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On transcendental automorphisms of algebraic foliations

by

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Abstract. We study the group $\mathrm{Aut}(\mathcal{F})$ of (self) isomorphisms of a holomorphic foliation $\mathcal{F}$ with singularities on a complex manifold. We prove, for instance, that for a polynomial foliation on $\mathbb{C}^2$ this group consists of algebraic elements provided that the line at infinity $\mathbb{C}P(2) \setminus \mathbb{C}^2$ is *not* invariant under the foliation. If in addition $\mathcal{F}$ is of *general type* (cf. [20]) then $\mathrm{Aut}(\mathcal{F})$ is finite. For a foliation with hyperbolic singularities at infinity, if there is a transcendental automorphism then the foliation is either linear logarithmic, Riccati or *chaotic* (cf. Definition 1). We also give a description of foliations admitting an invariant algebraic curve $C \subset \mathbb{C}^2$ with a transcendental foliation automorphism.

1. Introduction and main results. In this paper we study the group of (self) isomorphisms of a foliation. Given a codimension one holomorphic foliation $\mathcal{F}$ with singularities on a complex manifold M we denote by $\mathrm{Aut}(\mathcal{F})$ the maximal subgroup of $\mathrm{Aut}(M)$ whose elements preserve the foliation $\mathcal{F}$. This object has been studied in [20] where it is proven that $\mathrm{Aut}(\mathcal{F})$ is finite for $\mathcal{F}$ of general type on a (compact) projective surface. We recall (cf. [20]) that a foliation $\mathcal{F}$ on a projective surface M^2 is of *general type* if its Kodaira dimension (cf. [17]) is equal to 2. Our results extend in a way the result above to open algebraic surfaces. Indeed, we are concerned with the non-compact case and its applications to the classification of holomorphic flows on $\mathbb{C}^2$, for instance. Our first result is the following:

Proposition 1. *Let $\mathcal{F}$ be a foliation with hyperbolic singularities on $\mathbb{C}P(2)$. Suppose there exists some* affine *open subset $U \subset \mathbb{C}P(2)$ such that the restriction $\mathcal{F}|_U$ contains a holomorphic flow in its group of automorphisms. Then $\mathcal{F}$ is, in suitable affine coordinates, a linear hyperbolic foliation.*

By an *affine set* $U \subset \mathbb{C}P(2)$ we mean the complement of an algebraic curve $\Gamma \subset \mathbb{C}P(2)$. This preliminary result suggests that the complexity of the transverse structure and that of the tangent structure are, in a certain

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sense, in inverse proportion. This idea is enforced by our next results. The first is the following extension lemma:

LEMMA 1 (Extension Lemma). *An entire automorphism* $\varphi\colon \mathbb{C}^2 \to \mathbb{C}^2$ *of an algebraic foliation* $\mathcal{F}$ *on* $\mathbb{C}P(2)$ *is algebraic provided that the infinity* $\mathbb{P}^1_\infty = \mathbb{C}P(2) \setminus \mathbb{C}^2$ *is not invariant under* $\mathcal{F}$.

The above result actually holds for bimeromorphic maps $\varphi\colon \mathbb{C}^2 \setminus \Gamma \dashrightarrow \mathbb{C}^2 \setminus \Gamma$ if we replace $\mathbb{P}^1_\infty$ by an irreducible (non-invariant) algebraic curve $\Gamma \subset \mathbb{C}P(2)$, as will be clear from the proof we give. We point out that a foliation automorphism $\varphi \in \mathrm{Aut}(\mathbb{C}^2)$ may extend to a bimeromorphic non-linear map of $\mathbb{C}P(2)$ as is the case for $\mathcal{F} : dx = 0$ and $\varphi(x, y) = (x, y + x^n)$. Denote by $\mathrm{Bim}(\mathcal{F})$ the group of bimeromorphic maps $\Phi\colon M \dashrightarrow M$ which preserve $\mathcal{F}$ in M (these maps take leaves of $\mathcal{F}$ onto leaves of $\mathcal{F}$ wherever defined). Combining the extended Lemma 1 with Theorem 1 of [20] we immediately obtain:

COROLLARY 1. *Let* $\mathcal{F}$ *be a codimension one singular holomorphic foliation on* $\mathbb{C}P(2)$ *and* $\Gamma \subset \mathbb{C}P(2)$ *a non-invariant algebraic curve. Then we have natural group immersions* $\mathrm{Bim}(\mathcal{F}|_{\mathbb{C}P(2)\setminus\Gamma}) \subset \mathrm{Bim}(\mathcal{F}) \subset \mathrm{Bim}(\mathbb{C}P(2))$. *In particular*, $\mathrm{Bim}(\mathcal{F}|_{\mathbb{C}P(2)\setminus\Gamma})$ *is finite if* $\mathcal{F}$ *is generic.*

