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The twisted products of spheres that have the fixed point property

by

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Abstract. By a twisted product of S^n we mean a closed, 1-connected $2n$-manifold M whose integral cohomology ring is isomorphic to that of $S^n \times S^n$, $n \geq 3$. We list all such spaces that have the fixed point property.

1. Introduction. An *almost smooth manifold* is a pair (M, D_M) in which

(1) M is a closed, 1-connected topological manifold;
(2) $D_M \subset M$ is an embedded disc, $\dim D_M = \dim M$; and
(3) $M \setminus \operatorname{int} D_M$ is furnished with a fixed smooth structure.

A *homeomorphism* between two almost smooth manifolds (M, D_M) and (N, D_N) is a homeomorphism $F : M \to N$ that restricts to a diffeomorphism

$$M \setminus \operatorname{int} D_M \to N \setminus \operatorname{int} D_N.$$

It is clear that such homeomorphisms yield an equivalence relation among all almost smooth manifolds. Denote by W the set of equivalence classes of this relation.

The category W introduced above is of classical interest. C. T. C. Wall classified all $(n-1)$-connected $2n$- and $(2n+1)$-manifolds, $n \geq 3$, exactly in this category [W_1], [W_2]. In general, it may be considered as a category between the smooth and PL categories.

It has been forty years since a complete classification for $(n-1)$-connected $2n$-dimensional almost smooth manifolds was achieved by C. T. C. Wall [W_1]. It seems, however, that the corresponding investigation into the geometry of maps between such manifolds has not yet received as much attention

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as it deserves. In this paper, without attempting a thorough study of this broad subject, we just present an evidence indicating an interesting aspect of this topic in the context of fixed point theory.

A topological space X is said to have the *fixed point property* if the equation

$$f(x) = x, \quad x \in X,$$

has a solution for every self-map f of X. The classical Brouwer fixed point theorem asserts that the n-dimensional disc $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ has the fixed point property. During the past century it has served as one of the main technical tools in establishing existence results for highly nonlinear problems [Fo]. On the other hand, except for even dimensional projective spaces and certain complex Grassmannians, few examples of closed manifolds are known to have this striking but useful property [F], [H].

By a *twisted product of* S^n, $n \geq 3$, we mean a closed, 1-connected, almost smooth $2n$-manifold M whose integral cohomology ring is isomorphic to that of $S^n \times S^n$. The importance and generality of such spaces can be seen from the following facts due to Wall [W_1]. Let $S(n)$ be the set of all homeomorphism types of twisted products of S^n. Then

(i) if n is odd, connected sums of elements in $S(n)$ yield all almost smooth $(n-1)$-connected $2n$-manifolds;

(ii) if n is even and if $n \neq 4, 8$, the Grothendieck group of n-spaces is generated by elements in $S(n)$ together with the single n-space whose intersection form is given by E_8 (cf. Theorem 2 in [W_1]).

The standard product $S^n \times S^n$ clearly fails the fixed point property. However, this is no longer so for twisted products of S^n.

In order to describe our results, we need some notation to describe the homotopy type of manifolds in $S(4k)$. First recall that the Bernoulli numbers B_k are the rationals defined by

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k \geq 1} (-1)^{k-1} B_k \frac{x^{2k}}{(2k)!}.$$

Let d_k be the denominator of $B_k/4k$ (expressed in lowest terms). Put $\sigma_k = d_k/2$ if $k = 1, 2$ and let $\sigma_k = d_k$ if $k \geq 3$. The first 10 values of σ_k are

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}
12	120	504	480	264	65520	24	16320	28728	13200

It is shown in Section 4 that the set $S(4k)$ is indexed by pairs of integers as

$$S(4k) = \{M(a, b) \mid a, b \in \mathbb{Z}\},$$

where, with respect to a certain basis x, y for $H^{4k}(M(a,b)) = \mathbb{Z} \oplus \mathbb{Z}$, the parameter (a,b) is related to the Pontryagin class p_k for the stable tangent bundle of $M(a,b)$ by

$$p_k = 2^{\varepsilon(k)}(2k-1)!(ax+by), \qquad \varepsilon(k) = \begin{cases} 0 & \text{if } k \text{ is even,} \\ 1 & \text{if } k \text{ is odd.} \end{cases}$$

