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N-determined p-compact groups

by

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Abstract. One of the major problems in the homotopy theory of finite loop spaces is the classification problem for p-compact groups. It has been proposed to use the maximal torus normalizer (which at an odd prime essentially means the Weyl group) as the distinguishing invariant. We show here that the maximal torus normalizer does indeed classify many p-compact groups up to isomorphism when p is an odd prime.

Contents

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1. Introduction. This paper addresses the classification problem at odd primes for the p-compact groups introduced by W. G. Dwyer and C. W. Wilkerson in their seminal paper [30] (surveyed in [63, 53, 26]). A *p-compact group* is a connected, pointed, $H^*\mathbf{F}_p$-local space BX such that $H^*(X;\mathbf{F}_p)$ is finite where $X = \Omega BX$ is the loop space [30, §2]. It is customary, though ambiguous, to refer to BX by the name, X, for its underlying loop space.

It has been conjectured [53, 72, 26], in analogy with the classification theorem for compact Lie groups [25, 83], that p-compact groups are determined by their maximal torus normalizers. The *maximal torus normalizer* $N(X)$ for the p-compact group X is an extension

$$T(X) \to N(X) \to W(X) \tag{1.1}$$

of the maximal torus $T(X)$ by the Weyl group $W(X)$ [30, 9.8], and X is said to be *totally N-determined* [66, 7.1] if

- X is determined by $N(X)$, and
- the automorphisms of X are determined by their restrictions to $N(X)$.

We show here that almost all simple p-compact groups are totally N-determined at odd primes.

1.2. Theorem. *Let X be a simple p-compact group, where p is an odd prime. Assume that the rational Weyl group* $(r_0W(X)) \neq (r_0W(\mathrm{E}_8))$ *if* $p=3$ *and* $(r_0W(X)) \neq (r_0W(\mathrm{E}_j))$, $j = 6,7,8$, *if* $p = 5$. *Then X is totally N-determined.*

The Weyl group $W(X)$ [30, 9.7] of a connected p-compact group X is a finite group of automorphisms of the free, finitely generated $\mathbf{Z}_p$-module $L(X)=\pi_1T(X)$, i.e. $W(X)\subset \mathrm{GL}(L(X))$. The *rational Weyl group*, $r_0W(X)$, is the image of $W(X)$ in $\mathrm{GL}(L(X)\otimes_{\mathbf{Z}_p}\mathbf{Q}_p)$, and $r_pW(X)$, the *$\mathbf{F}_p$-Weyl group*, the image of $W(X)$ in $\mathrm{GL}(L(X)\otimes_{\mathbf{Z}_p}\mathbf{F}_p)$. As usual, $(r_0W(X))$ stands for the conjugacy class of the rational Weyl group. The connected p-compact group X is *simple* if $L(X)\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$ is an irreducible $r_0W(X)$-module [65, 5.4].

At an odd prime p, the maximal torus normalizer extension (1.1) for a connected p-compact group splits in an essentially unique way [5] and thus $N(X)$ is in fact completely determined by the reflection group $(W(X), L(X))$. This explains the first part, merely a reformulation of Theorem 1.2, of the the corollary below; see (4.3) for the precise meaning of the other statements.

1.3. Corollary. *Let X be a simple p-compact group as in Theorem* 1.2. *Then X is determined up to* (*local*) *isomorphism by its* (*rational*) *Weyl group*, *and the automorphism group* $\mathrm{Aut}(X)$ *is isomorphic to*

$N_{\mathrm{GL}(L(X))}(W(X))/W(X)$. Furthermore, if X is centerless or simply connected, then X is determined by its $\mathbf{F}_p$-Weyl group, and X is a cohomologically unique p-compact group.

For the bigger class of *connected* (but not necessarily simple) p-compact groups Theorem 1.2 takes on a particularly appealing form.

1.4. Corollary. *Assume that $p > 5$. The map*

$$\left\{\begin{matrix}\textit{Isomorphism classes of}\\ \textit{connected p-compact groups}\end{matrix}\right\} \xrightarrow{(W,L)} \left\{\begin{matrix}\textit{Similarity classes of}\\ \mathbf{Z}_p\textit{-reflection groups}\end{matrix}\right\}$$

is a bijection, and $\mathrm{Aut}(X)$ *is isomorphic to* $N_{\mathrm{GL}(L)}(W)/W$ *for the connected p-compact group X corresponding to the reflection group (W, L).*

In the general case, for the class of not necessarily connected p-compact groups, Theorem 1.2 takes the following form.

1.5. Corollary. *Let X be a p-compact group such that all its simple factors satisfy the assumptions of Theorem* 1.2. *Then X is totally N-determined and* $\mathrm{Out}(X) \cong \mathrm{Out}(N(X))$.

The *simple factors* of the p-compact group X are the simple, centerless p-compact groups in the splitting [32, 80] of $PX_0 = X_0/Z(X_0)$, the adjoint form of the identity component of X.