For a foliation $\mathcal{F}$ on a projective surface M admitting a minimal model $\mathcal{G}$ on $\widetilde{M}$ (cf. [6]), we have a natural group isomorphism $\mathrm{Bim}(\mathcal{F}) \simeq \mathrm{Aut}(\mathcal{G})$ [20]. Foliations without minimal model are listed as: rational fibrations, non-trivial Riccati foliations and the exceptional foliation $\mathcal{H}$ in [6] (see also Example 1.3 in [10]). In particular, if $\mathcal{F}$ is generic then it admits a minimal model ([20]). In [10] it is proved that for a foliation $\mathcal{F}$ on a projective surface if $\mathrm{Aut}(\mathcal{G}) \subsetneq \mathrm{Bim}(\mathcal{G})$ for every birational model $\mathcal{G}$ of $\mathcal{F}$ then we have the following possibilities: $\mathcal{F}$ is a rational fibration or $\mathcal{F}$ is the exceptional foliation $\mathcal{H}$ of [6], i.e., the foliation in Example 1.3 of [10]. Combining these results with Corollary 1 above we obtain:

COROLLARY 2. *Let* $\mathcal{F}$ *be a codimension one singular holomorphic foliation on* $\mathbb{C}P(2)$ *and* $\Gamma \subset \mathbb{C}P(2)$ *a non-invariant algebraic curve. Suppose that* $\mathrm{Bim}(\mathcal{F}|_{\mathbb{C}P(2)\setminus\Gamma})$ *is not finite. Then* $\mathcal{F}$ *is either a linear foliation*, *an elliptic fibration*, *a rational fibration*, *a Riccati foliation*, *or* (*birationally equivalent to*) *the exceptional foliation* $\mathcal{H}$ *of* [6].

Proof. If $\mathcal{F}$ has a minimal model then we may assume that $\mathcal{F}$ is minimal and therefore $\mathrm{Aut}(\mathcal{F}) = \mathrm{Bim}(\mathcal{F})$ is infinite. Theorem 1.1 of [10] then shows that $\mathcal{F}$ is either an elliptic fibration, a Riccati foliation or given by a global holomorphic vector field on $\mathbb{C}P(2)$, in which case $\mathcal{F}$ is necessarily linear. Now we may assume that $\mathcal{F}$ has no minimal model. Therefore, according to the above discussion $\mathcal{F}$ is either a non-trivial Riccati foliation, a rational fibration or the foliation $\mathcal{H}$ of [6] which is the foliation in Example 1.3 of [10]. ■

For the transcendental case we shall be more precise by introducing a dynamical concept:

DEFINITION 1 (chaotic foliation). We shall say that a foliation $\mathcal{F}$ on $\mathbb{C}P(2)$ which leaves invariant the line at infinity is *chaotic* if:

(ch-1) $\mathcal{F}|_{\mathbb{C}^2}$ has no parabolic leaf (in the sense of potential theory, cf. [4]);
(ch-2) the foliation $\mathcal{F}|_{\mathbb{C}^2}$ admits no (non-trivial) holonomy invariant measure, indeed the leaves of $\mathcal{F}|_{\mathbb{C}^2}$ have exponential growth for the Fubini–Study metric ([9]);
(ch-3) the holonomy group of the leaf $L_\infty = \mathbb{P}^1_\infty \setminus (\mathrm{sing}(\mathcal{F}) \cap \mathbb{P}^1_\infty)$ is non-solvable (see [19]);
(ch-4) all leaves of $\mathcal{F}$, except for L_∞, are dense in $\mathbb{C}P(2)$ ([15], [12]).
(ch-5) $\mathcal{F}|_{\mathbb{C}^2}$ (and therefore $\mathcal{F}$ on $\mathbb{C}P(2)$) is ergodic ([12]) and topologically rigid ([15], [23]).

Using this terminology we state our main results as follows:

THEOREM 1. *Let $\mathcal{F}$ be an algebraic foliation on* $\mathbb{C}P(2)$ *such that* $\mathrm{sing}(\mathcal{F}) \cap \mathbb{P}^1_\infty$ *is hyperbolic and* $\mathrm{Aut}(\mathcal{F}|_{\mathbb{C}^2})$ *contains a transcendental element* $\varphi\colon \mathbb{C}^2 \to \mathbb{C}^2$. *We have the following possibilities*: (i) $\mathcal{F}$ *is chaotic*; (ii) $\mathcal{F}$ *is a linear logarithmic foliation*; (iii) $\mathcal{F}$ *is a Riccati foliation.*

Examples of foliations of logarithmic and Riccati type with transcendental foliation automorphisms are given in §4. Also, consider the following construction: let

$$\varphi(x,y) = \left(x, \frac{y}{\alpha(x)y + \beta(x)}\right)$$

for entire functions $\alpha, \beta \in \mathcal{O}(\mathbb{C})$. We can obtain examples of φ-invariant Riccati equations of the form $\mathcal{F}\colon [(x-a_1)\ldots(x-a_r)]dy - [y^2A(x) + yB(x)]dx = 0$, where in the affine system $(x,y) \in \mathbb{C}^2 \subset \mathbb{C}P(2)$ we take the line at infinity to be $\mathbb{P}^1_\infty = \overline{(x=\infty)}$. It is then easy to obtain such examples with transcendental φ and $\mathrm{sing}(\mathcal{F}) \cap \mathbb{P}^1_\infty$ hyperbolic. A linear hyperbolic foliation $\mathcal{F} : xdy - \lambda ydx = 0$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, admits no transcendental foliation automorphisms (cf. Example 2).

