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Uncountable ω-limit sets with isolated points

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Abstract. We give two examples of tent maps with uncountable (as it happens, post-critical) ω-limit sets, which have isolated points, with interesting structures. Such ω-limit sets must be of the form $C \cup R$, where C is a Cantor set and R is a scattered set. Firstly, it is known that there is a restriction on the topological structure of countable ω-limit sets for finite-to-one maps satisfying at least some weak form of expansivity. We show that this restriction does not hold if the ω-limit set is uncountable. Secondly, we give an example of an ω-limit set of the form $C \cup R$ for which the Cantor set C is minimal.

1. Introduction. Let X be a space and $F : X \to X$ be continuous. For $x \in X$, the *ω-limit set* of x is the set

$$\omega(x) = \bigcap_{n\in\mathbb{N}} \overline{\{F^j(x) : j \geq n\}}.$$

The topological structure of the ω-limit set of x is an indication of the complexity of the orbit of x, and as such the topological structure and dynamical features of ω-limit sets is the subject of much study, [1], [2], [4], [7], [8], [10], [11], [14]. Of particular interest is the case where $X = [0,1]$ and f is a unimodal map with critical point c. In this setting we consider the ω-limit set of the critical point, $\omega(c)$. Typically (in the sense of Lebesgue measure) the orbit of c is dense, and so $\omega(c) = [0,1]$, [6], but $\omega(c)$ can be much more complicated.

If the ω-limit set of a point (in particular, the critical point) of a unimodal map with large enough gradient is not dense, then it is totally disconnected. By definition, these sets are compact and strongly invariant (i.e. $f(\omega(c)) = \omega(c)$). So it is common to think of such ω-limit sets as periodic orbits or invariant Cantor sets. However, there are many more varieties. For instance a sort of in-between case is when the ω-limit set is infinite yet contains

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isolated points. Suppose that A is an infinite, totally disconnected, compact subset of $[0,1]$. We can get an idea of the topological structure of A by considering its iterated derived set.

Let X be any non-empty topological space and let A be a subset of X. The *Cantor–Bendixson derivative* A' of A is the set of all limit points of A. Inductively, we can define the *iterated Cantor–Bendixson derivatives* of X by

$$\begin{aligned} X^{(0)} &= X, \\ X^{(\alpha+1)} &= (X^{(\alpha)})', \\ X^{(\lambda)} &= \bigcap_{\alpha<\lambda} X^{(\alpha)} \quad \text{if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

Clearly for some ordinal γ, $X^{(\gamma)} = X^{(\gamma+1)}$. If this set is non-empty, then it is called the *perfect kernel*, and if it is empty, then X is said to be *scattered*. In the scattered case, a point of X has a well-defined Cantor–Bendixson rank, often called the *scattered height* or *limit type* of x, defined by $\mathrm{lt}(x) = \alpha$ if and only if $x \in X^{(\alpha)} \setminus X^{(\alpha+1)}$. The αth level L_α of X (or, more formally, L_α^X) is then the set of all points of limit type α. Clearly L_α is the set of isolated points of $X^{(\alpha)}$.

Since the collection of $X^{(\alpha)}$s forms a decreasing sequence of closed subsets of X, if X is a compact scattered space, then it has a non-empty finite top level $X^{(\gamma)} = L_\gamma$.

We endow an ordinal (regarded as the set of its own predecessors) with the interval topology generated by its natural order. With this topology every ordinal is a scattered space.

The standard set-theoretic notation for the first infinite ordinal, i.e. the set of all natural numbers, is ω. The ordinal $\omega+1$, then, is the set of all ordinals less than *or equal to* ω, so $\omega+1$ is the set consisting of ω together with all natural numbers. Then $\omega+1$ with its order topology is homeomorphic to the convergent sequence $S_0 = \{0\} \cup \{1/n : 0 < n \in \mathbb{N}\}$ with the usual topology inherited from the real line. In fact, every countable ordinal is homeomorphic to a subset of $\mathbb{Q}$. The next limit ordinal is $\omega+\omega = \omega\cdot 2$. The space $\omega\cdot 2+1$ consists of all ordinals less than or equal to $\omega\cdot 2$, i.e. all natural numbers, ω, the ordinals $\omega+n$ for each $n\in\mathbb{N}$, and the limit ordinal $\omega\cdot 2$. The set $\omega\cdot 2+1$ with its order topology is homeomorphic to two disjoint copies of S_0. For each $n\in\mathbb{N}$, the ordinals n and $\omega+n$ $(0<n)$ have scattered height 0. On the other hand, ω and $\omega\cdot 2$ have scattered height 1, corresponding to the fact that 0 is a limit of isolated points in S_0 but is not a limit of limit points in S_0. The ordinal space ω^2+1 consists of all ordinals less than or equal to ω^2 (namely: 0; the successor ordinals n and $\omega\cdot n+j$ for each $j,n\in\mathbb{N}$; the limit ordinals $\omega\cdot n$ for each $n\in\mathbb{N}$; and the limit

