FUNDAMENTA
MATHEMATICAE
205 (2009)

Minimality of non-σ-scattered orders

by

Tetsuya Ishiu (Oxford, OH) and **Justin Tatch Moore** (Ithaca, NY)

Abstract. We will characterize—under appropriate axiomatic assumptions—when a linear order is minimal with respect to not being a countable union of scattered suborders. We show that, assuming PFA^+, the only linear orders which are minimal with respect to not being σ-scattered are either Countryman types or real types. We also outline a plausible approach to demonstrating the relative consistency of: *There are no minimal non-σ-scattered linear orders*. In the process of establishing these results, we will prove combinatorial characterizations of when a given linear order is σ-scattered and when it contains either a real or Aronszajn type.

1. Introduction. Recall that a linear order is *scattered* if it does not contain a copy of the rational line, and is *σ-scattered* if it is a countable union of scattered suborders. Laver proved the following structure theorem for σ-scattered orders.

Theorem 1.1 ([7]). *If L_i $(i < \omega)$ is a sequence of σ-scattered linear orders, then there are $i < j$ such that L_i embeds into L_j.*

We will be interested in the extent to which this theorem is sharp. Let us begin with a few observations. Hausdorff demonstrated that every uncountable scattered order contains a copy of either ω_1 or $-\omega_1$. Hence no σ-scattered order contains a *real type*—an uncountable linear order which is isomorphic to a suborder of the real line. Since an *Aronszajn type* is by definition an uncountable linear order which does not contain an uncountable scattered or real type, it follows from the definitions that no σ-scattered linear order contains an Aronszajn type. We will occasionally make mention of *Countryman types*; for our discussion it is sufficient to know that every Countryman type is Aronszajn.

Around the time of [7], Baumgartner established the following result which shows in particular that Laver's Theorem is consistently not sharp.

2000 *Mathematics Subject Classification*: 03E05, 03E75, 06A05.
Key words and phrases: Aronszajn line, Specker type, real type, σ-scattered, Proper Forcing Axiom, minimal type.

DOI: 10.4064/fm205-1-2

Theorem 1.2 ([1]). (PFA) *Any two $\aleph_1$-dense sets of reals are isomorphic. In particular, every set of reals of size $\aleph_1$ is minimal with respect to not being σ-scattered.*

Here *minimal* refers to the quasi-order of embeddability. We also have the following theorem of Todorcevic.

Theorem 1.3 ([12]). (PFA [1]) *There is a minimal Aronszajn type. In fact, every Countryman type is minimal.*

On the other hand, Dushnik and Miller proved the following classical result which suggests that CH might be relevant in obtaining a model of set theory in which Laver's Theorem is sharp.

Theorem 1.4 ([3]). *There is no minimal separable linear order of cardinality continuum. In particular, if* CH *is true, then there are no minimal real types.*

Recently, the second author proved the following result.

Theorem 1.5 ([10]). *It is consistent with* CH *that there are no minimal Aronszajn types.*

Hence it is consistent that the only minimal uncountable linear orders are ω_1 and $-\omega_1$. In this model, the only linear orders (if there are any at all) which are minimal with respect to not being σ-scattered do not contain any real or Aronszajn types.

The reader might now be wondering whether there are *any* non-σ-scattered orders which do not contain real or Aronszajn types. Baumgartner showed that this is indeed the case.

Theorem 1.6 ([2]). *There is a linear order which is not σ-scattered and yet has the property that all of its uncountable suborders contain a copy of ω_1.*

In fact, Baumgartner's example can be described as follows. Suppose that $\Xi \subseteq \omega_1$ is a stationary set of limit ordinals. If f_ξ ($\xi \in \Xi$) is a sequence such that f_ξ is a strictly increasing function from ω into ξ with cofinal range, then the lexicographic ordering on this sequence satisfies the conclusion of Theorem 1.6. Moreover, if $\Xi' \subseteq \Xi$ differ by a stationary set, then $\{f_\xi : \xi \in \Xi\}$ does not embed into $\{f_\xi : \xi \in \Xi'\}$. This motivated Galvin to ask the following question.

Question 1.7 ([2, Problem 4]). *Is there a linear order which is minimal with respect to not being σ-scattered and which has the property that all of its uncountable suborders contain a copy of ω_1?*

We will provide a consistent negative answer to Question 1.7.

---

[1] Actually MA_{ω_1} suffices.

THEOREM 1.8. (PFA$^+$) *If L is a minimal non-σ-scattered linear order, then L is either a real or Countryman type.*

In fact, the witnesses to non-minimality in Theorem 1.8 exist for essentially the same reasons as in Baumgartner's example. By the following theorem, it is sufficient to conclude in Theorem 1.8 that L contains either a real or Aronszajn type.

THEOREM 1.9 ([9]). *Every Aronszajn type contains a Countryman type.*

Theorems 1.2 and 1.3 imply that the conclusion in Theorem 1.8 is sharp in the presence of the full strength of PFA. We will go to extra lengths, however, to work within the class of *completely proper forcings* for which it is unknown whether the corresponding forcing axiom is consistent with CH. In particular, we will establish the following theorem.

THEOREM 1.10. (CPFA$^+$) *If L is a minimal non-σ-scattered linear order, then L is either a real or Aronszajn type.*

The reason is that if CPFA$^+$ is relatively consistent with CH—something which is plausible at present—then one would obtain a model in which Laver's Theorem is sharp.

The paper is organized as follows. In Section 2 we define an invariant of linear orders and prove a characterization of the σ-scattered orders in terms of this invariant. In Section 3 we isolate a property of Baumgartner's example which is useful in proving its non-minimality. In Section 4 we prove a combinatorial characterization of when a given linear order contains a real or Aronszajn type. Section 5 contains the metamathematical analysis and in particular the proof of Theorems 1.8 and 1.10. Information on complete properness (which we will not define) can be found in [10].

The notation and terminology in this paper is fairly standard. We will use [4] and [6] as general references for set theory and [13] as a reference for linear orders. Proper forcing is covered in [4, §31] and in greater detail in [11].