As an immediate consequence of Theorem 1 we obtain:

COROLLARY 3. *Let $\mathcal{F}$ be a foliation on $\mathbb{C}P(2)$ with hyperbolic singularities in $\mathbb{P}^1_\infty$ and such that* $\mathrm{Aut}(\mathcal{F}|_{\mathbb{C}^2})$ *contains a transcendental element. Suppose $\mathcal{F}$ has no algebraic invariant curve $C \subset \mathbb{C}^2$. Then $\mathcal{F}$ is chaotic.*

In general, for a foliation admitting an algebraic invariant curve and with a transcendental automorphism we have:

THEOREM 2. *Let $\mathcal{F}$ be an algebraic foliation on $\mathbb{C}P(2)$ such that* $\mathrm{Aut}(\mathcal{F}|_{\mathbb{C}^2})$ *contains a transcendental element $\varphi\colon \mathbb{C}^2 \to \mathbb{C}^2$. Suppose that $\mathcal{F}$ has an invariant algebraic curve $C \subset \mathbb{C}^2$. Then, up to replacing φ by some power φ^n, $n \in \mathbb{N}$, we have (only) the following possibilities*:

(tr.1) *$\varphi(C)$ is not algebraic and $\mathcal{F}|_{\mathbb{C}^2}$ is holomorphically equivalent to $dy = 0$ in $\mathbb{C}^2$. Up to some entire automorphism taking $\mathcal{F}|_{\mathbb{C}^2}$ to $dy = 0$ we have $\varphi(x, y) = (a_0(y)x + a_1(y), b_0 + b_1 y)$ with $a_0(y), a_1(y) \in \mathcal{O}(\mathbb{C}^2)$, $b_0, b_1 \in \mathbb{C}$ and some natural restrictions on the coefficients. In particular C is isomorphic to the affine line, $C \simeq \mathbb{C}$.*

(tr.2) *There is $n \in \mathbb{N}$ such that $\varphi^n(C) = C$ in $\mathbb{C}^2$, and C is, up to a polynomial automorphism of $\mathbb{C}^2$, in the following list*:

- (a) $\alpha(x) + y\beta(x) = 0$, $\alpha(x), \beta(x) \in \mathbb{C}[x]$;
- (b) *a union of irreducible components of fibers of $x^n y^m$, $n, m \in \mathbb{N}$, $\langle n, m\rangle = 1$;*
- (c) *a union of irreducible components of fibers of a polynomial $R(x, y) = x^n(x^l y + P(x))^m$ with $n, m, l \in \mathbb{N}$, $\langle n, m\rangle = 1$, $P(x) \in \mathbb{C}[x]$ of degree at most $l - 1$, $P(0) \neq 0$.*

In particular, $\overline{C}$ is rational and $\overline{C} \cap \mathbb{P}^1_\infty \leq 2$, so that each irreducible component $C_0 \subset C$ satisfies $C_0 \simeq \mathbb{C}$ or $\mathbb{C} \setminus \{0\}$.

(tr.3) *$\varphi^n(C)$ is algebraic for some $n \in \mathbb{N}$ but $\varphi^m(C) \neq C$ for all $m \in \mathbb{N}$. In this case $\mathcal{F}$ admits a rational first integral $R\colon \mathbb{C}P(2) \dashrightarrow \mathbb{P}^1$; in particular all the leaves of $\mathcal{F}$ are algebraic.*

Examples are given and we believe that one may go further in the discussion of Case (tr.2) above.

2. The case of flows—Proof of Proposition 1

2.1. *Motivations and the compact case.* Let $\mathcal{F}$ be a codimension k foliation on a manifold M with singular set $\mathrm{sing}(\mathcal{F})$. The *group of foliation automorphisms* of $\mathcal{F}$ is the subgroup $\mathrm{Aut}(\mathcal{F})$ of $\mathrm{Aut}(M)$ whose elements $\psi\colon M \to M$ satisfy $\psi(\mathrm{sing}(\mathcal{F})) = \mathrm{sing}(\mathcal{F})$ and $\psi^*\mathcal{F} = \mathcal{F}$, in other words, ψ fixes the singular set and takes (non-singular) leaves of $\mathcal{F}$ onto (non-singular) leaves of $\mathcal{F}$. If M is (complex) compact then $\mathrm{Aut}(M)$ is a (complex) Lie group, and since $\mathrm{Aut}(\mathcal{F})$ is a closed subgroup of $\mathrm{Aut}(M)$, it is a (complex) Lie subgroup. Thus we may consider the following general situation: $\mathrm{Aut}(\mathcal{F})$ is a complex Lie group and therefore there exists a Lie group action $\varphi\colon G \times M \to M$ such that each automorphism φ_g, $g \in G$, belongs to $\mathrm{Aut}(\mathcal{F})$. We have two main (antipodal) cases: (i) φ is transverse to $\mathcal{F}$; (ii) φ is tangent to $\mathcal{F}$. In the rest of this section we shall consider the two main cases above in the compact two-dimensional situation for singular foliations.

