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Definable Davies' theorem

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Abstract. We prove the following descriptive set-theoretic analogue of a theorem of R. O. Davies: Every Σ^1_2 function $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ can be represented as a sum of rectangular Σ^1_2 functions if and only if all reals are constructible.

1. Introduction. In [1], R. O. Davies proved that the continuum hypothesis, CH, is equivalent to the statement that every function $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ can be represented as a sum of "rectangular" functions as follows: There are $g_n, h_n : \mathbb{R} \to \mathbb{R}$, $n \in \omega$, such that

$$f(x, y) = \sum_{n=0}^{\infty} g_n(x)h_n(y),$$

where at each $(x, y) \in \mathbb{R}^2$ there are at most finitely many non-zero terms in the above sum. We call such a representation a *Davies representation* of f. Thus Davies' Theorem says that CH is equivalent to the statement that every function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has Davies representation.

The purpose of this paper is to prove the following descriptive set-theoretic analogue of Davies' Theorem:

Theorem 1. *Every Σ^1_2 function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has a Davies representation*

$$f(x, y) = \sum_{n=0}^{\infty} g(x, n)h(y, n),$$

where $g, h : \mathbb{R} \times \omega \to \mathbb{R}$ are Σ^1_2 functions and the sum has only finitely many non-zero terms at each $(x, y) \in \mathbb{R}^2$, if and only if all reals are constructible.

We will also show that it is not possible to find a Davies representation of $f(x, y) = e^{xy}$ using Baire or Lebesgue measurable functions g and h. Note though that e^{xy} *does* have a representation as an infinite power series in x

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and y. We will give an example of a Borel (in fact, Δ^1_1) function $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ which does not admit a rectangular sum representation as above with Baire or Lebesgue measurable g and h, even if we drop the pointwise finiteness condition on the sum, and only ask that at each (x, y) the sum converges pointwise.

Organization. In §2 below we show (Theorem 2) that if there is a strongly Δ^1_n well-ordering of $\mathbb{R}$ then every Σ^1_n function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ admits a representation

$$f(x, y) = \sum_{n=0}^{\infty} g(x, n)h(y, n),$$

with Σ^1_n functions $g, h : \mathbb{R} \times \omega \to \mathbb{R}$, and where the sum has only finitely many non-zero terms at each $(x, y) \in \mathbb{R}^2$.

In §3 we establish the converse to Theorem 2 in the case of Σ^1_2 functions (Theorem 3). We also establish a converse in the Σ^1_3 case, under the additional assumption that there is a measurable cardinal. Finally, we establish the two facts regarding representations using Baire and Lebesgue measurable functions mentioned after Theorem 1 above.

2. Inductive argument. The necessary descriptive set-theoretic background for this paper can be found in [9] and [8], in particular the definitions of the (lightface) point-classes Σ^1_n, Δ^1_n and Π^1_n. Here we recall the notions for Δ^1_n well-orderings that are the most important to us.

Following [2], we say that a Δ^1_n well-ordering $\prec$ of $\mathbb{R}$ is *strongly* Δ^1_n if it has length ω_1 and the following (equivalent) statements hold (cf. [9, Chapter 5]):

1. If $P \subseteq \mathbb{R} \times \mathbb{R}$ is Σ^1_n then
$$R(x, y) \iff (\forall z \prec y)\ P(x, z)$$
is Σ^1_n.
2. The initial segment relation $\mathrm{IS} \subseteq \mathbb{R} \times \mathbb{R}^{\leq\omega}$ defined by
$$\mathrm{IS}(x, y) \iff (\forall z \prec x)(\exists n)\ y(n) = z \wedge (\forall i, j)\ i = j \vee y(i) \neq y(j)$$
is Σ^1_n.

If all reals are constructible then there is a strongly Δ^1_2 well-ordering of $\mathbb{R}$ (see e.g. [6]).

It will often be necessary to work with recursively presented Polish spaces other than $\mathbb{R}$, such as ω^ω or $\mathbb{R}^{\leq\omega}$ (see below). Since all uncountable recursively presented Polish spaces are isomorphic in the sense that there is a Δ^1_1 bijection between them with a Δ^1_1 inverse (see [9, 3E.7]), once we have a strongly Δ^1_n well-ordering of $\mathbb{R}$ we have a strongly Δ^1_n well-ordering of all recursively presented Polish spaces. For convenience we will use the same symbol, usually $\prec$, for such a well-ordering in all the recursively presented spaces we consider. This minor ambiguity poses no real danger.

