

Characterization of diffeomorphisms that are symplectomorphisms

by

Stanisław Janeczko and Zbigniew Jelonek (Warszawa)

Abstract. Let (X, ω_X) and (Y, ω_Y) be compact symplectic manifolds (resp. symplectic manifolds) of dimension $2n > 2$. Fix $0 < s < n$ (resp. $0 < k \leq n$) and assume that a diffeomorphism $\Phi : X \rightarrow Y$ maps all $2s$ -dimensional symplectic submanifolds of X to symplectic submanifolds of Y (resp. all isotropic k -dimensional tori of X to isotropic tori of Y). We prove that in both cases Φ is a conformal symplectomorphism, i.e., there is a constant $c \neq 0$ such that $\Phi^* \omega_Y = c \omega_X$.

1. Introduction and main results. Let (X, ω_0) be the standard symplectic vector space over \mathbb{R} of dimension $2n$, i.e., $X \cong \mathbb{R}^{2n}$ and $\omega_0 = \sum_i dx_i \wedge dy_i$ is the standard non-degenerate skew-symmetric form on X . The group of automorphisms of (X, ω_0) is called the symplectic group and is denoted by $\mathbf{Sp}(X)$.

Linear symplectomorphisms of (X, ω_0) are characterized in [6] as linear automorphisms of X preserving some minimal, complete data defined by ω_0 on systems of linear subspaces. In this way the linear symplectic group $\mathbf{Sp}(X)$ may be characterized geometrically together with its natural conformal and anti-symplectic extensions. It is the natural task to put the linear considerations into a more general nonlinear context (cf. [5], [7]).

Let (X, ω_X) and (Y, ω_Y) be symplectic manifolds of dimension $2n$ (all manifolds in this paper are assumed to be smooth and connected). We say that a diffeomorphism $F : X \rightarrow Y$ is a *conformal symplectomorphism* (cf. [10]) if there is a non-zero constant $c \in \mathbb{R}$ such that $F^* \omega_Y = c \omega_X$.

Here is our main result:

THEOREM 1. *Let (X, ω_X) and (Y, ω_Y) be symplectic manifolds of dimension $2n > 2$ (resp. compact symplectic manifolds of dimension $2n > 2$). Fix a number $1 < k \leq n$ (resp. $0 < s < n$). Assume that $\Phi : X \rightarrow Y$ is a diffeomorphism which transforms all k -dimensional isotropic tori of X (resp.*

2s-dimensional symplectic, closed submanifolds of X) onto isotropic tori of Y (resp. symplectic, closed submanifolds of Y). Then Φ is a conformal symplectomorphism.

In other words, for any fixed k (or s) as above, the conformal symplectic structure on X is uniquely determined by the family of all k -dimensional isotropic tori (resp. 2s-dimensional, closed symplectic submanifolds) of X . The proof is based on a number of results which we state below.

First we show that any element of $\mathbf{Sp}(\mathbb{R}^{2n}, \omega_0)$ can be finitely decomposed into elements of a family of elementary automorphisms.

THEOREM 2. *The symplectic group $\mathbf{Sp}(\mathbb{R}^{2n}, \omega_0)$ is generated by the family of elementary symplectic automorphisms:*

$$\{L_i(c_i), L_{ij}(c_{ij}), R_i(d_i), R_{ij}(d_{ij}) : 0 < i < j \leq n, c_i, c_{ij}, d_i, d_{ij} \in \mathbb{R}\}$$

defined by

$$\begin{aligned} L_i(c_i)(x_1, \dots, x_n, y_1, \dots, y_n) &= (x_1, \dots, x_n, y_1, \dots, y_{i-1}, y_i + c_i x_i, y_{i+1}, \dots, y_n), \\ L_{ij}(c_{ij})(x_1, \dots, x_n, y_1, \dots, y_n) &= (x_1, \dots, x_n, y_1, \dots, y_{i-1}, y_i + c_{ij} x_j, y_{i+1}, \dots, y_{j-1}, y_j + c_{ij} x_i, y_{j+1}, \dots, y_n), \\ R_i(d_i)(x_1, \dots, x_n, y_1, \dots, y_n) &= (x_1, \dots, x_{i-1}, x_i + d_i y_i, x_{i+1}, \dots, x_n, y_1, \dots, y_n), \\ R_{ij}(d_{ij})(x_1, \dots, x_n, y_1, \dots, y_n) &= (x_1, \dots, x_{i-1}, x_i + d_{ij} y_j, x_{i+1}, \dots, x_{j-1}, x_j + d_{ij} y_i, x_{j+1}, \dots, x_n, y_1, \dots, y_n). \end{aligned}$$

Let $H : \mathbb{R}^{2n} \times \mathbb{R} \ni (z, t) \mapsto H(z, t) \in \mathbb{R}$ be a smooth function and consider the system of differential equations (Hamiltonian system)

$$\frac{d}{dt} \phi(t, z) = J_0(\nabla_z H)(\phi(t, z), t), \quad \phi(0, z) = z,$$

where $z = (x_1, \dots, x_n, y_1, \dots, y_n)$ and J_0 is the $2n \times 2n$ matrix of ω_0 . Then for the smooth solution $\phi(t, z)$ the map $\Phi(z) = \phi(1, z)$ is a diffeomorphism preserving ω_0 , i.e. a symplectomorphism, which is called the *hamiltonian symplectomorphism* with Hamiltonian H (cf. [11]). The next basic result we get is

THEOREM 3. *For any linear symplectomorphism $L : (\mathbb{R}^{2n}, \omega_0) \rightarrow (\mathbb{R}^{2n}, \omega_0)$ there exists a polynomial Hamiltonian*

$$H_L(z, t) = \sum_{i,j=1}^{2n} a_{i,j}(t) z_i z_j,$$

where $a_{i,j}(t) \in \mathbb{R}[t]$ are polynomials of t . Moreover, H_L can be computed effectively.