1. Let $\mathcal{F}$ be a singular codimension one foliation on a complex surface M and assume there is a flow $\varphi\colon \mathbb{C}\times M \to M$ such that $\varphi_t \in \mathrm{Aut}(\mathcal{F})$ for all $t\in\mathbb{C}$. If φ is transverse to $\mathcal{F}$ then we are in the situation considered in [22]; in particular there exists a closed holomorphic 1-form Ω, non-singular and defining $\mathcal{F}$ on M, and we may describe the dynamics of $\mathcal{F}$ according to the group $\mathrm{Per}(\Omega)\subset(\mathbb{C},+)$ (cf. [22]). For instance, if M is compact Kaehler with rank $H_1(M;\mathbb{R})\le 2$ then $\mathcal{F}$ is a Seifert fibration (i.e., a foliation by compact leaves having finite holonomy groups).

2. Assume now that there is a flow φ which is tangent to $\mathcal{F}$; this implies that given any $p\in M$ the leaf L_p of $\mathcal{F}$ containing p admits a holomorphic map $\varphi_p\colon \mathbb{C}\to L_p$ and L_p is holomorphically equivalent to one of the following Riemann surfaces: $\mathbb{C}$, $\mathbb{C}^*$ or a torus $\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}$. Moreover, $X(p)=\frac{\partial\varphi_t}{\partial t}(p)|_{t=0}$ defines a global holomorphic vector field (possibly with singularities) on M, tangent to $\mathcal{F}$. If X is singular but $M=\mathbb{C}P(2)$ then X must be linear: $X=\lambda x\,\frac{\partial}{\partial x}+\mu y\,\frac{\partial}{\partial y}$ or $X=\lambda x\,\frac{\partial}{\partial x}+(x+\lambda y)\frac{\partial}{\partial y}$ in some affine chart $(x,y)\in\mathbb{C}^2$ and $\mathcal{F}$ is linear as well.

3. Now we consider the non-singular case. If M is compact and X is non-singular then the second Chern class of M is zero: $c_2(M)=0$, and we may use the Enriques–Kodaira classification [1] in order to describe M. Actually, in [18] it is proved that such a surface must be one of the following: (1) a flat holomorphic fibre bundle over an elliptic curve with connected fibres; (2) a complex torus; (3) an elliptic surface without singular fibres or with singular fibres of type ${}_m\mathrm{I}_0$ only; (4) a non-elliptic Hopf surface; (5) an Inoue surface of type $S^{(+)}_{N,p,q,r,t}$.

4. Finally, concerning the existence of compact leaves in the non-singular case we find that if L_0 is a toral leaf then we have two possibilities: (i) L_0 is an isolated toral leaf; (ii) L_0 is not an isolated toral leaf; in this case the holonomy group $\mathrm{Hol}(L_0)\hookrightarrow\mathrm{Diff}(\mathbb{C},0)$ is abelian with some finite pseudo-orbits so that it cannot contain non-trivial flat elements (cf. [7]) and therefore it is a finite group of rational rotations. Since $\mathrm{Hol}(L_0)$ is finite the global stability theorem in [2] implies that every leaf of $\mathcal{F}$ is compact with finite holonomy, therefore $\mathcal{F}$ is a Seifert fibration by complex tori.

2.2. *Proof of Proposition 1.* We shall keep the notation of the preceding subsection. The case where φ_t is completely transverse to $\mathcal{F}$ in $\mathbb{C}P(2)\setminus\Gamma$ follows from [22]. Indeed, as we have seen above the foliation in $U=\mathbb{C}P(2)\setminus\Gamma$ is given by a closed holomorphic one-form Ω. Since the singularities of $\mathcal{F}$ in Γ are hyperbolic, Ω extends meromorphically with simple poles along Γ to a neighborhood of each singular point $p\in\mathrm{sing}(\mathcal{F})\in\Gamma$ [21]. Thus we conclude by the Hartogs Theorem that Ω extends to a meromorphic one-form $\widetilde{\Omega}$ in $\mathbb{C}P(2)$. This one-form has simple poles in $\mathbb{C}P(2)$ and by the

Integration Lemma is logarithmic; by [3] the foliation must be linear in some affine chart due to the fact that its singularities are hyperbolic.

Assume now that the flow is everywhere tangent to $\mathcal{F}$ in U. This means that the leaves of $\mathcal{F}|_U$ are the nonsingular orbits of the vector field $X(p) = (\partial\varphi_t(p)/\partial t)|_{t=0}$. In particular $\mathcal{F}|_U$ and therefore $\mathcal{F}$ is parabolic; since $\mathcal{F}$ has only hyperbolic singularities in $\mathbb{C}P(2)$, it follows from [4] that it is linear in some affine chart in $\mathbb{C}^2$.

Finally, assume that φ_t is not completely transverse to $\mathcal{F}$ in $\mathbb{C}P(2) \setminus \Gamma$ but $\mathcal{F}$ is not the foliation induced by the vector field X as above. If L is a leaf of $\mathcal{F}$ in U which is also invariant under X then L is covered by $\mathbb{C}$. If L is non-algebraic then $\mathcal{F}$ has a transcendental leaf covered by $\mathbb{C}$ and according to [5] and [9] it follows that $\mathcal{F}$ admits a non-trivial holonomy invariant measure; moreover, [9], $\mathcal{F}$ is a linear hyperbolic foliation due to its hyperbolic singularities. If the closure $\overline{L} \subset \mathbb{C}P(2)$ is an algebraic curve, say Γ', then $U' = \mathbb{C}P(2) \setminus (\Gamma \cup \Gamma')$ is also affine. Due to the hyperbolic singularities $\mathcal{F}$ admits no rational first integral and therefore by a theorem of Darboux [13], $\mathcal{F}$ has only finitely many algebraic invariant curves. This reduces the proof to the two cases above. In other words, we may assume that either $\mathcal{F}$ is completely transverse to X or it coincides with the foliation of X in $\mathbb{C}P(2) \setminus \Gamma$. In either case the foliation is linear in some affine chart in $\mathbb{C}^2$. ∎

