FUNDAMENTA
MATHEMATICAE
205 (2009)

Open maps having the Bula property

by

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Abstract. An open continuous map f from a space X onto a paracompact C-space Y admits two disjoint closed sets $F_0, F_1 \subset X$ with $f(F_0) = Y = f(F_1)$, provided all fibers of f are infinite and C^*-embedded in X. Applications are given to the existence of "disjoint" usco multiselections of set-valued l.s.c. mappings defined on paracompact C-spaces, and to special type of factorizations of open continuous maps from metrizable spaces onto paracompact C-spaces. This settles several open questions.

1. Introduction. All spaces in this paper are assumed to be completely regular topological spaces. Following Kato and Levin [12], a continuous surjective map $f\colon X \to Y$ is said to have the *Bula property* if there are disjoint closed subsets $F_0, F_1 \subset X$ such that $f(F_0) = Y = f(F_1)$. The pair (F_0, F_1) will be called a *Bula pair* for f. Bula [2] proved that every open continuous map f from a compact Hausdorff space X onto a finite-dimensional metrizable space Y has this property provided all fibers of f are dense in themselves. On the other hand, there are open continuous maps between compact metric spaces with all fibers dense in themselves, but without the Bula property [4] (see, also, [12]).

Bula's result [2] was generalized in [7] to Y countable-dimensional and X either a compact Hausdorff space or a metrizable space. Levin and Rogers [13] obtained a further generalization to the case of X compact metric and Y a C-space. The question whether the compactness condition in Levin–Rogers' result [13] could be removed was raised in [11, Problem 1514]. In this paper, we deal with this question by generalizing all these results from a common point of view where Y is supposed to be only a paracompact C-space rather than metrizable, and the fibers of f to be infinite rather than dense in themselves. The following theorem will be proved.

2000 *Mathematics Subject Classification*: Primary 54F45, 54F35, 54C60, 54C65; Secondary 55M10, 54C35, 54B20.
Key words and phrases: dimension, C-space, set-valued mapping, selection.

DOI: 10.4064/fm205-2-1

THEOREM 1.1. *Let X be a space, Y a paracompact C-space, and $f : X \to Y$ an open continuous surjection with all fibers infinite and C^*-embedded in X. Then f has the Bula property.*

The C-space property was originally defined by W. Haver [9] for compact metric spaces. Later on, Addis and Gresham [1] reformulated Haver's definition for arbitrary spaces: A space X has property C (or X is a *C-space*) if for every sequence $\{\mathscr{W}_n : n < \omega\}$ of open covers of X there exists a sequence $\{\mathscr{V}_n : n < \omega\}$ of pairwise disjoint open families in X such that each $\mathscr{V}_n$ refines $\mathscr{W}_n$, $n < \omega$, and $\bigcup\{\mathscr{V}_n : n < \omega\}$ is a cover of X. It is well-known that every finite-dimensional paracompact space, as well as every countable-dimensional metrizable space, is a C-space [1], but there exists a compact metric C-space which is not countable-dimensional [21]. Finally, let us recall that a subset $A \subset X$ is *C^*-embedded* in X if every bounded real-valued continuous function on A is continuously extendable to the whole of X.

Theorem 1.1 has several interesting applications. In Section 4, relying on the fact that the theorem involves no a priori restrictions on X, we apply it to the graph of an l.s.c. set-valued mapping defined on a paracompact C-space and having closed and infinite point-images in a completely metrizable space. We show that any such mapping has a pair of "disjoint" usco multiselections (see Corollaries 4.3 and 4.4), which provides a complete affirmative solution to [11, Problem 1515] and sheds some light on [11, Problem 1516]. In Section 5, we consider open continuous maps with all fibers dense in themselves, and apply Theorem 1.1 to show that every such map from a complete metric space (X, d) onto a paracompact C-space Y can be represented as the composition $P_Y \circ g$ of a continuous surjective map $g : X \to Y \times [0, 1]$ and the projection $P_Y : Y \times [0, 1] \to Y$ (see Theorem 5.1). This is a common generalization of [7, Theorem 1.1] and [13, Theorem 1.2], and provides a complete affirmative solution to [11, Problem 1512].

Finally, a word about the proof of Theorem 1.1 itself. It is based on an application of Uspenskij's selection theorem [23] that under the assumptions of Theorem 1.1 there exists a continuous function $g : X \to [0, 1]$ which is nonconstant on each fiber of f (see Lemma 2.1). The proof is then accomplished in Section 3 relying on a "parametric" version of an idea in the proof of [13, Theorem 1.3].

2. Bula property and fiber-constant maps. Suppose that (F_0, F_1) is a Bula pair for a (continuous) map $f : X \to Y$, where X is a normal space. Then there exists a continuous function $g : X \to [0, 1]$ such that both $g^{-1}(0)$ and $g^{-1}(1)$ intersect each fiber of f. Indeed, take $g : X \to [0, 1]$ such that $F_i \subset g^{-1}(i)$, $i = 0, 1$. In this section, we demonstrate that the map f in Theorem 1.1 has this property as well.

LEMMA 2.1. *Let X be a space, Y a paracompact C-space, and $f\colon X \to Y$ an open continuous surjection with all fibers infinite and C^*-embedded in X. Then there exists a continuous function $g : X \to [0,1]$ with $g^{-1}(0)$ and $g^{-1}(1)$ intersecting each fiber of f. In particular, $g{\restriction}f^{-1}(y)$ is nonconstant for every $y \in Y$.*

For the proof of Lemma 2.1, we need several statements. We use $C^*(X)$ to denote the Banach space of all continuous bounded functions on a space X equipped with the sup-metric

$$d(g,h) = \sup\{|g(x) - h(x)| : x \in X\}, \qquad g,h \in C^*(X).$$

In fact, we will be mainly interested in the closed convex subset $C(X,\mathbb{I})$ of $C^*(X)$ of all continuous functions from X to $\mathbb{I} = [0,1]$.

The next proposition is well-known and easy to prove.

