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Banach spaces of bounded Szlenk index II

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Abstract. For every $\alpha < \omega_1$ we establish the existence of a separable Banach space whose Szlenk index is $\omega^{\alpha\omega+1}$ and which is universal for all separable Banach spaces whose Szlenk index does not exceed $\omega^{\alpha\omega}$. In order to prove that result we provide an intrinsic characterization of which Banach spaces embed into a space admitting an FDD with Tsirelson type upper estimates.

1. Introduction. The added structure and rich theory of coordinate systems can be of significant help when studying Banach spaces. Because of this, it is often the case that Banach spaces are studied in the context of being a subspace or quotient of some space with a coordinate system. Two early results in this area are that every separable Banach space is the quotient of ℓ_1 and also every separable Banach space may be embedded as a subspace of $C[0,1]$. Both ℓ_1 and $C[0,1]$ have bases, and so in particular every separable Banach space is a quotient of a Banach space with a basis and may be embedded as a subspace of a Banach space with a basis. However, one often has a Banach space with a particular property, and one wishes that the coordinate system has some associated property. An important step in this direction was made by Zippin [21] who proved the following two major results: every separable reflexive Banach space may be embedded as a subspace of a space with a shrinking and boundedly complete basis, and every Banach space with separable dual may be embedded in a Banach space with a shrinking basis. Further results in this area give intrinsic characterizations of when a space may be embedded as a subspace of a reflexive space with unconditional basis [9], or a reflexive space with an asymptotic ℓ_p FDD [16]. These are only a portion of the recent results in this area. These new characterizations are all based on the relatively recent tool of weakly null

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trees. In particular, this has been used effectively to study Banach spaces of Szlenk index ω_0 (cf. [7], [11], [13], [14], and [15]). An important result in particular for us is the characterization of subspaces of reflexive spaces with an FDD satisfying subsequential V-upper block estimates and subsequential U-lower block estimates where V is an unconditional, block stable, and right dominant basic sequence and U is an unconditional, block stable, and left dominant basic sequence [17]. This characterization when applied to Tsirelson spaces was shown to have strong applications to the Szlenk index of reflexive spaces [18]. Our main result adds to this theory with the following theorem which extends the results in [17] and [18] to spaces with separable dual. The notions and concepts used will be introduced in the next section.

THEOREM 1.1. *Let X^* be separable and $V = (v_i)$ be a normalized*, 1-*unconditional*, *block stable*, *right dominant*, *and shrinking basic sequence. Then the following are equivalent.*

(1) *X has subsequential V-upper tree estimates.*
(2) *X is a quotient of a space Z with Z^* separable and Z has subsequential V-upper tree estimates.*
(3) *X is a quotient of a space Z with a shrinking FDD satisfying subsequential V-upper block estimates.*
(4) *There exists a w^*-to-w^* continuous embedding of X^* into Z^*, a space with boundedly complete FDD (F_i^*) (so $Z = \bigoplus F_i$ defines Z^*) satisfying subsequential V^*-lower block estimates.*
(5) *X is isomorphic to a subspace of a space Z with a shrinking FDD satisfying subsequential V-upper block estimates.*

Using our characterization, we are able to achieve the following universality result:

THEOREM 1.2. *Let $V = (v_i)$ be a normalized*, 1-*unconditional*, *shrinking*, *block stable*, *and right dominant basic sequence. There is a Banach space Z with a shrinking FDD (F_i) satisfying subsequential V-upper block estimates such that if a Banach space X with separable dual satisfies subsequential V-upper tree estimates*, *then X embeds into Z.*

We will apply Theorems 1.1 and 1.2 for the case that V is the canonical basis of $T_{\alpha,c}$, the Tsirelson space of order α and parameter c, which will allow us to prove some new results for the Szlenk index. As shown in [18], the Szlenk index is closely related to a space having subsequential $T_{\alpha,c}$-upper tree estimates for some $0 < c < 1$. In particular, for each $\alpha < \omega_1$ if a Banach space X with separable dual has Szlenk index at most $\omega^{\alpha\omega}$, then X satisfies subsequential $T_{\alpha,c}$-upper tree estimates for some $c \in (0, 1)$. In [18] the converse is also shown in the case that X is reflexive. Our characterization allows us to extend this to the class of spaces with separable dual. We give the following theorem.

THEOREM 1.3. *Let $\alpha < \omega_1$. For a space X with separable dual, the following are equivalent*:

(i) *X has Szlenk index at most $\omega^{\alpha\omega}$.*
(ii) *X satisfies subsequential $T_{\alpha,c}$-upper tree estimates for some $c \in (0,1)$.*
(iii) *X embeds into a space Z with an FDD (E_i) which satisfies subsequential $T_{\alpha,c}$-upper block estimates in Z for some $c \in (0,1)$.*

Note that the implication (iii)$\Rightarrow$(i) shows that the space Z in (iii) also has Szlenk index at most $\omega^{\alpha\omega}$. In particular, since the unit vector basis of $T_{\alpha,c}$ satisfies subsequential $T_{\alpha,c}$-upper block estimates, the same is true for $T_{\alpha,c}$. The above structure theorem then says that the Tsirelson spaces $T_{\alpha,c}$ form a sort of upper envelope for the class of spaces with separable dual and with Szlenk index at most $\omega^{\alpha\omega}$.

We are able to combine the previous two theorems using ideas in [18] to prove the following universality result.

THEOREM 1.4. *For each $\alpha < \omega_1$ there exists a Banach space Z with a shrinking FDD and Szlenk index at most $\omega^{\alpha\omega+1}$ such that Z is universal for the collection of spaces with separable dual and Szlenk index at most $\omega^{\alpha\omega}$.*

In particular, the universal space Z will be of the form $(\sum_{n\in\mathbb{N}} X_n)_{\ell_2}$, where X_n has an FDD satisfying subsequential $T_{\alpha,n/(n+1)}$-upper block estimates and X_n is universal for all Banach spaces with separable dual which satisfy subsequential $T_{\alpha,n/(n+1)}$-upper tree estimates.

Theorem 1.4 represents a quantitative version of a result first shown by Dodos and Ferenczi [5], which states that for every $\alpha < \omega_1$ there is a Banach space with separable dual which is universal for all separable Banach spaces whose Szlenk index does not exceed α. As well as finding a bound on the Szlenk index of this universal space, we also express, as mentioned above, the topological property of having a certain Szlenk index in terms of norm estimates in which the Tsirelson spaces play an essential role. While the proofs in [5] use methods of descriptive set theory first developed by Bossard [1], in the context of Banach spaces, our proofs will rely on concepts like *infinite asymptotic games*, trees and branches as introduced in [13] and [15] (see also Rosendal's excellent paper on the subject [19]).

2. Definitions and lemmas. Our main result characterizes subspaces and quotients of spaces having a shrinking FDD with subsequential V-upper block estimates, where V is an unconditional, right dominant, block stable, and shrinking basic sequence. The case when $V = T_{\alpha,c}$ is a Tsirelson space is intimately related to the Szlenk index, and has become an important property in the fertile area between descriptive set theory and the classification of Banach spaces [18]. For $\alpha < \omega_1$ and $c \in (0,1)$, the definition of the Tsirelson space $T_{\alpha,c}$ of order α and parameter c, and the properties of $T_{\alpha,c}$ relevant for us can also be found in [18].