Recall that a submanifold Z of a symplectic manifold X is *isotropic* if $\omega|_{TZ} = 0$. We will call Z a *symplectic submanifold* of X if it is closed and the

pair $(Z, \omega|_{TZ})$ is a symplectic manifold. Existence of isotropic or symplectic submanifolds with prescribed data in a compact symplectic manifold is a fundamental geometric property of symplectic structures (cf. [3]). In fact, we can find such submanifolds which satisfy some extra conditions:

THEOREM 4. *Let (X, ω) be a symplectic manifold of dimension $2n$ (resp. compact symplectic manifold of dimension $2n$). Let a_1, \dots, a_m be a family of points of X . Take $0 < k \leq n$ (resp. $0 < s \leq n$). For every $i = 1, \dots, m$ choose a linear k -dimensional isotropic subspace (resp. a $2s$ -dimensional symplectic subspace) $H_i \subset T_{a_i}X$. Then there is a closed isotropic k -dimensional torus (resp. a closed symplectic $2s$ -dimensional submanifold) $Y \subset X$ such that*

$$a_i \in Y, \quad T_{a_i}Y = H_i, \quad i = 1, \dots, m.$$

The purpose of this article is twofold: First, we provide the new results in still basic linear symplectic geometry, formulated above. Second, we characterize general symplectic manifolds and their structure groups through the family of isotropic or symplectic submanifolds and their basic invariance. This leads to a complete geometric characterization of symplectomorphisms and to a reinterpretation of symplectomorphisms as diffeomorphisms acting purely on isotropic or symplectic submanifolds (cf. [2], [9]). We prove an isotropic version of our results for isotropic tori as a smallest optimal (complete for the invariance considered) family of isotropic submanifolds. The corresponding invariance result on symplectic submanifolds of a compact symplectic manifold, following from the main theorem, reads as follows.

THEOREM 5. *Let (X, ω) be a compact symplectic manifold of dimension $2n > 2$. Fix a number $0 < k < n$. Assume that $\Phi : X \rightarrow X$ is a diffeomorphism which maps all $2k$ -dimensional symplectic submanifolds of X onto submanifolds of the same type. Then Φ is a symplectomorphism or an antisymplectomorphism, i.e., $\Phi^*\omega = \pm\omega$. If Φ preserves orientation and n is odd, then Φ is a symplectomorphism. Moreover, if n is even, then Φ has to preserve orientation.*

2. Generators of the group $\mathbf{Sp}(\mathbb{R}^{2n})$. Here we recall some basic facts about the linear symplectic group. Let (X, ω) be a symplectic vector space. There exists a basis of X , called a *symplectic basis*, $u_1, \dots, u_n, v_1, \dots, v_n$, such that

$$\omega(u_i, u_j) = \omega(v_i, v_j) = 0, \quad \omega(u_i, v_j) = \delta_{ij}.$$

Let (X, ω_X) and (Y, ω_Y) be symplectic vector spaces. We say that a linear isomorphism $F : X \rightarrow Y$ is a *symplectomorphism* (or is *symplectic* on X) if $F^*\omega_Y = \omega_X$, i.e., $\omega_X(x, y) = \omega_Y(F(x), F(y))$ for every $x, y \in X$. The group of symplectomorphisms of (X, ω) is called the *symplectic group* and is denoted by $\mathbf{Sp}(X, \omega)$. Via a symplectic basis, X can be identified with

the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$ and $\mathbf{Sp}(X, \omega)$ can be identified with the group of $2n \times 2n$ real matrices A which satisfy $A^T J_0 A = J_0$, where J_0 is the $2n \times 2n$ matrix of ω_0 (in the standard basis), i.e.,

$$J_0 = \begin{bmatrix} 0 & \dots & 0 & -1 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & -1 \\ 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \end{bmatrix}.$$

We can define the following “elementary” symplectomorphisms:

$$\begin{aligned} L_i(c_i)(x_1, \dots, x_n, y_1, \dots, y_n) &= (x_1, \dots, x_n, y_1, \dots, y_{i-1}, y_i + c_i x_i, y_{i+1}, \dots, y_n), \\ L_{ij}(c_{ij})(x_1, \dots, x_n, y_1, \dots, y_n) &= (x_1, \dots, x_n, y_1, \dots, y_{i-1}, y_i + c_{ij} x_j, y_{i+1}, \dots, y_{j-1}, y_j + c_{ij} x_i, y_{j+1}, \dots, y_n), \\ R_i(d_i)(x_1, \dots, x_n, y_1, \dots, y_n) &= (x_1, \dots, x_{i-1}, x_i + d_i y_i, x_{i+1}, \dots, x_n, y_1, \dots, y_n), \\ R_{ij}(d_{ij})(x_1, \dots, x_n, y_1, \dots, y_n) &= (x_1, \dots, x_{i-1}, x_i + d_{ij} y_j, x_{i+1}, \dots, x_{j-1}, x_j + d_{ij} y_i, x_{j+1}, \dots, x_n, y_1, \dots, y_n), \end{aligned}$$

where c_i, c_{ij}, d_i, d_{ij} are real numbers and $1 \leq i < j \leq n$.