ordinal ω^2). With its natural order topology, ω^2+1 is homeomorphic to the subset of the real line $S = \{0\} \cup \bigcup_{n\in\mathbb{N}} S_n$ defined in the Introduction. In this case, the ordinals $\omega \cdot n$, $n \in \mathbb{N}$, which have scattered height 1, correspond to the points $1/n$, which are limits of isolated points $1/n + 1/k$ but not of limit points. The ordinal ω^2 has scattered height 2 and corresponds to the point 0, which is a limit of the limit points $1/n$.

In general, the ordinal space $\omega^\alpha \cdot n + 1$ consists of n copies of the space $\omega^\alpha + 1$, which itself consist of a single point with limit type α as well as countably many points of every limit type β with $\beta < \alpha$. It is a standard topological fact that every countable, compact Hausdorff space X is not only scattered, but homeomorphic to a countable successor ordinal of the form $\omega^\alpha \cdot n + 1$ for some countable ordinal α. Of course every countable compact metric space is also homeomorphic to a subset of the rationals and, in this context, we can interpret the statement that $X \simeq \omega^\alpha \cdot n + 1$ as notation to indicate that X is homeomorphic to a compact subset of the rationals with n points of highest limit type α. For more on scattered spaces, see section G of [12].

If f is a unimodal map of the interval, then the ω-limit set of the critical point is a subset of $[0,1]$. In this case, $\omega(c)$ is a subset of $[0,1]$, the perfect kernel exists and γ is countable. Moreover, this "final level" of A contains no isolated points and is either empty or a Cantor set.

We show in [10] that if A is a scattered ω-limit set of a finite-to-one map on a compact metric space, with a weak form of expansivity, then the height of A is a countable ordinal not equal to a limit ordinal or the successor of a limit ordinal, i.e. the empty perfect kernel cannot occur at a limit ordinal. This result applies, for example to locally eventually onto unimodal maps of the interval, such as tent maps with gradient greater than $\sqrt{2}$. Conversely, given a compact scattered subset A of the interval with height not equal to a limit ordinal or the successor of a limit ordinal, there is a tent map for which the ω-limit set of the critical point is homeomorphic to A.

In this paper we address the case of non-scattered, i.e. uncountable, ω-limit sets that nevertheless have isolated points. Specifically, we build an ω-limit set of a tent map such that the perfect kernel for A occurs at a limit height (in fact, height ω). This demonstrates that the restriction on the height of scattered ω-limit sets [10] is not valid for uncountable ω-limit sets with isolated points. In this example the Cantor set perfect kernel contains a fixed point and is hence not minimal. Therefore, in response to a question of the referee, we construct a tent map with a critical point whose ω-limit set is the union of a minimal Cantor set and a scattered part (consisting of isolated points) that is dense in the ω-limit set. In such cases the scattered part is always a dense subset of the ω-limit set.

2. The construction of a perfect kernel at level ω. In this section we construct a particular unimodal map, f, with critical point c such that $\omega(c)$ is an infinite set with isolated points that violates the limit height restriction on scattered ω-limit sets. We make extensive use of symbolic dynamics and itineraries. For background definitions and results see [5] or [9].