For basic notions like (shrinking and boundedly complete) FDDs and their projection constants and blockings we refer to [17]. If Z is a Banach space with an FDD $E = (E_i)$, we denote by $c_{00}(\bigoplus E_i)$ the dense linear subspace of Z spanned by (E_i) and its closure by $[E_i] = [E_i]_Z$. We denote the closure of $c_{00}(\bigoplus E_i^*)$ inside Z^* by $Z^{(*)}$. If (E_i) is shrinking it follows that $Z^{(*)} = Z^*$, and if (E_i) is boundedly complete, then $Z^{(*)}$ is the predual of Z. If $A \subset \mathbb{N}$ is finite or cofinite, we denote the natural projection onto the closed span of $(E_i : i \in A)$ by P_A^E, i.e. $P_A^E : Z \to Z$,

$$P_A^E\Big(\sum_{i=1}^{\infty} x_i\Big) = \sum_{i\in A} x_i \text{ whenever } x_i \in E_i \text{ for } i \in \mathbb{N} \text{ so that } \sum_{i=1}^{\infty} x_i \in Z.$$

Let us also recall the following notion from [17].

Definition 2.1. Let Z be a Banach space with an FDD (E_n), let $V = (v_i)$ be a normalized 1-unconditional basis, and let $1 \le C < \infty$. We say that (E_n) *satisfies subsequential C-V-upper block estimates* if every normalized block sequence (z_i) of (E_n) in Z is C-dominated by (v_{m_i}), where $m_i = \min \operatorname{supp}_E(z_i)$ for all $i \in \mathbb{N}$. We say that (E_n) *satisfies subsequential C-V-lower block estimates* if every normalized block sequence (z_i) of (E_n) in Z C-dominates (v_{m_i}), where $m_i = \min \operatorname{supp}_E(z_i)$ for all $i \in \mathbb{N}$. We say that (E_n) *satisfies subsequential V-upper* (*or lower*) *block estimates* if it satisfies subsequential C-V-upper (or lower) block estimates for some $1 \le C < \infty$.

Note that if (E_i) satisfies subsequential C-V-upper block estimates and (z_i) is a normalized block sequence with $\max \operatorname{supp}_E(z_{i-1}) < k_i \le \min \operatorname{supp}_E(z_i)$ for all $i > 1$, then (z_i) is C-dominated by (v_{k_i}) (and a similar remark holds for lower estimates). This follows easily if we replace z_i by $(\varepsilon e_{k_i} + z_i)/\|\varepsilon e_{k_i} + z_i\|$ where $e_{k_i} \in S_{E_{k_i}}$ and letting $\varepsilon \to 0$.

Subsequential V-upper block estimates and subsequential V-lower block estimates are dual properties, as shown in the following proposition from [17].

Proposition 2.2 ([17, Proposition 2.14]). *Assume that Z has an FDD (E_i), and let $V = (v_i)$ be a 1-unconditional normalized basic sequence with biorthogonal functionals $V^* = (v_i^*)$. The following statements are equivalent*:

(a) *(E_i) satisfies subsequential V-upper block estimates in Z.*
(b) *(E_i^*) satisfies subsequential V^*-lower block estimates in $Z^{(*)}$.*

Moreover, if (E_i) is bimonotone in Z, then the equivalence holds true if one replaces, for some $C \ge 1$, V-upper estimates by C-V-upper estimates in (a) *and V^*-lower block estimates by C-V^*-lower block estimates in* (b).

It is important to note that if a Banach space Z has an FDD (E_n) which satisfies subsequential V-upper block estimates where $V=(v_i)$ is weakly null, then (E_n) is shrinking. Indeed, any normalized block sequence of (E_n) is then dominated by a weakly null sequence, and is thus weakly null. Thus if V is weakly null, a necessary condition for a Banach space X to be isomorphic to a quotient or subspace of a Banach space with an FDD satisfying subsequential V-upper block estimates is that X have separable dual. This is important as spaces with separable dual may be analyzed using *weakly null trees*. In this paper we will need in particular *weakly null even trees* (see [17]).

In order to index weakly null even trees, we define

$$T_\infty^{\text{even}} = \{(n_1,\dots,n_{2\ell}) : n_1 < \dots < n_{2\ell} \text{ are in } \mathbb{N} \text{ and } \ell \in \mathbb{N}\}.$$

Definition 2.3. If X is a Banach space, an indexed family $(x_\alpha)_{\alpha\in T_\infty^{\text{even}}} \subset X$ is called an *even tree*. Sequences of the form $(x_{n_1,\dots,n_{2\ell-1},k})_{k=n_{2\ell-1}+1}^{\infty}$ are called *nodes*. This should not be confused with the more standard terminology where a node would refer to an individual member of the tree. Sequences of the form $(x_{n_1,\dots,n_{2\ell}})_{\ell=1}^{\infty}$ are called *branches*. A *normalized tree*, i.e. one with $\|x_\alpha\|=1$ for all $\alpha\in T_\infty^{\text{even}}$, is called *weakly null* if every node is a weakly null sequence.

If $T'\subset T_\infty^{\text{even}}$ is closed under taking restrictions so that for each $\alpha\in T'\cup\{\emptyset\}$ and for each $m\in\mathbb{N}$ the set $\{n\in\mathbb{N}: (\alpha,m,n)\in T'\}$ is either empty or has infinite size, and moreover the latter occurs for infinitely many values of m, then we call $(x_\alpha)_{\alpha\in T'}$ a *full subtree* of $(x_\alpha)_{\alpha\in T_\infty^{\text{even}}}$. Note that $(x_\alpha)_{\alpha\in T'}$ could then be relabeled to a family indexed by T_∞^{even}, and note that the branches of $(x_\alpha)_{\alpha\in T'}$ are branches of $(x_\alpha)_{\alpha\in T_\infty^{\text{even}}}$, and the nodes of $(x_\alpha)_{\alpha\in T'}$ are subsequences of certain nodes of $(x_\alpha)_{\alpha\in T_\infty^{\text{even}}}$.

In the case that X has an FDD (E_i), we say that a normalized tree is a *block tree* (with respect to (E_i)) if every node is a block sequence with respect to (E_i). Note that every weakly null tree has a full subtree which is a perturbation of a block tree, and if (E_i) is shrinking, then every block tree is weakly null.

This motivates the following *coordinate free* definition.

Definition 2.4. Let X be a Banach space, $V=(v_i)$ be a normalized 1-unconditional basis, and $1\le C<\infty$. We say that X *satisfies subsequential C-V-upper tree estimates* if every weakly null even tree $(x_\alpha)_{\alpha\in T_\infty^{\text{even}}}$ in X has a branch $(x_{n_1,\dots,n_{2\ell}})_{\ell=1}^{\infty}$ which is C-dominated by $(v_{n_{2\ell-1}})_{\ell=1}^{\infty}$.

We say that X *satisfies subsequential V-upper tree estimates* if it satisfies subsequential C-V-upper tree estimates for some $1\le C<\infty$.

If X is a subspace of a dual space, we say that X *satisfies subsequential C-V-lower w^* tree estimates* if every w^* null even tree $(x_\alpha)_{\alpha\in T_\infty^{\text{even}}}$ in X has a branch $(x_{n_1,\dots,n_{2\ell}})_{\ell=1}^{\infty}$ which C-dominates $(v_{n_{2\ell-1}})_{\ell=1}^{\infty}$.

We have a property of trees and a property of FDDs, and our goal is to show how they are related. Zippin's theorem allows us to embed a Banach space with separable dual into a space with shrinking FDD. Our next step will be to pass information about trees in the space to information about $\bar{\delta}$-skipped blocks of the FDD, which we define here.

DEFINITION 2.5. Let $\mathbf{E} = (E_i)$ be an FDD for a Banach space Y and let $\bar{\delta} = (\delta_i)$ with $\delta_i \downarrow 0$. A sequence $(y_i) \subset S_Y$ is called a *$\bar{\delta}$-skipped block* with respect to (E_i) if there exist integers $1 = k_0 < k_1 < \cdots$ so that for all $i \in \mathbb{N}$,

$$\|P^E_{(k_{i-1},k_i)} y_i - y_i\| < \delta_i.$$

The following proposition is an adaptation of Proposition 2.18 in [17] for the case of (E_i) shrinking, but not necessarily boundedly complete and for the case where X is a w^* closed subspace of a dual space. We will need to first recall some notation introduced in [17].

