13 of [17].

So the Claim is proved, hence σ is true in the structure M, so that S, S_1 are conjugate in G. ∎

Note that Lemmas 4.1 and 4.2 give Theorem 1.1 in the linear case. Let us now prove Theorem 1.2 in the linear case, by a slight extension of the proof of Lemma 4.1.

Lemma 4.3. (G *linear.*) *Let $\mathfrak{s}$ be a semisimple Lie subalgebra of $L(G)$. Then there is a* unique *ind-definable semisimple subgroup S of G such that $L(S) = \mathfrak{s}$. Moreover S is definable.*

Proof. First let S_1 be a semialgebraic semisimple subgroup of G with $\mathfrak{s} = L(S_1)$. Let S be another ind-definable semisimple subgroup of G with $L(S) = \mathfrak{s}$. The proof of Lemma 4.1 shows that $S \leq S_1$. Let P be a semisimple centreless definable group with $\pi : S \to P$ witnessing the semisimplicity of S (according to Definition 2.5). By 1.5(vi) we may assume P to be linear and semialgebraic. Let us now work inside the linear semialgebraic group $S_1 \times P$. The graph of π, W say, is clearly an ind-definable semisimple subgroup of $S_1 \times P$. Let $\mathfrak{w}$ be its Lie algebra, which is semisimple by 2.8. By 1.5(v), there is a semialgebraic semisimple $W_1 \leq S_1 \times P$ such that $L(W_1) = \mathfrak{w}$. Again as in the proof of 4.1 one sees that $W \leq W_1$. Note that $\dim W_1 = \dim S_1 = \dim P$. So W_1 has finite cokernel. We will assume for simplicity that this cokernel is trivial so that W_1 is the graph of a definable (semialgebraic) homomorphism π_1 from S_1 to P, which has to have finite kernel. Hence $\ker(\pi)$ is also finite, from which it follows that π is definable, therefore S is definable. Hence $S = S_1$. ∎

We will say that G is *almost linear* if for some finite central subgroup N of G, G/N is linear (i.e. there is a definable homomorphism from G into some $\mathrm{GL}_n(K)$ with finite central kernel).

LEMMA 4.4. *Theorems* 1.1 *and* 1.2 *hold when* G *is almost linear. Moreover in this case any ind-definable semisimple subgroup of* G *is definable.*

Proof. Let G/N be linear where N is finite (central) and let $\pi : G \to G/N$ be the canonical surjective homomorphism. Note first that by Lemma 4.3 (and Lemma 2.8), any ind-definable semisimple subgroup of G/N is definable. So if $S \leq G$ is ind-definable semisimple then $\pi(S)$ is definable, and so thus is S. This proves the "moreover" clause. The rest easily follows from the previous lemmas. ∎

5. The general case. This is an easy consequence of the special cases in Sections 3 and 4 but we sketch the proofs nevertheless. First:

Proof of Theorem 1.1. We construct the obvious ind-definable semisimple subgroup S of G, observe the desired properties, then we prove its maximality and uniqueness up to conjugacy.

By Fact 1.5(iii), $G_1 = G/Z(G)^0$ is almost linear. Let $\pi : G \to G_1$ be the canonical surjective homomorphism. Also let R be the solvable radical of G (which contains $Z(G)^0$) and then $R_1 = \pi(R)$ will be the solvable radical of G_1.

By Lemma 4.4 let S_1 be a definable Levi subgroup of G_1 and let $H = \pi^{-1}(S_1)$, a definably connected definable subgroup of G. So H is definably an extension of the semisimple definable S_1 by the central subgroup $Z(G)^0$. So $Z(G)^0$ coincides with $Z(H)^0$ and will be the solvable radical of H.

Therefore Lemma 3.2 applies to H. Let $S = [H, H]$, which is in particular an ind-definable semisimple subgroup of G. As $S_1 \cap R_1$ is finite, and $S \cap Z(G)^0$ is finitely generated, $S \cap R$ is finitely generated (and normal, discrete, hence central in S).