PROPOSITION 2.2. *Let X be a space and $A \subset X$ a C^*-embedded subset. Then the restriction map $\pi_A : C(X,\mathbb{I}) \to C(A,\mathbb{I})$ is an open continuous surjection.*

For a nonempty subset A of a space X, let $\Theta_X(A,\mathbb{I})$ be the set of all members of $C(X,\mathbb{I})$ which are constant on A, and let $\Theta(A,\mathbb{I}) = \Theta_A(A,\mathbb{I})$. Note that $\Theta(A,\mathbb{I})$ is, in fact, homeomorphic to $\mathbb{I}$.

For spaces S and T, we will use $\Phi : S \rightsquigarrow T$ to denote that Φ is a set-valued mapping, i.e. a map from S into the nonempty subsets of T. A mapping $\Phi : S \rightsquigarrow T$ is *lower semicontinuous*, or l.s.c., if the set

$$\Phi^{-1}(U) = \{s \in S : \Phi(s) \cap U \neq \emptyset\}$$

is open in S for every open $U \subset T$. A map $g : S \to T$ is a *selection* for $\Phi : S \rightsquigarrow T$ if $g(s) \in \Phi(s)$ for every $s \in S$. Finally, recall that a space Z is C^m for some $m \geq 0$ if every continuous image of the k-dimensional sphere $\mathbb{S}^k$ ($k \leq m$) in Z is contractible in Z.

PROPOSITION 2.3. *Let X be a space and $A \subset X$ an infinite C^*-embedded subset. Then the set $C(X,\mathbb{I}) \setminus \Theta_X(A,\mathbb{I})$ is C^m for every $m \geq 0$.*

Proof. Consider the restriction map $\pi_A : C(X,\mathbb{I}) \to C(A,\mathbb{I})$, and take a continuous $g : \mathbb{S}^n \to C(X,\mathbb{I}) \setminus \Theta_X(A,\mathbb{I})$ for some $n \geq 0$. Then, by Proposition 2.2, the composition $\pi_A \circ g : \mathbb{S}^n \to C(A,\mathbb{I}) \setminus \Theta(A,\mathbb{I})$ is also continuous. However, $C(A,\mathbb{I})$ is an infinite-dimensional closed convex subset of $C^*(A)$ because A is infinite, while $\Theta(A,\mathbb{I})$ is one-dimensional, being homeomorphic to $\mathbb{I}$. Thus, by [17, Lemma 2.1], $C(A,\mathbb{I}) \setminus \Theta(A,\mathbb{I})$ is C^m for all $m \geq 0$. Hence, there exists a continuous extension $\ell : \mathbb{B}^{n+1} \to C(A,\mathbb{I}) \setminus \Theta(A,\mathbb{I})$ of $\pi_A \circ g$ over the $(n+1)$-dimensional ball $\mathbb{B}^{n+1}$. Consider the set-valued mapping $\Phi : \mathbb{B}^{n+1} \rightsquigarrow C(X,\mathbb{I})$ defined by $\Phi(t) = \{g(t)\}$ if $t \in \mathbb{S}^n$ and $\Phi(t) = \pi_A^{-1}(\ell(t))$ otherwise. Since g is a selection for $\pi_A^{-1} \circ \ell{\restriction}\mathbb{S}^n$ and, by Proposition 2.2, the restriction map π_A is open, the mapping Φ is l.s.c. (see [15, Examples

1.1* and 1.3*]). Also, Φ is closed and convex-valued in $C(X,\mathbb{I})$, hence in the Banach space $C^*(X)$ as well. Then, by Michael's selection theorem [15, Theorem 3.2″], Φ has a continuous selection $h:\mathbb{B}^{n+1}\to C(X,\mathbb{I})$ which is, in fact, a continuous extension of g over $\mathbb{B}^{n+1}$. Moreover, $\pi_A(h(t))=\ell(t)\notin\Theta(A,\mathbb{I})$ for all $t\in\mathbb{B}^{n+1}$, which completes the proof. ■

A function $\xi:X\to\mathbb{R}$ is *lower* (*upper*) *semicontinuous* if the set

$$\{x\in X:\xi(x)>r\}\qquad(\text{respectively, }\{x\in X:\xi(x)<r\})$$

is open in X for every $r\in\mathbb{R}$. Suppose that $f:X\to Y$ is a surjective map. Then to any $g:X\to\mathbb{I}$ we associate the functions $\inf[g,f],\sup[g,f]:Y\to\mathbb{I}$ defined for $y\in Y$ by

$$\inf[g,f](y)=\inf\{g(x):x\in f^{-1}(y)\},$$
$$\sup[g,f](y)=\sup\{g(x):x\in f^{-1}(y)\}.$$

Finally, we define a function $\operatorname{var}[g,f]:X\to\mathbb{I}$ by

$$\operatorname{var}[g,f](y)=\sup[g,f](y)-\inf[g,f](y),\qquad y\in Y.$$

Observe that $g:X\to\mathbb{I}$ is nonconstant on each fiber of f if and only if $\operatorname{var}[g,f]$ is positive-valued. The following property is well-known [10] (see also [5, 1.7.16]).

PROPOSITION 2.4 ([10]). *Let X and Y be spaces, $f:X\to Y$ an open continuous surjection, and $g\in C(X,\mathbb{I})$. Then* $\sup[g,f]$ *is lower semicontinuous, while* $\inf[g,f]$ *is upper semicontinuous. In particular,* $\operatorname{var}[g,f]$ *is lower semicontinuous.*

Finally, a set-valued mapping $\Phi:S\rightsquigarrow T$ has an *open* (*closed*) graph if its graph

$$\operatorname{Graph}(\Phi)=\{(s,t)\in S\times T:t\in\Phi(s)\}$$

is open (respectively, closed) in $S\times T$.