Given a Banach space X, we let $\mathcal{U}(X) \subset (\mathbb{N} \times S_X)^\omega$ denote the set of all sequences (k_i, x_i), where $k_1 < k_2 < \cdots$ are positive integers, and (x_i) is a sequence in S_X. We equip $\mathcal{U}(X)$ with the relative topology of $(\mathbb{N} \times S_X)^\omega$ which is given the product topology of the discrete topologies of $\mathbb{N}$ and S_X. Given $A \subset \mathcal{U}(X)$ and $\varepsilon > 0$, we let

$$A_\varepsilon = \{((\ell_i, y_i) : i \in \mathbb{N}) \in \mathcal{U}(X) : \exists ((k_i, x_i) : i \in \mathbb{N}) \in A \quad k_i \le \ell_i,\ \|x_i - y_i\| < \varepsilon 2^{-i}\ \forall i \in \mathbb{N}\},$$

and we let $\overline{A}$ be the closure of A in $\mathcal{U}(X)$.

Given $A \subset \mathcal{U}(X)$, we say that an even tree $(x_\alpha)_{\alpha \in T_\infty^{\mathrm{even}}}$ in X *has a branch in A* if there exist $n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that $((n_{2i-1}, x_{(n_1,n_2,\ldots,n_{2i})}) : i \in \mathbb{N}) \in A$.

Let Z be a Banach space with an FDD (E_i) and assume that Z contains X. Let C be the projection constant of (E_i) in Z. For each $m \in \mathbb{N}$ we set $Z_m = \overline{\bigoplus_{i>m} E_i}$. Given $\varepsilon > 0$, we consider the following game between players S (subspace chooser) and P (point chooser). The game has an infinite sequence of moves; on the nth move ($n \in \mathbb{N}$) S picks $k_n, m_n \in \mathbb{N}$ and P responds by picking $x_n \in S_X$ with $d(x_n, Z_{m_n}) < \varepsilon' \cdot 2^{-n}$, where $\varepsilon' = \min\{\varepsilon, 1\}$. S wins the game if the sequence (k_i, x_i) the players generate ends up in $\overline{A_{5C\varepsilon}}$, otherwise P is declared the winner. We will refer to this as the (A,ε)-*game*. Note that the definition of S winning is slightly different from the one given in [17]. This is because of the extra complication of dealing with the nonreflexive case.

PROPOSITION 2.6. *Let X be an infinite-dimensional closed subspace of a space Z with an FDD (E_i). Let $A \subset \mathcal{U}(X)$. If (E_i) is shrinking, or if Z is a dual space with (E_i) boundedly complete and X w^* closed in Z, then the following are equivalent.*

(a) *For all* $\varepsilon > 0$ *there exists* $(K_i) \subset \mathbb{N}$ *with* $K_1 < K_2 < \cdots$, $\overline{\delta} = (\delta_i) \subset (0,1)$ *with* $\delta_i \searrow 0$, *and a blocking* $F = (F_i)$ *of* (E_i) *such that if* $(x_i) \subset S_X$ *is a* $\overline{\delta}$*-skipped block sequence of* (F_n) *in* Z *with* $\|x_i - P^F_{(r_{i-1},r_i)}x_i\| < \delta_i$ *for all* $i \in \mathbb{N}$, *where* $1 \le r_0 < r_1 < r_2 < \cdots$, *then* $(K_{r_{i-1}}, x_i) \in \overline{A_\varepsilon}$.

(b) *For all* $\varepsilon > 0$, S *has a winning strategy for the* (A,ε)*-game.*

If (E_i) *is shrinking, then* (a) *and* (b) *are equivalent to*

(c) *for all* $\varepsilon > 0$ *every normalized, weakly null even tree in* X *has a branch in* $\overline{A_\varepsilon}$.

If Z *is a dual space with* (E_i) *boundedly complete and* X w^* *closed in* Z, *then* (a) *and* (b) *are equivalent to*

(d) *for all* $\varepsilon > 0$ *every normalized,* w^* *null even tree in* X *has a branch in* $\overline{A_\varepsilon}$.

Proof. The proofs of the implications (b)⇒(a)⇒(d)⇒(b) shown in the reflexive case [17, Proposition 2.18] still hold in the nonreflexive case when Z is a dual space with (E_i) boundedly complete and X w^* closed in Z: we use w^* compactness of B_X, and the fact that (E_i) is biorthogonal to a shrinking FDD of a predual of Z (instead of weak compactness of B_X and the shrinking property of (E_i) as in [17]).

For the case in which (E_i) is shrinking, the proofs of the implications (b)⇒(a)⇒(c) shown for the reflexive case still work. The proof for the implication (c)⇒(b) requires some adaptation which we provide here.

We start with a preliminary result: Let $Z_m = \overline{\bigoplus_{i>m} E_i}$ as before, and let C be the projection constant of (E_i) in Z. Then for every $\eta > 0$ with $(1 + C)\eta < 1$ and for every sequence $(z_i) \subset S_X$ with $d(z_i, Z_i) < \eta$ for all $i \in \mathbb{N}$, there is a subsequence (x_i) of (z_i) and a weakly null sequence $(y_i) \subset S_X$ such that $\|x_i - y_i\| < 2(1+C)\eta$ for all $i \in \mathbb{N}$. Indeed, we can pass to a weakly Cauchy subsequence (x_i) of (z_i) such that

$$\|P^E_{[1,n]}(x_i - x_j)\| < \eta 2^{-j} \quad \forall n \in \mathbb{N} \text{ and } i > j \ge n. \tag{1}$$

As $d(x_i, Z_i) < \eta$, we have $\|P^E_{[1,i]}x_i\| < C\eta$ for all $i \in \mathbb{N}$. Since (E_i) is shrinking, $(P^E_{(i,\infty)}x_i)$ is weakly null, and so for each $n \in \mathbb{N}$ there exists $(a_i^{(n)})_{i=n}^{K(n)} = (a_i)_{i=n}^K \subset [0,1]$ such that $\sum_{i=n}^K a_i = 1$ and $\|\sum_{i=n}^K a_i P^E_{(i,\infty)}x_i\| < \eta$. Set

$$y_n = \frac{x_n - \sum a_i x_i}{\|x_n - \sum a_i x_i\|}.$$

We have

$$\Big\|\sum_{i=n}^K a_i x_i\Big\| \le \sum_{i=n}^K a_i \|P^E_{[1,i]}x_i\| + \Big\|\sum_{i=n}^K a_i P^E_{(i,\infty)}x_i\Big\| < \sum_{i=n}^K a_i C\eta + \eta = (1+C)\eta,$$

which implies that $\|x_n - y_n\| < 2(1+C)\eta$. We also have

$$\|P_{[1,n]}y_n\| \le 2\Big\|P_{[1,n]}\Big(x_n - \sum a_i x_i\Big)\Big\| \le 2\sum a_i\|P_{[1,n]}(x_n - x_i)\| < 2\eta 2^{-n},$$

which tends to zero as $n \to \infty$. Hence $(y_n) \subset S_X$ is weakly null as (E_i) is shrinking.