The main prerequisite for this paper is a proficiency in stationary sets and countable elementary submodels. We will now review some of the essentials. If θ is a regular uncountable cardinal, then we will use $H(\theta)$ to denote the collection of sets whose transitive closure has cardinality less than θ. We will also use $H(\theta)$ to denote the structure $(H(\theta), \in)$. It will be convenient to adopt the convention that θ always denotes a regular uncountable cardinal.

FACT 1.11. *$H(\theta)$ satisfies all of the axioms of* ZFC *except the power-set axiom.*

FACT 1.12. *If X is in $H(\theta^+)$, then $\mathscr{P}(X)$ and $H(\theta)$ are elements of $H(2^{\theta^+})$.*

If X is a set, we will let θ_X denote the least regular cardinal θ such that all finite iterates of the power set operation applied to the transitive closure of X are in $H(\theta_X)$. Similarly, $\mathscr{E}(X)$ will be used to denote the collection of all countable elementary submodels of $H(\theta_X)$ which contain X as an element.

If X is an uncountable set, then $[X]^\omega$ will be used to denote the collection of all countable subsets of X. This collection is equipped with a topology—the *Ellentuck topology*—generated by the basic open sets

$$[x, M] = \{N \in [X]^\omega : x \subseteq N \subseteq M\}$$

where M is in $[X]^\omega$ and x is a finite subset of M. A subset of $[X]^\omega$ which is closed in this topology and $\subseteq$-cofinal is said to be a *club*. A subset of $[X]^\omega$ which intersects every club is said to be *stationary*.

FACT 1.13. *If $E \subseteq [X]^\omega$ is club, then there is a function $f : X^{<\omega} \to X$ such that if Z is in $[X]^\omega$ and $f''Z^{<\omega} \subseteq Z$, then Z is in E. Moreover, the collection of those countable Z which are closed under f is a club.*

FACT 1.14. *If Y is a countable subset of $H(\theta_X)$, then the set of all $M \cap X$ such that M is in $\mathscr{E}(L)$ with $Y \subseteq M$ is a club in $[X]^\omega$.*

FACT 1.15. *If M is a countable elementary submodel of some $H(\theta)$ and X is in M, then X is countable iff $X \subseteq M$.*

FACT 1.16. *If $S \subseteq [X]^\omega$ and M is a countable elementary submodel of some $H(\theta)$ such that S is in M and $M \cap X$ is in S, then S is stationary. Equivalently, $M \cap X$ is in every club in $[X]^\omega$ which is in M.*

FACT 1.17. *If M is an elementary submodel of $H(\theta)$ and X is an element of $H(\theta)$ which is definable from parameters in M, then X is in M.*

FACT 1.18 (Pressing Down Lemma). *If $S \subseteq [X]^\omega$ is stationary and $r : S \to X$ satisfies $r(Z) \in Z$ for all Z in S, then there is an x in X such that $r^{-1}(x)$ is stationary.*

2. The invariant $\Omega(L)$. If L is a linear order, then we will let $\widehat{L}$ denote its completion. While we will not require a rigorous definition of $\widehat{L}$, we will define it formally as follows: z is in $\widehat{L}$ iff z is in L or z is an initial segment of L such that $\sup z$ is not in L. Hence $\widehat{L}$ has a first and last endpoint regardless of whether L does. Intervals in L will also be construed as intervals in $\widehat{L}$ and vice versa.

Our goal of this section is to define a quantity $\Omega(L)$ for each linear order L which provides a measure of how close the linear order is to being σ-scattered. Informally, $\Omega(L)$ is the set of all countable subsets Z of $\widehat{L}$ which contain all the cuts of $Z \cap L$ induced by an element of L. The following definitions allow this to be made more precise.

Definition 2.1. If Z is a subset of a linear order L, then define the equivalence relation $\sim_Z$ on $\widehat{L}$ by $x \sim_Z y$ iff the set of z in Z which are between x and y is finite. If there is a need to clarify which linear order L is used in this definition, then we will write $\sim_Z^L$. It will also be convenient to let $\sim_M$ denote $\sim_{M \cap L}$ if M is an arbitrary set. Note in particular that, if L is clear from the context and $Z \subseteq \widehat{L}$, then the default meaning of $\sim_Z$ is $\sim_{Z \cap L}$ and not $\sim_Z^{\widehat{L}}$.

Definition 2.2. If Z is a subset of $\widehat{L}$ and x is an element of $\widehat{L}$, then we say that Z *captures* x if there is a z in Z such that $z \sim_Z x$.

The following observations are useful.

Fact 2.3. *Suppose L is a linear order and M is an elementary submodel of $H(\theta)$ for some θ and $\widehat{L}$ is in M. If x is in $\widehat{L} \setminus M$, then M captures x iff there is a unique z in $M \cap \widehat{L}$ such that no element of $M \cap L$ is between x and z.*

Proof. Let M and x be given as in the statement of the fact. By assumption there is a z in Z such that there are only finitely many elements of $L \cap M$ between x and z. By replacing z if necessary, we may assume that there are no elements of $L \cap M$ between x and z. Now suppose for contradiction that $z_0 < z_1$ are two such elements of $\widehat{L} \cap M$. Since x is not in M, neither z_0 nor z_1 are equal to x. If $z_0 < z_1 < x$, then by elementarity of M and the fact that L is dense in $\widehat{L}$, there is a y in $L \cap M \cap (z_0, x)$, contradicting our choice of z_0. One similarly obtains a contradiction if either $x < z_0 < z_1$ or $z_0 < x < z_1$. ■

Fact 2.4. *A linear order L contains a real type iff there is a countable Z such that uncountably many $\sim_Z$-equivalence classes intersect L.*

Proof. If $X \subseteq L$ is uncountable and contains a countable dense set D, then different elements of X are $\sim_D$-inequivalent. If Z is countable and there are uncountably many $\sim_Z$-equivalence classes which intersect L, let $D = Z \cap L$ and let $X \subseteq L$ contain a unique element of each $\sim_Z$-equivalence class which intersects L. If $a < b$ are in X and $(a, b) \cap X$ contains some c, then there is a d in D which is strictly less than b but not strictly less than c. It follows that d is in $D \cap (a, b)$ and hence X is a real type as witnessed by the countable dense set D. ■

Definition 2.5. Define $\Omega(L)$ to be the set of all countable subsets of $\widehat{L}$ which capture every element of L.

