On transcendental automorphisms of algebraic foliations

by

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Abstract. We study the group Aut(\mathcal{F}) of (self) isomorphisms of a holomorphic foliation \mathcal{F} with singularities on a complex manifold. We prove, for instance, that for a polynomial foliation on \mathbb{C}^2 this group consists of algebraic elements provided that the line at infinity \mathbb{C}P(2) \setminus \mathbb{C}^2 is not invariant under the foliation. If in addition \mathcal{F} is of general type (cf. [20]) then Aut(\mathcal{F}) is finite. For a foliation with hyperbolic singularities at infinity, if there is a transcendental automorphism then the foliation is either linear logarithmic, Riccati or chaotic (cf. Definition 1). We also give a description of foliations admitting an invariant algebraic curve \mathcal{C} \subset \mathbb{C}^2 with a transcendental foliation automorphism.

1. Introduction and main results. In this paper we study the group of (self) isomorphisms of a foliation. Given a codimension one holomorphic foliation \mathcal{F} with singularities on a complex manifold \mathcal{M} we denote by Aut(\mathcal{F}) the maximal subgroup of Aut(\mathcal{M}) whose elements preserve the foliation \mathcal{F}. This object has been studied in [20] where it is proven that Aut(\mathcal{F}) is finite for \mathcal{F} of general type on a (compact) projective surface. We recall (cf. [20]) that a foliation \mathcal{F} on a projective surface \mathcal{M}^2 is of general type if its Kodaira dimension (cf. [17]) is equal to 2. Our results extend in a way the result above to open algebraic surfaces. Indeed, we are concerned with the non-compact case and its applications to the classification of holomorphic flows on \mathbb{C}^2, for instance. Our first result is the following:

**Proposition 1.** Let \mathcal{F} be a foliation with hyperbolic singularities on \mathbb{C}P(2). Suppose there exists some affine open subset \mathcal{U} \subset \mathbb{C}P(2) such that the restriction \mathcal{F}|_{\mathcal{U}} contains a holomorphic flow in its group of automorphisms. Then \mathcal{F} is, in suitable affine coordinates, a linear hyperbolic foliation.

By an affine set \mathcal{U} \subset \mathbb{C}P(2) we mean the complement of an algebraic curve \Gamma \subset \mathbb{C}P(2). This preliminary result suggests that the complexity of the transverse structure and that of the tangent structure are, in a certain
sense, in inverse proportion. This idea is enforced by our next results. The first is the following extension lemma:

**Lemma 1 (Extension Lemma).** An entire automorphism \( \varphi : \mathbb{C}^2 \to \mathbb{C}^2 \) of an algebraic foliation \( \mathcal{F} \) on \( \mathbb{C}P(2) \) is algebraic provided that the infinity \( \mathbb{P}^1_\infty = \mathbb{C}P(2) \setminus \mathbb{C}^2 \) is not invariant under \( \mathcal{F} \).

The above result actually holds for bimeromorphic maps \( \varphi : \mathbb{C}^2 \setminus \Gamma \to \mathbb{C}^2 \setminus \Gamma \) if we replace \( \mathbb{P}^1_\infty \) by an irreducible (non-invariant) algebraic curve \( \Gamma \subset \mathbb{C}P(2) \), as will be clear from the proof we give. We point out that a foliation automorphism \( \varphi \in \text{Aut}(\mathbb{C}^2) \) may extend to a bimeromorphic non-linear map of \( \mathbb{C}P(2) \) as is the case for \( \mathcal{F} : dx = 0 \) and \( \varphi(x, y) = (x, y + x^n) \). Denote by \( \text{Bim}(\mathcal{F}) \) the group of bimeromorphic maps \( \Phi : M \to M \) which preserve \( \mathcal{F} \) in \( M \) (these maps take leaves of \( \mathcal{F} \) onto leaves of \( \mathcal{F} \) wherever defined). Combining the extended Lemma 1 with Theorem 1 of [20] we immediately obtain:

**Corollary 1.** Let \( \mathcal{F} \) be a codimension one singular holomorphic foliation on \( \mathbb{C}P(2) \) and \( \Gamma \subset \mathbb{C}P(2) \) a non-invariant algebraic curve. Then we have natural group immersions \( \text{Bim}(\mathcal{F}|_{\mathbb{C}P(2) \setminus \Gamma}) \subset \text{Bim}(\mathcal{F}) \subset \text{Bim}(\mathbb{C}P(2)) \). In particular, \( \text{Bim}(\mathcal{F}|_{\mathbb{C}P(2) \setminus \Gamma}) \) is finite if \( \mathcal{F} \) is generic.