PROPOSITION 2.5. *Let X and Y be spaces, and let $f:X\to Y$ be an open continuous surjection. Then the set-valued mapping $\Theta:Y\rightsquigarrow C(X,\mathbb{I})$ defined by $\Theta(y)=\Theta_X(f^{-1}(y),\mathbb{I})$, $y\in Y$, has a closed graph.*

Proof. Choose $y\in Y$ and $g\notin\Theta(y)$. Then, $\operatorname{var}[g,f](y)>2\delta$ for some $\delta>0$. By Proposition 2.4, there exists a neighbourhood V of y such that $\operatorname{var}[g,f](z)>2\delta$ for every $z\in V$. Let $B^d_\delta(g)=\{h\in C(X,\mathbb{I}):d(g,h)<\delta\}$, the open δ-neighbourhood of g in $C(X,\mathbb{I})$. Then $V\times B^d_\delta(g)$ is an open set in $Y\times C(X,\mathbb{I})$ disjoint from $\operatorname{Graph}(\Theta)$. Indeed, take $z\in V$ and $h\in B^d_\delta(g)$. Since $\operatorname{var}[g,f](z)>2\delta$, there are $x,t\in f^{-1}(z)$ such that $|g(x)-g(t)|>2\delta$. Since $h\in B^d_\delta(g)$, we have $|h(x)-g(x)|<\delta$ and $|h(t)-g(t)|<\delta$. Hence, $h(x)\neq h(t)$, which implies that $\operatorname{var}[h,f](z)>0$. That is, $h\notin\Theta(z)$. ■

Proof of Lemma 2.1. Define $\Phi : Y \rightsquigarrow C(X,\mathbb{I})$ by $\Phi(y) = C(X,\mathbb{I}) \setminus \Theta(y)$, $y \in Y$, where Θ is as in Proposition 2.5. Then, by Proposition 2.5, Φ has an open graph, while, by Proposition 2.3, each $\Phi(y)$, $y \in Y$, is C^m for all $m \geq 0$. Since Y is a paracompact C-space, by Uspenskij's selection theorem [23, Theorem 1.3], Φ has a continuous selection $\varphi : Y \to C(X,\mathbb{I})$. Define $g : X \to \mathbb{I}$ by $g(x) = [\varphi(f(x))](x)$, $x \in X$. Since f and φ are continuous, so is g (see the proof of [8, Theorem 6.1]). Since $g\restriction f^{-1}(y) = \varphi(y)\restriction f^{-1}(y)$ and $\varphi(y) \notin \Theta(y)$ for every $y \in Y$, g is as required. ∎

3. Proof of Theorem 1.1. Suppose that X, Y and f are as in Theorem 1.1. By Lemma 2.1, there exists a function $g \in C(X,\mathbb{I})$ such that $\inf[g,f](y) < \sup[g,f](y)$ for every $y \in Y$. By Proposition 2.4, $\inf[g,f]$ is upper semicontinuous and $\sup[g,f]$ is lower semicontinuous. Since Y is paracompact, there are continuous functions $\gamma_0, \gamma_1 : Y \to \mathbb{I}$ such that

$$\inf[g,f](y) < \gamma_0(y) < \gamma_1(y) < \sup[g,f](y), \qquad y \in Y$$

(see, e.g., [5, 5.5.20]). Let $\alpha_i = \gamma_i \circ f : X \to \mathbb{I}$, $i = 0, 1$. Then

$$\inf[g,f](f(x)) < \alpha_0(x) < \alpha_1(x) < \sup[g,f](f(x)) \quad \text{for every } x \in X. \tag{3.1}$$

Next, define a continuous function $\ell : X \times \mathbb{I} \to \mathbb{R}$ by

$$\ell(x,t) = \frac{t - \alpha_0(x)}{\alpha_1(x) - \alpha_0(x)}, \qquad (x,t) \in X \times \mathbb{I}.$$

Since $\ell(x, \alpha_0(x)) = 0$ and $\ell(x, \alpha_1(x)) = 1$, and since $\ell\restriction\{x\} \times \mathbb{I}$ is increasing, we have

$$\ell(\{x\} \times [\alpha_0(x), \alpha_1(x)]) = [0,1] \quad \text{for every } x \in X. \tag{3.2}$$

Finally, define a continuous function $h : X \to \mathbb{R}$ by $h(x) = \ell(x, g(x))$, $x \in X$. According to (3.1) and (3.2), for every $y \in Y$,

$$h^{-1}((-\infty, 0]) \cap f^{-1}(y) \neq \emptyset \neq f^{-1}(y) \cap h^{-1}([1, +\infty)).$$

Thus, $F_0 = h^{-1}((-\infty, 0])$ and $F_1 = h^{-1}([1, +\infty))$ are as required. The proof of Theorem 1.1 is complete.

4. Bula pairs and multiselections. In this section, we present several applications of Theorem 1.1 to multiselections of l.s.c. mappings. Recall that a set-valued mapping $\varphi : Y \rightsquigarrow Z$ is called a *multiselection* for $\Phi : Y \rightsquigarrow Z$ if $\varphi(y) \subset \Phi(y)$ for every $y \in Y$.

COROLLARY 4.1. *Let Y be a paracompact C-space, Z a normal space, and $\Phi : Y \rightsquigarrow Z$ an l.s.c. mapping such that each $\Phi(y)$, $y \in Y$, is infinite and closed in Z. Then there exists a closed-graph mapping $\theta : Y \rightsquigarrow Z$ such that $\Phi(y) \cap \theta(y) \neq \emptyset \neq \Phi(y) \setminus \theta(y)$ for every $y \in Y$.*

Proof. Let $X = \mathrm{Graph}(\Phi)$, and let $f : X \to Y$ be the projection. Then f is an open continuous map (because Φ is l.s.c.) with all fibers infinite. Also, each fiber of f is C^*-embedded in X. Indeed, take $y \in Y$ and a continuous function $g : f^{-1}(y) \to \mathbb{I}$. Since $f^{-1}(y) = \{y\} \times \Phi(y)$, we may define a continuous function $g_0 : \Phi(y) \to \mathbb{I}$ by $g_0(z) = g(y, z)$, $z \in \Phi(y)$. Since Z is normal, g_0 has a continuous extension $h_0 : Z \to \mathbb{I}$. Finally, define $h : X \to \mathbb{I}$ by $h(t, z) = h_0(z)$ for every $t \in Y$ and $z \in \Phi(t)$. Then h is a continuous extension of g. Thus, by Theorem 1.1, there are disjoint closed subsets $F_0, F_1 \subset X$ such that $f(F_0) = Y = f(F_1)$. Finally, take a closed set $F \subset Y \times Z$ with $F \cap X = F_0$, and define $\theta : Y \rightsquigarrow Z$ by $\mathrm{Graph}(\theta) = F$. This θ is as required. ■

To prepare for our next application, we need the following simple observation about l.s.c. multiselections of l.s.c. mappings.