We will need the following ordering on families of countable sets.

Definition 2.6. Suppose that A and B are collections of countable sets with $X = \bigcup A$ and $Y = \bigcup B$. Define $A \leq B$ iff there is an injection

$\iota : X \to Y$ such that for a closed and unbounded set of M in $[Y]^\omega$, if M is in B, then $\iota^{-1}M$ is in A. If $A \le B$ and $B \le A$, then we write $A \equiv B$.

We leave the following proposition for the reader to verify.

PROPOSITION 2.7. *If $A \equiv B$, then $|\bigcup A| = |\bigcup B|$ and any witnessing bijection sends A to a set which differs from B on a non-stationary subset of $[Y]^\omega$.*

The following proposition shows that, after identifying $\equiv$-equivalent sets, $\Omega(L)$ is an invariant of linear orders.

PROPOSITION 2.8. *If L_0 and L_1 are linear orderings and L_0 embeds into L_1, then $\Omega(L_0) \le \Omega(L_1)$.*

Proof. Without loss of generality, $L_0 \subseteq L_1$ and the embedding is the inclusion map. By Fact 1.14, it is sufficient to show that whenever M is in $\mathscr{E}(L_1)$ and $M \cap \widehat{L}_1$ is in $\Omega(L_1)$, $M \cap \widehat{L}_0$ is in $\Omega(L_0)$. To see this, let M be given such that $M \cap \widehat{L}_1$ is in $\Omega(L_1)$. Suppose that x is in L_0. By assumption, there is a y in $M \cap \widehat{L}_1$ such that $x \sim_M y$. Assume for simplicity that $x < y$. By elementarity of M, $y' = \sup\{x' \in L_0 : x' < y\}$ is in $\widehat{L}_0 \cap M$. Since $x \le y' \le y$ and $L_1 \cap (x, y) \cap M$ is finite, $L_0 \cap (x, y') \cap M$ is finite. ∎

While we will prove a stronger result later, let us note the following proposition now.

PROPOSITION 2.9. *If L contains either a real or Aronszajn suborder, then $\Omega(L)$ is non-stationary.*

This is a consequence of Fact 2.4 and the following claim.

CLAIM 2.10. *If $X \subseteq L$ is an Aronszajn suborder and M is in $\mathscr{E}(L)$ with X in M, then $\widehat{L} \cap M$ does not capture any element of X not in the closure of $X \cap M$ in the order topology on L.*

Proof. Let x be an element of X which is not in the closure of $X \cap M$. Suppose for contradiction that there is a z in $M \cap \widehat{L}$ such that $z \sim_M x$. By Fact 2.3, we may assume that there is no element of $L \cap M$ between x and z. Since the argument is similar in the other case, we may assume that $x < z$. By Fact 1.17, $A = \{y \in X : y < z\}$ is in M. If A' is a countable subset of A in M, then $A' \subseteq M$ by Fact 1.15. By our choice of z, every element of A' is less than x. Since x is not in the closure of A', $\sup(A') < x < z$. It follows that A has uncountable cofinality and therefore X contains a copy of ω_1, a contradiction. ∎

We will now prove the main theorem in this section.

THEOREM 2.11. *For every linear order L the following are equivalent:*

(1) *L is σ-scattered.*
(2) *$\Omega(L)$ contains a club.*

Proof. We will first prove that (1) implies (2). Let L be a σ-scattered linear order and let M be in $\mathscr{E}(L)$. It suffices to show that $Z = M \cap \widehat{L}$ is in $\Omega(L)$. To this end, suppose that x is an element of L and let $S \subseteq L$ be a scattered suborder in M such that x is in S. If x is in M, then there is nothing to show. Observe that if $\{z \in S \cap M : z < x\}$ is empty, then $\inf(S)$ is in M by Fact 1.17. Since $\inf(S) \sim_M x$, M captures x. We are similarly finished unless $\{z \in S \cap M : z < x\}$ and $\{z \in S \cap M : x < z\}$ are non-empty and have no greatest/least elements, respectively.

Let $\equiv_\alpha$ be the equivalence relation on L recursively defined as follows. Start by letting $\equiv_0$ be the equality relation on L. If $\equiv_\alpha$ has been defined for all $\alpha < \beta$, then $x \equiv_\beta y$ iff there is an $\alpha < \beta$ for which there are only finitely many $\equiv_\alpha$-equivalence classes between x and y which contain an element of S. Notice that for each β, a $\equiv_\beta$-equivalence class is an interval in L. Furthermore, since S is scattered, there is a β such that all elements of L are $\equiv_\beta$-equivalent—the existence of such a β is equivalent to S being scattered.

Let β be minimal such that there are $a < x < b$ with a and b in $L \cap M$ and $a \equiv_\beta b$. It follows from the definition of $\equiv_\beta$ that $\beta = \alpha + 1$ for some α. Notice that α is in M since it is definable from S, L, a, and b. By definition of $\equiv_\beta$, there are only a finite number of $\equiv_\alpha$-equivalence classes between $[a]_\alpha$ and $[b]_\alpha$. By revising our choice of a and b if necessary, we may assume that $[a]_\alpha$ and $[b]_\alpha$ are adjacent. Let $z \in \widehat{L}$ be the common boundary point of these two intervals. By Fact 1.17, z is in M. Notice that if there were an element y of $L \cap M$ between z and x, then either $a < x < y < z$ and $a \equiv_\alpha y$, or $z < y < x < b$ and $y \equiv_\alpha b$. In either case this would contradict our assumption that β was minimal.

Now we will prove that (2) implies (1) by induction on the cardinality of L. Observe that if L is countable, then it is trivially σ-scattered and there is nothing to prove. Now suppose that $|L| = \kappa$ is uncountable, and applying Fact 1.13, let $f : \widehat{L}^{<\omega} \to \widehat{L}$ be such that if Z is a countable subset of $\widehat{L}$ which is closed under f, then Z is in $\Omega(L)$. Let M_ξ $(\xi < \kappa)$ be a continuous $\in$-chain of elementary submodels of $H(\theta_L)$ such that for all $\xi < \kappa$, L and f are in M_ξ, $\xi \subseteq M_\xi$, and $|M_\xi| = |\xi| + \aleph_0$.