We will say that a function $f : X \to Y$ from one recursively presented Polish space X to another, Y, is Σ^1_n (respectively Π^1_n and Δ^1_n) if its graph is Σ^1_n (respectively Π^1_n and Δ^1_n). A function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to have a Σ^1_n *Davies representation* if there are Σ^1_n functions $g, h : \mathbb{R} \times \omega \to \mathbb{R}$ such that

$$\sum_{n=0}^{\infty} g(x,n)h(y,n)$$

and the sum has only finitely many non-zero terms at each (x, y). The notions of Π^1_n and Δ^1_n Davies representation are defined similarly.

THEOREM 2. *If there is a strongly Δ^1_n well-ordering of $\mathbb{R}$ then every Σ^1_n function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has a Σ^1_n Davies representation. In particular, if all reals are constructible then every Σ^1_2 function has a Σ^1_2 Davies representation.*

To prove this, we will need to verify that Davies' proof, which uses Zorn's Lemma, produces functions $g, h : \mathbb{R} \times \omega \to \mathbb{R}$ that are Σ^1_n and witness that f has a Σ^1_n Davies representation. This in turn requires that we produce Σ^1_n predicates (in the sense of [8, p. 3] or [6, p. 152–157]) that define g and h. These predicates will essentially be formulas defining g and h by transfinite recursion as in the usual proof of the transfinite recursion theorem (see e.g. [4, p. 22, (2.6)]).

If X is a set, we write $X^{\leq\omega}$ for the set of functions $g : \alpha \to X$ for some $\alpha \in \omega + 1$, and we set $\mathrm{lh}(g) = |\mathrm{dom}(g)|$, the cardinality of $\mathrm{dom}(g)$. For $g \in \mathbb{R}^{\leq\omega}$ we let

$$\mathrm{supp}(g) = \{n \in \omega : n \in \mathrm{dom}(g) \wedge g(n) \neq 0\}.$$

It is convenient for the proof to work relative to a fixed countable sequence $x_n \in \mathscr{P}(\omega)$ of almost disjoint infinite subsets of ω. The sequence (x_n) will be used to make sure that certain almost disjoint families that are finite are not maximal, because they will be constructed so that they are almost disjoint from all x_n, $n \in \omega$. We will assume that the map $n \mapsto x_n$ is recursive.

DEFINITION. The set $S \subseteq (\mathbb{R}^\omega)^{\leq\omega} \times (\mathbb{R}^\omega)^{\leq\omega}$ is defined by $(g, h) \in S$ if and only if

(a) The sets $\mathrm{supp}(g(k))$, $\mathrm{supp}(h(m))$ and x_n ($k \in \mathrm{dom}(g)$, $m \in \mathrm{dom}(h)$, $n \in \omega$) form an almost disjoint sequence of sets.
(b) For all $m \in \mathrm{dom}(g)$ there are infinitely many k such that $g(m)(k) = 1$.
(c) For all $n \in \mathrm{dom}(h)$ there are infinitely many k such that $h(n)(k) = 1$.

Note that S is Δ^1_1. We need the following lemma to encode the inductive step.

2.1. Lemma. *Suppose $f \in \mathbb{R}^{\leq\omega}$ is given and $(g,h) \in S$ is such that $\mathrm{lh}(h) = \mathrm{lh}(f)$. Then there is $\theta = \theta(f,g,h) : \omega \to \mathbb{R}$ such that:*

(1) *For all $k \in \mathrm{dom}(f)$,*

$$f(k) = \sum_{l=0}^{\infty} \theta(l)h(k)(l),$$

and the sum has only finitely many non-zero terms.

(2) *For all $n \in \mathrm{dom}(h)$, $\mathrm{supp}(\theta) \cap \mathrm{supp}(h(n))$ is finite.*

(3) *For all $n \in \mathrm{dom}(g)$, $\mathrm{supp}(\theta) \cap \mathrm{supp}(g(n))$ is finite.*

(4) *For all $n \in \omega$, $\mathrm{supp}(\theta) \cap x_n$ is finite.*

(5) *For infinitely many k we have $\theta(k) = 1$.*

Moreover, θ may be found recursively in the given data. In particular, there is a Δ^1_1 function $\theta : \mathbb{R}^{\leq\omega} \times S \to \mathbb{R}^\omega$ such that $\theta(f,g,h)$ satisfies (1)–(5) *for all $(f,g,h) \in \mathbb{R}^{\leq\omega} \times S$.*