A homotopy classification of elements of $S(4k)$ is given in

THEOREM 2. *$M(a_1, a_2)$ is homotopy equivalent to $M(b_1, b_2)$ if and only if one of the following eight congruence systems is satisfied*:

$$\begin{cases} a_1 \pm b_1 \equiv 0 \pmod{\sigma_k}, \\ a_2 \pm b_2 \equiv 0 \pmod{\sigma_k}, \end{cases} \qquad \begin{cases} a_1 \pm b_2 \equiv 0 \pmod{\sigma_k}, \\ a_2 \pm b_1 \equiv 0 \pmod{\sigma_k}. \end{cases}$$

Consequently, the subset $T(k) = \{M(a_1, a_2) \mid 0 \le a_1 \le a_2 \le \sigma_k/2\}$ of $S(4k)$ consists of all distinct homotopy types of twisted products of S^{4k}.

Since the fixed point property is invariant with respect to homotopy equivalence of closed manifolds, a combination of Theorem 2 with the next result classifies, with respect to homeomorphism type, all twisted products of S^n that have the fixed point property.

For $a \in \mathbb{Z}$ let $o(a) \in \mathbb{Z}$ be the order of a in the cyclic group $\mathbb{Z}_{\sigma_k}$.

THEOREM 3. *If $n = 4k$, then $M = M(a_1, a_2) \in T(k)$ has the fixed point property if and only if $a_1 a_2 \neq 0$ and*

$$\gcd\{o(a_1), o(a_2)\} \neq 1,$$
$$\gcd\{2o(a_1), o(a_2)\} \neq 2,$$
$$\gcd\{o(a_1), 2o(a_2)\} \neq 2.$$

If $n \neq 4k$, then every $M \in S(n)$ fails the fixed point property.

Let $J(k)$ be the subset of $T(k)$ consisting of all the homotopy types that have the fixed point property, and let c_k be the cardinality of $J(k)$. Computation based on Theorem 3 shows that most of the elements in $T(k)$ have the fixed point property.

Table 1. $J(1)$

(1,1)				
(1,2)	(2,2)			
(1,3)		(3,3)		
(1,4)	(2,4)		(4,4)	
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)

Table 2. $J(7)$

(1,1)										
(1,2)	(2,2)									
(1,3)	(2,3)	(3,3)								
(1,4)	(2,4)		(4,4)							
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)						
(1,6)	(2,6)	(3,6)		(5,6)	(6,6)					
(1,7)	(2,7)	(3,7)	(4,7)	(5,7)	(6,7)	(7,7)				
(1,8)	(2,8)		(4,8)	(5,8)		(7,8)	(8,8)			
(1,9)	(2,9)	(3,9)		(5,9)	(6,9)	(7,9)		(9,9)		
(1,10)	(2,10)	(3,10)	(4,10)	(5,10)	(6,10)	(7,10)	(8,10)	(9,10)	(10,10)	
(1,11)	(2,11)	(3,11)	(4,11)	(5,11)	(6,11)	(7,11)	(8,11)	(9,11)	(10,11)	(11,11)

Table 3. c_k, $k \leq 10$

k	1	2	3	4	5
c_k	13	1672	31104	28222	8410
k	6	7	8	9	10
c_k	469532700	60	33250102	103080204	21744712

For $M \in S(n)$ denote by H_M the cohomology $H^n(M;\mathbb{Z})$ in the middle dimension.

The paper is organized as follows. Section 2 recalls from [W$_1$] the constructions of elements in $S(n)$ both in terms of handle decomposition and cell decomposition. In Theorem 1 (Section 3) we answer the question which homomorphism $H_N \to H_M$ can be induced by a continuous map $f : M \to N$ with $M, N \in S(n)$. Combining Theorem 1 with the results of Adams [A] and Quillen [Q] of late 60's, we obtain in Section 4 a homotopy type classification for elements in $S(n)$, for n even, which was incomplete in [W$_1$] due to lack of information on J-homomorphisms. The proof of Theorem 3 is given in Section 5; finally, Section 6 discusses some numerical phenomena arising from the previous computation.