For a foliation \( \mathcal{F} \) on a projective surface \( M \) admitting a minimal model \( \mathcal{G} \) on \( \tilde{M} \) (cf. [6]), we have a natural group isomorphism \( \text{Bim}(\mathcal{F}) \simeq \text{Aut}(\mathcal{G}) \) [20]. Foliations without minimal model are listed as: rational fibrations, non-trivial Riccati foliations and the exceptional foliation \( \mathcal{H} \) in [6] (see also Example 1.3 in [10]). In particular, if \( \mathcal{F} \) is generic then it admits a minimal model ([20]). In [10] it is proved that for a foliation \( \mathcal{F} \) on a projective surface if \( \text{Aut}(\mathcal{G}) \subset \text{Bim}(\mathcal{G}) \) for every birational model \( \mathcal{G} \) of \( \mathcal{F} \) then we have the following possibilities: \( \mathcal{F} \) is a rational fibration or \( \mathcal{F} \) is the exceptional foliation \( \mathcal{H} \) of [6], i.e., the foliation in Example 1.3 of [10]. Combining these results with Corollary 1 above we obtain:

**Corollary 2.** Let \( \mathcal{F} \) be a codimension one singular holomorphic foliation on \( \mathbb{C}P(2) \) and \( \Gamma \subset \mathbb{C}P(2) \) a non-invariant algebraic curve. Suppose that \( \text{Bim}(\mathcal{F}|_{\mathbb{C}P(2) \setminus \Gamma}) \) is not finite. Then \( \mathcal{F} \) is either a linear foliation, an elliptic fibration, a rational fibration, a Riccati foliation, or (birationally equivalent to) the exceptional foliation \( \mathcal{H} \) of [6].

**Proof.** If \( \mathcal{F} \) has a minimal model then we may assume that \( \mathcal{F} \) is minimal and therefore \( \text{Aut}(\mathcal{F}) = \text{Bim}(\mathcal{F}) \) is infinite. Theorem 1.1 of [10] then shows that \( \mathcal{F} \) is either an elliptic fibration, a Riccati foliation or given by a global holomorphic vector field on \( \mathbb{C}P(2) \), in which case \( \mathcal{F} \) is necessarily linear. Now we may assume that \( \mathcal{F} \) has no minimal model. Therefore, according to the above discussion \( \mathcal{F} \) is either a non-trivial Riccati foliation, a rational
fibration or the foliation $\mathcal{H}$ of [6] which is the foliation in Example 1.3 of [10].

For the transcendental case we shall be more precise by introducing a dynamical concept:

**Definition 1** (chaotic foliation). We shall say that a foliation $\mathcal{F}$ on $\mathbb{C}P(2)$ which leaves invariant the line at infinity is chaotic if:

1. $\mathcal{F}|_{\mathbb{C}^2}$ has no parabolic leaf (in the sense of potential theory, cf. [4]);
2. $\mathcal{F}|_{\mathbb{C}^2}$ admits no (non-trivial) holonomy invariant measure, indeed the leaves of $\mathcal{F}|_{\mathbb{C}^2}$ have exponential growth for the Fubini–Study metric ([9]);
3. The holonomy group of the leaf $L_\infty = \mathbb{P}_\infty^1 \setminus (\text{sing}(\mathcal{F}) \cap \mathbb{P}_\infty^1)$ is non-solvable (see [19]);
4. All leaves of $\mathcal{F}$, except for $L_\infty$, are dense in $\mathbb{C}P(2)$ ([15], [12]).
5. $\mathcal{F}|_{\mathbb{C}^2}$ (and therefore $\mathcal{F}$ on $\mathbb{C}P(2)$) is ergodic ([12]) and topologically rigid ([15], [23]).

Using this terminology we state our main results as follows:

**Theorem 1.** Let $\mathcal{F}$ be an algebraic foliation on $\mathbb{C}P(2)$ such that sing($\mathcal{F}$) $\cap \mathbb{P}_\infty^1$ is hyperbolic and $\text{Aut}(\mathcal{F}|_{\mathbb{C}^2})$ contains a transcendental element $\varphi: \mathbb{C}^2 \to \mathbb{C}^2$. We have the following possibilities: (i) $\mathcal{F}$ is chaotic; (ii) $\mathcal{F}$ is a linear logarithmic foliation; (iii) $\mathcal{F}$ is a Riccati foliation.

Examples of foliations of logarithmic and Riccati type with transcendental foliation automorphisms are given in §4. Also, consider the following construction: let

$$\varphi(x, y) = \left( x, \frac{y}{\alpha(x)y + \beta(x)} \right)$$

for entire functions $\alpha, \beta \in \mathcal{O}(\mathbb{C})$. We can obtain examples of $\varphi$-invariant Riccati equations of the form $\mathcal{F}: [(x-a_1) \cdots (x-a_r)]dy - [y^2A(x) + yB(x)]dx = 0$, where in the affine system $(x, y) \in \mathbb{C}^2 \subset \mathbb{C}P(2)$ we take the line at infinity to be $\mathbb{P}_\infty^1 = \overline{(x = \infty)}$. It is then easy to obtain such examples with transcendental $\varphi$ and sing($\mathcal{F}$) $\cap \mathbb{P}_\infty^1$ hyperbolic. A linear hyperbolic foliation $\mathcal{F}: xdy - \lambda ydx = 0, \lambda \in \mathbb{C} \setminus \mathbb{R}$, admits no transcendental foliation automorphisms (cf. Example 2).

As an immediate consequence of Theorem 1 we obtain:

**Corollary 3.** Let $\mathcal{F}$ be a foliation on $\mathbb{C}P(2)$ with hyperbolic singularities in $\mathbb{P}_\infty^1$ and such that $\text{Aut}(\mathcal{F}|_{\mathbb{C}^2})$ contains a transcendental element. Suppose $\mathcal{F}$ has no algebraic invariant curve $C \subset \mathbb{C}^2$. Then $\mathcal{F}$ is chaotic.