3. Entire automorphisms—Proof of Lemma 1 and Theorem 1. Let $\mathcal{F}$ be an algebraic foliation on $\mathbb{C}^2$. Then $\mathcal{F}$ admits an extension of algebraic type to a foliation (also denoted by) $\mathcal{F}$ on $\mathbb{C}P(2) = \mathbb{C}^2 \cup \mathbb{P}^1_\infty$. Let $\varphi\colon \mathbb{C}^2 \to \mathbb{C}^2$ be an entire (algebraic or transcendental) automorphism of $\mathbb{C}^2$ that *preserves* $\mathcal{F}$. *Under which conditions on* $\mathcal{F}$ *is* φ *algebraic?*

3.1. *Case of* $\mathbb{P}^1_\infty$ *is not* $\mathcal{F}$*-invariant—Proof of Extension Lemma 1.* Assume that $\mathbb{P}^1_\infty$ is not $\mathcal{F}$-invariant; the set of points $p \in \mathbb{P}^1_\infty$ such that $\mathcal{F}$ is not transverse to $\mathbb{P}^1_\infty$ in a neighborhood of p is a finite subset of $\mathbb{P}^1_\infty$, say $T(\mathcal{F}, \mathbb{P}^1_\infty) = \{p_1, \dots, p_r\}$. Let $p \in \mathbb{P}^1_\infty \setminus T(\mathcal{F}, \mathbb{P}^1_\infty)$ be given and choose a flow-box neighborhood U of p in $\mathbb{C}P(2)$ such that in local coordinates (x, y) in U we have $x(p) = y(p) = 0$, $\mathbb{P}^1_\infty \cap U = \{x = 0\}$ and $\mathcal{F}|_U : dy = 0$. Also we may assume that $\{y = 0\} \subset L_p$. Denote by L_p the leaf of $\mathcal{F}$ that contains p and by $L^*_p = L_p \setminus \mathbb{P}^1_\infty$ the corresponding leaf of $\mathcal{F}|_{\mathbb{C}^2}$. Since $\varphi\colon \mathbb{C}^2 \to \mathbb{C}^2$ is an automorphism that preserves $\mathcal{F}$ we have $L^*_p = \varphi(L^*)$, where $L^* = L \setminus \mathbb{P}^1_\infty \subset \mathbb{C}^2$ for some leaf L of $\mathcal{F}$ on $\mathbb{C}P(2)$. Now, using the flow-box in U we may choose a disk $\Sigma \subset U \cap \mathbb{C}^2$ transverse to $\mathcal{F}$ with the properties that each plaque of $\mathcal{F}|_U$ cuts Σ at most once and the same holds for the disk $\mathbb{P}^1_\infty \cap U =: \Sigma_\infty$.

Let $p' \in \Sigma \cap L_p$, $p' \in \{y=0\}$; choose a simple path $\alpha\colon [0,1] \to L_p \cap U$ such that $\alpha(t) \in \{y=0\}$ for all $t \in [0,1]$, $\alpha(0)=p'$ and $\alpha(1)=p$, and denote by $h_\alpha\colon (\Sigma, p') \to (\Sigma_\infty, p)$ the holonomy map corresponding to α. Let Σ_1 be the inverse image $\Sigma_1 = \varphi^{-1}(\Sigma) \subset \mathbb{C}^2$. Then Σ_1 is also a transverse disk to $\mathcal{F}$ in $\mathbb{C}^2$ and $\Sigma_1 \cap L \ni p_1' = \varphi^{-1}(p')$. Now we choose any simple path $\beta\colon [0,1] \to L$ with the following properties: $\beta(0) = p_1$ and $\beta(1) = q' \in \mathbb{C}^2$ is a point which belongs to a flow-box $V \subset \mathbb{C}P(2)$ centered at a point $q \in \mathbb{P}^1_\infty \cap L$.

Fig. 1

In particular we may assume that q' and q belong to the same plaque P of $\mathcal{F}|_V$, which is a disk $P \approx \mathbb{D}$. We fix a transverse disk $\Sigma_2 \subset V$ to $\mathcal{F}$ with $\Sigma_2 \cap P = q'$ and choose any simple path $\alpha_1\colon [0,1] \to P$ with $\alpha_1(0) = q'$, $\alpha_1(1) = q$.

Denote by $h_\beta\colon (\Sigma_1, p_1') \to (\Sigma_2, q')$ and $h_{\alpha_1}\colon (\Sigma_2, q') \to (\Sigma'_\infty, q)$ the holonomy diffeomorphisms corresponding to β and α_1 respectively, where $\Sigma'_\infty = \mathbb{P}^1_\infty \cap V$ is transverse to $\mathcal{F}$.