Let me also mention the following partial classification result for connected finite loop spaces with maximal tori [70, 1.1].

1.6. Corollary (cf. [96], [70, 1.6]). *Let X be a connected finite loop spaces with a maximal torus. Assume that X has the same Weyl group as the compact, connected simple Lie group G and that no simple factor of G is locally isomorphic to* E_6, E_7, *or* E_8. *Then* $(BX)[1/2]$ *and* $(BG)[1/2]$ *are homotopy equivalent spaces.*

In light of the observation by C. Wilkerson [97] that the Weyl group of any connected finite loop space with maximal torus must agree with the Weyl group of a compact connected Lie group, this proves the maximal torus conjecture [49, Conjecture D, p. 68], [99] away from the prime 2 in a number of particular cases.

The proof that the simple p-compact groups of Theorem 1.2 are N-determined goes in outline as follows. Consider some connected p-compact group X with maximal torus normalizer $j\colon N \to X$ and assume that the same extended p-compact torus N can also serve as the maximal torus normalizer $j'\colon N \to X'$ for some other p-compact group X'. Starting with the configuration

$$X \xleftarrow{j} N \xrightarrow{j'} X'$$

our task is to construct an isomorphism $f\colon X \to X'$ under N. We observe (3.7) that it suffices to consider the centerless form of X. According to the Homology Decomposition Theorem [31, §8], BX is (the p-completion of) the homotopy colimit of the $\mathbf{A}(X)^{\mathrm{op}}$-space of centralizers $BC_X(E,\nu)$ of non-trivial elementary abelian p-subgroups $\nu\colon E \to X$ of X. For any monomorphism $\nu\colon E \to X$ it is possible to find (non-uniquely) a preferred lift $\mu\colon E \to N$ of ν such that the morphisms

$$C_X(E,\nu) \xleftarrow{C_j} C_N(E,\mu) \xrightarrow{C_{j'}} C_{X'}(E,\nu')$$

are again maximal torus normalizers for the centralizers of (E,ν) and (E,ν') where $\nu' = j'\mu$ [67]. As the center of X is trivial, the centralizer of (E,ν) will have smaller cohomological dimension than that of X [30, 6.14, 6.15]. Assuming, as part of an induction argument, that $C_X(E,\nu)$ (which may very well be disconnected) is totally N-determined, there will therefore be an isomorphism $f(E,\mu)\colon C_X(E,\nu) \to C_{X'}(E,\nu')$ under $C_N(E,\mu)$. It remains to show that these locally defined isomorphisms $f(E,\mu)$ do not depend on the choice of preferred lifts and that they combine to yield a morphism $f\colon X \to X'$ under N.

N-determinism is actually not a property of the p-compact group X itself but rather a property of the extended p-compact torus $N(X)$: If X is N-determined, so is, by the very nature of the concept, any other p-compact group that admits $N(X)$ for a maximal torus normalizer.

Most of the time the prime p will be assumed to be odd. Some modifications will be needed to handle the case where $p = 2$ [58]. Even the formulation of the N-conjecture itself will have to be refined as $N(\mathrm{O}(2)) = \mathrm{O}(2) = N(\mathrm{SO}(3))$ but $\mathrm{O}(2)$ and $\mathrm{SO}(3)$ are distinct 2-compact groups.

Organization of the paper. In Section 3, I set up the general theory that will be applied in a case-by-case verification of the N-conjecture for the simple, centerless p-compact groups. We deal with an A-family, represented by the p-compact groups $\mathrm{PGL}(n,\mathbf{C}) = \mathrm{PSL}(n,\mathbf{C})$, in Section 5, and with the polynomial case, which includes nearly all remaining compact simple Lie groups and all the exotic (non-Lie) simple p-compact groups, in Section 7. The proofs of (1.2–1.6) are in Section 8. Sections 2 and 13 contain material dealing with the general problem of computing cohomology groups of categories. (There is no claim to originality here as the vanishing result of (2.4) was proved in [34] and the spectral sequence of (13.2) seems to be that of Lück [54, 17.28] or Słomińska [87].)

Notation. Write $\mathbf{Z}_p$ for the ring of p-adic integers, $\mathbf{Q}_p$ for the field of p-adic numbers, and $\mathbf{F}_p$ for the field with p elements. For a p-compact group X, let

- $T(X)$ denote the maximal torus of X [30, 8.9],
- $L(X) = \pi_2(BT(X))$ the lattice of X,
- $\check{T}(X) = L(X) \otimes \mathbf{Z}/p^\infty$ the p-discrete maximal torus of X [30, §6],
- $t(X) = L(X) \otimes \mathbf{Z}/p$ the maximal elementary abelian subgroup of $\check{T}(X)$,
- $W(X)$ the Weyl group of X [30, 9.6], $r_0W(X)$ the rational and $r_pW(X)$ the mod p Weyl group of X (Section 4),
- $N(X)$ the maximal torus normalizer of X [30, 9.8],
- $Z(X)$ the center of X [31, 69],
- $r(X)$ the rational rank (of the identity component) of X [30, 5.11],
- $\mathrm{Aut}(X)$ the group of invertible elements in the monoid $\mathrm{End}(X) = [BX, *; BX]$ [66, §3] of based homotopy classes of based self-maps of BX, and $\mathrm{Out}(X) = \mathrm{Aut}(X)/\pi_0(X)$ the corresponding group in the un-based category, and
- $\mathbf{A}(X)$ the Quillen category of X.