In general, for a foliation admitting an algebraic invariant curve and with a transcendental automorphism we have:
Theorem 2. Let $\mathcal{F}$ be an algebraic foliation on $\mathbb{C}P(2)$ such that $\text{Aut}(\mathcal{F}|_{\mathbb{C}^2})$ contains a transcendental element $\varphi: \mathbb{C}^2 \to \mathbb{C}^2$. Suppose that $\mathcal{F}$ has an invariant algebraic curve $C \subset \mathbb{C}^2$. Then, up to replacing $\varphi$ by some power $\varphi^n$, $n \in \mathbb{N}$, we have (only) the following possibilities:

(tr.1) $\varphi(C)$ is not algebraic and $\mathcal{F}|_{\mathbb{C}^2}$ is holomorphically equivalent to $dy = 0$ in $\mathbb{C}^2$. Up to some entire automorphism taking $\mathcal{F}|_{\mathbb{C}^2}$ to $dy = 0$ we have $\varphi(x, y) = (a_0(y)x + a_1(y), b_0 + b_1y)$ with $a_0(y), a_1(y) \in \mathcal{O}(\mathbb{C}^2)$, $b_0, b_1 \in \mathbb{C}$ and some natural restrictions on the coefficients. In particular $C$ is isomorphic to the affine line, $C \cong \mathbb{C}$.

(tr.2) There is $n \in \mathbb{N}$ such that $\varphi^n(C) = C$ in $\mathbb{C}^2$, and $C$ is, up to a polynomial automorphism of $\mathbb{C}^2$, in the following list:

(a) $\alpha(x) + y\beta(x) = 0$, $\alpha(x), \beta(x) \in \mathbb{C}[x]$;

(b) a union of irreducible components of fibers of $x^ny^m$, $n, m \in \mathbb{N}$, \langle n, m \rangle = 1;

(c) a union of irreducible components of fibers of a polynomial $R(x, y) = x^n(x^l y + P(x))^m$ with $n, m, l \in \mathbb{N}$, \langle n, m \rangle = 1$, $P(x) \in \mathbb{C}[x]$ of degree at most $l - 1$, $P(0) \neq 0$.

In particular, $\bar{C}$ is rational and $\bar{C} \cap \mathbb{P}^1_\infty \leq 2$, so that each irreducible component $C_0 \subset C$ satisfies $C_0 \cong \mathbb{C}$ or $\mathbb{C} \setminus \{0\}$.

(tr.3) $\varphi^n(C)$ is algebraic for some $n \in \mathbb{N}$ but $\varphi^m(C) \neq C$ for all $m \in \mathbb{N}$.

In this case $\mathcal{F}$ admits a rational first integral $R: \mathbb{C}P(2) \dashrightarrow \mathbb{P}^1$; in particular all the leaves of $\mathcal{F}$ are algebraic.

Examples are given and we believe that one may go further in the discussion of Case (tr.2) above.

2. The case of flows—Proof of Proposition 1

2.1. Motivations and the compact case. Let $\mathcal{F}$ be a codimension $k$ foliation on a manifold $M$ with singular set $\text{sing}(\mathcal{F})$. The group of foliation automorphisms of $\mathcal{F}$ is the subgroup $\text{Aut}(\mathcal{F})$ of $\text{Aut}(M)$ whose elements $\psi: M \to M$ satisfy $\psi(\text{sing}(\mathcal{F})) = \text{sing}(\mathcal{F})$ and $\psi^*\mathcal{F} = \mathcal{F}$, in other words, $\psi$ fixes the singular set and takes (non-singular) leaves of $\mathcal{F}$ onto (non-singular) leaves of $\mathcal{F}$. If $M$ is (complex) compact then $\text{Aut}(M)$ is a (complex) Lie group, and since $\text{Aut}(\mathcal{F})$ is a closed subgroup of $\text{Aut}(M)$, it is a (complex) Lie subgroup. Thus we may consider the following general situation: $\text{Aut}(\mathcal{F})$ is a complex Lie group and therefore there exists a Lie group action $\varphi: G \times M \to M$ such that each automorphism $\varphi_g$, $g \in G$, belongs to $\text{Aut}(\mathcal{F})$. We have two main (antipodal) cases: (i) $\varphi$ is transverse to $\mathcal{F}$; (ii) $\varphi$ is tangent to $\mathcal{F}$. In the rest of this section we shall consider the two main cases above in the compact two-dimensional situation for singular foliations.
1. Let $\mathcal{F}$ be a singular codimension one foliation on a complex surface $M$ and assume there is a flow $\varphi_t \in \text{Aut}(\mathcal{F})$ for all $t \in \mathbb{C}$. If $\varphi$ is transverse to $\mathcal{F}$ then we are in the situation considered in [22]; in particular there exists a closed holomorphic 1-form $\Omega$, non-singular and defining $\mathcal{F}$ on $M$, and we may describe the dynamics of $\mathcal{F}$ according to the group $\text{Per}(\Omega) \subset (\mathbb{C}, +)$ (cf. [22]). For instance, if $M$ is compact Kaehler with $\text{rank } H_1(M; \mathbb{R}) \leq 2$ then $\mathcal{F}$ is a Seifert fibration (i.e., a foliation by compact leaves having finite holonomy groups).