It remains to prove maximality and uniqueness up to conjugacy of S. Suppose S_2 is an ind-definable semisimple subgroup of G containing S. Then $\pi(S_2)$ is an ind-definable, semisimple, subgroup of G_1 containing S_1, so as S_1 was a Levi subgroup of G_1, we see that $\pi(S_2) = S_1$. Hence S_2 is contained in H, and by Lemma 3.2 applied to H, equals S.

Now let S_2 be another maximal ind-definable semisimple subgroup of G. Let $\pi(S_2) = S_3 \leq G_1$. Clearly S_3 is an ind-definable semisimple subgroup of G_1, which is definable by Lemma 4.4. Let S_4 be a maximal definable semisimple subgroup of G_1 containing S_3. Noting that $G_1 = R_1 \cdot S_4$, and as (by Lemma 2.7) S_2 is perfect, we see by Lemma 3.2 that $[\pi^{-1}(S_4), \pi^{-1}(S_4)]$ is ind-definable semisimple and contains S_2, hence by maximality of S_2 we have equality, and $S_3 = S_4$ and moreover $S_2 = [\pi^{-1}(S_3), \pi^{-1}(S_3)]$. By Lemma 4.4 again S_3 is conjugate in G_1 to S_1 by g_1 say. Hence (as $\ker(\pi)$ is central in G) for any lift g of g_1 to a point of G, $S_2 = [\pi^{-1}(S_3), \pi^{-1}(S_3)]$ is conjugate via g to $S = [\pi^{-1}(S_1), \pi^{-1}(S_1)]$, and we have proved conjugacy.

This completes the proof of Theorem 1.1. ∎

Proof of Theorem 1.2. Again we let $\pi : G \to G/Z(G)^0 = G_1$ be the canonical surjective homomorphism. If $\mathfrak{s}$ is semisimple and $\mathfrak{s}_1 = d\pi_e(\mathfrak{s})$ (the image of $\mathfrak{s}$ under the differential of π at the identity), then $\mathfrak{s}_1$ is also semisimple and so by Lemma 4.4 is the Lie algebra of the unique definable semisimple subgroup S_1 of G_1. Let $H = \pi^{-1}(S_1)$. Then by Lemma 3.2 applied to H we see that $\mathfrak{s}$ is the unique Levi factor of $L(H)$ and $S = [H, H]$ is the unique ind-definable semisimple subgroup of H with $L(S) = \mathfrak{s}$.

If S_2 is another ind-definable semisimple subgroup of G with $L(S_2) = \mathfrak{s}$, then by Lemma 4.4, $\pi(S_2) = S_1$, and by perfectness of S_2 and definable connectedness of $[H, H] = S$ we see that $S_2 = S$.

Theorem 1.2 is proved. ∎

Finally we mention cases when some (any) ind-definable Levi subgroup of G is definable. G remains a definably connected group definable in M.

PROPOSITION 5.1. *Suppose either of the following hold:*

(i) *G is affine Nash,*
(ii) *G/N is linear for some finite central N,*
(iii) *the semisimple part P of G has finite o-minimal fundamental group,*
(iv) *the semisimple part P of G is definably compact.*

Then any ind-definable Levi subgroup S of G is definable (so that $G = R \cdot S$ with $R \cap S$ finite).

Proof. (i) G being affine Nash means that G is definable in the RCF language and with its unique structure as a Nash manifold it has a Nash embedding in some K^n. See [12]. In fact in the latter paper it is proved that G is a finite cover of an "algebraic group" (namely of $H(K)^0$ where H is an algebraic group defined over K). Remark 2.9 of [4] says that $H(K)^0$ has a definable Levi subgroup, and this lifts to G.

(ii) This is already part of Lemma 4.4 above.