The objects (E, ν) of $\mathbf{A}(X)$ are conjugacy classes of monomorphisms $\nu : E \to X$ of non-trivial elementary abelian p-groups E into X. The morphisms $(E_0, \nu_0) \to (E_1, \nu_1)$ of $\mathbf{A}(X)$ are group homomorphisms $f \colon E_0 \to E_1$ such that $(E_0, \nu_0) = (E_0, \nu_1 f)$. An object (E, ν) of $\mathbf{A}(X)$ is *toral* if $\nu \colon E \to X$ factors through the maximal torus $T(X) \to X$. Let

- $\mathbf{A}(X)^{\leq t}$ denote the full subcategory of all toral objects, and
- $\mathbf{A}(X)_{\not\leq t}$ the full subcategory of all objects with a morphism to some non-toral object.

The notation for categories is:

- **pcg** is the category of p-compact groups,
- **Grp** is the category of groups,
- **Ab** is the category of abelian groups,
- **Sp** is the category of simplicial sets, and
- **Top** is the category of topological spaces.

In [**pcg**], [**Grp**], [**Sp**] the objects are p-compact groups, groups, topological spaces and the morphisms are conjugacy classes of p-compact group morphisms, conjugacy classes of group homomorphisms, homotopy classes of continuous maps.

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2. Higher limits of center functors. This section contains a vanishing result (2.4) for the derived limits of a certain functor, defined in purely algebraic terms, which informs on the obstruction theory associated to the Jackowski–McClure centralizer homology decomposition [45, 31] of BX.

Let W be a finite group and t a non-trivial $\mathbf{F}_pW$-module which is finite-dimensional as an $\mathbf{F}_p$-vector space. For non-trivial subgroups E_0 and E_1 of t, put

$$\begin{aligned}\overline{W}(E_0,E_1)&=\{w\in W\mid w(E_0)\subseteq E_1\},\\ W(E_0)&=\{w\in W\mid we=e\text{ for all }e\in E_0\},\end{aligned}\tag{2.1}$$

and note that the set of orbits for the action of the pointwise stabilizer group $W(E_0)$ on the set $\overline{W}(E_0,E_1)$ is the set of group homomorphisms $E_0\to E_1$ induced by elements of W. The stabilizer subgroup $\overline{W}(E_0,E_0)$ of E_0 will also be written as $\overline{W}(E_0)$.

2.2. Definition. $\mathbf{A}(W,t)$ is the category with

- objects: non-trivial elementary abelian subgroups E of t, and
- morphisms: group homomorphisms $E_0\to E_1$ induced by elements of W.

For any $\mathbf{Z}_pW$-module L, $L_j\colon\mathbf{A}(W,t)\to\mathbf{Ab}$, $j\ge0$, is the functor that takes the object $E\subseteq t$ to the cohomology group $H^j(W(E);L)$ and the morphism $E_0\xrightarrow{w}E_1$ in $\mathbf{A}(W,t)$ to the homomorphism $H^j(W(E_0);L)\xrightarrow{\mathrm{res}\circ w^*}H^j(W(E_1);L)$.

Here is a more detailed explanation of the functors L_j: Any morphism $E_0\to E_1$ in $\mathbf{A}(W,t)$, represented by an element $w\in\overline{W}(E_0,E_1)$, can be factored into an isomorphism $w\colon E_0\to wE_0$ followed by an inclusion. Consider the corresponding group homomorphisms

$$W(E_0)\xrightarrow{c(w)}W(E_0)^w\supseteq W(E_1)$$

where $c(w)w_0=ww_0w^{-1}$ is conjugation by w and $W(E_0)^w=wW(E_0)w^{-1}=W(wE_0)$ and let, as usual [37, 4.1.1], $w^*:H^j(W(E_0);L)\to H^j(W(E_0)^w;L)$ be the isomorphism induced by $c(w)^{-1}$ and multiplication by w on L. Define $L_j(w)$ as the composition

$$H^j(W(E_0);L)\xrightarrow{w^*}H^j(W(E_0)^w;L)\xrightarrow{\mathrm{res}}H^j(W(E_1);L)$$

of w^* followed by the restriction morphism. Since for all $w_0\in W(E_0)$ we have $W(E_0)^{ww_0}=W(E_0)^w$ and cohomology is insensitive to inner conjugation [55, IV.5.6], $L_j(ww_0)=L_j(w)$ for all $w_0\in W(E_0)$ and thus this morphism is independent of the choice of the representative for $wW(E_0)\in\overline{W}(E_0,E_1)/W(E_0)=\mathbf{A}(W,t)(E_0,E_1)$ (cf. [45, 7.6]).