We begin by constructing a kneading sequence that "encodes" a Cantor set, $C \subseteq \{0,1\}^{\mathbb{N}}$, in the sense that $\omega(c)$ is made up of all the points with itineraries in C. (As usual, $\{0,1\}^{\mathbb{N}}$ has the product topology, so that two sequences are close if they agree on a long initial segment.) Inside this Cantor set we designate a countable collection of sets, Δ_n, each of which is countable and has limit height n such that the sets Δ_n accumulate on a finite collection of points in C. Then we use this countable collection of subsets of C to encode another kneading sequence that also encodes C but now with homeomorphic copies of Δ_n, Δ_n^*, that are not in the Cantor set, but accumulate on the same finite subset of C. Since these sets are not in C we will see that the ω-limit set of this new kneading sequence is of the form $C \cup R$ where $R = \bigcup_{n\in\mathbb{N}} \Delta_n^*$, C is the largest Cantor set in $\omega(c)$, and for each n there are points in R with limit type n but the points in $\omega(c)$ with limit type ω are in C.

Let $\{0,1\}^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}}\{0,1\}^n$ be the collection of all finite words in the alphabet $\{0,1\}$. Let

$$A = 10^5 1$$

and for every $n > 5$ let

$$B_{n,0} = 10^3 1^{2n} 0^3 1 \quad \text{and} \quad B_{n,1} = 10^3 1^{2n+1} 0^3 1.$$

For every $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_k) \in \{0,1\}^{<\mathbb{N}}$ and $n > 5$ let

$$C_{n,\gamma} = B_{n,\gamma_0} B_{n,\gamma_1} \ldots B_{n,\gamma_k}.$$

Extend this definition to all $\gamma \in \{0,1\}^{\mathbb{N}}$ in the obvious way. Notice that the set

$$\Gamma_n^* = \{C_{n,\gamma}\}_{\gamma\in\{0,1\}^{\mathbb{N}}}$$

is a Cantor set in $\{0,1\}^{\mathbb{N}}$. Let

$$\Gamma_n = \overline{\bigcup_{m\in\mathbb{N}} \sigma^m(\Gamma_n^*)}$$

where σ is the shift map. Then

$$\overline{\bigcup_{n\in\mathbb{N}} \Gamma_n} = \bigcup_{n\in\mathbb{N}} \Gamma_n \cup \{\sigma^j(1^k 0^3 110^3 1^\infty) : 0 \le k, j \le 8\} \cup \{1^\infty\},$$

which is a Cantor set. The advantage of considering a countable collection of Cantor sets comes later in the paper.

Let $\mathcal{L} = \{B_{m,0} : m > 5\} \cup \{10^4\}$. Let Σ be the set of all finite length words made up of words from $\mathcal{L}$. Let Σ_∞ be the set of all infinite length words of that form, and let

$$\overline{\Sigma} = \overline{\bigcup_{n\in\mathbb{N}} \sigma^n(\Sigma_\infty)}.$$

It is easy to see that $\overline{\Sigma} \subseteq \{0,1\}^\infty$ is a Cantor set. Since Σ is a collection of finite length words, it is countable. Let $(R_i)_{i\in\mathbb{N}}$ be some enumeration of Σ. Enumerate $\{0,1\}^{<\mathbb{N}}$ by

$$\{\gamma_j = (\gamma_{j,0}, \gamma_{j,1}, \ldots, \gamma_{j,k})\}_{j=1}^\infty = \{0,1\}^{<\mathbb{N}}.$$

Let $\mathcal{S} = \{1^n 0 C_{n,\gamma_j} : n, j \in \mathbb{N},\ n > 5\}$, and, since $\mathcal{S}$ is countable, let $\{S_m\}_{m\in\mathbb{N}}$ be an enumeration of $\mathcal{S}$. Define $k(m)$ to be the unique n such that $S_m = 1^n 0 C_{n,j}$ for some $j \in \mathbb{N}$. Let

$$T_m = (10^4)^m R_m (10^4)^m S_m B^m_{k(m),0}.$$

Finally, define a kneading sequence by

$$S = AAT_1T_2T_3\ldots.$$

It is easy to check that S is the kneading sequence of a tent map, f (see [9, Lemma III.1.6]).

Theorem 2.1. *Let f be the tent map with kneading sequence S and critical point c_f. Then $x \in \omega(c_f)$ if and only if the itinerary of x, $I(x)$, is a shift of one of the following*:

(1) $U \in \overline{\Sigma}$,
(2) $(10^4)^K 1^t 0 U_t$ *where* $U_n \in \Gamma_n$ *and* $K \in \mathbb{N}$,
(3) $(10^4)^K 1^\infty$ *where* $K \in \mathbb{N}$,
(4) $1^K 010^3 1^\infty$ *where* $K \in \mathbb{N}$.