2. Constructions. Let $J : \pi_{n-1}(SO(n)) \to \pi_{2n-1}(S^n)$ be the J-homomorphism [Wh], and let $H : \pi_{2n-1}(S^n) \to \mathbb{Z}$ be the Hopf invariant. We recall from [W$_1$] that elements in $S(n)$ are parameterized by pairs of elements in the group

$$G_n = \operatorname{Ker}\{H \circ J : \pi_{n-1}(SO(n)) \to \mathbb{Z}\}.$$

Let D^{2n} be the standard $2n$-disc. Fix two smooth embeddings

$$h_i : S^{n-1} \times D^n \to S^{2n-1} = \partial D^{2n} \subset D^{2n}, \qquad i = 1, 2,$$

with disjoint images so that the linking number of the restrictions $h_1|S^{n-1}\times 0$ and $h_2|S^{n-1} \times 0$ in S^{2n-1} is 1. For two $\alpha_i \in G_n$, $i = 1, 2$, let $N(\alpha_1, \alpha_2)$ be the handle body

$$D^{2n} \bigcup_{\alpha'_1 \sqcup \alpha'_2} (D^n \times D^n \sqcup D^n \times D^n)$$

with the attaching maps

$$D^n \times D^n \supset \partial(D^n \times D^n) \supset S^{n-1} \times D^n \xrightarrow{\alpha'_i} S^{2n-1} = \partial D^{2n} \subset D^{2n}$$

defined by $\alpha'_i(x, y) = h_i(x, \alpha_i(x)y)$, $i = 1, 2$. Then $N(\alpha_1, \alpha_2)$ is a smooth manifold whose boundary is topologically a $(2n-1)$-sphere (cf. Corollary to Lemma 3 in [W_1]), so a $2n$-dimensional disc D^{2n} can be added to yield a closed almost smooth $2n$-manifold $M(\alpha_1, \alpha_2) = (N(\alpha_1, \alpha_2) \cup D^{2n}, D^{2n})$. Since $M(\alpha_1, \alpha_2)$ is simply connected (because $n \geq 3$) and its intersection form is seen to be

$$\begin{pmatrix} 0 & (-1)^n \\ 1 & 0 \end{pmatrix},$$

it follows that $M(\alpha_1, \alpha_2) \in S(n)$. Conversely, all elements in $S(n)$ are obtained in this way.

The space $M(\alpha_1, \alpha_2)$ admits a cell decomposition

$$M(\alpha_1, \alpha_2) = \bigvee_{i=1,2} S_i^n \cup_\alpha D^{2n},$$

with the attaching map $\alpha \in \pi_{2n-1}(\bigvee_{i=1,2} S_i^n)$ related to $\alpha_1, \alpha_2 \in G_n \subseteq \pi_{n-1}(SO(n))$ as follows. Let $\iota_i : S^n \to \bigvee_{i=1,2} S_i^n \subset M(\alpha_1, \alpha_2)$ be the inclusion onto the ith copy of the bouquet $\bigvee_{i=1,2} S_i^n$, $i = 1, 2$. By a result of Hilton, there is a canonical splitting

$$\pi_{2n-1}\Big(\bigvee_{i=1,2} S_i^n\Big) = \bigoplus_{i=1,2} \pi_{2n-1}(S_i^n) \oplus \pi_{2n-1}(S^{2n-1}). \tag{2.1}$$

LEMMA 1. *With respect to the splitting* (2.1), $\alpha = \iota_1 \circ J(\alpha_1) + \iota_2 \circ J(\alpha_2) + [\iota_1, \iota_2]$, *where* $[\,,]$ *stands for the Whitehead product* [W_1], [Wh].

REMARK 1. It follows from Lemma 1 that $\pi_r(M(\alpha_1, \alpha_2)) \cong \pi_r(S^n \times S^n)$, $r \geq 0$.