Claim 2.12. *For every $\xi < \kappa$ and x in L there is a z in $M_\xi \cap \widehat{L}$ such that $z \sim_{M_\xi} x$.*

Proof. For countable ξ, this follows from the fact that since $\Omega(L)$ contains a club and is in M_ξ, $M_\xi \cap \widehat{L}$ is in $\Omega(L)$. Now suppose that ξ is uncountable. By elementarity of M_ξ, $Z = \widehat{L} \cap M_\xi$ is closed under f. Now suppose for contradiction that there is an x in L such that there is no z in Z with $z \sim_{M_\xi} x$. Define a function $g : Z \to Z \cap L$ so that if z is in Z, then $g(z)$ is strictly between z and $g(z)$. It follows that any subset of Z which is closed under g is not in $\Omega(L)$. This is a contradiction, however, since there are countable subsets of Z which are closed under both f and g. ∎

If $\xi < \kappa$, define $\widehat{L}_\xi$ to be those elements of z of $\widehat{L} \cap M_\xi$ such that for some x in L, there are no elements of $L \cap M_\xi$ between x and z. Let y_ξ $(\xi < \kappa)$ enumerate $\bigcup_{\xi<\kappa} \widehat{L}_\xi$ without repetition in such a way that if $\xi < \xi'$, then elements of $\widehat{L}_\xi$ are indexed before elements of $\widehat{L}_{\xi'} \setminus \widehat{L}_\xi$. If y is in L, define $f_y : \kappa \to \bigcup_{\xi<\kappa} \widehat{L}_\xi$ by setting $f_y(\xi) = y$ if y is in $\widehat{L}_\xi$ and otherwise setting $f_y(\xi) = z$ where z is the unique element of $\widehat{L}_\xi$ such that there is no element of $M_\xi \cap L$ which is between z and y; such an element exists by Fact 2.3.

Claim 2.13. *The mapping $y \mapsto f_y$ is an order-preserving function where $\{f_y : y \in L\}$ is given the lexicographic order.*

Proof. First observe that $f_y(\xi) = y$ whenever y is in $\widehat{L}_\xi$ and hence the mapping is an injection. Suppose that $y < y'$ are elements of Y and let ζ be the least ordinal such that $f_y(\zeta) \neq f_{y'}(\zeta)$. We know that there are no elements of $M_\zeta \cap L$ which are between $f_y(\zeta)$ and y or between $f_{y'}(\zeta)$ and y'. It follows that there is a z in $M_\zeta \cap L$ such that $f_y(\zeta) < z < f_{y'}(\zeta)$. ∎

Claim 2.14. *For every y in L and limit ordinal $\delta < \kappa$, there is a $\delta_0 < \delta$ such that f_y is constant on the interval $(\delta_0, \delta]$.*

Proof. Let $h : \kappa \to \kappa$ be such that $h(\xi)$ is the index of $f_y(\xi)$. It follows immediately from the definitions that h is non-decreasing. Hence it is sufficient to show that h has finite range. Suppose for contradiction that there is a least ordinal ν such that h takes infinitely many values on ν. Let ζ be the supremum of $h(\xi)$ as ξ ranges over ν. Let z in $\widehat{L}_\nu$ be such that no element of $M_\nu \cap L$ is between z and y. Let η be the index of such a z. By continuity of M_ξ $(\xi < \kappa)$, there is a $\nu_0 < \nu$ such that y_η is in $\widehat{L}_{\nu_0}$. But then h takes the constant value η on the interval $[\nu_0, \nu]$ and, by assumption that ν was minimal, h has finite range on $[0, \nu_0)$ and hence on all of $[0, \nu)$, a contradiction. ∎

Claim 2.15. *For all $\xi < \kappa$, $\widehat{L}_\xi$ is σ-scattered.*

Proof. By Proposition 2.8, $\Omega(L')$ contains a club whenever L' is a suborder of L. Hence, by our inductive assumption, every suborder of L of cardinality less than κ is σ-scattered. Define $g : \widehat{L}_\xi \to L$ so that $g(z) = z$ if z is in L and $g(z) = x$ if x is some element of $L \setminus M_\xi$ such that z is the unique element of $\widehat{L}_\xi$ with no element of $M_\xi \cap L$ between x and z. Notice that g is an order-preserving map from $\widehat{L}_\xi$ into L whose range has cardinality less than κ. Hence $\widehat{L}_\xi$ is σ-scattered. ∎

If y is in L, then there is a unique strictly increasing sequence $\xi_y(i)$ ($i < k$) such that $\xi_y(0) = 0$ and $\xi_y(i+1)$ is the least ordinal greater than $\xi_y(i)$ such that $f_y(\xi_y(i+1)) \neq f_y(\xi_y(i))$. Let σ_y be the sequence of length k such that

$$\sigma_y(i+1) = \begin{cases} +1 & \text{if } f_y(\xi_y(i)) < f_y(\xi_y(i+1)), \\ -1 & \text{if } f_y(\xi_y(i)) > f_y(\xi_y(i+1)), \end{cases}$$

and $\sigma_y(0) = 0$. It is easily verified that the map sending y to the sequence

$$f_y(\xi_y(0)), \sigma_y(1) \cdot \xi_y(1), f_y(\xi_y(1)), \ldots, \sigma_y(k-1) \cdot \xi_y(k-1), f_y(\xi_y(k-1))$$

is order-preserving, where $\{-1, +1\} \times \kappa$ is ordered lexicographically. Since the set of all such sequences, for a fixed k, is an iterated sum of σ-scattered orders, it is σ-scattered itself. This finishes the proof of the theorem. ■

3. Amenable linear orders. In this section we will define a combinatorial property of linear orders which will be used in our proof of Theorem 1.8.

Definition 3.1. If M is in $\mathscr{E}(L)$ and x is in L, then we say that x is *internal* (resp. *external*) *to* M if there is a club $E \subseteq [\widehat{L}]^\omega$ in M such that every (resp. no) element of $E \cap M$ captures x. If every element of L is internal to every element of $\mathscr{E}(L)$, we will say that L is *amenable*.