We now continue with the proof of the implication (c)⇒(b). Assume that S does not have a winning strategy for the (A,ε) game for some $\varepsilon > 0$. As this game is closed and hence determined [12], there exists a winning strategy ϕ for the point chooser. The function ϕ takes values in S_X: if $(k_i),(m_i) \in [\mathbb{N}]^\omega$ are the choices by player S and if $z_n = \phi(k_1, m_1, \ldots, k_n, m_n)$ for all $n \in \mathbb{N}$, then $d(z_i, Z_{m_i}) < \varepsilon 2^{-i}$ for all $i \in \mathbb{N}$ and $(k_i, z_i) \notin \overline{A_{5C\varepsilon}}$. We can, of course, assume that $(1+C)\varepsilon < 1$. For each $\alpha \in T_\infty^{\mathrm{even}}$ set $z_\alpha = \phi(\alpha)$. Then $(z_\alpha)_{\alpha \in T_\infty^{\mathrm{even}}}$ is a normalized even tree in X no branch of which is in $\overline{A_{5C\varepsilon}}$. Its nodes $(z_i) = (z_{(k_1,\ldots,k_{2\ell-1},i)})_{i > k_{2\ell-1}}$ satisfy $d(z_i, Z_i) < \varepsilon 2^{-\ell}$ for all $i > k_{2\ell-1}$. By repeated applications of our preliminary observation we can find a full subtree $(x_\alpha)_{\alpha \in T_\infty^{\mathrm{even}}}$ of $(z_\alpha)_{\alpha \in T_\infty^{\mathrm{even}}}$ and a weakly null even tree $(y_\alpha)_{\alpha \in T_\infty^{\mathrm{even}}}$ such that $\|x_\alpha - y_\alpha\| < 2(1+C)\varepsilon 2^{-\ell}$ for all $\ell \in \mathbb{N}$ and $\alpha = (k_1, \ldots, k_{2\ell}) \in T_\infty^{\mathrm{even}}$. Since no branch of $(x_\alpha)_{\alpha \in T_\infty^{\mathrm{even}}}$ is in $\overline{A_{5C\varepsilon}}$, it follows that no branch of $(y_\alpha)_{\alpha \in T_\infty^{\mathrm{even}}}$ is in $\overline{A_{C\varepsilon}}$. Thus (c) fails. ∎

Remark. We will be applying Proposition 2.6 for $A = \{(n_i, x_i)_{i=1}^\infty \mid (v_{n_i}) \text{ dominates } (x_i)\}$ where (v_i) is a 1-unconditional basic sequence. We will also be repeatedly applying the technique of blocking FDDs. For this reason it is important that properties of $\bar\delta$-skipped blocks are preserved by blockings. This follows at once from the following simple observation: Assume that (E_i) is an FDD with projection constant K, and (H_i) is a blocking of (E_i). Then a $\bar\delta$-skipped block of (H_i) is a $2K\bar\delta$-skipped block of (E_i).

We will be concerned with a space X which satisfies subsequential V-upper tree estimates. However, the nature of our proofs requires us to work with X^* as well. This is because some of the blocking techniques which we use depend on the FDD being boundedly complete. Before stating a duality result on upper tree estimates, we need the following definition: A basic sequence $V = (v_i)$ is *C-right dominant* (respectively, *C-left dominant*) if for all sequences $m_1 < m_2 < \cdots$ and $n_1 < n_2 < \cdots$ of positive integers with $m_i \le n_i$ for all $i \in \mathbb{N}$ the sequence (v_{m_i}) is C-dominated by (respectively, C-dominates) (v_{n_i}). We say that (v_i) is *right dominant* or *left dominant* if for some $C \ge 1$ it is C-right dominant or C-left dominant, respectively.

Lemma 2.7. *Let X be a Banach space with separable dual, and let $V = (v_i)$ be a normalized, 1-unconditional, right dominant basis. If X satisfies subsequential V-upper tree estimates, then X^* satisfies subsequential V^*-lower w^* tree estimates.*

Proof. X has separable dual, so by [4, Corollary 8] there exists a space Z with a shrinking and bimonotone FDD (F_i) for which there is a quotient map $Q : Z \to X$. After renorming X we may assume that it has the quotient norm $\|x\| = \inf_{Qy=x} \|y\|$ for all $x \in X$. This shows that Q^* is an isometric embedding of X^* into Z^*. Furthermore, (F_i^*) is a boundedly complete FDD for Z^* as (F_i) is shrinking.

We next show that if $(x_i) \subset S_{X^*}$ is w^* null, then there is a subsequence (x_i') of (x_i) and a weakly null sequence $(y_i) \subset S_X$ such that $x_i'(y_i) > 3/4$ for all $i \in \mathbb{N}$.

The sequence $(Q^* x_i)$ is w^* null in Z^* as Q^* is w^*-to-w^* continuous. Hence there is a subsequence (x_i') of (x_i) such that $\|P^{F^*}_{[1,i)} Q^* x_i'\| < 1/4$ for all i. As (F_i) is bimonotone, there exists $(z_i) \subset S_Z$ such that $\|P^F_{[1,i)} z_i\| = 0$ and $Q^* x_i'(z_i) > 3/4$. Note that $\|Q(z_i)\| > 3/4$ for all i, and the sequence (z_i) is coordinatewise null and hence weakly null as (F_i) is shrinking. It follows that $y_i = Q(z_i)/\|Q(z_i)\|$ defines a weakly null sequence in S_X with $x_i'(y_i) > 3/4$ for all i.

Now let $(x_\alpha)_{\alpha \in T_\infty^{\mathrm{even}}} \subset S_{X^*}$ be a w^* null tree. By repeated applications of the above result, there is a full subtree (x_α') of (x_α) and a weakly null tree (y_α) in X such that $x_\alpha'(y_\alpha) > 3/4$ for all $\alpha \in T_\infty^{\mathrm{even}}$. Passing to further subtrees if necessary, we can also assume that for $k < \ell$ in $\mathbb{N}$ and for $\alpha = (n_1, \ldots, n_{2k})$, $\beta = (n_1, \ldots, n_{2\ell})$ in T_∞^{even} we have $\max\{|x_\alpha'(y_\beta)|, |x_\beta'(y_\alpha)|\} < 4^{-\ell}$.

Let $(n_{2k-1}, y_{(n_1,\ldots,n_{2k})})_{k=1}^\infty$ be a branch of the weakly null tree $(y_\alpha)_{\alpha \in T_\infty^{\mathrm{even}}}$ such that $(v_{n_{2k-1}})_{k=1}^\infty$ C-dominates $(y_{(n_1,\ldots,n_{2k})})_{k=1}^\infty$ for some $C \geq 1$. Let $(a_i) \in c_{00}$ be such that $\|\sum a_i v_i^*\| = 1$. There exists $(b_i) \in c_{00}$ such that $\|\sum b_i v_i\| = 1$ and $\sum a_i b_i = 1$ as (v_i) is bimonotone. Furthermore, $\operatorname{sign}(a_i) = \operatorname{sign}(b_i)$ as (v_i) is 1-unconditional. We have

$$\begin{aligned}
1 = \Big\| \sum_{i=1}^\infty a_i v_i^* \Big\| &= \sum_{i=1}^\infty a_i b_i \leq \frac{4}{3} \sum_{i=1}^\infty a_i b_i x'_{(n_1,\ldots,n_{2i})}(y_{(n_1,\ldots,n_{2i})}) \\
&\leq \frac{4}{3} \Big(\sum_{k=1}^\infty a_k x'_{(n_1,\ldots,n_{2k})} \Big) \Big(\sum_{\ell=1}^\infty b_\ell y_{(n_1,\ldots,n_{2\ell})} \Big) \\
&\quad + \frac{4}{3} \sum_{k=1}^\infty \sum_{\ell \neq k} |x'_{(n_1,\ldots,n_{2k})}(y_{(n_1,\ldots,n_{2\ell})})| \\
&\leq \frac{4}{3} \Big(\sum_{k=1}^\infty a_k x'_{(n_1,\ldots,n_{2k})} \Big) \Big(\sum_{\ell} b_\ell y_{(n_1,\ldots,n_{2\ell})} \Big) + \frac{2}{3} \\
&< C \frac{4}{3} \Big\| \sum_{k=1}^\infty a_k x'_{(n_1,\ldots,n_{2k})} \Big\| + \frac{2}{3}.
\end{aligned}$$