3. Realization of a cohomology homomorphism by a map. For two $M, N \in S(n)$, sending a continuous map $f : M \to N$ to the induced cohomology homomorphism yields a representation

$$r : [M, N] \to \mathrm{Hom}(H_N, H_M),$$

where $[M, N]$ is the set of all homotopy classes of maps $M \to N$. This section is devoted to a description of $\mathrm{Im}(r)$, the image of r in $\mathrm{Hom}(H_N, H_M)$.

Assume, by the discussion in the previous section, that

$$M = M(\alpha_1, \alpha_2) = \bigvee_{i=1,2} S_i^n \cup_\alpha D^{2n}, \qquad N = M(\beta_1, \beta_2) = \bigvee_{i=1,2} S_i^n \cup_\beta D^{2n}$$

with $\iota_i : S^n \to \bigvee_{i=1,2} S_i^n \subset M$ (resp. $\iota'_i : S^n \to \bigvee_{i=1,2} S_i^n \subset N$) being the inclusion onto the ith component of the bouquet $\bigvee_{i=1,2} S_i^n$, $i = 1, 2$. Let $e_i \in H_M$ (resp. $e'_i \in H_N$) be the image of $\iota_{i*}[S^n] \in H_n(M; \mathbb{Z})$ (resp. $\iota'_{i*}[S^n] \in H_n(N; \mathbb{Z})$) under Poincaré duality. Then $H_M = \operatorname{span}\{e_1, e_2\}$ (resp. $H_N = \operatorname{span}\{e'_1, e'_2\}$). In view of this we may equally well regard r as a representation into the set $M(2)$ of all 2×2 integer matrices,

$$r : [M, N] \to M(2),$$

by $r(f) = (a_{ij})_{2\times 2}$, where $f^*(e'_i) = a_{i1}e_1 + a_{i2}e_2$, $i = 1, 2$.

THEOREM 1. $A = (a_{ij})_{2\times 2} \in \operatorname{Im}(r)$ *if and only if the following equations hold in* $\pi_{2n-1}(S^n)$:

$$\text{(3.1)} \qquad kJ(\beta_i) = a_{i1}J(\alpha_1) + a_{i2}J(\alpha_2) + a_{i1}a_{i2}[\kappa_n, \kappa_n], \qquad i = 1, 2,$$

where $k = a_{11}a_{22} + (-1)^n a_{12}a_{21}$, *and where* $\kappa_n \in \pi_n(S^n)$ *is the class of the identity.*

If n is even, applying the Hopf invariant H to (3.1) gives

$$a_{11}a_{12} = a_{21}a_{22} = 0$$

(since $\alpha_i, \beta_i \in G_n$ and $H([\kappa_n, \kappa_n]) = 2$). Theorem 1 implies

COROLLARY 1. *Let* n *be even. Then* $A = (a_{ij})_{2\times 2} \in \operatorname{Im}(r)$ *if and only if one of the following constraints is satisfied*:

(i) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ *with* $\begin{cases} abJ(\beta_1) = aJ(\alpha_1) \\ abJ(\beta_2) = bJ(\alpha_2) \end{cases}$ *in* $\pi_{2n-1}(S^n)$;

(ii) $A = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}$ *with* $\begin{cases} abJ(\beta_1) = bJ(\alpha_2) \\ abJ(\beta_2) = aJ(\alpha_1) \end{cases}$ *in* $\pi_{2n-1}(S^n)$;

(iii) $A = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ *with* $\begin{cases} aJ(\alpha_1) = 0 \\ bJ(\alpha_1) = 0 \end{cases}$ *in* $\pi_{2n-1}(S^n)$;

(iv) $A = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$ *with* $\begin{cases} aJ(\alpha_2) = 0 \\ bJ(\alpha_2) = 0 \end{cases}$ *in* $\pi_{2n-1}(S^n)$.