For instance, for a connected p-compact group X, the functors

$$L(X)_{2-j}\colon\mathbf{A}(W(X),t(X))\to\mathbf{Ab},\qquad j=1,\,2,\tag{2.3}$$

take the non-trivial elementary abelian p-subgroup E of $t(X)$ to the group $H^{2-j}(W(X)(E);L(X))$.

2.4. Lemma [34, 8.1]. *$L_j\colon \mathbf{A}(W,t) \to \mathbf{Ab}$ is an acyclic functor in the sense that*

$$\lim{}^i(\mathbf{A}(W,t);L_j) = \begin{cases} H^j(W;L), & i=0,\\ 0, & i>0, \end{cases}$$

for all $j \geq 0$.

For a p-compact group X, let $BC_X\colon \mathbf{A}(X)^{\mathrm{op}} \to \mathbf{pcg}$ and $BZC_X : \mathbf{A}(X) \to \mathbf{Top}$ be the functors that take the object (E,ε) of $\mathbf{A}(X)$ to

$$BC_X(E,\varepsilon) = \mathrm{map}(BE,BX)_{B\varepsilon}, \tag{2.5}$$

$$BZC_X(E,\varepsilon) = \mathrm{map}(BC_X(E,\varepsilon),BX)_{Be(\varepsilon)}, \tag{2.6}$$

where $Be(\varepsilon)\colon BC_X(E,\varepsilon) \to BX$ is the evaluation map, and define

$$\pi_j(BZC_X)\colon \mathbf{A}(X) \to \mathbf{Ab}, \quad j=1,2, \tag{2.7}$$

to be the composition of BZC_X with the jth homotopy functor. (There is no basepoint problem here since only abelian p-compact groups are involved.)

2.8. Lemma. *Let p be an odd prime and X a connected p-compact group. Assume that the identity component $C_X(E)_0$ of the centralizer of any non-trivial elementary abelian p-subgroup E of $T(X)$ has N-determined automorphisms* (3.1). *Then there is an equivalence of categories*

$$\mathbf{A}(W(X),t(X)) \to \mathbf{A}(X)^{\leq t}$$

such that the functors $\pi_j(BZC_X)$ when restricted to $\mathbf{A}(X)^{\leq t}$ correspond to the functors $L(X)_{2-j}$, $j=1$, 2, of (2.3).

Proof. Take $wW(E_0)\colon E_0 \to E_1$ in $\mathbf{A}(W(X),t(X))$ to the morphism $w|E_0\colon (E_0,ie_0) \to (E_1,ie_1)$ in $\mathbf{A}(X)^{\leq t}$ (where $e_j\colon E_j \to t(X)$, $j=0,1$, is the inclusion and i the p-compact group morphism $t(X) \to T(X) \to X$). This provides a functor

$$\mathbf{A}(W(X),t(X)) \to \mathbf{A}(X)^{\leq t}. \tag{2.9}$$

Since the natural map $W\backslash[BE,BT(X)] \to [BE,BX]$, induced by $BT(X) \to BX$, is injective for any elementary abelian p-group E [65, 3.4], [32, 3.4], this functor is full and as it is also clearly faithful, (2.9) is an equivalence of categories.

Let now $\check{N}(X)$ be a discrete approximation [31, 3.12] to the maximal torus normalizer $N(X)$. For any elementary abelian p-subgroup E of $\check{T}(X)$, $C_{\check{N}(X)}(E)$ is a discrete approximation to $C_{N(X)}(E)$ and $ZC_{\check{N}(X)}(E)$ is a discrete approximation to $ZC_{N(X)}(E)$ which is isomorphic to $ZC_X(E)$ (2.19) [66, 4.12]. Since the prime p is odd, $\check{N}(X) = \check{T}(X) \rtimes W(X)$ is a semidirect product [5, 2.1], so

$$C_{\check{N}(X)}(E) = C_{\check{T}(X)\rtimes W(X)} = \check{T}(X) \rtimes W(X)(E), \tag{2.10}$$
$$Z(C_{\check{N}(X)}(E)) = \check{T}(X)^{W(X)(E)},$$

and hence it follows that $\pi_j(BZC_X)(E) = \pi_j((BH^0(W(X)(E); \check{T}(X)))^\wedge_p) = H^{2-j}(W(X)(E); L(X)) = L(X)_{2-j}(E)$.