Notice that if L is a linear order, x is in L, and M is in $\mathscr{E}(L)$, then x is internal to M if it is captured by $M \cap \widehat{L}$. The converse, however, is false. Baumgartner's example is easily seen to be amenable, though it is not σ-scattered.

The next proposition shows that amenable linear orders of size $\aleph_1$ which are not σ-scattered behave in a similar manner to Baumgartner's example.

Proposition 3.2. *Suppose that L is a linear order of size $\aleph_1$ which is amenable but not σ-scattered. Then there is a suborder L' of L which is not σ-scattered such that $\Omega(L') < \Omega(L)$.*

Proof. Let N_ξ ($\xi < \omega_1$) be a continuous $\in$-chain of countable elementary submodels of $H(2^{|\widehat{L}|^+})$ which contain L. Since L does not contain a real type, there are functions f_n ($n < \omega$) from ω_1 into L such that each f_n is in N_0 and if $\xi < \omega_1$ and y is an element of $L \setminus N_\xi$, then there is an $n < \omega$ such that $f_n(N_\xi \cap \omega_1)$ is the unique z provided by Fact 2.3 such that there is no element of $N_\xi \cap L$ between z and y. Observe that if M is in $\mathscr{E}(L)$ and $\langle N_\xi : \xi < \omega_1 \rangle$ is in M, but $M \cap \widehat{L}$ is not in $\Omega(L)$, then there is an $n < \omega$ such that $f_n(M \cap \omega_1)$ is not captured by $M \cap \widehat{L}$.

Claim 3.3. *Suppose M is in $\mathscr{E}(L)$ such that $\langle N_\xi : \xi < \omega_1 \rangle$ is in M and let $\delta = M \cap \omega_1$. If y is in L, then $M \cap \widehat{L}$ captures y iff $N_\delta \cap \widehat{L}$ does.*

Proof. First observe that, by continuity of $\langle N_\xi : \xi < \omega_1\rangle$ and elementarity of M, N_δ is a subset of M and $N_\delta \cap L = M \cap L$. It follows that any element of L which is captured by $N_\delta \cap \widehat{L}$ is also captured by $M \cap \widehat{L}$.

Now suppose that $M \cap \widehat{L}$ captures y as witnessed by $z \in \widehat{L} \cap M$ with either $z = y$ or else z is the unique element of $M \cap \widehat{L}$ such that no element of $M \cap L$ is between y and z. Set $\delta = M \cap \omega_1$ and let $E \subseteq [\widehat{L}]^\omega$ be a club in N_δ such that every element of $E \cap N_\delta$ captures y. Notice that for sufficiently large $\nu < \delta$, E is in N_ν by continuity of $\langle N_\xi : \xi < \omega_1\rangle$. Therefore $N_\nu \cap \widehat{L}$ is in E by Fact 1.16 and hence N_ν captures y. Let $\overline{N}$ be a countable elementary submodel of $H(2^{|\widehat{L}|^+})$ such that z, E, and $\langle N_\xi : \xi < \omega_1\rangle$ are in $\overline{N}$, and $\overline{N}$ is in M. Set $\nu = \overline{N} \cap \omega_1$, observing that, by continuity of $\langle N_\xi : \xi < \omega_1\rangle$, $N_\nu \subseteq \overline{N}$ with $N_\nu \cap L = \overline{N} \cap L$. By Fact 2.3, z is in $N_\nu \subseteq N_\delta$. It follows that N_δ captures y, since $N_\delta \cap L = M \cap L$. ■

Let Ξ_0 be the set of all $\xi < \omega_1$ such that $N_\xi \cap \omega_1 = \xi$ and there is an $n_\xi < \omega$ with $f_{n_\xi}(\xi)$ not captured by N_ξ. Since $\Omega(L)$ does not contain a club, the previous claim implies that Ξ_0 is stationary. By pressing down and refining Ξ_0, we can find a stationary $\Xi \subseteq \Xi_0$ such that

(1) there is an n such that if ξ is in Ξ, then $n_\xi = n$,
(2) there is a club $E \subseteq [\widehat{L}]^\omega$ such that if ξ is in Ξ, then E is in N_ξ and Z captures $f_{n_\xi}(\xi)$ whenever Z is in $N_\xi \cap E$.

Observe that if $\xi \neq \xi'$, then $N_\xi \cap \widehat{L}$ captures $f_n(\xi')$, since either $\xi' < \xi$ and $f_n(\xi')$ is in N_ξ, or $\xi < \xi'$ and $N_\xi \cap \widehat{L}$ is in $E \cap N_{\xi'}$.

Fix a stationary subset Ξ' of Ξ such that $\Xi \setminus \Xi'$ is stationary. Let L' be the set of all $f_{n_\xi}(\xi)$ such that ξ is in Ξ'. Observe that if M is in $\mathscr{E}(L)$ and L' is in M, then if $M \cap \omega_1$ is in $\Xi \setminus \Xi'$, then $M \cap \widehat{L}$ is not in $\Omega(L)$. On the other hand, $M \cap \widehat{L}'$ is in $\Omega(L')$ by our observation. It follows that L cannot embed into L' by Proposition 2.8.

It therefore suffices to show that $\Omega(L')$ is not a club. To this end, suppose that M is in $\mathscr{E}(L')$ with Ξ' and L in M and $\delta = M \cap \omega_1$ in Ξ'. Our goal is to show that $f_n(\delta)$ is not captured by $M \cap \widehat{L}'$. Let z be in $M \cap \widehat{L}'$ and identify z with the corresponding element in $\widehat{L}$. By Fact 2.3, it suffices to show that there is an x in $L' \cap M$ which is between $f_n(\delta)$ and z. Since $M \cap \widehat{L}$ does not capture $f_n(\delta)$, there must be a y in $M \cap L$ such that y is between $f_n(\delta)$ and z. For simplicity suppose $z < y < f_n(\delta)$.