We complete this section by proving Theorem 1. For a homomorphism $h : H_N \to H_M$ one constructs a map $g : \bigvee_{i=1,2} S_i^n \to \bigvee_{i=1,2} S_i^n$ so that the induced g^* on cohomology fits in the commutative diagram

$$\begin{array}{ccc} H_N & \xrightarrow{\quad h \quad} & H_M \\ \cong\downarrow & & \downarrow\cong \\ H^n(\bigvee_{i=1,2} S_i^n) & \xrightarrow{\quad g^* \quad} & H^n(\bigvee_{i=1,2} S_i^n) \end{array}$$

where the vertical isomorphisms are induced by the inclusions $\bigvee_{i=1,2} S_i^n \subset N$ and $\bigvee_{i=1,2} S_i^n \subset M$. A standard discussion in homotopy theory yields

LEMMA 2. *g is extendible to a map $f : M \to N$ (of degree k) if and only if the induced homomorphism $g_* : \pi_{2n-1}(\bigvee_{i=1,2} S_i^n) \to \pi_{2n-1}(\bigvee_{i=1,2} S_i^n)$ satisfies*

$$g_*(\alpha) = k\beta. \tag{3.2}$$

Assume that, with respect to the basis $\{e'_1, e'_2\}$ for H_N and $\{e_1, e_2\}$ for H_M, $h : H_N \to H_M$ has the representation

$$h(e'_i) = \sum_{j=1,2} a_{ij} e_j, \qquad a_{ij} \in \mathbb{Z}. \tag{3.3}$$

Equivalently $g_* : \pi_n(\bigvee_{i=1,2} S_i^n) \to \pi_n(\bigvee_{i=1,2} S_i^n)$ is given by $g_*(\iota_i) = \sum_{j=1,2} a_{ji}\iota'_j$, $i = 1, 2$. With these notations we compute

$$\begin{aligned} g_*(\alpha) &= \sum_{i=1,2} g_*(\iota_i) \circ J(\alpha_i) + [g_*(\iota_1), g_*(\iota_2)] \\ &= \sum_{j=1,2} \iota'_j \circ (a_{j1} J(\alpha_1) + a_{j2} J(\alpha_2) + a_{j1}a_{j2}[\kappa_n, \kappa_n]) \\ &\quad + (a_{11}a_{22} + (-1)^n a_{12}a_{21})[\iota'_1, \iota'_2], \end{aligned}$$

where we have made use of the $(-1)^n$-symmetry and bilinearity of the Whitehead product $[\,,]$, the bilinearity of the composition operator $\circ$ (note that $\circ$ is linear with respect the first factor since $\alpha_i \in G_n$, cf. formula (1.16) in [Wh, p. 494]), as well as the obvious relation

$$[\iota'_i, \iota'_i] = \iota'_i \circ [\kappa_n, \kappa_n]$$

(in $\pi_{2n-1}(\bigvee_{i=1,2} S_i^n)$). Now comparing the coefficients of ι'_i and $[\iota'_1, \iota'_2]$ on both sides of (3.2) yields

LEMMA 3. *g is extendible to a map $f : M \to N$ (of degree k) if and only if the homomorphism h defined by* (3.3) *satisfies*

$$kJ(\beta_i) = a_{i1} J(\alpha_1) + a_{i2} J(\alpha_2) + a_{i1}a_{i2}[\kappa_n, \kappa_n], \qquad i = 1, 2,$$

in $\pi_{2n-1}(S^n)$, where $k = a_{11}a_{22} + (-1)^n a_{12}a_{21}$.

This clearly finishes the proof of Theorem 1.

4. Homotopy type classification in $S(n)$ (for n even). Assume throughout this section that n is even. We need information on the groups G_n, as well as the restriction of the J-homomorphism to G_n. In the statement and proof of the next result we use a section of the homotopy sequence

$$\pi_n(S^n) \xrightarrow{\partial} \pi_{n-1}(SO(n)) \xrightarrow{i_*} \pi_{n-1}(SO(n+1)) \to 0$$

of the fibration $SO(n+1) \to S^n$. The number σ_k is as defined in Section 1.

LEMMA 4. *The groups G_n and the restriction of the J-homomorphism to G_n can be classified into the following four cases.*

Case 1. $n = 4k$, $k \le 2$: *Let* $\delta \in \pi_{n-1}(SO(n-1)) = \mathbb{Z}$ *be a generator, and put* $x = i_*(\delta)$. *Then*

(1-1) $G_n = \mathbb{Z}$ *is generated by* x;
(1-2) $J(x)$ *generates a direct cyclic summand of* $\pi_{2n-1}(S^n)$ *of order* σ_k.