2. Assume now that there is a flow $\varphi$ which is tangent to $\mathcal{F}$; this implies that given any $p \in M$ the leaf $L_p$ of $\mathcal{F}$ containing $p$ admits a holomorphic map $\varphi_p : \mathbb{C} \to L_p$ and $L_p$ is holomorphically equivalent to one of the following Riemann surfaces: $\mathbb{C}$, $\mathbb{C}^*$ or a torus $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}$. Moreover, $X(p) = \frac{\partial \varphi_t}{\partial t}(p)|_{t=0}$ defines a global holomorphic vector field (possibly with singularities) on $M$, tangent to $\mathcal{F}$. If $X$ is singular but $M = \mathbb{C}P(2)$ then $X$ must be linear: $X = \lambda x \frac{\partial}{\partial x} + \mu y \frac{\partial}{\partial y}$ or $X = \lambda x \frac{\partial}{\partial x} + (x + \lambda y) \frac{\partial}{\partial y}$ in some affine chart $(x, y) \in \mathbb{C}^2$ and $\mathcal{F}$ is linear as well.

3. Now we consider the non-singular case. If $M$ is compact and $X$ is non-singular then the second Chern class of $M$ is zero: $c_2(M) = 0$, and we may use the Enriques–Kodaira classification [1] in order to describe $M$. Actually, in [18] it is proved that such a surface must be one of the following: (1) a flat holomorphic fibre bundle over an elliptic curve with connected fibres; (2) a complex torus; (3) an elliptic surface without singular fibres or with singular fibres of type $mI_0$ only; (4) a non-elliptic Hopf surface; (5) an Inoue surface of type $S_{n,p,q,r,t}$. 

4. Finally, concerning the existence of compact leaves in the non-singular case we find that if $L_0$ is a toral leaf then we have two possibilities: (i) $L_0$ is an isolated toral leaf; (ii) $L_0$ is not an isolated toral leaf; in this case the holonomy group $\text{Hol}(L_0) \hookrightarrow \text{Diff}(\mathbb{C}, 0)$ is abelian with some finite pseudo-orbits so that it cannot contain non-trivial flat elements (cf. [7]) and therefore it is a finite group of rational rotations. Since $\text{Hol}(L_0)$ is finite the global stability theorem in [2] implies that every leaf of $\mathcal{F}$ is compact with finite holonomy, therefore $\mathcal{F}$ is a Seifert fibration by complex tori.

2.2. Proof of Proposition 1. We shall keep the notation of the preceding subsection. The case where $\varphi_t$ is completely transverse to $\mathcal{F}$ in $\mathbb{C}P(2) \setminus \Gamma$ follows from [22]. Indeed, as we have seen above the foliation in $U = \mathbb{C}P(2) \setminus \Gamma$ is given by a closed holomorphic one-form $\Omega$. Since the singularities of $\mathcal{F}$ in $\Gamma$ are hyperbolic, $\Omega$ extends meromorphically with simple poles along $\Gamma$ to a neighborhood of each singular point $p \in \text{sing}(\mathcal{F}) \in \Gamma$ [21]. Thus we conclude by the Hartogs Theorem that $\Omega$ extends to a meromorphic one-form $\tilde{\Omega}$ in $\mathbb{C}P(2)$. This one-form has simple poles in $\mathbb{C}P(2)$ and by the
Integration Lemma is logarithmic; by [3] the foliation must be linear in some affine chart due to the fact that its singularities are hyperbolic.

Assume now that the flow is everywhere tangent to \( \mathcal{F} \) in \( U \). This means that the leaves of \( \mathcal{F}|_U \) are the nonsingular orbits of the vector field \( X(p) = (\partial \varphi_t(p)/\partial t)|_{t=0} \). In particular \( \mathcal{F}|_U \) and therefore \( \mathcal{F} \) is parabolic; since \( \mathcal{F} \) has only hyperbolic singularities in \( \mathbb{CP}(2) \), it follows from [4] that it is linear in some affine chart in \( \mathbb{C}^2 \).

Finally, assume that \( \varphi_t \) is not completely transverse to \( \mathcal{F} \) in \( \mathbb{CP}(2) \) \( \setminus \Gamma \) but \( \mathcal{F} \) is not the foliation induced by the vector field \( X \) as above. If \( L \) is a leaf of \( \mathcal{F} \) in \( U \) which is also invariant under \( X \) then \( L \) is covered by \( \mathbb{C} \). If \( L \) is non-algebraic then \( \mathcal{F} \) has a transcendental leaf covered by \( \mathbb{C} \) and according to [5] and [9] it follows that \( \mathcal{F} \) admits a non-trivial holonomy invariant measure; moreover, [9], \( \mathcal{F} \) is a linear hyperbolic foliation due to its hyperbolic singularities. If the closure \( \overline{L} \subset \mathbb{CP}(2) \) is an algebraic curve, say \( \Gamma' \), then \( U' = \mathbb{CP}(2) \setminus (\Gamma \cup \Gamma') \) is also affine. Due to the hyperbolic singularities \( \mathcal{F} \) admits no rational first integral and therefore by a theorem of Darboux [13], \( \mathcal{F} \) has only finitely many algebraic invariant curves. This reduces the proof to the two cases above. In other words, we may assume that either \( \mathcal{F} \) is completely transverse to \( X \) or it coincides with the foliation of \( X \) in \( \mathbb{CP}(2) \) \( \setminus \Gamma \). In either case the foliation is linear in some affine chart in \( \mathbb{C}^2 \).