If the Weyl group element $w \in W(X)$ takes the elementary abelian p-subgroup $E_0 \subseteq \check{T}(X)$ into the elementary abelian p-subgroup $E_1 \subseteq \check{T}(X)$, then w represents a morphism $w\colon E_0 \to E_1$ in $\mathbf{A}(W(X), t(X))$. We want to determine the effect of w on the centralizer centers. Choose a lift $\check{w} \in \check{N}(X)$ of $w \in \overline{W}(X)(E_0, E_1) \subseteq W(X) = \check{N}(X)/\check{T}(X)$. Conjugation by $\check{w}$, given by $c(\check{w})(n) = \check{w}n\check{w}^{-1}$, $n \in \check{N}(X)$, takes E_0 into E_1 and conjugation by $\check{w}^{-1}$, $c(\check{w}^{-1})$, takes $C_{\check{N}(X)}(E_1)$ into $C_{\check{N}(X)}(E_0)$ in such a way that the diagram

$$\begin{tikzcd}
C_{\check{N}(X)}(E_0) \times E_0 \arrow[d, "e"'] & C_{\check{N}(X)}(E_1) \times E_0 \arrow[l, "c(\check{w}^{-1})\times 1"'] \arrow[d, "1\times c(\check{w})"] \\
\check{N}(X) & C_{\check{N}(X)}(E_1) \times E_1 \arrow[l, "e"]
\end{tikzcd}$$

where e is group multiplication, commutes up to inner automorphism of $\check{N}(X)$ (as $c(\check{w}) \circ e \circ (c(\check{w}^{-1}) \times 1) = e \circ (1 \times c(\check{w}))$). Therefore, the diagram of adjoint maps between spaces

$$\begin{tikzcd}
BC_{\check{N}(X)}(E_0) \arrow[d, "\simeq"'] & BC_{\check{N}(X)}(E_1) \arrow[l, "Bc(\check{w}^{-1})"'] \arrow[d, "\simeq"] \\
\mathrm{map}(BE_0, B\check{N}(X))_{B\mu_1} & \mathrm{map}(BE_1, B\check{N}(X))_{B\mu_2} \arrow[l, "\overline{Bc(\check{w})}"]
\end{tikzcd}$$

is homotopy commutative. (The vertical maps are equivalences by [40, Lemma 2].) This shows that the map $C_X(E_1) \to C_X(E_0)$ induced by the $\mathbf{A}(X)$-morphism $E_0 \to E_1$ represented by w lifts to the map $c(\check{w}^{-1}) : C_{\check{N}(X)}(E_1) \to C_{\check{N}(X)}(E_0)$ between maximal torus normalizers. ∎

2.11. Corollary. *Let p be an odd prime and X a connected p-compact group. Then*

$$\lim{}^i(\mathbf{A}(X)^{\leq t}, \pi_j(BZC_X)) = \begin{cases} \pi_j(BZ(X)), & i = 0, \\ 0, & i > 0. \end{cases}$$

for $j = 1, 2$. In particular, $\lim^*(\mathbf{A}(X)^{\leq t}, \pi_*(BZC_X)) = 0$ *if and only if X is centerless.*

Proof. By (2.4, 2.8, 3.12(2)),

$$\begin{aligned}\lim^0(\mathbf{A}(X)^{\le t}, \pi_j(BZC_X)) &= \lim^0(\mathbf{A}(W(X), t(X)), L(X)_{2-j})\\ &= H^{2-j}(W(X); L(X)) = \pi_j(BZ(X))\end{aligned}$$

and, similarly, $\lim^i(\mathbf{A}(X)^{\le t}, \pi_j(BZC_X)) = 0$ for $i > 0$. ∎

Let $\pi_j(BZC_X)_{\not\le t}$ be the subfunctor of $\pi_j(BZC_X)$ which vanishes on all toral objects of $\mathbf{A}(X)$ and has the same value as $\pi_j(BZC_X)$ on all non-toral objects of $\mathbf{A}(X)$. (To see that this is indeed a functor, observe that there can be no morphism from a non-toral object to a toral object of the Quillen category.)

2.12. COROLLARY. *Let p be an odd prime and X a connected p-compact group. Then there is an exact sequence*

$$\begin{aligned}0 \to \lim^0(\mathbf{A}(X)_{\not\le t}; \pi_j(BZC_X)_{\not\le t}) \to \lim^0(\mathbf{A}(X); \pi_j(BZC_X)) \to \pi_j(BZ(X))\\ \to \lim^1(\mathbf{A}(X)_{\not\le t}; \pi_j(BZC_X)_{\not\le t}) \to \lim^1(\mathbf{A}(X); \pi_j(BZC_X)) \to 0\end{aligned}$$

while $\lim^i(\mathbf{A}(X); \pi_j(BZC_X)) = \lim^i(\mathbf{A}(X)_{\not\le t}; \pi_j(BZC_X)_{\not\le t})$ *for* $i \ge 2$. *In particular*,

$$\begin{aligned}\lim^*(\mathbf{A}(X); \pi_j(BZC_X)_{\not\le t}) &\cong \lim^*(\mathbf{A}(X)_{\not\le t}; \pi_j(BZC_X)_{\not\le t})\\ &\cong \lim^*(\mathbf{A}(X); \pi_j(BZC_X))\end{aligned}$$

if and only if X is centerless.