Hence $(x'_{(n_1,\dots,n_{2k})})_{k=1}^{\infty}$ $4C$-dominates $(v^*_{n_{2k-1}})_{k=1}^{\infty}$. Finally, the branch $(n_{2k-1}, x'_{(n_1,\dots,n_{2k})})$ corresponds to a branch $(m_{2k-1}, x_{(m_1,\dots,m_{2k})})$ in the original tree with $n_i \le m_i$ for all $i \in \mathbb{N}$. Since (v_i) is right dominant, (v_i^*) is left dominant, and hence $(x_{(m_1,\dots,m_{2k})})$ dominates $(v^*_{m_{2k-1}})$. Thus X^* satisfies subsequential V^*-lower w^* tree estimates. ∎

Proposition 2.6 allows us to pass from information about trees to information about $\bar{\delta}$-skipped blocks of an FDD (E_n). To go from information about $\bar{\delta}$-skipped blocks to blocks in general, we will renorm the FDD (E_n) to form a new space.

Let Z be a space with an FDD $E = (E_n)$ and let $V = (v_i)$ be a normalized 1-unconditional basic sequence. The space $Z^V = Z^V(E)$ is defined to be the completion of $c_{00}(\bigoplus E_n)$ with respect to the following norm $\|\cdot\|_{Z^V}$:

$$\|z\|_{Z^V} = \max_{k\in\mathbb{N},\, 1\le n_0<n_1<\dots<n_k} \Big\| \sum_{j=1}^{k} \|P^E_{[n_{j-1},n_j)}(z)\|_Z \cdot v_{n_{j-1}} \Big\|_V \quad \text{for } z \in c_{00}(E_i).$$

We note that if $\|\cdot\|$ and $\|\cdot\|'$ are equivalent norms on Z then the corresponding norms $\|\cdot\|_{Z^V}$ and $\|\cdot\|'_{Z^V}$ are equivalent on $c_{00}(\bigoplus E_n)$. This allows us, when examining the space Z^V, to assume that (E_n) is bimonotone in Z. The following proposition from [17] is what makes the space Z_V essential for us. Recall that in [17] a basic sequence is called *C-block stable* for some $C \ge 1$ if any two normalized block bases (x_i) and (y_i) with

$$\max(\mathrm{supp}(x_i) \cup \mathrm{supp}(y_i)) < \min(\mathrm{supp}(x_{i+1}) \cup \mathrm{supp}(y_{i+1})) \quad \text{for all } i \in \mathbb{N}$$

are C-equivalent. We say that (v_i) is *block stable* if it is C-block stable for some constant C. This property has been considered before in various forms and under different names. In particular, it has been called the blocking principle [2] and the shift property [3] (see [6] for alternative forms).

The following proposition recalls some properties of Z^V which were shown in [17].

Proposition 2.8 ([17, Corollary 3.2, Lemmas 3.3 and 3.5]). *Let $V = (v_i)$ be a normalized, 1-unconditional, and C-block stable basic sequence. If Z is a Banach space with an FDD (E_i), then (E_i) satisfies $2C$-V-lower block estimates in $Z^V(E)$.*

If the basis (v_i) is boundedly complete then (E_i) is a boundedly complete FDD for $Z^V(E)$.

If the basis (v_i) is shrinking and (E_i) is shrinking in Z, then (E_i) is a shrinking FDD for $Z^V(E)$.

In proving our main theorem we will show that if X satisfies subsequential V-upper tree estimates then it is isomorphic to a subspace of some $Z^V(E)$ and to a quotient of some $Z^V(F)$.

3. Proofs of the main results

Proof of Theorem 1.1. (1)⇒(4). (v_i) is D-right dominant for some $D \geq 1$, from which we can easily deduce that (v_i^*) is D-left dominant. By [4, Corollary 8] there exists a space Z with a shrinking and bimonotone FDD (E_i) for which there is a quotient map $Q : Z \to X$. The map $Q^* : X^* \to Z^*$ is an into isomorphism. After renorming X if necessary, we can assume that X has the quotient norm induced by Q, and so Q^* is an isometric embedding. By Lemma 2.7, X^* satisfies subsequential C-V^*-lower w^* tree estimates for some $C \geq 1$. As $Q^*X^* \subset Z^*$ is w^* closed, we may apply Proposition 2.6 with $A = \{(n_i, x_i)_{i=1}^\infty \in (\mathbb{N} \times S_{Q^*X^*})^\omega \mid (x_i)\ C\text{-dominates } (v_{n_i})\}$ and $\varepsilon > 0$ such that $\overline{A_\varepsilon} \subset \{(n_i, x_i)_{i=1}^\infty \in (\mathbb{N} \times S_{Q^*X^*})^\omega \mid (x_i)\ 2CD\text{-dominates } (v_{n_i})\}$. This gives sequences $(K_i) \in [\mathbb{N}]^\omega$ and $\bar{\delta} = (\delta_i) \subset (0,1)$ and a blocking (F_i) of (E_i^*) such that if $(x_i) \subset S_{Q^*X^*}$ and $\|x_i - P^F_{(r_{i-1}, r_i)}(x_i)\| < 2\delta_i$ for some $(r_i) \in [\mathbb{N}]^\omega$ then $(K_{r_{i-1}}, x_i) \in \overline{A_\varepsilon}$. Hence, the sequence (x_i) $2CD$-dominates $(v_{K_{r_{i-1}}})$.

We choose a blocking $G = (G_i)$ of (F_i) defined by $G_i = \sum_{j=m_{i-1}+1}^{m_i} F_j$ for some $(m_i) \in [\mathbb{N}]^\omega$ such that there exists $(e_n) \subset S_{Q^*X^*}$ with $\|e_n - P_n^G(e_n)\| < \delta_n/2$ for all $n \in \mathbb{N}$.

In order to continue we need a result from [13] which is based on an argument due to W. B. Johnson [8]. [13, Corollary 4.4] was stated for reflexive spaces. Here we state it for w^* closed subspaces of dual spaces with a boundedly complete FDD: the proof is easily seen to work in this situation. Also note that conditions (d) and (e) which were not stated in [13] follow easily from the proof.

Proposition 3.1 ([13, Lemma 4.3 and Corollary 4.4]). *Let Y be a w^* closed subspace of a dual Banach space Z with a boundedly complete FDD $A = (A_i)$ having projection constant K. Let $\bar{\eta} = (\eta_i) \subset (0,1)$ with $\eta_i \downarrow 0$. Then there exists $(N_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$ such that the following holds. Given $(k_i)_{i=0}^\infty \in [\mathbb{N}]^\omega$ and $x \in S_Y$, there exists $x_i \in Y$ and $t_i \in (N_{k_{i-1}-1}, N_{k_{i-1}})$ for all $i \in \mathbb{N}$ with $N_0 = 0$ and $t_0 = 0$ such that*

(a) $x = \sum_{i=1}^\infty x_i$,

and for all $i \in \mathbb{N}$ we have

(b) *either* $\|x_i\| < \eta_i$ *or* $\|x_i - P^A_{(t_{i-1}, t_i)} x_i\| < \eta_i \|x_i\|$,
(c) $\|x_i - P^A_{(t_{i-1}, t_i)} x\| < \eta_i$,
(d) $\|x_i\| < K + 1$,
(e) $\|P^A_{t_i} x\| < \eta_i$.