3. Entire automorphisms—Proof of Lemma 1 and Theorem 1.

Let \( \mathcal{F} \) be an algebraic foliation on \( \mathbb{C}^2 \). Then \( \mathcal{F} \) admits an extension of algebraic type to a foliation (also denoted by) \( \mathcal{F} \) on \( \mathbb{CP}(2) = \mathbb{C}^2 \cup \mathbb{P}^1_\infty \). Let \( \varphi : \mathbb{C}^2 \to \mathbb{C}^2 \) be an entire (algebraic or transcendental) automorphism of \( \mathbb{C}^2 \) that preserves \( \mathcal{F} \). Under which conditions on \( \mathcal{F} \) is \( \varphi \) algebraic?

3.1. Case of \( \mathbb{P}^1_\infty \) is not \( \mathcal{F} \)-invariant—Proof of Extension Lemma 1. Assume that \( \mathbb{P}^1_\infty \) is not \( \mathcal{F} \)-invariant; the set of points \( p \in \mathbb{P}^1_\infty \) such that \( \mathcal{F} \) is not transverse to \( \mathbb{P}^1_\infty \) in a neighborhood of \( p \) is a finite subset of \( \mathbb{P}^1_\infty \), say \( T(\mathcal{F}, \mathbb{P}^1_\infty) = \{p_1, \ldots, p_r\} \). Let \( p \in \mathbb{P}^1_\infty \setminus T(\mathcal{F}, \mathbb{P}^1_\infty) \) be given and choose a flow-box neighborhood \( U \) of \( p \) in \( \mathbb{CP}(2) \) such that in local coordinates \((x,y)\) in \( U \) we have \( x(p) = y(p) = 0, \mathbb{P}^1_\infty \cap U = \{x = 0\} \) and \( \mathcal{F}|_U : dy = 0 \). Also we may assume that \( \{y = 0\} \subset L_p \). Denote by \( L_p \) the leaf of \( \mathcal{F} \) that contains \( p \) and by \( L^*_p = L_p \setminus \mathbb{P}^1_\infty \) the corresponding leaf of \( \mathcal{F}|_{\mathbb{C}^2} \). Since \( \varphi : \mathbb{C}^2 \to \mathbb{C}^2 \) is an automorphism that preserves \( \mathcal{F} \) we have \( L^*_p = \varphi(L^*_p) \), where \( L^* = L \setminus \mathbb{P}^1_\infty \subset \mathbb{C}^2 \) for some leaf \( L \) of \( \mathcal{F} \) on \( \mathbb{CP}(2) \). Now, using the flow-box in \( U \) we may choose a disk \( \Sigma \subset U \cap \mathbb{C}^2 \) transverse to \( \mathcal{F} \) with the properties that each plaque of \( \mathcal{F}|_U \) cuts \( \Sigma \) at most once and the same holds for the disk \( \mathbb{P}^1_\infty \cap U =: \Sigma_\infty \).
Let \( p' \in \Sigma \cap L_p, p' \in \{ y = 0 \} \); choose a simple path \( \alpha : [0, 1] \to L_p \cap U \) such that \( \alpha(t) \in \{ y = 0 \} \) for all \( t \in [0, 1] \), \( \alpha(0) = p' \) and \( \alpha(1) = p \), and denote by \( h_\alpha : (\Sigma, p') \to (\Sigma_\infty, p) \) the holonomy map corresponding to \( \alpha \). Let \( \Sigma_1 \) be the inverse image \( \Sigma_1 = \varphi^{-1}(\Sigma) \subset \mathbb{C}^2 \). Then \( \Sigma_1 \) is also a transverse disk to \( \mathcal{F} \) in \( \mathbb{C}^2 \) and \( \Sigma_1 \cap L \ni p' = \varphi^{-1}(p') \). Now we choose any simple path \( \beta : [0, 1] \to L \) with the following properties: \( \beta(0) = p_1 \) and \( \beta(1) = q' \in \mathbb{C}^2 \) is a point which belongs to a flow-box \( V \subset \mathbb{C}P(2) \) centered at a point \( q \in \mathbb{P}^1_\infty \cap L \).

In particular we may assume that \( q' \) and \( q \) belong to the same plaque \( P \) of \( \mathcal{F}|_V \), which is a disk \( P \cong \mathbb{D} \). We fix a transverse disk \( \Sigma_2 \subset V \) to \( \mathcal{F} \) with \( \Sigma_2 \cap P = q' \) and choose any simple path \( \alpha_1 : [0, 1] \to P \) with \( \alpha_1(0) = q' \), \( \alpha_1(1) = q \).

Denote by \( h_\beta : (\Sigma_1, p'_1) \to (\Sigma_2, q') \) and \( h_{\alpha_1} : (\Sigma_2, q') \to (\Sigma'_\infty, q) \) the holonomy diffeomorphisms corresponding to \( \beta \) and \( \alpha_1 \) respectively, where \( \Sigma'_\infty = \mathbb{P}^1_\infty \cap V \) is transverse to \( \mathcal{F} \).
Given a point $a \in \Sigma_\infty$ close enough to $p$, we define $\varphi^{-1}_\beta(a) := h_{\alpha_1} \circ h_\beta \circ h^{-1}_\alpha(a)$.