We have the following basic result:

THEOREM 2.1. *Let $X = (\mathbb{R}^{2n}, \omega_0)$ be the standard symplectic vector space. Then the group $\mathbf{Sp}(X)$ is generated by the following family of elementary symplectomorphisms:*

$\{L_i(c_i), L_{ij}(c_{ij}), R_i(d_i), R_{ij}(d_{ij}) : 0 < i < j \leq n \text{ and } c_i, c_{ij}, d_i, d_{ij} \in \mathbb{R}\}$,
i.e. if $g \in \mathbf{Sp}(X)$ then $g = \prod_{i=1}^m e_i$, where e_i is one of the elementary symplectomorphisms and $m \in \mathbb{N}$.

Proof. We reason by induction. For $n = 1$ we have $\mathbf{Sp}(\mathbb{R}^2) = \mathbf{SL}(2)$ and the result is well known from linear algebra. Assume $n > 1$.

Let $S : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be a linear symplectomorphism. Denote the coordinates by $x_1, \dots, x_n, y_1, \dots, y_n$ (where $\omega_0 = \sum_i dx_i \wedge dy_i$). We have

$$S(x_1, y_1, \dots, x_n, y_n) = \left(\sum_i a_{1i} x_i + \sum_j b_{1j} y_j, \dots, \sum_i a_{2n,i} x_i + \sum_j b_{2n,j} y_j \right).$$

Observe how the rows of the matrix of S are transformed under the composition $S \circ L$ with an elementary symplectomorphism L (for simplicity we consider only the first row). After composition with $L_i(c)$ we have

$$(1) \quad (a_{11}, \dots, a_{1n}, b_{11}, \dots, b_{1n}) \mapsto (a_{11}, \dots, a_{1i} + cb_{1i}, \dots, a_{1n}, b_{11}, \dots, b_{1n}),$$

with $L_{ij}(c)$ we have

$$(2) \quad (a_{11}, \dots, a_{1n}, b_{11}, \dots, b_{1n}) \\ \mapsto (a_{11}, \dots, a_{1i} + cb_{1j}, \dots, a_{1j} + cb_{1i}, \dots, a_{1n}, b_{11}, \dots, b_{1n}),$$

with $R_i(c)$ we have

$$(3) \quad (a_{11}, \dots, a_{1n}, b_{11}, \dots, b_{1n}) \mapsto (a_{11}, \dots, a_{1n}, b_{11}, \dots, b_{1i} + ca_{1i}, \dots, b_{1n}),$$

and with $R_{ij}(c)$ we have

$$(4) \quad (a_{11}, \dots, a_{1n}, b_{11}, \dots, b_{1n}) \\ \mapsto (a_{11}, \dots, a_{1n}, b_{11}, \dots, b_{1i} + ca_{1j}, \dots, b_{1j} + ca_{1i}, \dots, b_{1n}).$$

Transformations (1)–(4) will be called *elementary operations*. Now we show that using only elementary operations we can transform the first row of S into $(1, 0, \dots, 0)$ and the second into $(0, \dots, 0, 1, 0, \dots, 0)$ (here the unit corresponds to b_{1n}).

First note that the rows $\mathbf{r}_1, \dots, \mathbf{r}_{2n}$ of the matrix S form a symplectic basis. Now, consider the first row. Of course it has a non-zero element, say b_{1s} . Using $L_s(c)$ we can assume that also $a_{1s} \neq 0$. Now using $L_{is}(c)$ and $R_{js}(d)$ for sufficiently general c and d we can assume that all elements of the first row are non-zero. Again applying $R_i(c)$ for $i > 1$ we can now transform the first row into $(a_{11}, \dots, a_{1n}, 1, 0, \dots, 0)$. Using $L_{1j}(c)$ we can transform this row into $(1, 0, \dots, 0, 1, 0, \dots, 0)$ and finally using $R_1(-1)$ we obtain $(1, 0, \dots, 0)$. Now consider the row \mathbf{r}_{n+1} (after these transformations): $\mathbf{r}_{n+1} = (a_{n+1,1}, \dots, a_{n+1,n}, b_{n+1,1}, \dots, b_{n+1,n})$. We can apply our method to the subrow $(a_{n+1,2}, \dots, a_{n+1,n}, b_{n+1,2}, \dots, b_{n+1,n})$ (if it is non-zero) and finally obtain the row $(a_{n+1,1}, 1, 0, \dots, 0, b_{n+1,1}, 0, \dots, 0)$ (or $(a_{n+1,1}, 0, \dots, 0, b_{n+1,1}, 0, \dots, 0)$). Since the value of ω_0 on these two rows is 1 we conclude that $b_{n+1,1} = 1$. Now (in the first case) we can use $L_{12}(-1)$ to obtain a row of the form $(a_{n+1,1}, 0, \dots, 0, 1, 0, \dots, 0)$. Finally, applying $L_1(-a_{12})$ we get $(0, \dots, 0, 1, 0, \dots, 0)$.