Case 2. $n = 4k$, $k \ge 3$: *Let* $y \in \pi_{n-1}(SO(n))$ *be a class such that* $i_*(y)$ *generates* $\pi_{n-1}(SO(n+1)) = \mathbb{Z}$, *and put* $x = y - \frac{1}{2}HJ(y)\partial\kappa_n$. *Then*

(2-1) $G_n = \mathbb{Z}$ *is generated by* x;
(2-2) $J(x)$ *generates a direct cyclic summand of* $\pi_{2n-1}(S^n)$ *of order* σ_k.

Case 3. $n \equiv 2 \pmod 8$, $n > 8$: *Let* $y \in \pi_{n-1}(SO(n))$ *be a class so that* $i_*(y)$ *generates* $\pi_{n-1}(SO(n+1)) = \mathbb{Z}_2$, *and put* $x = y - \frac{1}{2}HJ(y)\partial\kappa_n$. *Then*

(3-1) $G_n = \mathbb{Z}_2$ *is generated by* x;
(3-2) $J : G_n \to \pi_{2n-1}(S^n)$ *is monomorphic.*

Case 4. $n = 8s + 6$: $G_n = \{0\}$.

Remark 2. In Cases 2 and 3, $HJ(y) \in \mathbb{Z}$ must be even for dimensional reasons.

Proof. All the statements above can be found in [W$_1$] except for (1-2), (2-2) and (3-2), due essentially to Adams [A] and Quillen [Q].

The J-homomorphisms induce the commutative diagram

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \pi_{4k}(S^{4k}) & \xrightarrow{\partial} & \pi_{4k-1}(SO(4k)) & \xrightarrow{i_*} & \pi_{4k-1}(SO(4k+1)) & \longrightarrow 0 \\ & & & \downarrow J & & \downarrow J & \\ 0 \longrightarrow & \pi_{8k+1}(S^{8k+1}) & \xrightarrow{P} & \pi_{8k-1}(S^{4k}) & \xrightarrow{E} & \pi_{8k}(S^{4k+1}) & \longrightarrow \cdots \end{array}$$

in which the bottom is a section of the EHP sequence [Wh, p. 548]. It is known that

(1) i_* maps $G_{4k} = \mathbb{Z}$ isomorphically onto $\pi_{4k-1}(SO(4k+1)) = \mathbb{Z}$ if $k \ge 3$, and monomorphically onto the subgroup of index 2 if $k \le 2$.

By Adams [A] and Quillen [Q] we have

(2) $J(\pi_{4k-1}(SO(4k+1))) \subset \pi_{8k}(S^{4k+1})$ is a cyclic subgroup of order d_k (cf. Section 1).

Combining these with the obvious fact that

(3) the composition $H \circ P : \pi_{8k+1}(S^{8k+1}) \to \mathbb{Z}$ is monomorphic

proves (1-2) and (2-2).

Assume now that $n \equiv 2 \pmod 8$ and $n > 8$ (i.e. Case 3). By Adams [A], $J(i_*(x)) \in \pi_{2n}(S^{n+1})$ is of order 2. This clearly implies (3-2). ■

If n is even and $n \ne 4k$, then J restricts to a monomorphism $G_n \to \pi_{2n-1}(S^n)$ by Lemma 4. The homotopy classification of elements in $S(n)$

now coincides with the homeomorphism classification, hence was done by Wall [W_1]:

COROLLARY 2. *If n is even and $n \neq 4k$, we have*

(1) *for* $n \equiv 2 \pmod 8$ *and* $n > 8$: $S(n) = \{M(0,0), M(x,0)\}$;
(2) *for* $n = 8s+6$: $S(n) = \{M(0,0)\}$,

where $M(0,0) = S^n \times S^n$.