Proposition 4.2. *Let Y be a paracompact space, Z a space, $\Phi : Y \rightsquigarrow Z$ an l.s.c. closed-valued mapping, and $\Psi : Y \rightsquigarrow Z$ an open-graph mapping with $\Phi(y) \cap \Psi(y) \neq \emptyset$ for every $y \in Y$. Then there exists a closed-valued l.s.c. mapping $\varphi : Y \rightsquigarrow Z$ such that $\varphi(y) \subset \Phi(y) \cap \Psi(y)$ for every $y \in Y$.*

Proof. Since Φ is l.s.c. and Ψ has an open graph, for every $y \in Y$ there are open sets $V_y \subset Y$ and $W_y \subset Z$ such that $y \in V_y \subset \Phi^{-1}(W_y)$ and $V_y \times \overline{W_y} \subset \mathrm{Graph}(\Psi)$. Whenever $y \in Y$, define a mapping $\varphi_y : V_y \rightsquigarrow Z$ by $\varphi_y(t) = \overline{\Phi(t) \cap W_y}$, $t \in V_y$. According to [15, Propositions 2.3 and 2.4], each φ_y, $y \in Y$, is l.s.c. Since Y is paracompact, there exists a locally finite open cover $\mathscr{U}$ of Y and a map $p : \mathscr{U} \to Y$ such that $U \subset V_{p(U)}$, $U \in \mathscr{U}$. Finally, define a mapping $\varphi : Y \rightsquigarrow Z$ by

$$\varphi(y) = \bigcup\{\varphi_{p(U)}(y) : U \in \mathscr{U} \text{ and } y \in U\}, \qquad y \in Y.$$

This φ is as required. ■

A mapping $\psi : Y \rightsquigarrow Z$ is called *upper semicontinuous*, or u.s.c., if the set

$$\Phi^{\#}(U) = \{y \in Y : \Phi(y) \subset U\}$$

is open in Y for every open $U \subset Z$. We say that a pair $\varphi, \psi : Y \rightsquigarrow Z$ is a *Michael pair* for $\Phi : Y \rightsquigarrow Z$ if φ is compact-valued and l.s.c., ψ is compact-valued and u.s.c., and $\varphi(y) \subset \psi(y) \subset \Phi(y)$ for every $y \in Y$.

The following result provides a complete affirmative solution to [11, Problem 1515].

Corollary 4.3. *Let (Z, ρ) be a metric space, Y a paracompact C-space, and $\Phi : Y \rightsquigarrow Z$ an l.s.c. mapping such that each $\Phi(y)$, $y \in Y$, is infinite and ρ-complete. Then Φ has a Michael pair $(\varphi, \psi) : Y \rightsquigarrow Z$ such that $\Phi(y) \setminus \psi(y) \neq \emptyset$ for every $y \in Y$.*

Proof. By Corollary 4.1, there is a closed-graph mapping $\theta : Y \rightsquigarrow Z$ such that $\Phi(y) \cap \theta(y) \neq \emptyset \neq \Phi(y) \setminus \theta(y)$ for every $y \in Y$. Define $\Psi : Y \rightsquigarrow Z$ by $\mathrm{Graph}(\Psi) = (Y \times Z) \setminus \mathrm{Graph}(\theta)$. Then, by the properties of θ, we have $\Phi(y) \cap \Psi(y) \neq \emptyset$ for every $y \in Y$. Also, Φ has closed and ρ-complete values. Hence, by Proposition 4.2, there exists a closed-valued l.s.c. mapping $\Phi_0 : Y \rightsquigarrow Z$ such that $\Phi_0(y) \subset \Phi(y) \cap \Psi(y)$ for every $y \in Y$. Then Φ_0 also has ρ-complete values and, by a result of [16], it has a Michael pair (φ, ψ). This (φ, ψ) is as required. ∎

We conclude this section with the following further application of Theorem 1.1 that sheds some light on [11, Problem 1516].

COROLLARY 4.4. *Let (Z, ρ) be a metric space, Y a paracompact C-space, and $\Phi : Y \rightsquigarrow Z$ an l.s.c. mapping such that each $\Phi(y)$, $y \in Y$, is infinite and ρ-complete. Then Φ has Michael pairs $(\varphi_i, \psi_i) : Y \rightsquigarrow Z$, $i = 0, 1$, such that $\psi_0(y) \cap \psi_1(y) = \emptyset$ for every $y \in Y$.*

Proof. According to Corollary 4.3, Φ has a Michael pair $(\varphi_0, \psi_0) : Y \rightsquigarrow Z$ such that $\Phi(y) \setminus \psi_0(y) \neq \emptyset$ for every $y \in Y$. Note that ψ_0 has a closed-graph, being u.s.c. Then, just in the proof of Corollary 4.3, there exists a Michael pair $(\varphi_1, \psi_1) : Y \rightsquigarrow Z$ for Φ such that $\psi_1(y) \subset \Phi(y) \setminus \psi_0(y)$, $y \in Y$. These (φ_i, ψ_i), $i = 0, 1$, are as required. ∎

5. Open maps looking like projections. A continuous map $f : X \to Y$ has *dimension* $\leq k$ if all fibers of f have dimension $\leq k$. A continuous map $f : X \to Y$ is *light* if it is 0-dimensional.

Suppose that $f : X \to Y$ is a surjective map. A subset $F \subset X$ will be called a *section* for f if $f(F) = Y$. In particular, we shall say that a section F for f is *open* (*closed*) if F is an open (respectively, closed) subset of X. Let $\Omega(f)$ and $\mathscr{F}(f)$ be the sets of all open (respectively, closed) sections for f.

In this section, we prove the following factorization theorem which is a partial generalization of [7, Theorem 1.1]. It provides a complete affirmative solution to [11, Problem 1512].