Thus under all these compositions the matrix of S has the form

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & \dots & 0 \\ a_{21} & a_{22} & a_{23} & \dots & b_{21} & \dots & b_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & b_{31} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \dots & 0 \\ a_{n+1,1} & a_{n+1,2} & a_{n+1,3} & \dots & b_{n+1,1} & \dots & b_{n+1,n} \\ a_{n+2,1} & a_{n+2,2} & a_{n+2,3} & \dots & b_{n+2,1} & \dots & b_{4n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2n,1} & a_{2n,2} & a_{2n,3} & \dots & b_{2n,1} & \dots & b_{2n,n} \end{bmatrix}.$$

For $j \neq 1, n + 1$ we have $\omega_0(\mathbf{r}_1, \mathbf{r}_j) = 0$ and $\omega_0(\mathbf{r}_{n+1}, \mathbf{r}_j) = 0$. We can easily conclude that for all such j the elements a_{j1} and b_{j1} in S are 0. This implies that the matrix

$$\begin{bmatrix} a_{22} & a_{23} & \dots & b_{22} & \dots & b_{2n} \\ a_{32} & a_{33} & \dots & b_{32} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n2} & a_{n3} & \dots & b_{n2} & \dots & b_{nn} \\ a_{n+2,2} & a_{n+2,3} & \dots & b_{n+2,2} & \dots & b_{n+2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2n,2} & a_{2n,3} & \dots & b_{2n,2} & \dots & b_{2n,n} \end{bmatrix}$$

is a symplectic matrix and we can apply the induction hypothesis. ■

We conclude this section by recalling (and extending) some result from [6].

DEFINITION 2.2. Let $\mathcal{A}_{l,2r} \subset G(l, 2n)$ denote the set of all l -dimensional linear subspaces of X on which the form ω has rank $\leq 2r$.

Of course $\mathcal{A}_{l,2r} \subset \mathcal{A}_{l,2r+2}$ if $2r + 2 \leq l$. We have the following important (see [6, Theorem 6.2]):

THEOREM 2.3. *Let (X, ω) be a symplectic vector space of dimension $2n$ and let $F : X \rightarrow X$ be a linear automorphism. Let $0 < 2r < 2n$. Assume F maps $\mathcal{A}_{2r,2r-2}$ into $\mathcal{A}_{2r,2r-2}$. Then there is a non-zero constant c such that $F^*\omega = c\omega$.*

From Theorem 2.3 we can deduce the following interesting facts:

PROPOSITION 2.4. *Let (X, ω_X) and (Y, ω_Y) be symplectic vector spaces of dimension $2n$ and let $F : X \rightarrow Y$ be a linear isomorphism. Fix a number s with $0 < s < n$ and assume that F maps all $2s$ -dimensional symplectic subspaces of X onto symplectic subspaces of Y . Then there is a non-zero constant c such that $F^*\omega_Y = c\omega_X$.*

Proof. Via a symplectic basis we can assume that $(X, \omega_X) \cong (\mathbb{R}^{2n}, \omega_0) \cong (Y, \omega_Y)$. By assumption the mapping F^* induced by F maps the set $A = \mathcal{A}_{2s,2s} \setminus \mathcal{A}_{2s,2s-2}$ into A . Of course $F^* : A \rightarrow A$ is an injection. Since A is a smooth algebraic variety and F^* is regular, the Borel Theorem (see [1]) implies that F^* is a bijection. This means that F maps $\mathcal{A}_{2s,2s-2}$ into the same set, and we conclude the proof by applying Theorem 2.3. ■

PROPOSITION 2.5. *Let (X, ω_X) and (Y, ω_Y) be symplectic vector spaces of dimension $2n$ and let $F : X \rightarrow Y$ be a linear isomorphism. Fix a number k with $1 < k \leq n$ and assume that F maps all k -dimensional isotropic sub-*

spaces of X onto isotropic subspaces of Y . Then there is a non-zero constant c such that $F^*\omega_Y = c\omega_X$.

Proof. For $k = 2$ this follows immediately from Theorem 2.3. Assume that $k > 2$. Take a plane H belonging to $\mathcal{A}_{2,0}$. Since H is isotropic, we can extend H to a k -dimensional isotropic subspace L . By the assumption, L is mapped onto an isotropic subspace $F(L)$. Observe that $F(H)$ is contained in $F(L)$, so $F(H)$ is also isotropic. In particular, $F(H) \in \mathcal{A}_{2,0}$. Then Theorem 2.3 yields the statement. ■

We end this section with:

PROPOSITION 2.6. *Let X be a vector space of dimension $2n$ and let ω_1, ω_2 be two symplectic forms on X . If $\mathbf{Sp}(X, \omega_1) \subset \mathbf{Sp}(X, \omega_2)$, then there exists a non-zero constant c such that $\omega_2 = c\omega_1$.*

Proof. If $n = 1$, then the conclusion is obvious. Assume that $n > 1$. Let \mathcal{A}_1 (resp. \mathcal{A}_2) be the set of all ω_1 (resp. ω_2) symplectic 2-dimensional subspaces of X . These sets are open and dense in the Grassmannian $G(2, 2n)$. Hence $\mathcal{A}_1 \cap \mathcal{A}_2 \neq \emptyset$. Take $H \in \mathcal{A}_1 \cap \mathcal{A}_2$. We have $\mathcal{A}_1 = \mathbf{Sp}(X, \omega_1)H \subset \mathbf{Sp}(X, \omega_2)H = \mathcal{A}_2$. Now apply Proposition 2.4 to $X = (X, \omega_1)$, $Y = (X, \omega_2)$ and $F = \text{identity}$. ■

3. Hamiltonian symplectomorphisms. Let $X = (\mathbb{R}^{2n}, \omega_0)$ be the standard symplectic vector space. In X we consider the norm $\|(a_1, \dots, a_{2n})\| = \max_{i=1}^{2n} |a_i|$. Take a smooth function $H : X \times \mathbb{R} \ni (z, t) \mapsto H(z, t) \in \mathbb{R}$ and consider a system of differential equations (cf. [4])

$$\frac{d}{dt}\phi(t, z) = J_0(\nabla_z H)(\phi(t, z), t), \quad \phi(0, z) = z.$$

Assume that this system has a solution $\phi(t, z)$ for all z and t (this holds, e.g., if the supports of all functions H_t , $t \in \mathbb{R}$, are contained in a compact set). Then we can define the diffeomorphism

$$(3.1) \quad \Phi(z) = \phi(1, z).$$

It is not difficult to check that Φ is a symplectomorphism.