Fig. 2

Given a point $a \in \Sigma_\infty$ close enough to p, we define $\varphi_\beta^{-1}(a) := h_{\alpha_1} \circ h_\beta \circ h_\alpha^{-1}(a)$.

LEMMA 2. *If the holonomy of L is trivial then φ_β^{-1} does not depend on the choice of β.*

Proof. This is clear. ∎

Proof of Lemma 1. We keep the above notation. Since $\mathcal{F}|_U$ and $\mathcal{F}|_V$ are trivial it follows that for a leaf L with trivial holonomy as above φ_β^{-1} is an extension of φ^{-1} to $\Sigma_\infty = U \cap \mathbb{P}^1_\infty$ and by the Hartogs Extension Theorem φ^{-1} extends to a meromorphic map $\overline{\varphi}^{-1}\colon \mathbb{C}P(2) \dashrightarrow \mathbb{C}P(2)$. Since the same holds for φ and such extensions must clearly satisfy $\overline{\varphi} \circ \overline{\varphi}^{-1} = \mathrm{Id}_{\mathbb{C}P(2)}$ we conclude that φ extends to a bimeromorphism $\overline{\varphi}$ of $\mathbb{C}P(2)$; the bimeromorphism $\overline{\varphi}$ still preserves $\mathcal{F}$ on $\mathbb{C}P(2)$. Using now the fact that for $\mathcal{F}$ arbitrary (with $\mathbb{P}^1_\infty$ not invariant) the set of leaves with trivial holonomy is residual we obtain Lemma 1. ∎

3.2. *Proof of Theorem 1.* Let $\mathcal{F}$ be an algebraic foliation on $\mathbb{C}P(2)$ such that $\mathrm{sing}(\mathcal{F}) \cap \mathbb{P}^1_\infty$ is hyperbolic and $\mathrm{Aut}(\mathcal{F}|_{\mathbb{C}^2})$ contains a transcendental element $\varphi\colon \mathbb{C}^2 \to \mathbb{C}^2$. According to Lemma 1 the line at infinity is invariant. Denote by $L_\infty = \mathbb{P}^1_\infty \setminus (\mathbb{P}^1_\infty \cap \mathrm{sing}(\mathcal{F}))$ the corresponding leaf of $\mathcal{F}$. If the holonomy group of L_∞ is solvable then, according to [21] and [8], due to the fact that the singularities in $\mathrm{sing}(\mathcal{F}) \cap \mathbb{P}^1_\infty$ are hyperbolic, the foliation is either a Riccati foliation or a logarithmic foliation. Thus we may assume that $\mathrm{Hol}(\mathcal{F}, L_\infty)$ is a non-solvable group. Since by the Index Theorem $\mathrm{sing}(\mathcal{F}) \cap \mathbb{P}_\infty \neq \emptyset$, the group $\mathrm{Hol}(\mathcal{F}, L_\infty)$ contains hyperbolic elements. Using now [19], [15], [9], [12], [23] we shall prove that $\mathcal{F}$ is chaotic:

LEMMA 3. *In the situation above, $\mathcal{F}$ has no algebraic leaf $C \subset \mathbb{C}^2$.*

Proof. Assume that $\mathcal{F}$ has some algebraic invariant curve $C \subset \mathbb{C}^2$; denote by $L = C \setminus \mathrm{sing}(\mathcal{F})$ the corresponding leaf of $\mathcal{F}|_{\mathbb{C}^2}$ (we may work with C irreducible). The leaf L is then a parabolic leaf of $\mathcal{F}|_{\mathbb{C}^2}$; denote by $L' = \varphi(L) \subset \mathbb{C}^2$ its image under φ. If L' is not algebraic (i.e., the closure $\overline{L'} \subset \mathbb{C}^2$ is not an algebraic curve) then the intersection $L' \cap D$ accumulates to the origin $o = D \cap L_\infty$ of any disk D transverse to $\mathcal{F}$ and to L_∞. Since C is closed in $\mathbb{C}^2$ the same holds for L' and therefore it induces a pseudo-orbit in the holonomy group $\mathrm{Hol}(\mathcal{F}, l_\infty, D)$, which is closed off the origin $o \in D$. According to the density theorem in [19], $\mathrm{Hol}(\mathcal{F}, L_\infty, D)$ must be solvable, a contradiction. Thus necessarily L' is algebraic. If $\mathcal{F}$ has infinitely many algebraic leaves then by a theorem of Darboux [13], $\mathcal{F}$ is given by a rational map $R\colon \mathbb{C}P(2) \dashrightarrow \overline{\mathbb{C}}$; this is not compatible with the hyperbolic singularities. We may therefore assume that $\varphi(C) = C$. Now we recall the following notions from [14]:

Let $C \subset \mathbb{C}^2$ be an irreducible algebraic affine curve. We say that C is *general* if any analytic automorphism of $\mathbb{C}^2$ which transforms C into an algebraic curve is algebraic.