THEOREM 5.1. *Let (X, d) a metric space, Y a paracompact C-space, and $f : X \to Y$ an open continuous surjection with each fiber dense in itself and d-complete. Then for every $U \in \Omega(f)$ there exists $H \in \mathscr{F}(f)$ with $H \subset U$, a continuous surjective map $g : X \to Y \times \mathbb{I}$, and a copy $\mathfrak{C} \subset \mathbb{I}$ of the Cantor set such that*

(a) *$f = P_Y \circ g$, where $P_Y : Y \times \mathbb{I} \to Y$ is the projection.*
(b) *$g(H) = Y \times \mathbb{I}$ and each $g^{-1}(y, c) \cap H$, $(y, c) \in Y \times \mathfrak{C}$, is compact and 0-dimensional.*

In particular, $H_{\mathfrak{C}} = H \cap g^{-1}(Y \times \mathfrak{C}) \in \mathscr{F}(f)$ *and* $f \restriction H_{\mathfrak{C}}$ *is a light map with compact fibers.*

To prepare for the proof of Theorem 5.1, we need several statements.

Proposition 5.2. *Let* (X, d) *be a metric space*, Y *a paracompact* C-*space*, *and* $f : X \to Y$ *an open continuous surjection with each fiber dense in itself and* d-*complete. Then for every* $U \in \Omega(f)$ *there are disjoint open sections* $U_0, U_1 \in \Omega(f)$ *such that* $\overline{U_i} \subset U$, $i = 0, 1$.

Proof. Endow U with the compatible metric

$$\rho(x, y) = d(x, y) + \left| \frac{1}{d(x, X \setminus U)} - \frac{1}{d(y, X \setminus U)} \right|, \qquad x, y \in U.$$

Next, define an l.s.c. mapping $\Phi : Y \rightsquigarrow U$ by $\Phi(y) = f^{-1}(y) \cap U$, $y \in Y$. Then each $\Phi(y)$, $y \in Y$, is infinite and ρ-complete in U because each fiber of f is dense in itself and d-complete. Hence, by Corollary 4.4, Φ has compact-valued u.s.c. multiselections $\psi_0, \psi_1 : Y \rightsquigarrow U$ such that $\psi_0(y) \cap \psi_1(y) = \emptyset$ for every $y \in Y$. In fact, ψ_0 and ψ_1 are compact-valued and u.s.c. as mappings from Y into the subsets of X. Hence, each $F_i = \bigcup\{\psi_i(y) : y \in Y\}$, $i = 0, 1$, is a closed subset of X, with $F_i \subset U$ and $f(F_i) = Y$. Since $F_0 \cap F_1 = \emptyset$, we can take disjoint open sets $U_0, U_1 \subset X$ such that $F_i \subset U_i \subset \overline{U_i} \subset U$, $i = 0, 1$. ∎

For convenience, all metrics on metrizable spaces other than the real line will be implicitly assumed to be bounded by 1. Let (X, d) be a metric space and $A \subset X$ a nonempty subset. Whenever $\varepsilon > 0$, let

$$B^d_\varepsilon(A) = \{x \in X : d(x, A) < \varepsilon\}.$$

For every such $A \subset X$ we define

$$\mathrm{td}_d(A) = \sup\{\mathrm{diam}_d(C) : C \subset A \text{ is connected}\},$$
$$\delta(A) = \inf\{\varepsilon > 0 : A \subset B^d_\varepsilon(S) \text{ for some finite } S \subset A\}.$$

Finally, for a surjective map $f : X \to Y$ and a section $U \in \Omega(f)$, let

$$\mathrm{td}_d(U, f) = \sup\{\mathrm{td}_d(f^{-1}(y) \cap U) : y \in Y\},$$
$$\mathrm{mesh}_d(U, f) = \sup\{\delta(f^{-1}(y) \cap U) : y \in Y\}.$$

Lemma 5.3. *Let* (X, d) *be a metric space*, Y *a paracompact* C-*space*, *and* $f : X \to Y$ *an open continuous surjection. Then*, *for every* $\varepsilon > 0$, *each* $G \in \Omega(f)$ *contains an* $U \in \Omega(f)$ *with* $\mathrm{mesh}_d(U, f) \le \varepsilon$ *and* $\mathrm{td}_d(U, f) \le \varepsilon$.

Proof. Let $\varepsilon > 0$ and $G \in \Omega(f)$. For any $y \in Y$ and $n < \omega$, take an open subset $W^n_y \subset G$ such that $y \in f(W^n_y)$ and $\mathrm{diam}_d(W^n_y) < \varepsilon \cdot 2^{-(n+1)}$. Since f is open, each family $\mathscr{W}_n = \{f(W^n_y) : y \in Y\}$, $n < \omega$, is an open cover of Y. Since Y is a paracompact C-space, there now exists a sequence $\{\mathscr{V}_n : n < \omega\}$ of pairwise disjoint open families of Y such that each $\mathscr{V}_n$ refines $\mathscr{W}_n$, $n < \omega$, and $\mathscr{V} = \bigcup\{\mathscr{V}_n : n < \omega\}$ is a locally finite cover of Y. For convenience, for every $n < \omega$, define a map $p_n : \mathscr{V}_n \to Y$ by $V \subset f(W^n_{p_n(V)})$, $V \in \mathscr{V}_n$, and set $U_{p_n(V)} = f^{-1}(V) \cap W^n_{p_n(V)}$. We are going to show that

$$U = \bigcup\{U_{p_n(V)} : V \in \mathscr{V}_n \text{ and } n < \omega\}$$

is as required. Since $\mathscr{V}$ is a cover of Y, U is a section for f, and clearly it is open. For any $y \in Y$, set $\mathscr{V}_y = \{V \in \mathscr{V} : y \in V\}$. Then $\mathscr{V}_y$ is finite and $|\mathscr{V}_y \cap \mathscr{V}_n| \leq 1$ for every $n < \omega$ because each family $\mathscr{V}_n$, $n < \omega$, is pairwise disjoint. Hence, we can enumerate the elements of $\mathscr{V}_y$ as $\{V_k : k \in K(y)\}$ so that $V_k \in \mathscr{V}_k$, $k \in K(y)$, where $K(y) = \{n < \omega : \mathscr{V}_y \cap \mathscr{V}_n \neq \emptyset\}$. Next, set $U_k = U_{p_k(V_k)}$, $k \in K(y)$. Since

$$\operatorname{diam}_d(U_k) < \varepsilon \cdot 2^{-(k+1)} \quad \text{for every } k \in K(y), \tag{5.1}$$

$f^{-1}(y) \cap U \subset B^d_\varepsilon(S)$ for every finite set $S \subset f^{-1}(y) \cap U$ with $S \cap U_k \neq \emptyset$ for all $k \in K(y)$. Thus, $\delta(f^{-1}(y) \cap U) \leq \varepsilon$, which implies that $\operatorname{mesh}_d(U, f) \leq \varepsilon$.