DEFINITION 3.1. Let $\Phi : X \rightarrow X$ be a symplectomorphism. We say that Φ is a *hamiltonian symplectomorphism* if it is given by the formula (3.1) for some smooth function H . We also say that H is a *Hamiltonian* of Φ .

LEMMA 3.2. *All elementary linear symplectomorphisms are hamiltonian symplectomorphisms.*

Proof. Indeed:

- $L_i(c)$ is given by the Hamiltonian $H(x, y) = (c/2)x_i^2$,
- $L_{ij}(c)$ is given by the Hamiltonian $H(x, y) = cx_ix_j$,

- $R_i(c)$ is given by the Hamiltonian $H(x, y) = -(c/2)y_i^2$,
- $R_{ij}(c)$ is given by the Hamiltonian $H(x, y) = -cy_i y_j$.

We notice that the hamiltonian flows generating the corresponding elementary symplectomorphisms are defined on the whole $\mathbb{R}^{2n} \times \mathbb{R}$. ■

Now we show how to compute a Hamiltonian of a linear symplectomorphism:

THEOREM 3.3. *Let $L : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be a linear symplectomorphism. Then L has a polynomial Hamiltonian*

$$(3.2) \quad H_L(z, t) = \sum_{i,j=1}^{2n} a_{i,j}(t) z_i z_j,$$

where $a_{i,j}(t) \in \mathbb{R}[t]$ are polynomials of t . Moreover, we can compute H_L effectively.

Proof. Let $L = L_m \circ \cdots \circ L_1$ where L_i are elementary symplectomorphisms. We proceed by induction on m . If $m = 1$ then we can use Lemma 3.2. In this case the flow $L_1(t)$ depends linearly on t .

Now consider $L' = L_{m-1} \circ \cdots \circ L_1$. By the induction hypothesis $L'(t) = L_{m-1}(t) \circ \cdots \circ L_1(t)$ is given by a Hamiltonian H' of the form (3.2). Let H'' be the Hamiltonian of L_m (as in Lemma 3.2). Now the flow $L(t) = L_m(t) \circ L'(t)$ is given by the Hamiltonian

$$H(z, t) = H''(z) + H'(L_m(t)^{-1}(z), t).$$

Of course it has also the form (3.2). Since we can decompose L into the product $L = L_m \circ \cdots \circ L_1$ effectively (see the proof of Theorem 2.1), we can also compute H in an effective way. ■

PROPOSITION 3.4. *Let $L : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be a hamiltonian symplectomorphism given by the flow $z \mapsto \phi(t, z)$, $t \in \mathbb{R}$. Assume that $\phi(t, 0) = 0$ for $t \in [0, 1]$. For every $\eta > 0$ there is an $\epsilon > 0$ and a hamiltonian symplectomorphism $\Phi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ such that*

- $\Phi(z) = L(z)$ for all z with $\|z\| \leq \epsilon$,
- $\Phi(z) = z$ for all z with $\|z\| \geq \eta$.

Proof. We know that $L(z) = \phi(1, z)$, where $\phi(t, z)$ is the solution of some differential equation

$$\frac{d}{dt} \phi(t, z) = J_0(\nabla_z H)(\phi(t, z), t), \quad \phi(0, z) = z.$$

Since $\phi(t, 0) = 0$ for every $t \in [0, 1]$, we can find $\epsilon > 0$ so small that all trajectories $\{\phi(t, z) : 0 \leq t \leq 1\}$ which start from the ball $B(0, \epsilon)$ are contained in the ball $B(0, \eta/2)$. Let $\sigma : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be a smooth function such

that

$$\sigma(z) = \begin{cases} 1 & \text{if } \|z\| \leq \eta/2, \\ 0 & \text{if } \|z\| \geq \eta. \end{cases}$$

Take $S = \sigma H$. The hamiltonian symplectomorphism Φ given by the differential equation

$$\frac{d}{dt}\phi(t, z) = J_0(\nabla_z S)(\phi(t, z), t), \quad \phi(0, z) = z,$$

is well defined on the whole of \mathbb{R}^{2n} and

$$\Phi(z) = \begin{cases} L(z) & \text{if } \|z\| \leq \epsilon, \\ z & \text{if } \|z\| \geq \eta. \blacksquare \end{cases}$$

Now Theorem 3.3 easily yields the following:

COROLLARY 3.5. *Let $L : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be a linear symplectomorphism. For every $\eta > 0$ there is an $\epsilon > 0$ and a hamiltonian symplectomorphism $\Phi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ such that*

- $\Phi(z) = L(z)$ for all z with $\|z\| \leq \epsilon$,
- $\Phi(z) = z$ for all z with $\|z\| \geq \eta$.