**Lemma 2.** If the holonomy of $L$ is trivial then $\varphi^{-1}_\beta$ does not depend on the choice of $\beta$.

**Proof.** This is clear. ■

**Proof of Lemma 1.** We keep the above notation. Since $\mathcal{F}|_U$ and $\mathcal{F}|_V$ are trivial it follows that for a leaf $L$ with trivial holonomy as above $\varphi^{-1}$ is an extension of $\varphi^{-1}$ to $\Sigma_\infty = U \cap \mathbb{P}^1_\infty$ and by the Hartogs Extension Theorem $\varphi^{-1}$ extends to a meromorphic map $\overline{\varphi}^{-1}: \mathbb{C}P(2) \to \mathbb{C}P(2)$. Since the same holds for $\varphi$ and such extensions must clearly satisfy $\overline{\varphi} \circ \overline{\varphi}^{-1} = \text{Id}_{\mathbb{C}P(2)}$ we conclude that $\varphi$ extends to a bimeromorphism $\overline{\varphi}$ of $\mathbb{C}P(2)$; the bimeromorphism $\overline{\varphi}$ still preserves $\mathcal{F}$ on $\mathbb{C}P(2)$. Using now the fact that for $\mathcal{F}$ arbitrary (with $\mathbb{P}^1_\infty$ not invariant) the set of leaves with trivial holonomy is residual we obtain Lemma 1. ■

**3.2. Proof of Theorem 1.** Let $\mathcal{F}$ be an algebraic foliation on $\mathbb{C}P(2)$ such that $\text{sing}(\mathcal{F}) \cap \mathbb{P}^1_\infty$ is hyperbolic and $\text{Aut}(\mathcal{F}|_{\mathbb{C}^2})$ contains a transcendental element $\varphi: \mathbb{C}^2 \to \mathbb{C}^2$. According to Lemma 1 the line at infinity is invariant. Denote by $L_\infty = \mathbb{P}^1_\infty \setminus (\mathbb{P}^1_\infty \cap \text{sing}(\mathcal{F}))$ the corresponding leaf of $\mathcal{F}$. If the holonomy group of $L_\infty$ is solvable then, according to [21] and [8], due to the fact that the singularities in $\text{sing}(\mathcal{F}) \cap \mathbb{P}^1_\infty$ are hyperbolic, the foliation is either a Riccati foliation or a logarithmic foliation. Thus we may assume that $\text{Hol}(\mathcal{F}, L_\infty)$ is a non-solvable group. Since by the Index Theorem $\text{sing}(\mathcal{F}) \cap \mathbb{P}^\infty_\infty \neq \emptyset$, the group $\text{Hol}(\mathcal{F}, L_\infty)$ contains hyperbolic elements. Using now [19], [15], [9], [12], [23] we shall prove that $\mathcal{F}$ is chaotic:

**Lemma 3.** In the situation above, $\mathcal{F}$ has no algebraic leaf $C \subset \mathbb{C}^2$.

**Proof.** Assume that $\mathcal{F}$ has some algebraic invariant curve $C \subset \mathbb{C}^2$; denote by $L = C \setminus \text{sing}(\mathcal{F})$ the corresponding leaf of $\mathcal{F}|_{\mathbb{C}^2}$ (we may work with $C$ irreducible). The leaf $L$ is then a parabolic leaf of $\mathcal{F}|_{\mathbb{C}^2}$; denote by $L' = \varphi(L) \subset \mathbb{C}^2$ its image under $\varphi$. If $L'$ is not algebraic (i.e., the closure $\overline{L'} \subset \mathbb{C}^2$ is not an algebraic curve) then the intersection $L' \cap D$ accumulates to the origin $o = D \cap L_\infty$ of any disk $D$ transverse to $\mathcal{F}$ and to $L_\infty$. Since $C$ is closed in $\mathbb{C}^2$ the same holds for $L'$ and therefore it induces a pseudo-orbit in the holonomy group $\text{Hol}(\mathcal{F}, l_\infty, D)$, which is closed off the origin $o \in D$. According to the density theorem in [19], $\text{Hol}(\mathcal{F}, L_\infty, D)$ must be solvable, a contradiction. Thus necessarily $L'$ is algebraic. If $\mathcal{F}$ has infinitely many algebraic leaves then by a theorem of Darboux [13], $\mathcal{F}$ is given by a rational map $R: \mathbb{C}P(2) \dashrightarrow \overline{\mathbb{C}}$; this is not compatible with the hyperbolic singularities. We may therefore assume that $\varphi(C) = C$. Now we recall the following notions from [14]:
Let $C \subset \mathbb{C}^2$ be an irreducible algebraic affine curve. We say that $C$ is general if any analytic automorphism of $\mathbb{C}^2$ which transforms $C$ into an algebraic curve is algebraic.