To show finally that $\operatorname{td}_d(U, f) \leq \varepsilon$, take a nonempty connected subset $C \subset f^{-1}(y) \cap U$ and points $x, z \in C$. Since C is connected and $C \subset \bigcup\{U_k : k \in K(y)\}$, there is a sequence $k_1, \ldots, k_m$ of distinct elements of $K(y)$ such that $x \in U_{k_1}$, $z \in U_{k_m}$, and $U_{k_i} \cap U_{k_j} \neq \emptyset$ if and only if $|i - j| \leq 1$ (see [5, 6.3.1]). Therefore, by (5.1),

$$\begin{aligned} d(x, z) &\leq \sum_{i=1}^m \operatorname{diam}_d(U_{k_i}) \leq \sum_{k \in K(y)} \operatorname{diam}_d(U_k) \\ &< \sum_{k \in K(y)} \varepsilon \cdot 2^{-(k+1)} < \varepsilon \cdot \sum_{k=0}^{\infty} 2^{-(k+1)} = \varepsilon. \end{aligned}$$

Consequently, $\operatorname{diam}_d(C) \leq \varepsilon$, which completes the proof. ∎

A partially ordered set $(T, \preceq)$ is called a *tree* if the set $\{s \in T : s \prec t\}$ is well-ordered for every $t \in T$. Here, "$s \prec t$" means that $s \preceq t$ and $s \neq t$. A *chain* η in a tree $(T, \preceq)$ is a subset $\eta \subset T$ which is linearly ordered by $\preceq$. A maximal chain η in T is called a *branch* in T. Let $\mathscr{B}(T)$ denote the set of all branches in T. Following Nyikos [18], for every $t \in T$, we set

$$U(t) = \{\beta \in \mathscr{B}(T) : t \in \beta\}, \tag{5.2}$$

and $\mathscr{U}(T) = \{U(t) : t \in T\}$. It is well-known that $\mathscr{U}(T)$ is a base for a non-Archimedean topology on $\mathscr{B}(T)$ (see [18, Theorem 2.10]). In fact, one can easily see that $s \prec t$ if and only if $U(t) \subset U(s)$, while s and t are incomparable if and only if $U(s) \cap U(t) = \emptyset$. We will refer to $\mathscr{B}(T)$ as a *branch space* if it is endowed with this topology.

For a tree $(T, \preceq)$, let $T(0)$ be the set of all minimal elements of T. Given an ordinal α, if $T(\beta)$ is defined for every $\beta < \alpha$, then we let

$$T\restriction\alpha = \bigcup\{T(\beta) : \beta < \alpha\},$$

and we will use $T(\alpha)$ to denote the minimal elements of $T \setminus (T\restriction\alpha)$. The set $T(\alpha)$ is called the αth *level* of T. The *height* of T is the least ordinal α such that $T\restriction\alpha = T$. In particular, we will say that T is an *α-tree* if its height is α. Finally, we can also define the *height* of an element $t \in T$, denoted by $\mathrm{ht}(t)$, which is the unique ordinal α such that $t \in T(\alpha)$.

In fact, we will be mainly interested in ω-trees, and the following realization of the Cantor set as a branch space. Let S be a set with at least two elements, S^n be the set of all maps $t : n \to S$ (i.e., the nth power of S), and let

$$S^{<\omega} = \bigcup\{S^{n+1} : n < \omega\}.$$

Whenever $t \in S^{<\omega}$, let $\mathrm{dom}(t)$ be the *domain* of t. Consider the partial order $\preceq$ on $S^{<\omega}$ defined for $s, t \in S^{<\omega}$ by $s \preceq t$ if and only if

$$\mathrm{dom}(s) \subset \mathrm{dom}(t) \quad \text{and} \quad t\restriction\mathrm{dom}(s) = s.$$

Then, $(S^{<\omega}, \preceq)$ is an ω-tree whose branch space $\mathscr{B}(S^{<\omega})$ is the Baire space S^ω. In particular, $\mathscr{B}(2^{<\omega})$ is the Cantor set 2^ω.

By Proposition 5.2 and Lemma 5.3, using induction on the levels of the tree $(2^{<\omega}, \preceq)$, we get the following immediate consequence.

Corollary 5.4. *Let (X, d), Y, $f : X \to Y$ and $U \in \Omega(f)$ be as in Theorem* 5.1. *Then there exists a map $h : 2^{<\omega} \to \Omega(f)$ such that, for any distinct $s, t \in 2^{<\omega}$,*

(a) $\overline{h(t)} \subset h(s) \subset U$ *if $s \prec t$,*
(b) $h(s) \cap h(t) = \emptyset$ *if s and t are incomparable,*
(c) $\mathrm{mesh}_d(h(t), f) \le 2^{-\mathrm{ht}(t)}$ *and* $\mathrm{td}_d(h(t), f) \le 2^{-\mathrm{ht}(t)}$.

We first prove the following special case of Theorem 5.1.