4. Characterization of symplectomorphisms. Before we formulate our next result we need the following (well-known)

LEMMA 4.1. *Let $X = (\mathbb{R}^{2n}, \omega_0)$ be the standard symplectic vector space. Fix $\eta > 0$ and let $a, b \in B(0, \eta)$. Then there exists a symplectomorphism $\Phi : X \rightarrow X$ such that*

$$\Phi(a) = b \quad \text{and} \quad \Phi(z) = z \quad \text{for } \|z\| \geq 2\eta.$$

Proof. Let $c = (c_1, \dots, c_{2n}) = b - a$. Define a sequence of points as follows:

$$a_0 = a, \quad a_i = a_{i-1} + (0, \dots, 0, c_i, 0, \dots, 0).$$

Of course $a_i \in B(0, \eta)$ and $a_{2n} = b$. Now consider the translation

$$T_i : \mathbb{R}^{2n} \ni (x, y) \mapsto (x, y) + (0, \dots, 0, c_i, 0, \dots, 0) \in \mathbb{R}^{2n}.$$

We have $T_i(a_{i-1}) = a_i$ for $i = 1, \dots, 2n$.

The translation T_i is a hamiltonian symplectomorphism given by the Hamiltonian

$$H_i(x, y) = \begin{cases} -c_i y_i & \text{if } i \leq n, \\ c_i x_{i-n} & \text{if } i > n. \end{cases}$$

Let V_i be the symplectic vector field which is determined by the Hamiltonian H_i . Since the ball $B(0, r)$ is a convex set, all trajectories $\phi(t)$, $0 \leq t \leq 1$, of the symplectic vector fields V_i which begin at a_i lie in the ball $B(0, \eta)$.

Let $\sigma : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be a smooth function such that

$$\sigma(z) = \begin{cases} 1 & \text{if } \|z\| \leq \eta, \\ 0 & \text{if } \|z\| \geq 2\eta. \end{cases}$$

Now let $F_i : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be the hamiltonian symplectomorphism given by the Hamiltonian $G_i = \sigma H_i$. Then

$$G_i(a_{i-1}) = a_i \quad \text{and} \quad G_i(z) = z \quad \text{if } \|z\| \geq 2\eta.$$

Now it is enough to take $\Phi = G_{2n} \circ G_{2n-1} \circ \dots \circ G_1$. ■

We apply Proposition 3.4 to the general case:

THEOREM 4.2. *Let (X, ω) be a symplectic manifold. Let a_1, \dots, a_m and b_1, \dots, b_m be two families of points of X . For every $i = 1, \dots, m$ choose a linear symplectomorphism $L_i : T_{a_i}X \rightarrow T_{b_i}X$. Then there is a symplectomorphism $\Phi : X \rightarrow X$ such that*

$$\Phi(a_i) = b_i, \quad d_{a_i}\Phi = L_i, \quad i = 1, \dots, m.$$

Proof. By the Darboux Theorem every point $z \in X$ has an open neighborhood V_z which is symplectically isomorphic to the ball $B(0, r_z)$ in the standard vector space $(\mathbb{R}^{2n}, \omega_0)$. Denote by $U_z \subset V_z$ the open set which corresponds to the ball $B(0, r_z/3)$.

Since $\dim X \geq 2$ the manifold $X \setminus \{a_2, \dots, a_m\}$ is also connected. Hence there exists a smooth path $\gamma : I \rightarrow X$ such that $\gamma(0) = a_1$, $\gamma(1) = b_1$ and $\{a_2, \dots, a_m\} \cap \gamma(I) = \emptyset$. Additionally we can assume that the sets V_z which cover $\gamma(I)$ are also disjoint from $\{a_2, \dots, a_m\}$.

Let ϵ be a Lebesgue number for the function $\gamma : I \rightarrow X$ with respect to the cover $\{U_z\}_{z \in X}$ and choose an integer N with $1/N < \epsilon$. If $I_k := [k/N, (k+1)/N]$, then $\gamma(I_k)$ is contained in some U_z ; denote it by U_k , the set V_z by V_k , and r_z by r_k . Let $A_k := \gamma(k/N)$, in particular $A_0 = a_1, A_N = b_1$.

Since $V_k \cong B(0, r_k)$ and $A_k, A_{k+1} \in B(0, r_k/3)$ we can apply Lemma 4.1 to obtain a symplectomorphism $\Phi : B(0, r_k) \rightarrow B(0, r_k)$ such that

$$\Phi(A_k) = A_{k+1} \quad \text{and} \quad \Phi(z) = z \quad \text{for } \|z\| \geq (2/3)r_k.$$

We can extend Φ to the whole of X (we glue it with the identity); denote this extension by Φ_k . Put

$$\Psi = \Phi_N \circ \Phi_{N-1} \circ \dots \circ \Phi_0.$$

Then $\Psi(a_1) = b_1$ and $\Psi(a_i) = a_i$ for $i > 1$. Repeating this process, we finally arrive at a symplectomorphism $\Sigma : X \rightarrow X$ such that $\Sigma(a_i) = b_i$ for $i = 1, \dots, m$. In a similar way using Proposition 3.5 we can construct a symplectomorphism $\Pi : X \rightarrow X$ such that

$$\Pi(b_i) = b_i, \quad d_{b_i}\Pi = L_i \circ (d_{a_i}\Sigma)^{-1}.$$

Now it is enough to take $\Phi = \Pi \circ \Sigma$. ■

Since for a compact symplectic manifold (X, ω) of dimension $2n$ it is well known that for a fixed number $0 < s \leq n$ there exists a closed $2s$ -dimensional symplectic submanifold $Z \subset X$ (which is a nontrivial result by S. Donaldson [3, Corollary 6, p. 671]), we can use Theorem 4.2 to obtain:

COROLLARY 4.3. *Let (X, ω) be a compact symplectic manifold of dimension $2n$. Let a_1, \dots, a_m be a family of points of X . Take $0 < s \leq n$. For every $i = 1, \dots, m$ choose a linear $2s$ -dimensional symplectic subspace $H_i \subset T_{a_i}X$. Then there is a closed symplectic $2s$ -dimensional submanifold $Y \subset X$ such that*

$$a_i \in Y, \quad T_{a_i}Y = H_i, \quad i = 1, \dots, m.$$

In contrast to the previously mentioned result on existence of symplectic submanifolds, the existence of isotropic tori is easy to prove in Darboux local coordinates. In a similar way to Corollary 4.3 we get:

COROLLARY 4.4. *Let (X, ω) be a symplectic manifold of dimension $2n$. Let a_1, \dots, a_m be a family of points of X . Take $0 < k \leq n$. For every $i = 1, \dots, m$ choose a linear k -dimensional isotropic subspace $H_i \subset T_{a_i}X$. Then there is a closed isotropic k -dimensional torus $Y \subset X$ such that*

$$a_i \in Y, \quad T_{a_i}Y = H_i, \quad i = 1, \dots, m.$$

5. Diffeomorphisms that are symplectomorphisms. Finally, we show that a symplectomorphism can be described as a diffeomorphism which preserves symplectic or isotropic submanifolds of a given fixed dimension.

THEOREM 5.1. *Let (X, ω_X) and (Y, ω_Y) be compact symplectic manifolds of dimension $2n > 2$. Fix a number $0 < s < n$. Assume that $\Phi : X \rightarrow Y$ is a diffeomorphism which maps all $2s$ -dimensional symplectic submanifolds of X onto symplectic submanifolds of Y . Then Φ is a conformal symplectomorphism, i.e., there exists a non-zero number $c \in \mathbb{R}$ such that*

$$\Phi^* \omega_Y = c \omega_X.$$

Proof. Fix $z \in X$ and let $H \subset T_zX$ be a $2s$ -dimensional symplectic subspace of T_zX . By Corollary 4.3 (applied for $m = 1$, $a_1 = z$ and $H_1 = H$) there exists a $2s$ -dimensional symplectic submanifold M of X such that $z \in M$ and $T_zM = H$.

Let $\Phi(M) = M'$ and $z' = \Phi(z)$. By assumption the submanifold $M' \subset Y$ is symplectic. This means that the space $d_z\Phi(H) = T_{z'}M'$ is symplectic. Hence $d_z\Phi$ maps all linear $2s$ -dimensional symplectic subspaces of T_zX onto subspaces of the same type. By Proposition 2.4 this implies that $d_z\Phi$ is a conformal symplectomorphism. i.e.,

$$(d_z\Phi)^* \omega_Y = \lambda(z) \omega_X,$$

where $\lambda(z) \neq 0$. This means that there is a smooth function $\lambda : X \rightarrow \mathbb{R}^*$ ($= \mathbb{R} \setminus \{0\}$) such that

$$\Phi^*\omega_Y = \lambda\omega_X.$$

But since the form ω_Y is closed, so is $\Phi^*\omega_Y$. Since $n > 1$ this implies that the derivative $d\lambda$ vanishes, i.e., the function λ is constant. ■

THEOREM 5.2. *Let (X, ω_X) and (Y, ω_Y) be symplectic manifolds of dimension $2n > 2$. Fix a number $1 < k \leq n$. Assume that $\Phi : X \rightarrow Y$ is a diffeomorphism which maps all k -dimensional isotropic tori of X onto isotropic tori of Y . Then Φ is a conformal symplectomorphism, i.e., there exists a non-zero constant $c \in \mathbb{R}$ such that*

$$\Phi^*\omega_Y = c\omega_X.$$

Proof. Fix $z \in X$ and let $H \subset T_zX$ be a k -dimensional isotropic subspace of T_zX . By Corollary 4.4 (applied for $m = 1$, $a_1 = z$ and $H_1 = H$) there exists a k -dimensional isotropic torus M of X such that $z \in M$ and $T_zM = H$.

Let $\Phi(M) = M'$ and $z' = \Phi(z)$. By assumption the torus $M' \subset Y$ is isotropic. This means that the space $d_z\Phi(H) = T_{z'}M'$ is isotropic. Hence $d_z\Phi$ maps all linear k -dimensional isotropic subspaces of T_zX onto subspaces of the same type. By Proposition 2.5 this implies that $d_z\Phi$ is a conformal symplectomorphism. The rest of the proof is the same as in the case of Theorem 5.1 above. ■

REMARK 5.3. Let us note that in particular if Φ maps Lagrangian tori onto tori of the same type then Φ is a conformal symplectomorphism.

COROLLARY 5.4. *Let X be a compact manifold of dimension $2n > 2$. Let ω_1 and ω_2 be two symplectic forms on X . Fix a number $0 < k < n$. Assume that every $2k$ -dimensional ω_1 -symplectic submanifold of X is ω_2 -symplectic. Then there exists a non-zero number $c \in \mathbb{R}$ such that*

$$\omega_1 = c\omega_2.$$

Proof. It is enough to apply Theorem 5.1 to $X = (X, \omega_1)$, $Y = (X, \omega_2)$ and $\Phi = \text{identity}$. ■

COROLLARY 5.5. *Let (X, ω) be a compact symplectic manifold of dimension $2n > 2$. Fix a number $0 < k < n$. Assume that $\Phi : X \rightarrow X$ is a diffeomorphism which maps all $2k$ -dimensional symplectic submanifolds of X onto submanifolds of the same type. Then Φ is a symplectomorphism or an antisymplectomorphism, i.e., $\Phi^*\omega = \pm\omega$. If Φ preserves orientation and n is odd, then Φ is a symplectomorphism. Moreover, if n is even, then Φ has to preserve orientation.*