Proof. We define by induction on $k \in \omega$ an increasing sequence $n_k \in \omega$ and $\theta \restriction n_k + 1$ such that

(1′) For all $m \in \mathrm{dom}(f) \cap (k+1)$,

$$f(m) = \sum_{l=0}^{n_m} \theta(l)h(m)(l).$$

(2′) For all $m \in \mathrm{dom}(h) \cap (k+1)$, $\mathrm{supp}(\theta \restriction n_k+1) \cap \mathrm{supp}(h(m)) \subseteq n_m+1$.

(3′) For all $m \in \mathrm{dom}(g) \cap (k+1)$, $\mathrm{supp}(\theta \restriction n_k+1) \cap \mathrm{supp}(g(m)) \subseteq n_m+1$.

(4′) For all $m \leq k$, $\mathrm{supp}(\theta \restriction n_k + 1) \cap x_m \subseteq n_m + 1$.

(5′) $\theta(n_k) = 1$.

Assuming this can be done, θ will be defined on all of ω, since n_k is increasing. By (1′) and (2′) it follows that for $m \in \mathrm{dom}(f)$ we will have

$$f(m) = \sum_{l=0}^{\infty} \theta(l)h(m)(l)$$

and by (2′) it is the case that $\theta(l)h(m)(l) = 0$ for $l > n_m$. Thus (1) and (2) of the statement of the lemma hold. Finally, (3′), (4′) and (5′) ensure (3), (4) and (5).

To see that we can satisfy (1′)–(5′), suppose n_k and $\theta \restriction n_k + 1$ have been defined.

Case 1: $k + 1 \notin \mathrm{dom}(f)$. Then we let $p > n_k$ be the least number greater than n_k such that $p \notin \mathrm{supp}(g(m))$, $p \notin \mathrm{supp}(h(m))$ and $p \notin x_m$ for $m \leq k$. The number p exists because of condition (a) in the definition of S.

Define $n_{k+1} = p$, for $n_k < l < n_{k+1}$ let $\theta(l) = 0$, and $\theta(n_{k+1}) = 1$. Clearly conditions (1′)–(5′) are satisfied.

Case 2: $k+1 \in \operatorname{dom}(f)$. Then let $p > n_k$ be the least number greater than n_k such that $p \notin \operatorname{supp}(g(m))$, $p \notin \operatorname{supp}(h(m))$ and $p \notin x_m$ for $m \leq k$, and $h(k+1)(p) = 1$. The number p exists because of conditions (a) and (c) in the definition of S. We let $q > p$ be least such that $q \notin \operatorname{supp}(g(m))$, $q \notin \operatorname{supp}(h(m))$ and $q \notin x_m$ for $m \leq k+1$. Let $n_{k+1} = q$ and define, for $n_k < l \leq n_{k+1}$,

$$\theta(l) = \begin{cases} 1 & \text{if } l = q = n_{k+1}, \\ f(k+1) - \displaystyle\sum_{m=0}^{p-1} \theta(m)h(k+1)(m) & \text{if } l = p, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that (2′)–(5′) are satisfied. To see (1′), note that

$$\begin{aligned} \sum_{l=0}^{n_{k+1}} \theta(l)h(k+1)(l) &= \sum_{l=0}^{p} \theta(l)h(k+1)(l) \\ &= f(k+1) - \sum_{m=0}^{p-1} \theta(m)h(k+1)(m) + \sum_{l=0}^{p-1} \theta(l)h(k+1)(l) \\ &= f(k+1). \end{aligned}$$

This ends Case 2. It is clear from the construction that θ is recursive in the given data (f, g, h). Thus the map $(f, g, h) \mapsto \theta(f, g, h)$ is in particular Δ^1_1. ■