Moreover, $\omega(c_f)$ *is a Cantor set.*

We will call the collection of all such itineraries $\mathcal{I}_S$.

Proof. Notice that $x \in \omega(c_f)$ if, and only if every initial segment of the itinerary of x occurs infinitely often in S. So we see that if the itinerary of x is of one of the forms above then $x \in \omega(c_f)$. So suppose that $x \in \omega(c_f)$. We will show that the itinerary of x, $I(x)$, is one of the sequences listed above.

Either every initial segment of I of $I(x)$ occurs across the boundary between T_m and T_{m+1} for infinitely many m, or it occurs inside T_m for infinitely many m.

In the first case, I actually occurs in $B^m_{k(m),0}(10^4)^{m+1}$ for large enough m, and hence $I(x)$ is of type (1). In the second case, for infinitely many m, I occurs in words of the form

(a) $(10^4)^m R_m (10^4)^m$,
(b) $(10^4)^m 1^{k(m)} 0 C_{k(m),\gamma_j}$, or
(c) $C_{k(m),\gamma_j} B^m_{k(m),0}$.

By the definition of $\overline{\Sigma}$, (a) implies that $I(x)$ is of type (1). Notice that (c) is a special case of (b), so we consider (b). As $m \to \infty$, $j \to \infty$ but $k(m)$ can either remain fixed or increase. If $k(m)$ is fixed, $I(x)$ is of type (2). If $k(m)$ increases, $I(x)$ is either of type (3) or (4).

Finally, to see that $\omega(c_f)$ is a Cantor set, we show that it has no isolated points. Let $x \in \omega(c_f)$. Since $\overline{\Sigma}$ is a Cantor set, if x has itinerary of type (1), then x is not isolated. The same is true for type (2) since Γ_t is a Cantor set for all $t \in \mathbb{N}$. If x has itinerary of type (3) or (4) then it is a limit of points with itinerary of type (2). ■

For each positive integer $r > 5$, there is a subset $\Delta_r \subseteq \Gamma_r$ which is countable and has a single point, $B^{\infty}_{r,0}$, with limit type r such that for every $x \in \Delta_r$ there is an integer k with $\sigma^k(x) = B^{\infty}_{r,0}$. In fact, Δ_r is homeomorphic to the ordinal $\omega^r + 1$. So if $h : \Delta_r \to \omega^r + 1$ is the homeomorphism we see that whenever $h(x) = \alpha$ then x and α must have the same limit type. So we use $\omega^r + 1$ to index

$$\Delta_r = \{x_{r,\alpha}\}_{\alpha \in \omega^r + 1}.$$

For each $r > 5$ and for each $\alpha \in \omega^r + 1$ there is an infinite word $\delta_{r,\alpha} \in \{0,1\}^{\mathbb{N}}$ such that $x_{r,\alpha} = C_{r,\delta_{r,\alpha}}$ in the above notation. Let $(W_{r,n})_{n\in\mathbb{N}}$ enumerate all of the finite words in points of Δ_r.

In order to alter the previous kneading sequence to obtain one with postcritical ω-limit set with the topological structure that we are after, we will insert the finite words that make up each Δ_r carefully into S in such a way that we can see a homeomorphic copy of each Δ_r isolated from the Cantor set but limiting to one point in Γ_r with limit type r. For each $r \in \mathbb{N}$, let p_r be the rth prime number. We will insert a copy of each finite word of Δ_r sandwiched between the words

$$B_{p_r^n,0} = 10^3 1^{2p_r^n} 0^3 1 \quad \text{and} \quad B^{p_r^n}_{n,0} = (10^3 1^{2n} 0^3 1)^{p_r^n}.$$

But we need to do this in such a way that we still have $B_{p_r^n,0} B^{p_r^n}_{n,0}$ occurring in the kneading sequence infinitely often. To accomplish this, for each $r \in \mathbb{N}$ let $(R_{r,n})_{n\in\mathbb{N}} \subseteq (R_m)_{m\in\mathbb{N}} = \Sigma$ be chosen such that