THEOREM 3 (Kizuka [14]). *Let $C \subset \mathbb{C}^2$ be an affine algebraic curve. Then C is general provided that the closure $\overline{C}$ of C on $\mathbb{C}P(2)$ intersects the line at infinity $\mathbb{P}^1_\infty$ at more than two points. If C is not general then we have the following possibilities for C after an algebraic affine change of coordinates*:

(K.1) $C : \alpha(x) + y\beta(x) = 0$ (*special case*);
(K.2) *C is a sum of several prime surfaces of a monomial $f = x^m y^n$ where $m, n \in \mathbb{N}$;*
(K.3) *C is a sum of several prime surfaces of a polynomial $f = x^m(x^l y + P_{l-1}(x))^n$, where P_{l-1} is a one-variable polynomial of degree at most $l-1$ with $P_{l-1}(0) \neq 0$, and $m, n \in \mathbb{N}$.*

In each case above straightforward calculations show that $\mathcal{F}$ has either some separatrix at infinity which is not smooth at the singularity or some separatrix which is not transverse to the line at infinity. This contradicts the invariance of $\mathbb{P}^1_\infty$ and the hyperbolicity of the singularities in $\mathbb{P}^1_\infty$. Lemma 3 is now proved. ∎

Now we may finish the proof of Theorem 1:

(i) $\mathcal{F}|_{\mathbb{C}^2}$ has no parabolic leaf: otherwise according to [16], [4], [21], $\mathcal{F}|_{\mathbb{C}^2}$ would have some algebraic leaf, contradicting Lemma 3.

(ii) By (i) the leaves of $\mathcal{F}|_{\mathbb{C}^2}$ are hyperbolic (in the sense of potential theory) and covered by the disk $\mathbb{D}$. Moreover, the leaves of $\mathcal{F}|_{\mathbb{C}^2}$ have exponential growth (otherwise, according to [9], $\mathcal{F}$ would admit some holonomy invariant measure for the holonomy group of $L_\infty = \mathbb{P}^1_\infty$ and this measure would imply that the holonomy group $\mathrm{Hol}(\mathcal{F}, L_\infty)$ is solvable, giving a contradiction).

(iii) Since the holonomy group of $\mathbb{P}^1_\infty$ is non-solvable and contains hyperbolic elements, the foliations $\mathcal{F}|_{\mathbb{C}^2}$ and $\mathcal{F}$ on $\mathbb{C}P(2)$ are topologically rigid and, as follows from the techniques in [15], we may also conclude that $\mathcal{F}|_{\mathbb{C}^2}$ and $\mathcal{F}$ on $\mathbb{C}P(2)$ are ergodic.

(iv) The density of the leaves in $\mathbb{C}^2$ is proved as in [15] as a consequence of the density results in [19] and [23] for non-solvable groups of germs of one-dimensional complex diffeomorphisms at the origin. Theorem 1 is therefore proved. ∎

4. Case of an algebraic invariant curve in $\mathbb{C}^2$—Theorem 2. Let $\mathcal{F}$ be an algebraic foliation on $\mathbb{C}P(2)$ such that $\mathrm{Aut}(\mathcal{F}|_{\mathbb{C}^2})$ contains a transcendental element $\varphi\colon \mathbb{C}^2 \to \mathbb{C}^2$ and having an invariant algebraic curve

$C \subset \mathbb{C}^2$. As in the proof of Theorem 1 denote by $L = C \setminus \operatorname{sing}(\mathcal{F})$ the corresponding leaf of $\mathcal{F}|_{\mathbb{C}^2}$ and by $L' = \varphi(L) \subset \mathbb{C}^2$ its image under φ.

4.1. If L' is not algebraic (i.e., the closure $\overline{L'} \subset \mathbb{C}^2$ is not an algebraic curve) then by the Darboux Theorem [13], $\mathcal{F}$ has only a finite number of algebraic leaves; we may therefore assume that L' is a transcendental leaf of finite type and properly embedded so that, by the Corollary in [3], necessarily L' (and therefore L) is isomorphic to $\mathbb{C}$ and also $\mathcal{F}|_{\mathbb{C}^2}$ is holomorphically equivalent to the foliation $dy = 0$ on $\mathbb{C}^2$. In this case, up to some entire automorphism, we have $\varphi(x, y) = (a(x, y), b(y))$ and therefore $\varphi(x, y) = (a_0(y)x + a_1(y), b_0 + b_1 y)$ for some $b_0, b_1 \in \mathbb{C}$, $b_1 \neq 0$, $a_0(y), a_1(y) \in \mathcal{O}(\mathbb{C})$, with $a_0(y) \neq 0$ for all $y \in \mathbb{C}$. Again we may find holomorphic coordinates on $\mathbb{C}^2$ such that $\mathcal{F}|_{\mathbb{C}^2} : dy = 0$ and $\varphi(x, y) = (e^{u(y)}x + v(y), y)$ for some holomorphic $u, v \in \mathcal{O}(\mathbb{C})$.

Example 1. Let us see an example of the above situation: $X(x, y) = (1 + xy)\,\frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y}$ generates a foliation $\mathcal{F}$ on $\mathbb{C}^2$ with the entire first integral $f(x, y) = ye^{xy}$. The algebraic curve $C : \{y = 0\}$ is $\mathcal{F}$-invariant and we have an entire non-algebraic automorphism $\varphi(x, y) = (xe^{P(x,y)}, ye^{xy})$ for some polynomial $P(x, y)$ (see [24]) and clearly φ preserves $\mathcal{F}$. The vector field X is not complete.