2.2. *Davies' argument as an inductive construction.* For the remainder of this section of the paper, θ will be the function defined in Lemma 2.1. Using this lemma one can now produce a Davies representation of $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by induction as follows: Assuming CH, fix a well-ordering $\prec$ of $\mathbb{R}$ of order type ω_1. Suppose $g, h : \{y \in \mathbb{R} : y \prec x\} \times \omega \to \mathbb{R}$ have been defined such that for all $y, z \prec x$,

$$f(y, z) = \sum_{n=0}^{\infty} g(y, n)h(z, n)$$

and that further if (w_m) is an enumeration of the initial segment $\{y : y \prec x\}$ then the functions

$$g_0(m)(n) = g(w_m, n) \quad \text{and} \quad h_0(m)(n) = h(w_m, n)$$

are such that $(g_0, h_0) \in S$. If we let $f_0(m) = f(w_m, x)$ and define $g(x, n) = \theta(f_0, g_0, h_0)(n)$ then it is easy to check using Lemma 2.1 that for $y \prec x$,

$$f(x, y) = \sum_{n=0}^{\infty} g(x, n)h(y, n).$$

If (w'_m) enumerates $\{y : y \preceq x\}$ and we let $f_1(m) = f(x, w'_m)$ and

$$g_1(n) = \begin{cases} \theta(f_0, g_0, h_0) & \text{if } w'_n = x, \\ g_0(k) & \text{if } w'_n = w_k, \end{cases}$$

then $(h_0, g_1) \in S$, and if we let $h(x, n) = \theta(f_1, h_0, g_1))(n)$, it is again easy to check using the previous lemma that for all $y \preceq x$,

$$f(y, x) = \sum_{n=0}^{\infty} g(y, n) h(x, n).$$

Finally,

$$g_1(m)(n) = g(w'_m, n) \quad \text{and} \quad h_1(m)(n) = h(w'_m, n)$$

satisfy $(g_1, h_1) \in S$, thus allowing the induction to continue.

Our task is now to verify that if $\prec$ is a strongly Δ^1_n well-ordering of $\mathbb{R}$, then the construction we have described may be carried out in such a way that if $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Σ^1_n, then the functions $g, h : \mathbb{R} \times \omega \to \mathbb{R}$ will be Σ^1_n. This can be done since the strongly Δ^1_n well-ordering allows us to enumerate initial segments in a uniformly Δ^1_n way. However, in order to be able to write down Σ^1_n definitions of g and h we need a lemma which says that there is a Σ^1_n function which can correctly compute $g \restriction \{y : y \prec x\} \times \omega$ and $h \restriction \{y : y \prec x\} \times \omega$ for every x.

Before stating that lemma we introduce various functions and predicates. Fix a strongly Δ^1_n well-ordering $\prec$ of $\mathbb{R}$ and let $\mathrm{IS} \subseteq \mathbb{R} \times \mathbb{R}^{\leq\omega}$ be the initial segment relation as defined at the beginning of this section. Define $\mathrm{IS}^* : \mathbb{R} \to \mathbb{R}^{\leq\omega}$ by

$$\mathrm{IS}^*(x) = y \;\Leftrightarrow\; \mathrm{IS}(x, y) \wedge (\forall z \prec y)\, \neg\mathrm{IS}(x, z).$$

Note that IS^* is Δ^1_n. We also define a partial function $\mathrm{IS}^\# : \mathbb{R} \times \mathbb{R} \to \omega$ by

$$\mathrm{IS}^\#(x, y) = n \;\Leftrightarrow\; \mathrm{IS}^*(x)(n) = y.$$

Note that the graph of $\mathrm{IS}^\#$ is a Δ^1_n subset of $\mathbb{R} \times \mathbb{R} \times \omega$, and that if $y \prec x$ then $\mathrm{IS}^\#(x, y)$ computes the unique n which y corresponds to in the enumeration of the initial segment of x given by $\mathrm{IS}^*(x)$. Finally, we define

$$\mathrm{succ}(x) = y \;\Leftrightarrow\; (\forall z \prec y)\; z = x \vee z \prec x.$$

2.3. Lemma. *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be Σ^1_n and suppose there is a strongly Δ^1_n well-ordering $\prec$ of $\mathbb{R}$. Then there is a unique Σ^1_n function $F : \mathbb{R} \to (\mathbb{R}^\omega)^{\leq\omega} \times (\mathbb{R}^\omega)^{\leq\omega}$ satisfying $F(x) = (G, H)$ if and only if*