Lemma 5.5. *Let (X, d), Y, $f : X \to Y$ and $U \in \Omega(f)$ be as in Theorem* 5.1. *Then there exists $H \in \mathscr{F}(f)$ with $H \subset U$ and a surjective light map $\ell : H \to Y \times \mathfrak{C}$ with compact fibers such that $f\restriction H = P_Y \circ \ell$. In particular, $f\restriction H$ is also a light map with compact fibers.*

Proof. Let $h : 2^{<\omega} \to \Omega(f)$ be as in Corollary 5.4. For any $n < \omega$, consider the nth level of the tree $(2^{<\omega}, \preceq)$, which is, in fact, 2^{n+1}. Set $H_n = h(2^{n+1})$, $n < \omega$, and $H = \bigcap\{H_n : n < \omega\}$. By Corollary 5.4(a), $\overline{H}_{n+1} \subset H_n \subset U$ for every $n < \omega$ because each level of $2^{<\omega}$ is finite. Hence, H is a closed subset of X, with $H \subset U$.

Let us see that H is a section for f. Indeed, pick $y \in Y$ and a branch $\beta \in \mathscr{B}(2^{<\omega})$. Then each $H_t(y) = h(t) \cap f^{-1}(y)$, $t \in \beta$, is a nonempty subset of $f^{-1}(y)$ (because $h(t) \in \Omega(f)$) such that $\overline{H_t(y)} \subset H_s(y)$ for $s \prec t$ (by Corollary 5.4(a)) and $\lim_{t\in\beta} \delta(H_t(y)) = 0$ (by Corollary 5.4(c)). Hence, by [7, Lemma 3.2], $H_\beta(y) = \bigcap\{H_t(y) : t \in \beta\}$ is a nonempty compact subset of X. Clearly, $H_\beta(y) \subset H \cap f^{-1}(y)$, which completes the verification that H is a section for f.

In fact, this defines a compact-valued mapping $\varphi : Y \times \mathscr{B}(2^{<\omega}) \rightsquigarrow H$ by letting $\varphi(y,\beta) = H_\beta(y) = \bigcap\{h(t) \cap f^{-1}(y) : t \in \beta\}$ for $(y,\beta) \in Y \times \mathscr{B}(2^{<\omega})$. Since $\varphi(y,\beta) \subset f^{-1}(y)$ for $(y,\beta) \in Y \times \mathscr{B}(2^{<\omega})$, the mapping φ is the inverse ℓ^{-1} of a surjective single-valued map $\ell : H \to Y \times \mathscr{B}(2^{<\omega})$. Also, $\ell(x) = (y,\beta)$ if and only if $x \in \varphi(y,\beta) \subset f^{-1}(y)$, hence $f{\restriction}H = P_Y \circ \ell$.

To show that ℓ is continuous and light, take an open set $V \subset Y$ and $t \in 2^{<\omega}$, and let $U(t)$ be as in (5.2). Then $h(t)$ is an open set in X such that, by Corollary 5.4(b), $\ell^{-1}(y,\beta) = \varphi(y,\beta) \subset h(t)$ if and only if $t \in \beta$ (i.e., $\beta \in U(t)$). Consequently, $\ell^{-1}(V \times U(t)) = f^{-1}(V) \cap h(t) \cap H$ is open in H. Finally, take a nonempty connected subset $C \subset \ell^{-1}(y,\beta) = \varphi(y,\beta)$ for some $y \in Y$ and a branch $\beta \in \mathscr{B}(2^{<\omega})$. Then $C \subset h(t) \cap f^{-1}(y)$ for every $t \in \beta$, and therefore, by Corollary 5.4(c), $\operatorname{diam}_d(C) = 0$. Hence, C is a singleton, which implies that $\ell^{-1}(y,\beta)$ is 0-dimensional, being compact.

To show finally that $f{\restriction}H$ is a light map with compact fibers, pick $y \in Y$ and observe that $\ell{\restriction}(f^{-1}(y) \cap H)$ is perfect. Indeed, take a branch $\beta \in \mathscr{B}(2^{<\omega})$ and a neighbourhood W of $\ell^{-1}(y,\beta)$ in X. Then, by [7, Lemma 3.2], there exists $t \in \beta$ with $H_t(y) = h(t) \cap f^{-1}(y) \subset W$. In this case, $\ell^{-1}(y,\gamma) \subset W$ for every $\gamma \in U(t)$, where $U(t)$ is as in (5.2). Namely, $\gamma \in U(t)$ implies that $t \in \gamma$, and therefore $\ell^{-1}(y,\gamma) \subset H_t(y) \subset W$. Thus, $\ell{\restriction}(f^{-1}(y) \cap H)$ is perfect and $f^{-1}(y) \cap H = \ell^{-1}(\{y\} \times \mathscr{B}(2^{<\omega}))$ is compact because so is $\mathscr{B}(2^{<\omega})$. Since $\mathscr{B}(2^{<\omega})$ is zero-dimensional and ℓ is a light map, according to the classical Hurewicz theorem (see [6]), this also implies that $\dim(f^{-1}(y) \cap H) = 0$, which completes the proof. ∎

Proof of Theorem 5.1. We repeat the arguments of [2, Theorem 1]. Briefly, let (X,d), Y, $f : X \to Y$ and $U \in \Omega(f)$ be as in Theorem 5.1. By Lemma 5.5, there exists $H \in \mathscr{F}(f)$ with $H \subset U$ and a continuous surjective map $\ell : H \to Y \times \mathfrak{C}$ such that $f{\restriction}H$ is a light map with compact fibers, and $f{\restriction}H = P_Y \circ \ell$. Take a continuous surjective map $\rho : \mathfrak{C} \to \mathbb{I}$ such that the set

$$D = \{t \in \mathbb{I} : |\rho^{-1}(t)| > 1\}$$

is countable. Also, let $P_{\mathfrak{C}} : Y \times \mathfrak{C} \to \mathfrak{C}$ be the projection. Then, using the Tietze–Urysohn theorem, extend $\rho \circ P_{\mathfrak{C}} \circ \ell$ to a continuous map $u : X \to \mathbb{I}$. In this way, we have

$$u(f^{-1}(y) \cap H) = \mathbb{I} \quad \text{for every } y \in Y.$$

Then we can define our $g : X \to Y \times \mathbb{I}$ by $g(x) = (f(x), u(x))$, $x \in X$. As for the second part of Theorem 5.1, take a copy $\mathfrak{C}$ of the Cantor set in $\mathbb{I} \setminus D$, which is possible because D is countable. Then, by the properties of ρ, we have $g^{-1}(Y \times \{c\}) \cap H = \ell^{-1}(Y \times \{c\})$ for every $c \in \mathfrak{C}$. Hence, Lemma 5.5 completes the proof. ∎