Proof. The quotient functor $\pi_*(BZC_X)/\pi_*(BZC_X)_{\not\le t}$ vanishes on all non-toral objects so that, by (13.12), for all $i \ge 0$,

$$\begin{aligned}&\lim^i(\mathbf{A}(X); \pi_j(BZC_X)/\pi_j(BZC_X)_{\not\le t})\\ &\qquad = \lim^i(\mathbf{A}(X)^{\le t}; \pi_j(BZC_X)/\pi_j(BZC_X)_{\not\le t})\\ &\qquad = \lim^i(\mathbf{A}(X)^{\le t}; \pi_j(BZC_X))\end{aligned}$$

which was computed in (2.11). Combine this with the fact that restriction

$$\lim^*(\mathbf{A}(X)_{\not\le t}; \pi_j(BZC_X)_{\not\le t}) \leftarrow \lim^*(\mathbf{A}(X); \pi_j(BZC_X)_{\not\le t})$$

is an isomorphism by (13.12) again. ∎

Let $\mathrm{St}(\mathrm{GL}(E))$ denote the Steinberg representation for $\mathrm{GL}(E)$.

2.13. COROLLARY. *Let p be an odd prime, X a connected p-compact group with trivial center, and let j be equal to* 1 *or* 2. *If*

$$\mathrm{Hom}_{\mathbf{A}(X)(E,\nu)}(\mathrm{St}(\mathrm{GL}(E)), \pi_j(BZC_X(E,\nu))) = 0$$

for all non-toral objects (E,ν) of rank $j+1$ and $j+2$, then

$$\lim^j(\mathbf{A}(X); \pi_j(BZC_X)) = 0 = \lim^{j+1}(\mathbf{A}(X); \pi_j(BZC_X)).$$

Proof. Use (2.12) and Oliver's cochain complex [81] for computing higher limits over $\mathbf{A}(X)$. ∎

For example, when (X,p) is $(\mathrm{F}_4,3)$ or $(\mathrm{E}_8,5)$ we have

$$\lim{}^*(\mathbf{A}(X);\pi_j(BZC_X))=0$$

because the Quillen category $\mathbf{A}(X)$ contains, up to isomorphism, a unique non-toral object (V,ν); this V has order p^3, $V\cong C_X(V,\nu)$, and $\mathbf{A}(X)(V,\nu)=\mathrm{SL}(V)$ [41, 7.4, 10.3]. The situation is much more complicated for the other members of the E-family at $p=3$ [3].

2.14. *Relation between* $\mathbf{A}(W,t)$ *and the orbit category* $\mathcal{O}(W)$. Let $\mathcal{O}'(W)$ denote the full subcategory of the orbit category of W generated by all objects W/G with $t^G\neq 0$. There are obvious functors

$$\mathbf{A}(W,t) \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{O}'(W)^{\mathrm{op}}$$

given by

$$L(E_0\xrightarrow{wW(E_0)}E_1)=(W/W(E_0)\xleftarrow{wW(E_0)}W/W(E_1)),\quad w(E_0)\subset E_1,$$

$$R(W/G\xrightarrow{wH}W/H)=(t^G\xleftarrow{wW(t^H)}t^H),\quad w^{-1}Gw\subset H.$$

Using the fact that $G\subset W(t^G)$ and $E\subset t^{W(E)}$ we see that L and R are adjoint functors in that

$$\mathbf{A}(W,t)(E,R(W/G))=\mathcal{O}'(W)^{\mathrm{op}}(L(E),W/G)$$

for all objects E of $\mathbf{A}(W,t)$ and all objects W/G of $\mathcal{O}'(W)^{\mathrm{op}}$. Observe also that

$$N_W(G)\subset\overline{W}(t^G)\quad\text{and}\quad\overline{W}(E)\subset N_W(W(E)) \tag{2.15}$$

for all non-trivial subspaces $E\subset t$ and all subgroups $G\subset W$. In particular, the endomorphism monoid of E is the quotient

$$\mathbf{A}(W,t)(E)=\overline{W}(E)/W(E)$$

of the *group* $\overline{W}(E)$ by its normal subgroup $W(E)$. Thus $\mathbf{A}(W,t)$ is an *EI-category* [54], a category in which all endomorphisms are isomorphisms.

A *collection* is a set $\mathcal{C}$ of subgroups of W which is closed under conjugation. Let $\mathcal{O}_{\mathcal{C}}(W)$ denote the $\mathcal{C}$-*orbit category*, the full subcategory of $\mathcal{O}(W)$ generated by all objects W/G with $G\in\mathcal{C}$, and $\mathbf{A}_{\mathcal{C}}(W,t)$ the full subcategory of $\mathbf{A}(W,t)$ generated by all objects of the form t^G for $G\in\mathcal{C}$.

The collection $\mathcal{C}$ is said to be *subgroup-sharp* for the $\mathbf{Z}_pW$-module L [27, 1.13] if

$$\lim{}^i(\mathcal{O}_{\mathcal{C}}(W)^{\mathrm{op}};L_j)=\begin{cases}H^j(W;L), & i=0,\\ 0, & i>0,\end{cases}$$

where $L_j(W/G)=H^j(G;L)$ as in (2.14).