Proof. Indeed, we have $\Phi^*\omega = c\omega$. We can write

$$(5.3) \quad \text{vol}(X) = \int_X \omega^n = \pm \int_X \Phi^*\omega^n = \pm c^n \int_X \omega^n,$$

hence $c = \pm 1$. Moreover, if Φ preserves orientation and n is odd, then we get $c = 1$. If n is even then $(-\omega)^n = \omega^n$ and Φ has to preserve orientation. ■

REMARK 5.6. Corollaries like Corollary 5.4 and Corollary 5.5 are also true for a compact symplectic manifold X in the case of isotropic tori. A similar geometric characterization of symplectomorphisms has already been proved for diffeomorphisms preserving capacity, which turn out to be symplectic or antisymplectic (cf. [5],[8]).

EXAMPLE 5.7. We show that in the general case Φ need not to be a symplectomorphism. Let $Y = (S^2, \omega)$ (where ω is the standard volume form on the sphere) and let $(X_n, \omega_n) = \prod_{i=1}^n Y$ be a standard symplectic product. Further, let $\sigma : S^2 \ni (x, y, z) \mapsto (x, y, -z) \in S^2$ be a mirror symmetry. Of course $\sigma^*\omega = -\omega$. More generally, if $\Sigma = \prod_{i=1}^n \sigma : X_n \rightarrow X_n$, then $\Sigma^*\omega_n = -\omega_n$. Hence Φ from Corollary 5.5 may well be an antisymplectomorphism.

However, in any case either Φ or $\Phi \circ \Phi$ is a symplectomorphism.

Now let (X, ω) be a symplectic manifold and denote by $\mathbf{Symp}(X, \omega)$ the group of symplectomorphisms of X . To end this note we show that this group also determines the conformal symplectic structure of X :

THEOREM 5.8. *Let X be a smooth manifold of dimension $2n > 2$ and let ω_1, ω_2 be two symplectic forms on X . If $\mathbf{Symp}(X, \omega_1) \subset \mathbf{Symp}(X, \omega_2)$, then there exists a non-zero constant c such that $\omega_2 = c\omega_1$.*

Proof. Take $z \in X$ and consider the symplectic vector spaces $V_1 = (T_z X, \omega_1)$ and $V_2 = (T_z X, \omega_2)$. By Theorem 4.2, for every linear symplectomorphism S of V_1 , there is a symplectomorphism $\Phi_S \in \mathbf{Symp}(X, \omega_1)$, such that

$$\Phi_S(z) = z, \quad d_z \Phi_S = S.$$

Since $\mathbf{Symp}(X, \omega_1) \subset \mathbf{Symp}(X, \omega_2)$ we easily obtain $\mathbf{Sp}(V_1) \subset \mathbf{Sp}(V_2)$. Consequently, by Proposition 2.6 there exists a non-zero number $\lambda(z)$ such that $\omega_2(z) = \lambda(z)\omega_1(z)$. Now we finish the proof as in the proof of Theorem 5.1. ■

Acknowledgements. This research was partially supported by EC FP6 Marie Curie ToK programme SPADE2, MTKD-CT-2004-014508 and Polish MNiSW SPB-M.

References

- [1] A. Borel, *Injective endomorphisms of algebraic Nash varieties*, Arch. Math. (Basel) 20 (1969), 531–537.
- [2] E. Calabi, *On the group of automorphisms of a symplectic manifold*, in: Problems in Analysis, R. Gunning (ed.), Princeton Univ. Press, 1970, 1–26.
- [3] S. Donaldson, *Symplectic submanifolds and almost-complex geometry*, J. Differential Geom. 44 (1996), 666–705.
- [4] I. Ekeland and H. Hofer, *Symplectic topology and Hamiltonian dynamics*, Math. Z. 200 (1989), 355–378.
- [5] H. Hofer, *Dynamics, topology and holomorphic curves*, Documenta Math., Extra volume ICM, Vol. II (1998), 443–452.
- [6] S. Janeczko and Z. Jelonek, *Linear automorphisms that are symplectomorphisms*, J. London Math. Soc. 69 (2004), 503–517.
- [7] D. McDuff, *A survey of topological properties of groups of symplectomorphisms*, in: Topology, Geometry and Quantum Field Theory, Proc. of 2002 Symposium in honor of G. B. Segal, ed. U. L. Tillmann, Cambridge Univ. Press, 2004, 173–193.
- [8] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, 2nd ed., Oxford Univ. Press, 1998.
- [9] L. Polterovich, *The Geometry of the Group of Symplectic Transformations*, Lecture Notes in Math. ETH Zürich, Birkhäuser, 2001.
- [10] I. Vaisman, *Locally conformal symplectic manifolds*, Int. J. Math. Sci. 8 (1985), 521–536.
- [11] A. Weinstein, *Lectures on Symplectic Manifolds*, CBMS Reg. Conf. Ser. Math. 29, Amer. Math. Soc., Providence RI, 1977.

S. Janeczko
Instytut Matematyczny PAN
Śniadeckich 8
00-956 Warszawa, Poland
and
Wydział Matematyki i Nauk Informatycznych
Politechnika Warszawska
Pl. Politechniki 1
00-661 Warszawa, Poland
E-mail: janeczko@mini.pw.edu.pl

Z. Jelonek
Instytut Matematyczny PAN
Śniadeckich 8
00-956 Warszawa, Poland
E-mail: najelone@cyf-kr.edu.pl

*Received 17 September 2008;
in revised form 2 March 2009*