We apply Proposition 3.1 with $Y = Q^*X^*$, $A = G$ and $\bar{\eta} = \bar{\delta}$, which gives a sequence $(N_i) \in [\mathbb{N}]^\omega$. We set $H_j = \bigoplus_{i=N_{j-1}+1}^{N_j} G_i$ for each $j \in \mathbb{N}$. To make notation easier we let $V_M^* = (v_{M_i}^*)$ be the subsequence of (v_i^*) defined by $M_i = K_{m_{N_i}}$.

Fix $x \in S_{Q*X^*}$ and a sequence $(n_i)_{i=0}^\infty \in [\mathbb{N}]^\omega$. The proof in [17, Theorem 4.1(a)] shows

$$\Big\| \sum_{i=1}^\infty \|P^H_{[n_{i-1},n_i)}(x)\|_{Z^*} \cdot v^*_{M_{n_{i-1}}} \Big\|_{V^*} \leq 4D^2C(1+2\Delta+2)+2+3\Delta,$$

where $\Delta = \sum_{i=1}^\infty \delta_i$. Thus the norms $\|\cdot\|_{Z^*}$ and $\|\cdot\|_{(Z^*)^{V_M^*}}$ are equivalent on Q^*X^*. As the norm on each H_j is unchanged, a coordinatewise null sequence in $Q^*X^* \subset Z^*$ will still be coordinatewise null in $(Z^*)^{V_M^*}$. Hence the map $Q^* : X^* \to (Z^*)^{V_M^*}$ is still w^*-to-w^* continuous.

The space $(Z^*)^{V_M^*}$ has a boundedly complete FDD (H_j) which satisfies subsequential V_M^*-lower block estimates by Proposition 2.8. We can now fill in the FDD as in [17, Lemma 2.13]. We let $B_{M_j} = H_j$ for all $j \in \mathbb{N}$, and $B_j = \mathbb{R}$ for each $j \notin (M_i)$. For $x = (x_j) \in c_{00}(B_j)$ we define

$$\|x\| = \Big\| \sum_{j\in\mathbb{N}} x_{M_j} \Big\|_{(Z^*)^{V_M^*}} + \sum_{j \notin M} |x_j|.$$

We let Y be the completion of $c_{00}(\bigoplus B_j)$ under this norm. Then Y is clearly isometrically isomorphic to $(Z^*)^{V_M^*} \oplus \ell_1$ or $(Z^*)^{V_M^*} \oplus \ell_1^n$ for some $n \in \mathbb{N}_0$. Thus the natural embedding of $(Z^*)^{V_M^*}$ into Y is w^*-to-w^* continuous. Hence there is a w^*-to-w^* continuous embedding of X^* into Y. Finally, as (H_j) satisfies subsequential V_M^*-lower block estimates in $(Z^*)^{V_M^*}$, it is not hard to deduce that (B_j) satisfies subsequential V^*-lower block estimates in Y.

(4)⇒(3). This is clear because if (F_i^*) is a boundedly complete FDD of Z^* then (F_i) is a shrinking FDD of Z and a w^*-to-w^* continuous embedding $T : X^* \to Z^*$ must be the dual of some quotient map $Q : Z \to X$. Also, (F_i^*) having subsequential V^* lower block estimates is equivalent to (F_i) having subsequential V-upper block estimates due to Proposition 2.2.

(3)⇒(1). Let (F_i) be a bimonotone shrinking FDD which satisfies subsequential V-upper block estimates, and $Q : Z \to X$ be a quotient map. There exists $C > 0$ such that $B_X \subset Q(CB_Z)$. We will need the following lemma.

LEMMA 3.2. *Let X and Z be Banach spaces, $F = (F_i)$ be a bimonotone FDD for Z, and $Q : Z \to X$ be a quotient map. If $(x_i) \subset S_X$ is weakly null and $Q(CB_Z) \supseteq B_X$ for some $C > 0$ then for all $\varepsilon > 0$ and $n \in \mathbb{N}$ there exist $N \in \mathbb{N}$ and $z \in 2CB_Z$ such that $P_{[1,n]}z = 0$ and $\|Qz - x_N\| < \varepsilon$.*

Proof. Let $z_i \in CB_Z$ be such that $Qz_i = x_i$. After passing to a subsequence (z_{k_i}) and perturbing we may assume instead that $P^F_{[1,n]} z_{k_i} = z_0$ for some $z_0 \in CB_Z$, and that $\|Qz_{k_i} - x_{k_i}\| < \varepsilon/3$. As (x_{k_i}) is weakly null, zero must be in the closure of the convex hull of (x_{k_i}). Hence there is some finite sequence $(a_i)_{i=2}^m \subset [0,1]$ such that $\|\sum_{i=2}^m a_i x_{k_i}\| < \varepsilon/3$ and $\sum_{i=2}^m a_i = 1$. Let $z = z_{k_1} - \sum_{i=2}^m a_i z_{k_i}$. Then $z \in 2CB_Z$, $P^F_{[1,n]} z = 0$, and

$$\begin{aligned}\|Qz - x_{k_1}\| &= \Big\|Qz_{k_1} - x_{k_1} - \sum_{i=2}^m a_i Q z_{k_i}\Big\| \\ &\le \|Qz_{k_1} - x_{k_1}\| + \sum_{i=2}^m a_i \|Qz_{k_i} - x_{k_i}\| + \Big\|\sum_{i=2}^m a_i x_{k_i}\Big\| < \varepsilon. \ \blacksquare\end{aligned}$$

Continuation of proof of Theorem 1.1. Let $(x_t)_{t\in T_\infty^{\mathrm{even}}} \subset S_X$ be a weakly null even tree in X, and let $\eta \in (0,1)$. By Lemma 3.2 we may pass to a full subtree $(x'_t)_{t\in T_\infty^{\mathrm{even}}}$ of (x_t) such that there exists a block tree $(z_t)_{t\in T_\infty^{\mathrm{even}}} \subset 2CB_Z$ such that $\|Q(z_t) - x'_t\| < \eta 2^{-\ell}$ for all $\ell \in \mathbb{N}$ and $t = (k_1, \ldots, k_{2\ell}) \in T_\infty^{\mathrm{even}}$. Now choose $1 = k_1 < k_2 < \cdots$ such that $\max\operatorname{supp}(z_{(k_1,\ldots,k_{2i})}) < k_{2i+1} < \min\operatorname{supp}(z_{(k_1,\ldots,k_{2i+2})})$ for all $i \in \mathbb{N}$. Then $(z_{(k_1,\ldots,k_{2i})})$ is dominated by $(v_{k_{2i-1}})$, and hence $(x'_{(k_1,\ldots,k_{2i})})$ is dominated by $(v_{k_{2i-1}})$ provided η was chosen sufficiently small. Finally, the branch $(k_{2i-1}, x'_{(k_1,\ldots,k_{2i})})$ corresponds to a branch $(\ell_{2i-1}, x_{(\ell_1,\ldots,\ell_{2i})})$ in the original tree with $k_i \le \ell_i$ for all $i \in \mathbb{N}$. Since (v_i) is right dominant, it follows that $(x_{(\ell_1,\ldots,\ell_{2i})})$ is dominated by $(v_{\ell_{2i-1}})$. Thus X satisfies subsequential V-upper tree estimates.

(2)⇒(1). We assume that X is a quotient of a space Z with separable dual such that Z satisfies subsequential V-upper tree estimates. By the implication (1)⇒(3) applied to Z, Z is the quotient of a space Y with a shrinking FDD satisfying subsequential V-upper block estimates. X is then also a quotient of Y, so by the implication (3)⇒(1), X satisfies subsequential V-upper tree estimates.