(iii) The previous material shows that it suffices to look at central extensions G of a semisimple group P, in which case by Lemma 3.2, $S = [G, G]$ is the ind-definable Levi subgroup. The induced surjective homomorphism $S \to P$ is, by Remark 2.10, a quotient of the *o*-minimal universal cover of P, so if P had finite *o*-minimal fundamental group, then the kernel of $S \to P$ is finite, and it follows easily that S is definable.

(iv) If P is definably compact then it has finite *o*-minimal fundamental group (as this corresponds to the usual fundamental group of the associated compact semisimple Lie group discussed at the end of Section 3), so we can use (iii). But this case also follows from the results of [11]. ∎

Acknowledgements. This paper is closely related to our earlier papers [4], [5], to [11], and to themes in the first author's thesis [3]. The first author would like to thank her advisor Alessandro Berarducci, as well as Ya'acov Peterzil, for helpful conversations. The second author was supported by EPSRC grant EP/I002294/1.

References

[1] E. Baro and M. Otero, *Locally definable homotopy*, Ann. Pure Appl. Logic 161 (2010), 488–503.

[2] A. Berarducci and M. Otero, *o-minimal fundamental group, homology and manifolds*, J. London Math. Soc. 65 (2002), 257–270.

[3] A. Conversano, *On the connections between definable groups in o-minimal structures and real Lie groups: the non-compact case*, Ph.D. thesis, Univ. of Siena, 2009.

[4] A. Conversano and A. Pillay, *Connected components of definable groups and o-minimality I*, Adv. Math. 231 (2012), 605–623.

[5] A. Conversano and A. Pillay, *Connected components of definable groups and o-minimality II*, Ann. Pure Appl. Logic, to appear.

[6] M. J. Edmundo, *Covers of groups definable in o-minimal structures*, Illinois J. Math. 49 (2005), 99–120; Erratum, ibid. 51 (2007), 1037–1038.

[7] M. J. Edmundo, *Locally definable groups in o-minimal structures*, J. Algebra 301 (2006), 194–223; Corrigendum, ibid. 320 (2008), 3079–3080.

[8] M. J. Edmundo and P. E. Eleftheriou, *The universal covering homomorphism in o-minimal expansions of groups*, Math. Logic Quart. 53 (2007), 571–582.

[9] M. J. Edmundo, G. O. Jones and N. J. Peatfield, *Invariance results for definable extensions of groups*, Arch. Math. Logic 50 (2011), 19–31.

[10] E. Hrushovski and F. Loeser, *Nonarchimidean tame topology and stably dominated types*, Ann. of Math. Stud., to appear.

[11] E. Hrushovski, Y. Peterzil and A. Pillay, *On central extensions and definably compact groups in o-minimal structures*, J. Algebra 327 (2011), 71–106.

[12] E. Hrushovski and A. Pillay, *Affine Nash groups over real closed fields*, Confluentes Math. 3 (2011), 577–585.

[13] M. Otero, Y. Peterzil and A. Pillay, *On groups and rings definable in o-minimal expansions of real closed fields*, Bull. London Math. Soc. 28 (1996), 7–14.

[14] Y. Peterzil, A. Pillay and S. Starchenko, *Definably simple groups in o-minimal structures*, Trans. Amer. Math. Soc. 352 (2000), 4397–4419.

[15] Y. Peterzil, A. Pillay and S. Starchenko, *Linear groups definable in o-minimal stuctures*, J. Algebra 247 (2002), 1–23.

[16] A. Pillay, *An application of model theory to real and p-adic algebraic groups*, J. Algebra 126 (1989), 139–146.

[17] V. S. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*, Prentice-Hall, 1974.

Annalisa Conversano
Institute of Natural and Mathematical Sciences
Massey University
P/bag 102-904 NSMC
Auckland, NZ
E-mail: a.conversano@massey.ac.nz

Anand Pillay
Department of Pure Mathematics
University of Leeds
LS2 9JT Leeds, UK
E-mail: pillay@maths.leeds.ac.uk

Received 4 February 2012;
in revised form 20 May 2013