If $n = 4k$ (Cases 1 and 2), we write $M(a_1, a_2)$ instead of $M(a_1x, a_2x)$, $a_i \in \mathbb{Z}$. In view of the construction of $M(a_1, a_2)$ described in Section 2, the characteristic map for the normal bundle of the embedding $\iota_i : S^n \to M(a_1, a_2)$ is seen to be $a_ix \in \pi_{4k-1}(SO(4k))$, $i = 1, 2$. Thus, by the divisibility result of R. Bott [B], the Pontryagin class p_k for the stable tangent bundle of $M(a_1, a_2)$ is related to (a_1, a_2) by the formula of Section 1.

Proof of Theorem 2. Since $n = 4k$, the J-homomorphism restricts to the modulo-σ_k reduction $G_n = \mathbb{Z} \to \pi_{2n-1}(S^n)$ by Lemma 4. Let $f : M(a_1, a_2) \to M(b_1, b_2)$ be a homotopy equivalence. Then $r(f) \in M(2)$ must be unimodular. The congruence relations follow from (i) and (ii) of Corollary 1. ∎

5. Proof of Theorem 3. For a self-map f of a manifold M, let $L(f)$ be the Lefschetz number of f [Br]. We put

$$L(M) = \{L(f) \mid f : M \to M\}.$$

By the definition of Lefschetz number we deduce from Corollary 1 the following

LEMMA 5. *If $M = M(a_1, a_2) \in T(k)$, then $L(M) = L_1 \cup L_2 \cup L_3 \cup L_4$ with*

$$L_1 = \{(a+1)(b+1) \mid (ab-a)a_1 \equiv (ab-b)a_2 \equiv 0 \pmod{\sigma_k},\ a, b \in \mathbb{Z}\},$$
$$L_2 = \{1 + ab \mid aba_1 - ba_2 \equiv aba_2 - aa_1 \equiv 0 \pmod{\sigma_k},\ a, b \in \mathbb{Z}\},$$
$$L_3 = \{1 + a \mid aa_1 \equiv 0 \pmod{\sigma_k},\ a \in \mathbb{Z}\},$$
$$L_4 = \{1 + b \mid ba_2 \equiv 0 \pmod{\sigma_k},\ b \in \mathbb{Z}\}.$$

It is well known that if M is a simply connected manifold, then M has the fixed point property if and only if $0 \notin L(M)$.

Proof of Theorem 3. Assume $n = 4k$, and $M(a_1, a_2) \in T(k)$ (i.e. $0 \leq a_1 \leq a_2 \leq \sigma_k/2$). It is easy to see from Lemma 5 that the condition $a_1a_2 \neq 0$ is equivalent to $0 \notin L_3 \cup L_4$, and that $0 \in L_2$ implies $0 \in L_1$. We may assume below that $a_1a_2 \neq 0$. Consequently, $o(a_1) \neq 0$, $o(a_2) \neq 0$ (since $a_1, a_2 \leq \sigma_k/2$).

If $0 \in L_1$, then by Lemma 5 we have either

(i) $b = -1$ and $2aa_1 \equiv (a-1)a_2 \equiv 0$, or
(ii) $a = -1$ and $(b-1)a_1 \equiv 2ba_2 \equiv 0$.

However (i) implies $o(a_1) \mid 2a$ and $o(a_2) \mid a - 1$, and similarly (ii) implies $o(a_1) \mid b - 1$ and $o(a_2) \mid 2b$, both leading to either

$$\gcd\{o(a_1), o(a_2)\} = 1,\ \gcd\{2o(a_1), o(a_2)\} = 2,\ \text{or}\ \gcd\{o(a_1), 2o(a_2)\} = 2.$$

Conversely, if $\gcd\{o(a_1), o(a_2)\} = 1$, so that there are $s, t \in \mathbb{Z}$ such that

$$so(a_1) + to(a_2) = 1,$$

then $(a, b) = (-1, o(a_2)t)$ satisfies (ii). Alternatively, if $\gcd\{2o(a_1), o(a_2)\} = 2$ (say), so that there are $s, t \in \mathbb{Z}$ such that

$$2o(a_1)s + o(a_2)t = 2,$$

then $o(a_2)t$ is divisible by 2, and $(a, b) = (-1,\ o(a_2)t/2)$ satisfies (ii). Thus $0 \in L_1$. The first assertion of Theorem 3 is verified.