(1)⇒(5). Our proof will be based on the proof of [17, Theorem 4.1(b)]. By Zippin's theorem we may assume, after renorming X if necessary, that there exists a Banach space Z with a shrinking, bimonotone FDD (F_j) and an isometric embedding $i : X \hookrightarrow Z$. Also, by [4, Corollary 8] we know that there exists a Banach space W with a shrinking FDD (E_j) and a quotient map $Q : W \twoheadrightarrow X$. Thus we have a quotient map $i^* : [F_j^*] = Z^* \twoheadrightarrow X^*$ and an embedding $Q^* : X^* \hookrightarrow [E_j^*] = W^*$. We can assume, after renorming W if necessary, that Q^* is an isometric embedding. Note that (F_j^*) and (E_j^*) are boundedly complete FDDs of Z^* and W^*, respectively, and X^* has the quotient norm induced by i^*. Let K be the projection constant of (E_j) in W.

By Lemma 2.7, X^* satisfies subsequential C-V^*-lower w^* tree estimates for some $C \ge 1$. Choose $D \ge 1$ such that (v_i) is D-right dominant. Since Q^*X^* is w^* closed in W^*, we can apply Proposition 2.6 as in the proof of the implication (1)⇒(4): after blocking (E_j^*), we find sequences $(K_i) \in [\mathbb{N}]^\omega$ and $\bar\delta = (\delta_i) \subset (0,1)$ with $\delta_i \downarrow 0$ such that if $(x_i) \subset S_{Q^*X^*}$ is a $2K\bar\delta$-skipped block of (E_j^*) with $\|x_i - P^{E^*}_{(r_{i-1},r_i)} x_i\| < 2K\delta_i$ for all $i \in \mathbb{N}$, where

$1 \le r_0 < r_1 < r_2 < \cdots$, then $(v^*_{K_{r_{i-1}}})$ is $2CD$-dominated by (x_i), and moreover, using standard perturbation arguments and making $\bar\delta$ smaller if necessary, we can assume that if $(w_i) \subset W^*$ satisfies $\|x_i - w_i\| < \delta_i$ for all $i \in \mathbb{N}$, then (w_i) is a basic sequence equivalent to (x_i) with projection constant at most $2K$. We can also assume that $\Delta = \sum_{i=1}^\infty \delta_i < 1/7$.

Choose a sequence $(\varepsilon_i) \subset (0,1)$ with $\varepsilon_i \downarrow 0$ and $3K(K+1)\sum_{j=i}^\infty \varepsilon_j < \delta_i^2$ for all $i \in \mathbb{N}$. After blocking (E^*_j) if necessary, we can assume that for any subsequent blocking D of E^* there is a sequence (e_i) in $S_{Q^*X^*}$ such that $\|e_i - P^D_i(e_i)\| < \varepsilon_i/2K$ for all $i \in \mathbb{N}$.

Using Johnson and Zippin's blocking lemma [10] we may assume, after further blocking our FDDs (F^*_j) and (E^*_j) if necessary, that given $k < \ell$, if $z^* \in \bigoplus_{j\in(k,\ell)} F^*_j$, then $\|P^{E^*}_{[1,k)}Q^*i^*z^*\| < \varepsilon_k$ and $\|P^{E^*}_{[\ell,\infty)}Q^*i^*z^*\| < \varepsilon_\ell$, and moreover the same holds if one passes to any blocking of (F^*_j) and the corresponding blocking of (E^*_j). Note that although the conditions of the Johnson–Zippin lemma are not satisfied here, the proof is easily seen to apply because our FDDs are boundedly complete, and the map Q^*i^* is w^*-to-w^* continuous.

We now continue as in the proof of [17, Theorem 4.1(b)]: we replace F^*_j by the quotient space $\tilde F_j = i^*(F^*_j)$, we let $\tilde Z$ be the completion of $c_{00}(\tilde F_j)$ with respect to the norm $\|\!|\cdot|\!\|$ as defined in [17] and obtain a quotient map $\tilde\imath\colon \tilde Z \to X^*$. We note that the results corresponding to [17, Proposition 4.9(b),(c)] are valid here as their proof does not require reflexivity (part (a) is not required, and indeed neither valid, here).

Finally, we find a blocking $(\tilde G_j)$ of $(\tilde F_j)$ and a subsequence $V^*_N = (v^*_{n_i})$ of (v^*_i) such that $\tilde\imath$ is still a quotient map of $\tilde Z^{V^*_N}(\tilde G)$ onto X^* and it is still w^*-to-w^* continuous (note that $(\tilde G_j)$ is boundedly complete in $\tilde Z^{V^*_N}(\tilde G)$ by Proposition 2.8). Since $(\tilde G_j)$ satisfies subsequential V^*_N-lower block estimates in $\tilde Z^{V^*_N}(\tilde G)$, statement (5) will then follow by duality (after filling the FDD as in the proof of the implication (1)⇒(4)). To find suitable $\tilde G$ and (n_i) we now follow the proof of [17, Theorem 4.1(b)] verbatim. The only comment we need to make is that [17, Lemma 4.10] is valid since we are working with boundedly complete FDDs and w^*-to-w^* continuous maps.

Finally, since the missing implications (5)⇒(1) and (3)⇒(2) are trivial, we finished the proof of the theorem. ∎

The proof of the following result is an adaptation of the proof of Theorem 5.1 in [17] to the nonreflexive case.

Corollary 3.3. *Let $V = (v_i)$ be a 1-unconditional, shrinking, block stable, and right dominant normalized basic sequence. There is a Banach space Y with a shrinking FDD (E_i) satisfying subsequential V-upper block estimates such that if a Banach space X with separable dual has subsequential V-upper tree estimates, then X embeds into Y.*

Proof. By Schechtman's result [20] there exists a space W with a bimonotone FDD $E = (E_i)$ with the property that any space X with bimonotone FDD $F = (F_i)$ naturally almost isometrically embeds into W, i.e. for any $\varepsilon > 0$ there is a $(1+\varepsilon)$-embedding $T : X \to W$ and a sequence $(k_i) \in [\mathbb{N}]^\omega$ such that $T(F_i) = E_{k_i}$, and moreover $\sum_i P^E_{k_i}$ is a norm-1 projection of W.

Since V^* is boundedly complete it follows from Proposition 2.8 that the sequence (E_i^*) is a boundedly complete FDD of the space $(W^{(*)})^{V^*}$. It follows that (E_i) is a shrinking FDD of the space $Y = ((W^{(*)})^{V^*})^{(*)}$ and that $Y^* = (W^{(*)})^{V^*}$. We denote by $\|\cdot\|_W$, $\|\cdot\|_{W^{(*)}}$, $\|\cdot\|_Y$, $\|\cdot\|_{Y^*}$ the norms in W, $W^{(*)}$, Y and Y^*, respectively.

By Proposition 2.8, (E_i^*) satisfies subsequential V^*-lower block estimates in $(W^{(*)})^{V^*}$, and, thus, by Proposition 2.2, (E_i) satisfies subsequential V-upper block estimates in Y (recall that $Y^{(*)} = Y^* = (W^{(*)})^{V^*}$).

We now have to show that a space X with separable dual and with subsequential V-upper tree estimates embeds in Y. By Theorem 1.1 we can assume that X has a shrinking, bimonotone FDD (F_i) satisfying subsequential V-upper block estimates. By our choice of W we can assume that X is the complemented subspace of W generated by a subsequence (E_{k_i}) of (E_i). We need to show that on X the norms $\|\cdot\|_W$ and $\|\cdot\|_Y$ are equivalent.