(1) $R_{r,n}$ contains the word $B_{p_r^n,0} B^{p_r^n}_{n,0}$,
(2) $(R_{r,n})_{n\in\mathbb{N}} \cap (R_{r',n})_{n\in\mathbb{N}} = \emptyset$ for all $r \neq r'$,
(3) each $R_{r,n}$ occurs infinitely often as a subword of terms in

$$(R_i)_{i\in\mathbb{N}} \setminus \Big[\bigcup_{k\in\mathbb{N}} (R_{k,n})_{n\in\mathbb{N}}\Big].$$

Let $U_{r,n}$ be the word in $R_{r,n}$ before the first occurrence of $B_{p_r^n,0} B^{p_r^n}_{n,0}$, and let $U'_{r,n}$ be the word in $R_{r,n}$ that occurs after the first occurrence of $B_{p_r^n,0} B^{p_r^n}_{n,0}$

in $R_{r,n}$. So

$$R_{r,n} = U_{r,n} B_{p_r^n,0} B_{n,0}^{p_r^n} U'_{r,n}.$$

We alter each $R_{r,n}$ by inserting

$$10^3 1^{p_r^n} 0 W_{r,n}$$

in between $B_{p_r^n,0} B_{n,0}^{p_r^n}$ and define

$$R'_m = \begin{cases} U_{r,n} B_{p_r^n,0} 10^3 1^{p_r^n} 0 W_{r,n} B_{n,0}^{p_r^n} U'_{r,n} & \text{if } R_m = R_{r,n}, \\ R_m & \text{otherwise.} \end{cases}$$

Just as before, for each $m \in \mathbb{N}$ let

$$T'_m = (10^4)^m R'_m (10^4)^m S_m B_{k(m),0}^m$$

where the S_ms and $k(m)$s are defined as above. Let

$$S' = AAT'_1 T'_2 T'_3 \ldots .$$

Again, it is easy to check that S' is the kneading sequence of a tent map, g.

THEOREM 2.2. *Let g be the tent map with kneading sequence S' and critical point c_g. Then $x \in \omega(c_g)$ if and only if the itinerary of x, $I(x)$, is a shift of one of the following*:

(1) *U where $U \in \mathcal{I}_S$,*
(2) *$1^k 0 x_{r,\alpha}$ for $k \in \mathbb{N}$, $r > 5$, $\alpha \in \omega^r + 1$.*

Moreover, $\omega(c_g) = C \cup P$ where C is the largest Cantor set in $\omega(c_g)$ and $P = \bigcup_{r>5} P_r$ where P_r contains points with limit type r but not any points with higher limit type.

Proof. Clearly, if $x \in [0,1]$ and the itinerary of x is in $\mathcal{I}_S$ then $x \in \omega(c_g)$, because we ensured that every word that occurred infinitely often in S still occurs infinitely often in S'. The new points in $\omega(c_g)$ must occur due to the changed R'_ms. Note that r and n depend on m and as $m \to \infty$, we have $p_r^n \to \infty$, which can occur in two ways: either the p_rs are the same prime but with increasing powers in n, or the p_rs are an increasing sequence of primes.

This implies that every initial segment of $I(x)$ occurs in infinitely many R'_ms which are of the form

$$U_{r,n} B_{p_r^n,0} 10^3 1^{p_r^n} 0 W_{r,n} B_{n,0}^{p_r^n} U'_{r,n}.$$

Since $p_r^n \to \infty$, it follows that $|B_{p_r^n,0}| \to \infty$ and $|W_{r,n}| \to \infty$. So every initial segment of $I(x)$ occurs infinitely often in one of:

(1) $U_{r,n} B_{p_r^n,0}$,
(2) $B_{p_r^n,0} 10^3 1^{p_r^n}$,
(3) $1^{p_r^n} 0 W_{r,n}$,

(4) $W_{r,n}B_{n,0}^{p_r^n}$, or
(5) $B_{n,0}^{p_r^n}U'_{r,n}$.

Notice that (3) is the only possibly new form of an allowed initial segment. Recall that the words $W_{r,n}$ are finite subwords that describe Δ_r. Thus $I(x) = 1^k0x_{r,\alpha}$ for some $k \in \mathbb{N}, r > 5$ and $\alpha \in \omega^r + 1$.