(1) *$\mathrm{lh}(G) = \mathrm{lh}(H) = \mathrm{lh}(\mathrm{IS}^*(x))$ and $(G, H) \in S$.*
(2) *If $z, z' \prec x$, $\mathrm{IS}^\#(x, z) = k$ and $\mathrm{IS}^\#(x, z') = k'$ then*

$$f(z, z') = \sum_{n=0}^{\infty} G(k)(n) H(k')(n).$$

(3) *For all* $y \prec x$, *if we let* $w' = \mathrm{IS}^*(y)$, $w = \mathrm{IS}^*(x)$ *and* $f_0(k) = f(y, w'(k))$, *and define, for* $k \in \mathrm{dom}(w')$,

$$G'(k) = G(l) \iff w'(k) = w(l)$$

and

$$H'(k) = H(l) \iff w'(k) = w(l),$$

then $w(m) = y$ *implies that*

$$G(m) = \theta(f_0, G', H').$$

(4) *For all* $y \prec x$, *if we let* $w' = \mathrm{IS}^*(y)$, $w'' = \mathrm{IS}^*(\mathrm{succ}(y))$, $w = \mathrm{IS}^*(x)$ *and* $f_1(k) = f(w''(k), y)$, *and define, for* $k \in \mathrm{dom}(w'')$,

$$G''(k) = G(l) \iff w''(k) = w(l),$$

and for $k \in \mathrm{dom}(w')$,

$$H'(k) = H(l) \iff w'(k) = w(l),$$

then $w(m) = y$ *implies that*

$$H(m) = \theta(f_1, H', G'').$$

Proof. Conditions (1)–(4) express *exactly* that for $y \prec x$, if we let

$$g(y, n) = G(\mathrm{IS}^{\#}(x, y))(n) \quad \text{and} \quad h(y, n) = H(\mathrm{IS}^{\#}(x, y))(n)$$

then g and h are the functions we have constructed at stage x in the inductive construction described in 2.2 above, provided that at any stage of the induction we use the enumeration of the initial segments given by the function IS^*. Thus F is unique and defined for all x. Finally, we note that conditions (1)–(4) can be expressed using Σ^1_n predicates when f is a Σ^1_n function. For instance, (3) may be replaced by

$$\begin{aligned}
&(\forall y \prec x)(\exists w, w', f_0, G', H' \in \mathbb{R}^{\le\omega})(w' = \mathrm{IS}^*(y) \wedge w = \mathrm{IS}^*(x)\\
&\wedge \mathrm{lh}(f_0) = \mathrm{lh}(w') \wedge (\forall k \in \mathrm{dom}(w'))(f_0(k) = f(y, w'(k))\\
&\wedge (\forall l \in \mathrm{dom}(G))(G'(k) = G(l) \wedge H'(k) = H(l) \iff w'(k) = w(l)))\\
&\wedge (\forall m \in \mathrm{dom}(w))(w(m) \ne y \vee G(m) = \theta(f_0, G', H'))).
\end{aligned}$$

Thus (1)–(4) gives a Σ^1_n definition of the graph of F, and so the function F is Σ^1_n. ∎

Proof of Theorem 2. If $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Σ^1_n and $\prec$ is a strongly Δ^1_n well-ordering, let F be as in Lemma 2.3 and let $F(x) = (G(x), H(x))$ for all x. Then

$$g(x, n) = G(\mathrm{succ}(x))(\mathrm{IS}^{\#}(\mathrm{succ}(x), x))(n)$$

and

$$h(x, n) = H(\mathrm{succ}(x))(\mathrm{IS}^{\#}(\mathrm{succ}(x), x))(n)$$

define Σ^1_n functions that give us a Davies representation of f. ∎

Remark. If $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is Δ^1_n then conditions (1)–(4) define a Δ^1_n function F. Consequently, the functions g and h produced in the proof of Theorem 2 will be Δ^1_n. Therefore we have:

2.4. Corollary. *If there is a strongly Δ^1_n well-ordering of $\mathbb{R}$ then every Δ^1_n function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ has a Δ^1_n Davies representation.*

3. A definable converse. We now show the following converse to Theorem 2 for Σ^1_2 functions:

Theorem 3. *If there are Σ^1_2 functions $g,h:\mathbb{R}\times\omega\to\mathbb{R}$ such that*

$$e^{xy}=\sum_{n=0}^{\infty}g(x,n)h(y,n)$$

with only finitely many non-zero terms at each (x,y) then there is a Σ^1_2 well-ordering of $\mathbb{R}$.