2.16. Corollary. *If the collection $\mathcal{C}$ is subgroup-sharp for L and $t^G \neq 0$ for all g in $\mathcal{C}$ then L_j restricts to an acyclic functor on $\mathbf{A}_{\mathcal{C}}(W,t)$ with $\lim^0(\mathbf{A}_{\mathcal{C}}(W,t);L_j) = H^j(W;L)$.*

Proof. This is immediate from (13.11) as $\mathbf{A}_{\mathcal{C}}(W,t) = R\mathcal{O}_{\mathcal{C}}(W)$. ∎

It is known [45, §5] that the collection $\mathcal{C}(p)$ of all p-subgroups of W is subgroup-sharp for any $\mathbf{Z}_pW$-module L and, for general reasons, $t^P \neq 0$ for any p-group $P \in \mathcal{C}(p)$.

2.17. *Centralizers.* I close this section with a simplified proof of the following well-known result from [74, 3.9] which was used in connection with the mapping spaces of (2.6).

Let P be a *p-toral Lie group* (i.e. the identity component of P is a torus and $\pi_0(P)$ is a finite p-group), G a compact Lie group having a finite p-group as its component group, and $C_G(P)$ the Lie group centralizer, which also has a finite p-group as component group [47, A4], of a Lie group homomorphism $f\colon P \to G$. The standard Lie group multiplication homomorphism $C_G(P) \times P \to G$ extending f induces a p-compact group morphism $\widehat{C_G(P)} \times \hat{P} \to \hat{G}$ extending $\hat{f}\colon \hat{P} \to \hat{G}$. ($\hat{G}$ denotes the p-compact group $BG^\wedge_p$ obtained by p-completing the classifying space of the compact Lie group G.) We shall now see that $\widehat{C_G(P)} = C_{\hat{G}}(\hat{P})$ and in particular that $\widehat{Z(P)} = Z(\hat{P})$, i.e. that centralizers and centers of p-toral Lie groups can be computed either in the Lie group category or in the p-compact group category.

2.18. Lemma [35, 101, 73]. *The adjoint*

$$B\widehat{C_G(P)} \to \operatorname{map}(B\hat{P}, B\hat{G})_{B\hat{f}}$$

of the above standard morphism is a homotopy equivalence. In particular,

$$B\widehat{Z(P)} \simeq \operatorname{map}(B\hat{P}, B\hat{P})_{B1}$$

where $Z(P)$ is the Lie group center of P.

Proof. The p-toral Lie group P contains [48, 1.1] a dense p-discrete toral subgroup $\check{P} = \bigcup \check{P}_m$ which is the union of an ascending sequence of finite p-groups $\check{P}_m$. The inclusion of $\check{P}$ into P induces a discrete approximation $i\colon \check{P} \to \hat{P}$ to the p-compact toral group $\hat{P}$ and so we have homotopy equivalences [30, §6]

$$\operatorname{map}(B\hat{P}, B\hat{G})_{B\hat{f}} \simeq \operatorname{map}(B\check{P}, B\hat{G})_{B\hat{f}i} \simeq \operatorname{map}(B\check{P}_m, B\hat{G})_{B(\hat{f}i|\check{P}_m)}$$

for m large enough. In particular, the above mapping spaces are $\mathbf{F}_p$-complete [31, 2.5], [30, 6.20]. Furthermore, by Dwyer–Zabrodsky [35, 1.1] and [36, 2.5] or Lannes [52], the canonical map

$$BC_G(\check{P}_m) \to \operatorname{map}(B\check{P}_m, B\hat{G})_{B(\hat{f}i|\check{P}_m)}$$

is an $H^*\mathbf{F}_p$-equivalence and here

$$C_G(\check{P}_m) \cong C_G(\check{P}) \cong C_G(P)$$

when m is large enough and since $\check{P}$ is dense in P. ∎

Let now G be an extended p-compact torus and $\check{G}$ its discrete approximation [31, 3.12].

2.19. Lemma. *Let $\mu\colon \pi \to \check{G}$ be a homomorphism from a discrete group π into the extended p-discrete torus $\check{G}$.*

(1) *The group-theoretic centralizer $C_{\check{G}}(\mu)$ of μ is a discrete approximation to the extended p-compact torus $BC_G(\mu) = \mathrm{map}(B\pi, BG)_{B\mu}$.*

(2) *The group-theoretic center $Z(\check{G})$ of $\check{G}$ is a discrete approximation to the extended p-compact torus $BZ(G) = \mathrm{map}(BG, BG)_{B1}$*

Proof. The maps

$$BC_{\check{G}}(\mu) \to \mathrm{map}(B\pi, B\check{G})_{B\mu} \to \mathrm{map}(B\pi, BG)_{B\mu}$$

are $H^*\mathbf{F}_p$-equivalences: The first map is even a homotopy equivalence [40, Lemma 2] and the fibre of the second map is [60] a $K(V, 1)$, for some rational vector space V, because the fiber of $B\check{G} \to BG$ has this form [31, 3.1]. Taking μ to be the identity map of $\check{G}$, we obtain a discrete approximation to $Z(G)$. ∎

3. N-determinism. This section contains comments on and further development of the material in [66] concerning N-determined p-compact groups.