For each $r > 5$, let $P_r = \{x \in \omega(c_g) : I(x) = 1^k0x_{r,\alpha},\ k > r,\ \alpha \in \omega^r + 1\}$ and let $C = \omega(c_g) \setminus \bigcup_{r>5} P_r$. Since $1^k0x_{r,\alpha} \in \mathcal{I}_S$ if and only if $k \le r$, we see that the P_rs contain all of the points of $\omega(c_g)$ with itineraries that are not in $\mathcal{I}_S$. So C is a Cantor set that contains every point with itinerary that was an itinerary of some point in $\omega(c_f)$. If $x \in P_r$ then $I(x) = 1^k0x_{r,\alpha}$ with $k > r$ and $\alpha \in \omega^r + 1$. Let $V_{r,k}$ be the set of all points in $\omega(c_g)$ with itineraries that start with $1^k010^31^{2r}0^3$ or $1^k010^31^{2r+1}0^3$. Each $V_{r,k}$ is a subset of P_r that is homeomorphic to $\omega^r + 1$ and is open in $\omega(c_g)$ because it is a cylinder set. So we see that C is the largest Cantor set in $\omega(c_g)$, and that each P_r contains points with limit type r and none with higher limit type. ∎

Thus we have constructed an ω-limit set with isolated points that violates the restriction on ω-limit sets given in [10]. Notice that the specific construction we employed used subsets of the Cantor set with limit height n for each n but without anything of limit type ω. It is easy to see that the technique can be altered to allow the subsets Δ_r have any limit type structure. Thus for every countable ordinal γ, there exists a tent map such that $\omega(c)$ is uncountable and its perfect kernel occurs at level γ.

3. An uncountable ω-limit set with a minimal perfect kernel and a dense set of isolated points. In this section we address a question of the referee by constructing the following example.

Example 3.1. *There is a tent map h with critical point c_h such that $\omega(c_h) = C \cup R$ where C is a minimal Cantor set and R is a scattered set. Moreover, the set R is dense in $\omega(c_h)$.*

To begin, we let K' be the kneading sequence of a tent map $f : [0,1] \to [0,1]$ with critical point c_f such that $\omega(c_f)$ is minimal. An example of such a kneading sequence can be found in [7]; other examples are provided by strange adding machines [8], [13]. Consider the inverse limit of f, and let $\mathrm{Fd}(f)$ be the set of folding points in $\varprojlim\{[0,1], f\}$, [14]. It is known that $\mathrm{Fd}(f) = \varprojlim \omega_f(c_f) f|_{\omega_f(c_f)}$. Let

$$\widehat{x} = (x_1, x_2, \dots) \in \mathrm{Fd}(f) \setminus \bigcup_{n\in\mathbb{N}} \pi_n^{-1}(c_f)$$

be such that

$$x_1 \notin \bigcup_{n\in\mathbb{N}} f^{-n}(c_f).$$

Then x_1 has a unique itinerary made up of 0s and 1s which we denote by $I_f(x_1)$, and $\widehat{x}$ has a unique symbolic representation, $\mathcal{I}_f(\widehat{x}) \in \{0,1\}^{\mathbb{Z}}$, where

$$\mathcal{I}_f(\widehat{x}) = (\dots, i_f(x_3), i_f(x_2), i_f(x_1).i_f(f(x_1)), i_f(f^2(x_1)), i_f(f^3(x_1)), \dots)$$

and $i_f(z) = 0$ if $z < c_f$ but $i_f(z) = 1$ otherwise.

Let $V = (V^-.V^+)$ denote $\mathcal{I}_f(\widehat{x})$ and, for each $n \in \mathbb{N}$, let $V_n^-.V_n^+$ be the central segment of V of "diameter" $2n$. Notice that $V_n^-.V_n^+$ is a central segment of the full itinerary of x_n. Since $\widehat{x} \in \varprojlim \omega_f(c_f) f|_{\omega_f(c_f)}$, we see that $x_n \in \omega_f(c_f)$. Thus each $V_n^-.V_n^+$ occurs infinitely often in K' and we can write

$$K' = W_1 V_{m_1}^- V_{m_1}^+ W_2 V_{m_2}^- V_{m_2}^+ W_3 V_{m_3}^- V_{m_3}^+ \dots$$

where each W_i is a word in 0 and 1, and both $|W_i| \to \infty$ and $m_i \to \infty$ as $i \to \infty$.