Let $C \ge 1$ be chosen so that (v_i) is C-block stable and C-right dominant (thus (v_i^*) is C-block stable and C-left dominant) and such that $(E^*_{k_i})$ satisfies subsequential C-V^*-lower block estimates in X^*. Let $w^* \in c_{00}(\bigoplus F_i^*) = c_{00}(\bigoplus E^*_{k_i})$. Clearly, we have $\|w^*\|_{W^{(*)}} \le \|w^*\|_{Y^*}$. Choose $1 \le m_0 < m_1 < \cdots$ such that

$$\|w^*\|_{Y^*} = \Big\| \sum_{i=1}^{\infty} \|P^{E^*}_{[m_{i-1},m_i)}(w^*)\|_{W^{(*)}} v^*_{m_{i-1}} \Big\|_{V^*}.$$

We can assume that $m_0 = 1$ and that $P^{E^*}_{[m_{i-1},m_i)}(w^*) \neq 0$ for $i \in \mathbb{N}$. Since $w^* \in c_{00}(\bigoplus E^*_{k_i})$, we can choose $j_1 < j_2 < \cdots$ such that $k_{j_i} = \min \operatorname{supp} P^{E^*}_{[m_{i-1},m_i)}(w^*)$ and deduce

$$\begin{aligned}
\|w^*\|_{Y^*} &= \Big\| \sum_{i=1}^{\infty} \|P^{E^*}_{[m_{i-1},m_i)}(w^*)\|_{W^{(*)}} v^*_{m_{i-1}} \Big\|_{V^*} \\
&\le C \Big\| \sum_{i=1}^{\infty} \|P^{E^*}_{[m_{i-1},m_i)}(w^*)\|_{W^{(*)}} v^*_{k_{j_i}} \Big\|_{V^*} \\
&\le C^2 \Big\| \sum_{i=1}^{\infty} \|P^{F^*}_{[j_i,j_{i+1})}(w^*)\|_{W^{(*)}} v^*_{j_i} \Big\|_{V^*} \le C^3 \|w^*\|_{W^{(*)}}.
\end{aligned}$$

This proves that $\|\cdot\|_{W^{(*)}}$ and $\|\cdot\|_{Y^*}$ are equivalent on $c_{00}(\bigoplus E^*_{k_i})$. Since X is 1-complemented in W, and X^* is 1-complemented in $W^{(*)}$, and since

$\sum_i P_{k_i}^{E^*}$ is still a norm-1 projection from Y^* onto $\overline{c_{00}(\bigoplus(E_{k_i}))}^{Y^*}$, it follows for any $w \in c_{00}(\bigoplus E_{k_i})$ that $(1/C^3)\|w\|_W \le \|w\|_Y \le \|w\|_W$, which finishes the proof of our claim. ■

As an application of Theorem 1.1 we extend structural and universality results on classes of bounded Szlenk index from the reflexive case studied in [18] to the nonreflexive case.

COROLLARY 3.4. *Let $\alpha < \omega_1$. For a space X with separable dual, the following are equivalent:*

(i) *X has Szlenk index at most $\omega^{\alpha\omega}$.*
(ii) *X satisfies subsequential $T_{\alpha,c}$-upper tree estimates for some $c \in (0,1)$.*
(iii) *X embeds into a space Z with an FDD (E_i) which satisfies subsequential $T_{\alpha,c}$-upper block estimates in Z for some $c \in (0,1)$.*

Proof. The implication (i)⇒(ii) is proved in Corollary 19 and Theorem 21 of [18] (the reflexivity assumption there is not used for the relevant implication). The implication (iii)⇒(i) follows from [18, Proposition 17]. Finally, (ii)⇒(iii) follows from the implication (1)⇒(5) of Theorem 1.1. ■

COROLLARY 3.5. *For each $\alpha < \omega_1$ there exists a Banach space Z_α with a shrinking FDD and Szlenk index at most $\omega^{\alpha\omega+1}$ such that Z_α is universal for the collection of spaces with separable dual and Szlenk index at most $\omega^{\alpha\omega}$.*

Proof. By Corollary 3.3 for all $n \in \mathbb{N}$ there exists a Banach space X_n with an FDD satisfying subsequential $T_{\alpha,n/(n+1)}$-upper block estimates which is universal for all Banach spaces with separable dual which satisfy subsequential $T_{\alpha,n/(n+1)}$-upper tree estimates. Let $Z_\alpha = (\bigoplus X_n)_{\ell_2}$. The space Z_α is universal for the collection of spaces with separable dual and Szlenk index at most $\omega^{\alpha\omega}$ by Corollary 3.4. The Szlenk index of Z_α is at most $\omega^{\alpha\omega+1}$ as proven in [18]. ■

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References

[1] B. Bossard, *An ordinal version of some applications of the classical interpolation theorem*, Fund. Math. 152 (1997), 55–74.

[2] P. G. Casazza, W. B. Johnson and L. Tzafriri, *On Tsirelson's space*, Israel J. Math. 47 (1984), 81–98.

[3] P. G. Casazza and N. Kalton, *Unconditional bases and unconditional finite-dimensional decompositions in Banach spaces*, ibid. 95 (1996), 349–373.

[4] W. J. Davis, T. Figiel, W. B. Johnson and A. Pełczyński, *Factoring weakly compact operators*, J. Funct. Anal. 17 (1974), 311–327.

[5] P. Dodos and V. Ferenczi, *Some strongly bounded classes of Banach spaces*, Fund. Math. 193 (2007), 171–179.

[6] V. Ferenczi and Ch. Rosendal, *Ergodic Banach spaces*, Adv. Math. 195 (2005), 259–282.

[7] G. Godefroy, G. Lancien and N. Kalton, *Szlenk indices and uniform homeomorphisms*, Trans. Amer. Math. Soc. 353 (2001), 3895–3918.

[8] W. B. Johnson, *On quotients of L_p which are quotients of ℓ_p*, Compos. Math. 34 (1977), 69–89.

[9] W. B. Johnson and B. Zheng, *A characterization of subspaces and quotients of reflexive Banach spaces with unconditional bases*, Duke Math. J. 141 (2008), 505–518.

[10] W. B. Johnson and M. Zippin, *On subspaces of quotients of $(\sum G_n)_{l_p}$ and $(\sum G_n)_{c_0}$*, Israel J. Math. 13 (1972), 311 –316.

[11] H. Knaust, E. Odell and Th. Schlumprecht, *On asymptotic structure, the Szlenk index and UKK properties in Banach spaces*, Positivity 3 (1999), 173–199.

[12] D. A. Martin, *Borel determinacy*, Ann. of Math. (2) 102 (1975), 363–371.

[13] E. Odell and Th. Schlumprecht, *Trees and branches in Banach spaces*, Trans. Amer. Math. Soc. 354 (2002), 4085–4108.

[14] —, —, *A universal reflexive space for the class of uniformly convex Banach spaces*, Math. Ann. 335 (2006), 901–916.

[15] —, —, *Embedding into Banach spaces with finite dimensional decompositions*, Rev. R. Acad. Cien. Ser. A Mat. 100 (2006), 295–323.

[16] E. Odell, Th. Schlumprecht and A. Zsák, *On the structure of asymptotic ℓ_p spaces*, Quart. J. Math. 59 (2008), 85–122.

[17] —, —, —, *A new infinite game in Banach spaces with applications*, in: Proc. Int. Conf. in Honor of Nigel Kalton's Sixtieth Birthday (Miami, FL, 2006), de Gruyter, Berlin, 2007, 147–183.

[18] —, —, —, *Banach spaces of bounded Szlenk index*, Studia Math. 183 (2007), 63–97.

[19] Ch. Rosendal, *Infinite asymptotic games*, Ann. Inst. Fourier (Grenoble) 59 (2009), 1359–1384.

[20] G. Schechtman, *On Pełczyński's paper "Universal bases"*, Israel J. Math. 22 (1975), 181–184.

[21] M. Zippin, *Banach spaces with separable duals*, Trans. Amer. Math. Soc. 310 (1988), 371–379.

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