**Theorem 3** (Kizuka [14]). Let $C \subset \mathbb{C}^2$ be an affine algebraic curve. Then $C$ is general provided that the closure $\overline{C}$ of $C$ on $\mathbb{C}P(2)$ intersects the line at infinity $\mathbb{P}^1_{\infty}$ at more than two points. If $C$ is not general then we have the following possibilities for $C$ after an algebraic affine change of coordinates:

(K.1) $C : \alpha(x) + y\beta(x) = 0$ (special case);
(K.2) $C$ is a sum of several prime surfaces of a monomial $f = x^m y^n$ where $m, n \in \mathbb{N}$;
(K.3) $C$ is a sum of several prime surfaces of a polynomial $f = x^m (x^l y + P_{l-1}(x))^n$, where $P_{l-1}$ is a one-variable polynomial of degree at most $l - 1$ with $P_{l-1}(0) \neq 0$, and $m, n \in \mathbb{N}$.

In each case above straightforward calculations show that $\mathcal{F}$ has either some separatrix at infinity which is not smooth at the singularity or some separatrix which is not transverse to the line at infinity. This contradicts the invariance of $\mathbb{P}^1_{\infty}$ and the hyperbolicity of the singularities in $\mathbb{P}^1_{\infty}$. Lemma 3 is now proved. ■

Now we may finish the proof of Theorem 1:

(i) $\mathcal{F}|_{\mathbb{C}^2}$ has no parabolic leaf: otherwise according to [16], [4], [21], $\mathcal{F}|_{\mathbb{C}^2}$ would have some algebraic leaf, contradicting Lemma 3.

(ii) By (i) the leaves of $\mathcal{F}|_{\mathbb{C}^2}$ are hyperbolic (in the sense of potential theory) and covered by the disk $\mathbb{D}$. Moreover, the leaves of $\mathcal{F}|_{\mathbb{C}^2}$ have exponential growth (otherwise, according to [9], $\mathcal{F}$ would admit some holonomy invariant measure for the holonomy group of $L_\infty = \mathbb{P}^1_{\infty}$ and this measure would imply that the holonomy group $\text{Hol}(\mathcal{F}, L_\infty)$ is solvable, giving a contradiction).

(iii) Since the holonomy group of $\mathbb{P}^1_{\infty}$ is non-solvable and contains hyperbolic elements, the foliations $\mathcal{F}|_{\mathbb{C}^2}$ and $\mathcal{F}$ on $\mathbb{C}P(2)$ are topologically rigid and, as follows from the techniques in [15], we may also conclude that $\mathcal{F}|_{\mathbb{C}^2}$ and $\mathcal{F}$ on $\mathbb{C}P(2)$ are ergodic.

(iv) The density of the leaves in $\mathbb{C}^2$ is proved as in [15] as a consequence of the density results in [19] and [23] for non-solvable groups of germs of one-dimensional complex diffeomorphisms at the origin. Theorem 1 is therefore proved. ■

**4. Case of an algebraic invariant curve in $\mathbb{C}^2$—Theorem 2.** Let $\mathcal{F}$ be an algebraic foliation on $\mathbb{C}P(2)$ such that $\text{Aut}(\mathcal{F}|_{\mathbb{C}^2})$ contains a transcendental element $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ and having an invariant algebraic curve
$C \subset \mathbb{C}^2$. As in the proof of Theorem 1 denote by $L = C \setminus \text{sing}(\mathcal{F})$ the corresponding leaf of $\mathcal{F}|_{\mathbb{C}^2}$ and by $L' = \varphi(L) \subset \mathbb{C}^2$ its image under $\varphi$.

4.1. If $L'$ is not algebraic (i.e., the closure $\overline{L'} \subset \mathbb{C}^2$ is not an algebraic curve) then by the Darboux Theorem [13], $\mathcal{F}$ has only a finite number of algebraic leaves; we may therefore assume that $L'$ is a transcendental leaf of finite type and properly embedded so that, by the Corollary in [3], necessarily $L'$ (and therefore $L$) is isomorphic to $\mathbb{C}$ and also $\mathcal{F}|_{\mathbb{C}^2}$ is holomorphically equivalent to the foliation $dy = 0$ on $\mathbb{C}^2$. In this case, up to some entire automorphism, we have $\varphi(x, y) = (a(x, y), b(y))$ and therefore $\varphi(x, y) = (a_0(y)x + a_1(y), b_0 + b_1y)$ for some $b_0, b_1 \in \mathbb{C}$, $b_1 \neq 0$, $a_0(y), a_1(y) \in \mathcal{O}(\mathbb{C})$, with $a_0(y) \neq 0$ for all $y \in \mathbb{C}$. Again we may find holomorphic coordinates on $\mathbb{C}^2$ such that $\mathcal{F}|_{\mathbb{C}^2} : dy = 0$ and $\varphi(x, y) = (e^{u(y)}x + v(y), y)$ for some holomorphic $u, v \in \mathcal{O}(\mathbb{C})$.

**Example 1.** Let us see an example of the above situation: $X(x, y) = (1 + xy) \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y}$ generates a foliation $\mathcal{F}$ on $\mathbb{C}^2$ with the entire first integral $f(x, y) = ye^{xy}$. The algebraic curve $C : \{y = 0\}$ is $\mathcal{F}$-invariant and we have an entire non-algebraic automorphism $\varphi(x, y) = (xe^{P(x,y)}, ye^{xy})$ for some polynomial $P(x, y)$ (see [24]) and clearly $\varphi$ preserves $\mathcal{F}$. The vector field $X$ is not complete.