By elementarity of M, whenever (a, b) is an interval containing $f_n(\delta)$, there is a ξ in $\Xi' \cap M$ with $f_n(\xi)$ in (a, b). Let

$$A = \{f_n(\xi) : \xi \in \Xi' \text{ and } f_n(\xi) > y\}.$$

If every element of $A \cap M$ is greater than $f_n(\delta)$, then the infimum of A is in M and is $\sim^L_M$-equivalent to $f_n(\delta)$. Since this is assumed not to be the case, there is a ξ in $\Xi' \cap M$ such that $z < y < f_n(\xi) < f_n(\delta)$. Since $x = f_n(\xi)$ is in L' by definition, we are done. ■

4. A characterization of when a linear order contains a real or Aronszajn type. In this section, we will strengthen Proposition 2.9 and prove an appropriate converse.

THEOREM 4.1. *Suppose that L is a linear order. The following are equivalent*:

(1) *L contains a real or Aronszajn type.*
(2) *There is an M in $\mathscr{E}(L)$ and an x in L which is external to M.*

Proof. We will first show that (1) implies (2). Suppose that $X \subseteq L$ is either a real or Aronszajn type and that X is in M for some M in $\mathscr{E}(L)$. We will actually show that if x is any element of $X \setminus M$, then x is external to M. To this end, let x in $X \setminus M$ be given.

If X is a real type and $D \subseteq X$ is a countable dense subset of X, then $z \sim_D x$ implies $z = x$ for x and z in X. In particular, if D is in M, then the set $E = \{Z \in [\widehat{L}]^\omega : D \subseteq Z\}$ witnesses that x is external to M.

If X is Aronszajn, then the closure of any countable subset Z of X has countable intersection with X. Hence if Z is a countable subset of X which is an element of M, then the intersection of the closure of Z with X is a subset of M and therefore does not contain x. By Claim 2.10 there is a club $E \subseteq [\widehat{L}]^\omega$ in M such that if Z is in E and $x' \in X$ is not in the closure of Z, then Z does not capture x'. It follows that no element of $E \cap M$ captures x and hence that x is external to M.

We will now show that (2) implies (1). Let $E \subseteq [\widehat{L}]^\omega$ be a club in M which witnesses that x is external to M. Suppose that L does not contain a real type. Let $\mathscr{I}$ be the collection of all open intervals I with endpoints in L such that there is a stationary set of countable elementary submodels N of $H(|E|^+)$ such that E and L are in N and there is an x' in $I \cap L$ such that no Z in $E \cap N$ captures x'. By adding endpoints to L if necessary (and noting Fact 1.16), we may assume that L is in $\mathscr{I}$.

CLAIM 4.2. *If I is in $\mathscr{I}$, then the set of all $\{J \in \mathscr{I} : J \subseteq I\}$ is not σ-linked.*

Proof. Suppose for contradiction that the claim is false. Since linked families of intervals are actually centered and since $\widehat{L}$ is complete, there is a countable $D \subseteq \widehat{L}$ such that if $J \subseteq I$ and J is in $\mathscr{I}$, then $D \cap J$ is nonempty. Let N be a countable elementary submodel of $H(|2^E|^+)$ containing L, E, I, and D, and let x' be an element of $I \cap L$ such that no element of $E \cap N$ captures x'. Since L does not contain any real types, there are only countably many $\sim_D$-equivalence classes that intersect L and therefore each belongs to N. Since $N \cap \widehat{L}$ does not capture x', there must be elements $a < b$ of $[x']_D \cap L \cap N$ such that $a < x' < b$. Since $H(|E|^+)$ is in N, Facts 1.16 and 1.17 imply that $J = (a, b)$ is in $\mathscr{I}$, contained in I, and disjoint from D, a contradiction. ∎

Claim 4.3. *There is no sequence I_ξ $(\xi < \omega_1)$ of elements of $\mathscr{I}$ such that either* $\max I_\xi < \min I_\eta$ *for every $\xi < \eta$, or* $\min I_\xi > \max I_\eta$ *for every $\xi < \eta$.*

Proof. Suppose for contradiction that such a sequence exists. Let N_0 be a countable elementary submodel of $H(|E|^+)$ such that E, L, and $\langle I_\xi : \xi < \omega_1\rangle$ are in N_0. Let $\zeta < \omega_1$ be such that both endpoints of I_ζ are $\sim_{N_0}$-equivalent to those of any I_ξ with $\zeta < \xi < \omega_1$. Notice that the supremum and infimum of the endpoints of I_ξ $(\xi < \omega_1)$ are in N_0. Hence $Z = N_0 \cap \widehat{L}$ is an element of E which captures any element of I_ζ. By definition of membership in $\mathscr{I}$, there is a countable elementary submodel N of $H(|E|^+)$ such that L, E, Z, and ζ are in N and such that there is an x' in I_ζ which is not captured by any element of $E \cap N$. This is a contradiction, since Z is in $E \cap N$ and Z captures x'. ∎

It follows from Claim 4.2 that if I is in $\mathscr{I}$ and $D \subseteq L$ is countable, then there are J_0 and J_1 in $\mathscr{I}$ which are subsets of I, disjoint from each other, and disjoint from D. Using this observation and Claim 4.3, it is possible to construct a tree $\mathscr{T} \subseteq \mathscr{I}$ such that:

(1) All levels of $\mathscr{T}$ are countable and $\mathscr{T}$ is uncountable.
(2) If I is in $\mathscr{T}$, then there are disjoint J_0 and J_1 in $\mathscr{T}$ such that $J_0 \cup J_1 \subseteq I$.

The combination of the properties of $\mathscr{T}$ and Claim 4.3 implies that $\mathscr{T}$ has no uncountable branches and hence is an Aronszajn tree. It follows that the set X of left endpoints of elements of $\mathscr{T}$ is an Aronszajn suborder of L. ∎

5. PFA$^+$ and minimal non-σ-scattered orders. In the previous section we saw that we could draw the sorts of conclusions we are interested in if, for a given linear order L and M in $\mathscr{E}(L)$, every element of L was either internal or external to M. In this section we will see how to use axiomatic assumptions to influence these conditions.