Since by Mansfield's Theorem ([7], see also [4, 25.39]) the existence of a Σ^1_2 well-ordering or $\mathbb{R}$ is equivalent to the statement that all reals are constructible, Theorem 3 together with Theorem 2 proves Theorem 1 as stated in the introduction. The proof requires several lemmata:

3.1. Lemma. *Let $b_0,\dots,b_n\in\mathbb{R}$ be distinct reals and $c_0,\dots,c_n\in\mathbb{R}$. Then*

$$f(x)=\sum_{j=0}^{n}c_je^{xb_j}$$

has $n+1$ distinct roots if and only if $c_0=\cdots=c_n=0$.

Proof. By induction on n. If $f(x)$ has $n+1$ distinct roots then so does

$$g(x)=e^{-b_0x}f(x).$$

Using Rolle's Theorem from calculus it follows that $g'(x)$ has n distinct roots, and so by the inductive hypothesis must be constantly zero. Thus $f(x)$ is constantly zero. ∎

3.2. Lemma. *Let $a_0,\dots,a_n$ and $b_0,\dots,b_n$ be two distinct sequences of real numbers. Then there are no functions $g_l,h_l:\mathbb{R}\to\mathbb{R}$, $l<n$, such that*

$$e^{a_ib_j}=\sum_{l=0}^{n-1}g_l(a_i)h_l(b_j)$$

for all $0\le i,j\le n$.

Proof. If there are such functions then we have the matrix identity

$$[e^{a_i b_j}] = \begin{bmatrix} g_0(a_0) & \cdots & g_{n-1}(a_0) \\ \vdots & & \vdots \\ g_0(a_n) & \cdots & g_{n-1}(a_n) \end{bmatrix} \begin{bmatrix} h_0(b_0) & \cdots & h_0(b_n) \\ \vdots & & \vdots \\ h_{n-1}(b_0) & \cdots & h_{n-1}(b_n) \end{bmatrix}$$

and so $[e^{a_i b_j}]$ is a product of an $(n+1)\times n$ and an $n\times(n+1)$ matrix. It follows that $\operatorname{rank}([e^{a_i b_j}]) \le n$, which contradicts the previous lemma. ∎

3.3. Lemma. *Assume Σ^1_n uniformization holds and that there are Σ^1_n functions $g, h : \mathbb{R}\times\omega \to \mathbb{R}$ such that*

$$e^{xy} = \sum_{n=0}^{\infty} g(x,n)h(y,n)$$

with only finitely many non-zero terms at each (x,y). Suppose there is an uncountable Σ^1_n set $A \subseteq \mathbb{R}$ and a binary Σ^1_n relation $\prec$ on $\mathbb{R}$ such that $(A,\prec)$ is well-ordered. Then there is a Σ^1_n well-ordering of $\mathbb{R}$.

Proof. Define

$$N(x,y) = k \iff g(x,k)h(y,k) \neq 0 \wedge (\forall l > k)\ g(x,l)h(y,l) = 0.$$

Clearly $N : \mathbb{R}\times\mathbb{R} \to \omega$ is Σ^1_n. Also define $Q \subseteq \mathbb{R}\times\omega$ by

$$\begin{aligned} Q(x,n) \iff & (\exists a \in \mathbb{R}^\omega)(\forall i)(\forall j)(i = j \vee a(i) \neq a(j)) \\ & \wedge (\forall k)(a(k) \in A \wedge N(x,a(k)) = n), \end{aligned}$$

which is Σ^1_n. Let $Q^* : \mathbb{R} \to \omega$ be a Σ^1_n uniformization of Q. Note that Q^* is defined everywhere since A is uncountable.

Now define $R \subseteq \mathbb{R}\times[\mathbb{R}]^{<\omega}$, where $[\mathbb{R}]^{<\omega}$ denotes the set of finite subsets [1] of $\mathbb{R}$, by

$$R(x,s) \iff |s| = Q^*(x) + 2 \wedge (\forall y \in s)(y \in A \wedge N(x,y) = Q^*(x)).$$

Let $R^* : \mathbb{R} \to [\mathbb{R}]^{<\omega}$ be a Σ^1_n uniformization of R.