Now, by our assumptions on f, we know that $\omega_f(c_f)$ is minimal. In particular, the orientation reversing fixed point, p, with itinerary 1^∞ is not in $\omega_f(c_f)$. This implies that there is some least $N \in \mathbb{N}$ such that 1^N does not occur in K'. Let

$$B = 101^N01$$

and let

$$K = W_1 V_{m_1}^- B V_{m_1}^+ W_2 V_{m_2}^- V_{m_2}^+ W_3 V_{m_3}^- B V_{m_3}^+ \dots.$$

Specifically, in K', replace every odd occurrence

$$V_{m_{2i-1}}^- V_{m_{2i-1}}^+$$

with

$$V_{m_{2i-1}}^- B V_{m_{2i-1}}^+.$$

It is easy to check that K is shift maximal and primary (see, for example, [10] for the terminology) and is therefore the kneading sequence of a tent map, h, with critical point c_h. Let $x \in \omega(c_h)$. By construction, there are three possibilities for the itinerary of x:

(1) $\mathcal{I}_h(x) = \mathcal{I}_f(y)$ for some $y \in \omega(c_f)$,
(2) $\mathcal{I}_h(x)$ contains B, in which case $\mathcal{I}_h(x) = \sigma^m(V_n^-)BV^+$ for some $m < n$,
(3) $\mathcal{I}_h(x) = \sigma^k(BV^+)$.

Points of type (1) give rise to a minimal Cantor set C on which h acts in conjugate fashion to the action of f on $\omega(c_f)$. Points of type (2) are isolated, since every itinerary containing a B terminates with BV^+ (hence, any initial segment of the itinerary that contains B defines an open set that contains just this point). Points of type (3) are either isolated (in particular when $k < 3$, so that the itinerary contains 1^N, which is always followed by $01V^+$) or contained in C. Hence $\omega(c_h) = C \cup R$, where C is a minimal Cantor set and R is a collection of isolated points.

Since the only points of $\omega(c_h)$ that are not isolated are in C, by compactness there is at least one point $z \in C$ that is a limit point of a sequence (x_k) of isolated points, where (without loss) the itinerary of x_k is $V_{n_k}^- BV^+$. Since C is minimal, for any $y \in C$ and any $\varepsilon > 0$ there is some $m > 0$ such that $|h^m(z) - y| < \varepsilon$. This is equivalent to saying that the itineraries of $h^m(z)$ and y agree for m' terms for some $m' \in \mathbb{N}$. But then whenever k is chosen so that $n_k > m + m'$, the itineraries of $h^m(x_k)$, z and y will agree for the first m' terms. Since its itinerary contains B, it follows that $h^m(x_k)$ is isolated, and hence the isolated points of $\omega(c_h)$ are dense. As the referee points out, the scattered part of such an ω-limit set with a minimal perfect kernel will always form a dense set. In fact, this holds for any continuous function on a compact metric space and follows from Sharkovskiĭ's property of ω-limit sets (weak incompressibility): if F is a proper, non-empty closed subset of an ω-limit set W, then the closure, $\overline{f(W-F)}$, of $f(W-F)$ meets F (see [5]). Now if $W = C \cup R$, where C is a Cantor set and R is a scattered subset of W, then either $\overline{R} = W$, in which case we are done, or $\overline{f(W-\overline{R})}$ meets $\overline{R}$. If C is a minimal Cantor set then $\overline{f(W-\overline{R})}$ is a non-empty subset of C, so that $C \cap \overline{R}$ is non-empty. But $C \cap \overline{R}$ is a closed, forward invariant subset of the minimal Cantor set C, and is therefore equal to C and indeed R is dense. This shows the following.

PROPOSITION 3.2. *Let $f : X \to X$ be a continuous function on the compact metric space X. If $\omega(x) = C \cup R$, where C is a minimal Cantor set and R is a scattered subset of $\omega(x)$, then R is dense in $\omega(x)$.*

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