Recall that a forcing $\mathscr{Q}$ is *proper* if forcing with $\mathscr{Q}$ preserves stationary subsets of $[X]^\omega$ for arbitrary uncountable X. This is equivalent to the assertion that, whenever M is in $\mathscr{E}(\mathscr{Q})$ and q is in $\mathscr{Q} \cap M$, there is a $\bar{q} \leq q$ which is $(M, \mathscr{Q})$-*generic*: whenever $D \subseteq \mathscr{Q}$ is a dense set in M and $r \leq \bar{q}$, there is an s in $D \cap M$ which is compatible with r. Notice that if $\bar{q}$ has the property that $G = \{s \in \mathscr{Q} \cap M : \bar{q} \leq s\}$ is an M-generic filter, then $\bar{q}$ is $(M, \mathscr{Q})$-generic. The existence of extensions $\bar{q}$ with this stronger property is easily seen to be equivalent to the additional assertion that forcing with

$\mathscr{Q}$ does not adjoin new real numbers. We will be interested in the following strengthening of the Proper Forcing Axiom:

PFA$^+$: If $\mathscr{Q}$ is a proper forcing, D_α $(\alpha < \omega_1)$ is a collection of dense subsets of $\mathscr{Q}$, and $\dot{\Xi}$ is a $\mathscr{Q}$-name for a stationary subset of ω_1, then there is a filter $G \subseteq \mathscr{Q}$ such that $D_\alpha \cap G$ is non-empty for all $\alpha < \omega_1$ and $\{\xi < \omega_1 : \exists q \in G\ (q \Vdash \check{\xi} \in \dot{\Xi})\}$ is stationary.

Now we will recall some definitions from [8].

Definition 5.1. Let X be a fixed uncountable set and θ be a regular cardinal such that $[X]^\omega$ is in $H(\theta)$. If M is a countable elementary submodel of $H(\theta)$ and $\Sigma \subseteq [X]^\omega$ is such that $\Sigma \cap E \cap M$ is non-empty for every club $E \subseteq [X]^\omega$ in M, then we say that Σ is *M-stationary*.

Definition 5.2. An *open stationary set mapping* is a function Σ defined on a club of countable elementary submodels of $H(\theta)$ and such that for each M in the domain of Σ, $\Sigma(M)$ is an open M-stationary subset of $[X]^\omega$. The underlying set X for a given Σ will be referred to as X_Σ.

For us, the motivating example of an open stationary set mapping is as follows.

Example 5.3. Suppose that L is a linear order which does not contain real or Aronszajn types. Suppose that for each M in $\mathscr{E}(L)$, x_M is an element of L. Define $\Sigma(M)$ to be the set of all Z in $[\widehat{L}]^\omega$ which capture x_M. It is easily checked that $\Sigma(M)$ is open. It follows from Theorem 4.1 that $\Sigma(M)$ is also M-stationary for all M.

Definition 5.4. An open stationary set mapping Σ is said to *reflect* if there is a continuous $\in$-chain N_ξ $(\xi < \omega_1)$ in the domain of Σ such that for every ν, there is a $\nu_0 < \nu$ such that $N_\xi \cap X_\Sigma \in \Sigma(N_\nu)$ whenever $\nu_0 < \xi < \nu$. The sequence $\langle N_\xi : \xi < \omega_1 \rangle$ is called a *reflecting sequence* for Σ.

Suppose now that L is a fixed non-σ-scattered linear order which does not contain real or Aronszajn types. Let $\mathscr{Q} = \mathscr{Q}_L$ be the collection of all continuous $\in$-chains $\langle N_\xi : \xi \le \delta \rangle$ in $\mathscr{E}(L)$ of countable length such that for all $\nu \le \delta$ and x in L, there is a $\nu_x < \nu$ such that $N_\xi \cap \widehat{L}$ captures x whenever $\nu_x < \xi < \nu$. $\mathscr{Q}$ is ordered by $q \le p$ iff p is a restriction of q.

Assuming that $\mathscr{Q}$ is proper, let D_α be those conditions in $\mathscr{Q}$ which are sequences of length at least α. Define D'_ξ to be those q in D_ξ such that if $q(\xi)$ is not in $\Omega(L)$, then there is an x in $q(\eta)$ for some η in $\mathrm{dom}(q)$ such that x is not captured by $q(\xi)$. It is routine to verify that both D_ξ and D'_ξ are dense open subsets of $\mathscr{Q}$ (see [8, 3.1]). Finally, define a $\mathscr{Q}$-name $\dot{\Xi}$ so that q forces ξ in $\dot{\Xi}$ iff ξ is in the domain of q and $q(\xi)$ is not in $\Omega(L)$. By Theorem 2.11 and properness of $\mathscr{Q}$, $\dot{\Xi}$ is forced to be stationary.

Applying PFA$^+$, there is a continuous $\in$-chain N_ξ $(\xi < \omega_1)$ in $\mathscr{E}(L)$ such that:

(1) If $\nu < \omega_1$ and x is in $L_0 = \bigcup_{\xi<\omega_1} N_\xi \cap L$, then there is a $\nu_x < \nu$ such that $N_\xi \cap \widehat{L}$ captures x whenever $\nu_x < \xi < \nu$.
(2) The set Ξ of all $\xi < \omega_1$ such that $N_\xi \cap \widehat{L}$ is not in $\Omega(L)$ is stationary.
(3) If $\xi < \omega_1$ and $N_\xi \cap \widehat{L}$ is not in $\Omega(L)$, then there is an x in L_0 such that $N_\xi \cap \widehat{L}$ does not capture x.

It follows that L_0 is amenable and, since $\Omega(L_0)$ is disjoint from Ξ, not σ-scattered. Applying Proposition 3.2, we have reduced Theorem 1.8 to verifying that $\mathscr{Q}$ is proper. This is essentially the same proof as [8, Theorem 3.1], where it is shown that the forcing consisting of all countable partial reflecting sequences of a given open stationary set mapping is proper.

Definition 5.5. Suppose that M is in $\mathscr{E}(\mathscr{Q})$. If x is an element of the completion of $L \cap M$, then we say x is a *potential element of* L if whenever Z is a countable subset of L in M, there is an x' in M with $x' \sim_Z x$ (2).

Lemma 5.6. *Suppose that M is as above and F is a finite set of potential elements of L. Then the set of all Z in $[\widehat{L}]^\omega$ which are subsets of M and which capture every element of F is M-stationary.*

Proof. This is similar to [5, 4.1]. Let $E \subseteq [\widehat{L}]^\omega$ be a club in M and let M' be in $\mathscr{E}(L) \cap M$ such that E is in M'. For each x in F, let x' be an element of $L \cap M$ such that $x' \sim_{M'} x$, and set $F' = \{x' : x \in F\}$. Let $\{x_i : i < k\}$ enumerate F'.