The second assertion of Theorem 3 comes directly from the following observations:

(i) If n is odd, then for $M \in S(n)$ one has

$$L(\mathrm{Id}) = \chi(M) = 0 \in L(M),$$

where $\mathrm{Id} : M \to M$ is the identity and $\chi(M)$ is the Euler characteristic of M.

(ii) If $n = 8s + 6$, then $S(n)$ consists of the single element $S^n \times S^n$ by Corollary 2(2).

(iii) If $n \equiv 2 \pmod 8$ and $n > 8$, then $S(n) = \{S^n \times S^n, M(x, 0)\}$ by Corollary 2(1). The matrix

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

is realizable by a self-map f of $M(x, 0)$ by Corollary 1(ii); its Lefschetz number is seen to be zero. ■

6. Computational examples. We conclude this paper by describing some phenomena arising from the previous computation.

For every manifold M, the constant map and identity map of M contribute to the set $L(M)$ respectively 1 and $\chi(M)$ (the Euler characteristic). Therefore, the subset $L'(M) = L(M) \setminus \{1, \chi(M)\}$ can be viewed as *the set of non-obvious Lefschetz numbers* for self-maps of M.

If $M \in S(4k)$, the set $L(M)$ (hence $L'(M)$) may be computed by using Lemma 5. For instance, if $M = M(1, 2) \in S(4k)$, one can show that

$$|\lambda| \geq \sqrt{\sigma_k}/2 \quad \text{for all } \lambda \in L'(M).$$

This estimate points out an interesting phenomenon: *there exist twisted products of S^{4k} whose non-obvious Lefschetz numbers are arbitrarily large.*

A fundamental invariant for a map $f : M \to N$ between two closed oriented manifolds of the same dimension is its Brouwer degree, denoted by $\deg f$. It may be evaluated in the following manner. Let $[M] \in H_{\dim M}(M)$ be the fundamental class specified by the orientation, and let $f_* : H_*(M) \to H_*(N)$ be the induced homomorphism. In view of the fact that $H_{\dim M}(M) = \mathbb{Z}$ is generated by $[M]$, $\deg f$ is seen to be the unique integer satisfying $f_*[M] = \deg f \cdot [N]$ in $H_{\dim N}(N) = \mathbb{Z}$.

Given two closed oriented manifolds M, N of the same dimension we set

$$D(M,N) = \{\deg f \mid f : M \to N\}.$$

The problem of determining the set $D(M,N)$ for given M and N can be viewed as one of the realization problems in topology, and has been studied by many authors for certain classes of 3-manifolds (cf. [S] for the latest references).

Lemma 3 is sufficient to find the set $D(M,N)$ for $M, N \in S(n)$. For instance, from Corollary 1 (a special case of Lemma 3) one finds that if $M = M^{8k}(a_1,a_2), N = M^{8k}(b_1,b_2) \in S(4k)$, then

$$\begin{aligned} D(M,N) = \{xy \mid xyb_1 - xa_1 \equiv 0,\ xyb_2 - ya_2 \equiv 0 \pmod{\sigma_k}\} \\ \cup \{xy \mid xyb_1 - xa_2 \equiv 0,\ xyb_2 - ya_1 \equiv 0 \pmod{\sigma_k}\}. \end{aligned}$$

This indicates that the set $D(M,N)$ might possess interesting numerical features. Direct computations yield, as examples,

$$\begin{aligned} D(M^{8k}(1,1), M^{8k}(0,0)) &= \{\sigma_k^2 t \mid t \in \mathbb{Z}\}, \\ D(M^{48}(1,1), M^{48}(0,0)) &= \{65520^2 t \mid t \in \mathbb{Z}\}, \\ D(M^{16}(1,2), M^{16}(0,0)) &= \{7200t \mid t \in \mathbb{Z}\}, \\ D(M^{16}(1,3), M^{16}(0,0)) &= \{4800t \mid t \in \mathbb{Z}\}, \\ D(M^{16}(1,4), M^{16}(0,0)) &= \{3600t \mid t \in \mathbb{Z}\}. \end{aligned}$$

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