Claim. *R^* is finite-to-one.*

Proof. Suppose not. Then there is some $s = \{b_0,\dots,b_n\}$ such that $R^{*-1}(s)$ is infinite. Pick $a_0,\dots,a_n \in R^{*-1}(s)$ distinct. Note that since $R^*(b_i) = s$ we have $Q^*(b_i) = |s| - 2 = n - 1$. Thus

$$e^{a_i b_j} = \sum_{l=0}^{n-1} g(a_i,l)h(b_j,l),$$

contradicting the previous lemma. ∎

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[1] Formally we let $[\mathbb{R}]^{<\omega} = \{s \in \mathbb{R}^{<\omega} : (\forall k < \mathrm{lh}(s)-1)\ s(k) < s(k+1)\}$, where $<$ is the usual linear ordering of $\mathbb{R}$. Note that for $s \in [\mathbb{R}]^{<\omega}$, the quantifiers $(\forall x \in s)$ and $(\exists x \in s)$ can be replaced by number quantifiers in hierarchy calculations.

Let $\prec_{\text{lex}}$ be the lexicographic order on $[A]^{<\omega}$ that we obtain from the well-ordering $\prec$ of A. Then we define $<^*$ by

$$x <^* y \;\Leftrightarrow\; R^*(x) \prec_{\text{lex}} R^*(y) \vee (R^*(x) = R^*(y) \wedge x < y),$$

where $<$ is the usual linear ordering of $\mathbb{R}$. Since R^* is finite-to-one, $<^*$ is a Σ^1_n well-ordering of $\mathbb{R}$. ∎

3.4. Lemma. *There are no Baire or Lebesgue measurable functions* $g, h : \mathbb{R} \times \omega \to \mathbb{R}$ *such that*

$$e^{xy} = \sum_{n=0}^{\infty} g(x,n)h(y,n)$$

where the sum has finitely many non-zero terms at each (x, y).

Proof. Suppose there are Baire measurable $g, h : \mathbb{R}\times\omega \to \mathbb{R}$ representing e^{xy} as above. Then $N : \mathbb{R} \times \mathbb{R} \to \omega$ defined by

$$N(x,y) = k \;\Leftrightarrow\; g(x,k)h(y,k) \neq 0 \wedge (\forall l > k)\ g(x,l)h(y,l) = 0$$

is also Baire measurable. It follows that there is some n_0 such that

$$A = \{(x,y) \in \mathbb{R}^2 : N(x,y) = n_0\}$$

is non-meagre and has the property of Baire. Thus we may find $U, V \subseteq \mathbb{R}$ open and non-empty such that A is comeagre in $U \times V$. By Kuratowski–Ulam's Theorem it follows that

$$\{x \in U : A_x \text{ is comeagre in } V\}$$

is comeagre in U. Hence we may pick distinct elements $a_0, \ldots, a_{n_0+1} \in U$ such that the section A_{a_i} is comeagre in V for all $i = 0, \ldots, n_0+1$. But then we can find distinct elements

$$b_0, \ldots, b_{n_0+1} \in \bigcap_{i=0}^{n_0+1} A_{a_i},$$

which gives us that for $0 \leq i, j \leq n_0 + 1$,

$$e^{a_i b_j} = \sum_{n=0}^{n_0} g(a_i, n)h(b_j, n),$$

contradicting Lemma 3.2.

The proof of the Lebesgue measurable case is similar. ∎

Proof of Theorem 3. Suppose we have Σ^1_2 functions $g, h : \mathbb{R} \times \omega \to \mathbb{R}$ representing e^{xy}. By the previous lemma, g and h cannot be Baire measurable, and so $L \cap \mathbb{R}$ cannot be countable by [4, 26.21]. But then we can apply Lemma 3.3 with $A = L \cap \mathbb{R}$ and $\prec$ the canonical Σ^1_2 well-ordering of $L \cap \mathbb{R}$ to get a Σ^1_2 well-ordering of $\mathbb{R}$. ∎

REMARK. Assume Σ^1_3 uniformization. Suppose there is a measurable cardinal and let U be a normal ultrafilter witnessing this. Then the tree representation for Σ^1_3 (see [6, p. 201], also [4, 32.14]) and [6, 15.10] imply that if $\mathbb{R} \cap L[U]$ is countable then all Σ^1_3 functions have the property of Baire. Since by [11, 4.6] there is a Σ^1_3 well-ordering of $\mathbb{R} \cap L[U]$, the proof above then shows that if there is a Σ^1_3 Davies representation of e^{xy} then there is a Σ^1_3 well-ordering of $\mathbb{R}$. In fact, we obtain the following stronger result:

3.5. COROLLARY. *Assume Σ^1_3 uniformization. Suppose there is a measurable cardinal and let U be a normal ultrafilter witnessing this. Then if there are Σ^1_3 functions $g, h : \mathbb{R} \times \omega \to \mathbb{R}$ such that*

$$e^{xy} = \sum_{n=0}^{\infty} g(x,n)h(y,n)$$

with only finitely many non-zero terms at each (x,y) then $\mathbb{R} = \mathbb{R} \cap L[U]$ and so there is a strongly Δ^1_3 well-ordering of $\mathbb{R}$.