The proof of Theorem 4.1 actually shows that for sufficiently large $\lambda < \theta_L$, if N is a countable elementary submodel of $H(\lambda)$ containing $\mathscr{P}(\mathscr{P}(L))$ as a member, then no element of L is external to N. Let $\lambda_0 = \lambda$ and $\lambda_{i+1} = 2^{\lambda_i{}^+}$. We will inductively construct a sequence N_i $(i \le k)$ such that:

(1) N_{i+1} is a countable elementary submodel of $H(\lambda_{k-i})$;
(2) $E \in N_{i+1} \in N_i$;
(3) N_i captures x_j for all $j < i$, as witnessed by $y_j \in \widehat{L} \cap N_i$.

Begin by setting $N_0 = M'$. If N_i is given, then by Theorem 4.1 and our assumption that L does not contain a real or Aronszajn type, there is a countable elementary submodel N_{i+1} of $H(\lambda_{k-i-1})$ which contains E and y_j for $j < i$, and captures x_i. Notice that N_{i+1} captures x_j for $j < i$ as well, since it contains y_j. Finally, $Z = N_k \cap \widehat{L}$ is in E (since E is in N_k) and captures x_j for all $j \le k$. ∎

(2) The notion of a potential element of L is only necessary in verifying that $\mathscr{Q}$ is completely proper and does not play a role in the proof of properness.

LEMMA 5.7. *Suppose that q is in $\mathcal{Q} \cap M$, $D \subseteq \mathcal{Q}$ is a dense open set which is in M, and F is a finite set of potential elements of L. There is a $\bar{q} \leq q$ in $D \cap M$ such that $\bar{q}(\xi) \cap \widehat{L}$ captures x whenever x is in F and ξ is in* $\mathrm{dom}(\bar{q}) \setminus \mathrm{dom}(q)$.

Proof. This follows from Lemma 5.6 and the proof of [8, 3.1]. ∎

LEMMA 5.8. *Suppose that $q = \langle N_\xi : \xi \leq \alpha \rangle$ is in $\mathcal{Q} \cap M$ and S is a countable set of potential elements of L. Then there is a $\langle N_\xi : \xi \leq \delta \rangle$ in $\mathcal{Q}$ such that*:

(1) $G = \{\langle N_\xi : \xi \leq \beta \rangle : \beta \in \delta\}$ *is an M-generic filter in $\mathcal{Q}$.*
(2) *If x is in S, then there is a $\delta_x < \delta$ such that $N_\xi \cap \widehat{L}$ captures x whenever $\delta_x < \xi < \delta$.*

Proof. Enumerate the dense open subsets of $\mathcal{Q}$ which are in M and use Lemma 5.7 to construct $\langle N_\xi : \xi \leq \delta \rangle$ recursively. ∎

By Lemma 5.8, $\mathcal{Q}$ is proper. While the assertion that all open stationary set mappings reflect implies $2^{\aleph_0} = \aleph_2$ [8], the above arguments show that $\mathcal{Q}$ is moreover *completely proper* and hence it is plausible that the conclusion of Theorem 1.10 is consistent with CH. The reader is referred to [10] for the definition of complete properness and an example of a verification that a forcing is completely proper. This gives further motivation to the following open problem.

PROBLEM 5.9. *Is* CPFA^+ (*or even just* CPFA) *consistent with* CH*?*

By Theorem 1.10, it would follow that it is consistent that Laver's Theorem 1.1 is sharp.

The following problem is also open.

PROBLEM 5.10. *Suppose that S is stationary. Is the class of linear orders L with $\Omega(L) \equiv S$ well quasi-ordered?*

Notice that the requirement that S be stationary is necessary since by [3], the real types are not well quasi-ordered. If S contains a club, then a positive answer follows from Laver's Theorem and Theorem 2.11 of this paper.

Acknowledgements. The second author would like to acknowledge support from NSF grants DMS-0401893 and DMS-0200671.

References

[1] J. E. Baumgartner, *All $\aleph_1$-dense sets of reals can be isomorphic*, Fund. Math. 79 (1973), 101–106.
[2] —, *A new class of order types*, Ann. Math. Logic 9 (1976), 187–222.
[3] B. Dushnik and E. W. Miller, *Concerning similarity transformations of linearly ordered sets*, Bull. Amer. Math. Soc. 46 (1940), 322–326.

[4] T. Jech, *Set Theory*, Springer Monogr. Math., Springer, Berlin, 2003.
[5] B. König, P. Larson, J. T. Moore, and B. Veličković, *Bounding the consistency strength of a five element linear basis*, Israel J. Math. 164 (2008), 1–18.
[6] K. Kunen, *An Introduction to Independence Proofs*, Stud. Logic Found. Math. 102, North-Holland, 1983.
[7] R. Laver, *On Fraïssé's order type conjecture*, Ann. of Math. (2) 93 (1971), 89–111.
[8] J. T. Moore, *Set mapping reflection*, J. Math. Logic 5 (2005), 87–97.
[9] —, *A five element basis for the uncountable linear orders*, Ann. of Math. (2) 163 (2006), 669–688.
[10] —, ω_1 *and* $-\omega_1$ *may be the only minimal uncountable order types*, Michigan Math. J. 55 (2007), 437–457.
[11] S. Shelah, *Proper and Improper Forcing*, 2nd ed., Springer, Berlin, 1998.
[12] S. Todorcevic, *Lipschitz maps on trees*, J. Inst. Math. Jussieu 6 (2007), 527–556.
[13] —, *Trees and linearly ordered sets*, in: Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, 235–293.

Department of Mathematics and Statistics
Miami University
Oxford, OH 45056, U.S.A.
E-mail: ishiut@muohio.edu

Department of Mathematics
Cornell University
310 Malott Hall
Ithaca, NY 14853-4201, U.S.A.
E-mail: justin@math.cornell.edu

Received 9 April 2007;
in revised form 29 April 2009