We finish this paper with some applications of Theorem 5.1. For a space X, let $\mathscr{F}(X)$ be the set of all nonempty closed subsets of X. Recall that the *Vietoris topology* τ_V on $\mathscr{F}(X)$ is generated by all collections of the form

$$\langle \mathscr{V} \rangle = \left\{ S \in \mathscr{F}(X) : S \subset \bigcup \mathscr{V} \text{ and } S \cap V \neq \emptyset \text{ whenever } V \in \mathscr{V} \right\},$$

where $\mathscr{V}$ runs over the finite families of open subsets of X. In what follows, any subset $\mathscr{D} \subset \mathscr{F}(X)$ will carry the relative Vietoris topology τ_V as a subspace of $(\mathscr{F}(X), \tau_V)$. In fact, we will be mainly interested in the subset

$$\mathscr{F}(f) = \{H \in \mathscr{F}(X) : f(H) = Y\}$$

of all closed sections of a surjective map $f : X \to Y$.

Corollary 5.6. *Let (X, d) be a metric space, Y a paracompact C-space, and $f : X \to Y$ an open continuous surjection with each fiber dense in itself and d-complete. Then the set*

$$\mathscr{L}(f) = \{H \in \mathscr{F}(f) : f{\restriction}H \text{ is a light map with compact fibers}\}$$

is dense in $\mathscr{F}(f)$ with respect to the Vietoris topology τ_V.

Proof. Take a closed section $F \in \mathscr{F}(f)$ and a finite family $\mathscr{U}$ of open subsets of X with $F \in \langle \mathscr{U} \rangle$. Then $U = \bigcup \mathscr{U}$ is an open section for f, so, by Theorem 5.1, it contains a closed section $H \subset U$ such that $f{\restriction}H$ is a light map with compact fibers. Take a finite set $S \in \langle \mathscr{U} \rangle$, and then set $Z = H \cup S$. Clearly, $Z \in \mathscr{L}(f) \cap \langle \mathscr{U} \rangle$, which completes the proof. ∎

Proposition 5.7. *Whenever Y is a metrizable space, there exists a closed 0-dimensional subset $A \subset Y \times \mathfrak{C}$ such that $P_Y(A) = Y$, where $P_Y : Y \times \mathfrak{C} \to Y$ is the projection.*

Proof. We follow the idea of [22, Lemma 4.1]. Fix a 0-dimensional metrizable space M and a perfect surjective map $h \colon M \to Y$. By [19, Proposition 9.1], there exists a continuous map $g \colon M \to Q$, where Q is the Hilbert cube, such that the diagonal map $h \triangle g \colon M \to Y \times Q$ is an embedding. Next, take a *Milyutin map* $p \colon \mathfrak{C} \to Q$, i.e. a surjective continuous map admitting an averaging operator between the function spaces $C(\mathfrak{C}) = C^*(\mathfrak{C})$ and $C(Q) = C^*(Q)$ (see [20]). According to [3], there exists a compact-valued l.s.c. mapping $\varphi \colon Q \rightsquigarrow \mathfrak{C}$ such that $\varphi(z) \subset p^{-1}(z)$ for all $z \in Q$. By Michael's 0-dimensional selection theorem [14], there is a continuous map $\ell : M \to \mathfrak{C}$ with $\ell(x) \in \varphi(g(x))$ for any $x \in M$. Then $h \triangle \ell$ embeds M as a closed subset A of $Y \times \mathfrak{C}$. Obviously, A is 0-dimensional and $P_Y(A) = Y$. ∎

Corollary 5.8. *Let X be a metrizable space, Y a metrizable C-space, and $f : X \to Y$ an open continuous perfect surjection with each fiber dense in itself. Then the set*

$$\mathscr{F}_0(f) = \{H \in \mathscr{F}(f) : \dim(H) = 0\}$$

is dense in $\mathscr{F}(f)$ with respect to the Vietoris topology τ_V.

Proof. Take a closed section $F \in \mathscr{F}(f)$, and a finite family $\mathscr{U}$ of open subsets of X with $F \in \langle \mathscr{U} \rangle$. Then $U = \bigcup \mathscr{U}$ is an open section for f, so, by Theorem 5.1, there exists $H \in \mathscr{F}(f)$ with $H \subset U$, a continuous surjective map $g : X \to Y \times \mathbb{I}$, and a copy $\mathfrak{C} \subset \mathbb{I}$ of the Cantor set such that $f = P_Y \circ g$, $g(H) = Y \times \mathbb{I}$ and $f \restriction (H \cap g^{-1}(Y \times \mathfrak{C}))$ is a light map. By Proposition 5.7, $Y \times \mathfrak{C}$ contains a closed 0-dimensional set A with $P_Y(A) = Y$. Finally, take $B = H \cap g^{-1}(A)$, which is a closed section for f because $P_Y(A) = Y$. Since f is perfect, so is g. Hence, $g \restriction B$ is a perfect light map, and according to the classical Hurewicz theorem, $\dim(B) = 0$. Then $Z = B \cup S \in \mathscr{F}_0(f) \cap \langle \mathscr{U} \rangle$ for some (every) finite set $S \in \langle \mathscr{U} \rangle$. ■

Let us remark that Corollary 5.8 is related to a result of Levin and Rogers [13, Theorem 1.2] asserting that under the additional assumption of compactness of X the set $\mathscr{F}_0(f)$ is a dense G_δ-subset of $\mathscr{F}(f)$ with respect to the Vietoris topology. Indeed, the G_δ-property of $\mathscr{F}_0(f)$ for a compact metric space X follows by routine arguments, regardless of the properties of Y.

Acknowledgements. The authors would like to express their gratitude to the referee for several valuable remarks and suggestions.

Research of the first author is supported in part by the NRF of South Africa. The second author was partially supported by NSERC Grant 261914-03.

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Received 24 April 2007;
in revised form 19 July 2009