4.2. Suppose now $C$ is $\mathcal{F}$-invariant, algebraic and fixed by $\varphi$. According to Kizuka’s Theorem above [14], $\varphi$ is algebraic or, up to polynomial automorphism, $C$ belongs to the following list:

(K.1) $\alpha(x) + y\beta(x) = 0$, where $\alpha$ and $\beta$ are one-variable polynomials;

(K.2) a union of irreducible components of fibers of $x^ny^m$, where $n, m \in \mathbb{N}$ satisfy $\langle n, m \rangle = 1$;

(K.3) a union of irreducible components of fibers of a polynomial $Q$ of type $Q(x, y) = x^n(x^l y + P(x))^m$, with $n, m, l \in \mathbb{N}$, $\langle n, m \rangle = 1$, $P$ is a one-variable polynomial of degree at most $l - 1$, $P(0) \neq 0$.

In any of the cases (K.1), (K.2) and (K.3) above the irreducible components of the projective curve $\bar{C} \subset \mathbb{CP}(2)$ are rational and we have $\#\bar{C} \cap \mathbb{P}^1_{\infty} \leq 2$; in particular, each irreducible component of $C$ is diffeomorphic either to $\mathbb{C}$ or to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Examples of this situation are given by vector fields of the form

$$X = x(\mu + mC_1) \frac{\partial}{\partial x} - \left\{ \frac{x(x^l y + P(x))}{x^l} (\lambda + nC_1) + \frac{(lx^l y + xP'(x))(\mu + mC_1)}{x} \right\} \frac{\partial}{\partial y}$$

with $C_1 = x^n(x^l y + P(x))^m \cdot \tau'(x^n(x^l y + P(x))^m)$. These are complete vector fields (see [11]) whose flow maps $\varphi_t : \mathbb{C}^2 \to \mathbb{C}^2$ are not algebraic and preserve
both the foliation $\mathcal{F}|_{\mathbb{C}^2}$ and the algebraic invariant curve $\mathcal{C} = \{x^n(x^l y + P(x))^m = 0\}$.

4.3. Assume that $\mathcal{F}|_{\mathbb{C}^2}$ has some irreducible algebraic invariant curve $\mathcal{C}$ whose image $\varphi(\mathcal{C})$ is algebraic but is not $\mathcal{C}$. Denote by $\mathcal{C}_{\text{alg}}(\mathcal{F})$ the union of all irreducible algebraic curves $\mathcal{C}_1 \subset \mathbb{C}^2$ which are invariant under $\mathcal{F}$. We may assume that for any irreducible curve $\mathcal{C}_1 \in \mathcal{C}_{\text{alg}}(\mathcal{F})$ we have $\varphi^2(\mathcal{C}_1) \neq \mathcal{C}_1$. Then in particular we have $\varphi^n(\mathcal{C}) \neq \varphi^m(\mathcal{C})$ for all $n \neq m$ and thus $\mathcal{C}_{\text{alg}}(\mathcal{F}) = +\infty$. By the Darboux Theorem [13], $\mathcal{F}$ admits a rational first integral $R: \mathbb{C}^2 \rightarrow \overline{\mathbb{C}}$ and therefore $\mathcal{F}$ is a pencil of curves $\mathcal{F}: \{\lambda P + \mu Q = 0\}, (\lambda; \mu) \in \mathbb{P}^1$, for some $P, Q \in \mathbb{C}[x, y]$.

It remains therefore to consider the following case:

4.4. $\mathcal{F}|_{\mathbb{C}^2}$ has no algebraic leaf, and $\mathbb{P}_\infty^1$ is $\mathcal{F}$-invariant. This case contradicts the hypothesis of existence of the invariant curve $\mathcal{C}$. Theorem 2 is therefore proved.

Example 2. The linear foliation $\mathcal{F}_\lambda$ given on $\mathbb{C}^2 \subset \mathbb{C}P(2)$ by $\mathcal{F}_\lambda : xdy - \lambda ydx = 0$, $\lambda \in \mathbb{C} \setminus \{0\}$, admits the following subgroup of automorphisms: $\{\varphi: \mathbb{C}^2 \rightarrow \mathbb{C}^2; \varphi(x, y) = (xe^w, cye^\lambda w), \text{where } w \in \mathbb{C}, c \in \mathbb{C}^2\}$. Conversely, for $\lambda \notin \mathbb{Q}$, up to conjugacy with $\sigma(x, y) = (y, x)$, these are the only automorphisms of $\mathcal{F}_\lambda$ in $\mathbb{C}^2$: given $\varphi: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ preserving $\mathcal{F}_\lambda$ and the coordinate axis we may write $\varphi(x, y) = (xu, vu)$ for entire non-vanishing functions $u, v$ on $\mathbb{C}^2$. Then a straightforward calculation shows that $\varphi^*(\mathcal{F}_\lambda) = \mathcal{F}_\lambda$ iff $vu^{-\lambda} \in \mathcal{O}(\mathbb{C}^2)$ is an entire first integral for $\mathcal{F}_\lambda$ iff $vu^{-\lambda} \equiv \text{const}$. If we write $u = e^w$ then we obtain $\varphi(x, y) = (xe^w, cye^\lambda w)$ for $w \in \mathcal{O}(\mathbb{C}^2)$. Such a map is a foliation automorphism iff $w \in \mathbb{C}$ is constant.

Examples of transcendental foliation automorphisms are given by complete polynomial vector fields having non-polynomial flows (cf. [11]).

References


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Received 17 July 2003