Proof. By inspecting the proof of Lemma 3.3, there exists a finite-to-one Σ^1_3 function $\theta : \mathbb{R} \to \mathbb{R} \cap L[U]$. Since the relation $R \subseteq \mathbb{R} \times \mathbb{N}$ defined by

$$R(y,n) \;\Leftrightarrow\; (\exists x_1, \ldots, x_n) \bigwedge_{i=1}^{n} \theta(x_i) = y \wedge \bigwedge_{i \neq j} x_i \neq x_j$$

is Σ^1_3, it is absolute for transitive models containing U. Suppose that there is $x_1 \in \mathbb{R} \setminus L[U]$ and let $y = \theta(x_1)$. If $n = |\theta^{-1}(y) \cap L[U]|$ then $R(y, n+1)$ holds in V. By absoluteness it holds in $L[U]$, contradicting the fact that $n = |\theta^{-1}(y) \cap L[U]|$. Thus $\mathbb{R} = \mathbb{R} \cap L[U]$ and by [11, 5.2] there is a Δ^1_3 well-ordering of $\mathbb{R}$. ∎

In light of Theorem 3, it is natural to ask the following:

QUESTION 1. If there are Σ^1_2 functions g_n, h_n, $n \in \omega$, such that

$$e^{xy} = \sum_{n=0}^{\infty} g_n(x)h_n(y)$$

with the sum having only finitely many non-zero terms at each (x,y), does the conclusion of Theorem 3 still hold? That is, is it necessary in Theorem 3 that g_n, h_n are Σ^1_2 uniformly in n?

In [10], Shelah shows that the converse in Davies' original theorem does not remain true if we drop the assumption that the sum must have at most finitely many non-zero terms and only require the sum to converge pointwise. In a similar vein we ask:

Question 2. If we drop the finiteness condition, does Theorem 1 still hold?

Shelah also shows in [10] that if we add $\aleph_2$ Cohen reals then there is a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which does not allow a representation

$$f(x, y) = \sum_n^{\infty} g_n(x)h_n(y),$$

even when we allow for the sum to have infinitely many non-zero terms, requiring only that it converges pointwise. As a counterpoint to that result, we point out the following:

Theorem 4. *There is a Borel function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that for no $g_n, h_n : \mathbb{R} \to \mathbb{R}$ that are Baire measurable do we have*

$$f(x, y) = \sum_{n=0}^{\infty} g_n(x)h_n(y)$$

for all $(x, y) \in \mathbb{R}^2$, where the sum converges pointwise, but may have infinitely many non-zero terms. The same holds if we replace Baire measurable by Lebesgue measurable.

Proof. Let as usual E_0 denote the equivalence relation on 2^ω defined by

$$xE_0y \Leftrightarrow (\exists N)(\forall n \geq N)\ x(n) = y(n).$$

Let $\mathbf{1}_{E_0}$ be the characteristic function of E_0. Suppose now that there are Baire measurable $g_n, h_n : \mathbb{R} \to \mathbb{R}$ such that

$$\mathbf{1}_{E_0}(x, y) = \sum_{n=0}^{\infty} g_n(x)h_n(y).$$

Then we can find a dense G_δ set A on which all the functions g_n and h_n are continuous. But then for $x, y \in A$ we have

$$xE_0y \Leftrightarrow (\forall k)(\exists N \geq k)\ \sum_{n=0}^{N} g_n(x)h_n(y) > \frac{1}{2}.$$

This gives us a G_δ definition of E_0 on A, and hence E_0 must be a smooth equivalence relation on A by [3, Corollary 1.2]. But E_0 is not smooth on any comeagre set, and we have a contradiction.

The proof of the Lebesgue measurable case is similar. ∎

Remark. By [5] (see also [4, Exercise 26.2]), if there is a Cohen real (respectively random real) over L in V, then all Δ^1_2 functions are Baire measurable (respectively Lebesgue measurable). Thus it follows that in this setting, $\mathbf{1}_{E_0}$ cannot be represented as an infinite pointwise convergent sum of rectangular Δ^1_2 functions.

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