4.2. Suppose now C is $\mathcal{F}$-invariant, algebraic and fixed by φ. According to Kizuka's Theorem above [14], φ is algebraic or, up to polynomial automorphism, C belongs to the following list:

(K.1) $\alpha(x) + y\beta(x) = 0$, where α and β are one-variable polynomials;

(K.2) a union of irreducible components of fibers of $x^n y^m$, where $n, m \in \mathbb{N}$ satisfy $\langle n, m\rangle = 1$;

(K.3) a union of irreducible components of fibers of a polynomial Q of type $Q(x, y) = x^n(x^l y + P(x))^m$, with $n, m, l \in \mathbb{N}$, $\langle n, m\rangle = 1$, P is a one-variable polynomial of degree at most $l - 1$, $P(0) \neq 0$.

In any of the cases (K.1), (K.2) and (K.3) above the irreducible components of the projective curve $\overline{C} \subset \mathbb{C}P(2)$ are rational and we have $\sharp\overline{C} \cap \mathbb{P}^1_\infty \leq 2$; in particular, each irreducible component of C is diffeomorphic either to $\mathbb{C}$ or to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Examples of this situation are given by vector fields of the form

$$X = x(\mu + mC_1)\,\frac{\partial}{\partial x} - \left\{\frac{(x^l y + P(x))(\lambda + nC_1)}{x^l} + \frac{(lx^l y + xP'(x))(\mu + mC_1)}{x}\right\}\frac{\partial}{\partial y}$$

with $C_1 = x^n(x^l y + P(x))^m \cdot \tau'(x^n(x^l y + P(x))^m)$. These are complete vector fields (see [11]) whose flow maps $\varphi_t \colon \mathbb{C}^2 \to \mathbb{C}^2$ are not algebraic and preserve both the foliation $\mathcal{F}_X|_{\mathbb{C}^2}$ and the algebraic invariant curve $C : \{x^n(x^l y + P(x))^m = 0\}$.

4.3. Assume that $\mathcal{F}|_{\mathbb{C}^2}$ has some irreducible algebraic invariant curve C whose image $\varphi(C)$ is algebraic but is not C. Denote by $C_{\text{alg}}(\mathcal{F})$ the union of all irreducible algebraic curves $C_1 \subset \mathbb{C}^2$ which are invariant under $\mathcal{F}$. We may assume that for any irreducible curve $C_1 \in C_{\text{alg}}(\mathcal{F})$ we have $\varphi(C_1) \in C_{\text{alg}}(\mathcal{F})$ but $\varphi(C_1) \neq C_1$. Then in particular we have $\varphi^n(C) \neq \varphi^m(C)$ for all $n \neq m$ and thus $\sharp C_{\text{alg}}(\mathcal{F}) = +\infty$. By the Darboux Theorem [13], $\mathcal{F}$ admits a rational first integral $R\colon \mathbb{C}^2 \dashrightarrow \overline{\mathbb{C}}$ and therefore $\mathcal{F}$ is a pencil of curves $\mathcal{F}\colon \{\lambda P + \mu Q = 0\}$, $(\lambda;\mu) \in \mathbb{P}^1$, for some $P, Q \in \mathbb{C}[x,y]$.

It remains therefore to consider the following case:

4.4. $\mathcal{F}|_{\mathbb{C}^2}$ has no algebraic leaf, and $\mathbb{P}^1_\infty$ is $\mathcal{F}$-invariant. This case contradicts the hypothesis of existence of the invariant curve C. Theorem 2 is therefore proved. ∎

Example 2. The linear foliation $\mathcal{F}_\lambda$ given on $\mathbb{C}^2 \subset \mathbb{C}P(2)$ by $\mathcal{F}_\lambda : xdy - \lambda ydx = 0$, $\lambda \in \mathbb{C} \setminus \{0\}$, admits the following subgroup of automorphisms: $\{\varphi\colon \mathbb{C}^2 \to \mathbb{C}^2;\ \varphi(x,y) = (xe^w, cye^{\lambda w}),\ \text{where } w \in \mathbb{C},\ c \in \mathbb{C}^2\}$. Conversely, for $\lambda \notin \mathbb{Q}$, up to conjugacy with $\sigma(x,y) = (y,x)$, these are the only automorphisms of $\mathcal{F}_\lambda$ in $\mathbb{C}^2$: given $\varphi\colon \mathbb{C}^2 \to \mathbb{C}^2$ preserving $\mathcal{F}_\lambda$ and the coordinate axis we may write $\varphi(x,y) = (x.u, y.v)$ for entire non-vanishing functions u, v on $\mathbb{C}^2$. Then a straightforward calculation shows that $\varphi^*(\mathcal{F}_\lambda) = \mathcal{F}_\lambda$ iff $vu^{-\lambda} \in \mathcal{O}(\mathbb{C}^2)$ is an entire first integral for $\mathcal{F}_\lambda$ iff $vu^{-\lambda} \equiv \text{const}$. If we write $u = e^w$ then we obtain $\varphi(x,y) = (x.e^w, c.ye^{\lambda w})$ for $w \in \mathcal{O}(\mathbb{C}^2)$. Such a map is a foliation automorphism iff $w \in \mathbb{C}$ is constant.

Examples of transcendental foliation automorphisms are given by complete polynomial vector fields having non-polynomial flows (cf. [11]).

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