3.1. *N-determined automorphisms.* Let $j\colon N(X) \to X$ be the maximal torus normalizer for a p-compact group X. Turn this maximal torus normalizer $Bj\colon BN(X) \to BX$ into a fibration. Any automorphism $f\colon X \to X$ of the p-compact group X restricts to an automorphism $\mathrm{AM}(f)\colon N(X) \to N(X)$ of the maximal torus normalizer, unique up to the action of the Weyl group $W(X_0) = \pi_1(X/N(X))$ of the identity component X_0 of X, such that the diagram

$$\begin{tikzcd} BN(X) \arrow[r, "B(\mathrm{AM}(f))"] \arrow[d, "Bj"'] & BN(X) \arrow[d, "Bj"] \\ BX \arrow[r, "Bf"'] & BX \end{tikzcd}$$

commutes up to based homotopy [66, §3], [1], [101, Theorem C]. The *Adams–Mahmud homomorphism* is the resulting homomorphism

$$\mathrm{AM}\colon \mathrm{Aut}(X) \to \mathrm{Aut}(N(X))/W(X_0) \tag{3.2}$$

of automorphism groups, and X is said to have *N-determined automorphisms* if this homomorphism is injective [66, 3.10].

The following lemma, collecting results from [66, 4.2, 4.3, 4.8] and (9.4), reduces the problem of determining which p-compact groups have N-determined automorphisms to the connected and centerless case. (The *simple factors* of the p-compact group X are the simple, centerless p-compact groups in the splitting [32, 80] of $PX_0 = X_0/Z(X_0)$, the adjoint form of the identity component of X.)

3.3. Lemma. *Let p be any prime number.*

(1) *The connected p-compact group X has N-determined automorphisms if its adjoint form PX does.*

(2) *The p-compact group X has N-determined automorphisms if its identity component X_0 does.*

(3) *The p-compact group X has N-determined automorphisms if all of its simple factors do.*

In the connected, centerless case we use an inductive procedure based on homology decomposition [31, 8.1] and preferred lifts [67].

3.4. Proposition [66, 4.9]. *Suppose that the p-compact group X is connected and centerless. Suppose that*

(1) *$C_X(L,\nu)$ has N-determined automorphisms for each rank 1 object (L,ν) of $\mathbf{A}(X)$.*

(2) $\lim^1(\mathbf{A}(X); \pi_1(BZC_X)) = 0 = \lim^2(\mathbf{A}(X); \pi_2(BZC_X))$.

Then X has N-determined automorphisms.

Proof. Let $f\colon X \to X$ be an automorphism of X such that $\mathrm{AM}(f) : N \to N$ is conjugate to the identity map of N. Then $(E, f\nu) = (E,\nu)$ for each object (E,ν) of $\mathbf{A}(X)$, for if $\mu\colon E \to N$ is a lift of $\nu\colon E \to X$ we have

$$f\nu = fj\mu = j \circ \mathrm{AM}(f) \circ \mu = j \circ \mu = \nu.$$

Thus composition with f determines an automorphism

$$C_f\colon C_X(E,\nu) \to C_X(E,\nu)$$

of each centralizer in the homology decomposition $\operatorname{hocolim} BC_X \to BX$ [31, §8]. In particular, when (L,ν) is a rank 1 object with preferred lift $\mu\colon L \to T \to N$ [67, 4.10], we obtain a commutative diagram

$$\begin{tikzcd} & C_N(L,\mu) \arrow[dl, "C_j"'] \arrow[dr, "C_{j'}"] & \\ C_X(L,\nu) \arrow[rr, "\cong", "C_f"'] & & C_X(L,\nu) \end{tikzcd}$$

which implies, using the first assumption, that C_f is conjugate to the identity [66, 3.9]. But then C_f is conjugate to the identity for all $(E,\nu) \in \mathrm{Ob}(\mathbf{A}(X))$. To see this, choose any line $L < E$ and let $\overline{\nu}\colon E \to C_X(L,\nu|L)$ be the canonical factorization (3.18) of ν through the centralizer of L. Then note that under the isomorphism $C_{C_X(L,\nu|L)}(E,\overline{\nu}(L)) \cong C_X(E,\nu)$ the isomorphism C_f induced by f on X corresponds (3.20) to the isomorphism C_{C_f} induced by C_f on $C_X(L,\nu|L)$.

The second assumption of the lemma assures that there are no further obstructions to conjugating f to the identity [100], [66, 4.9]. ∎

3.5. *N-determined p-compact groups.* Let $j\colon N \to X$ be the maximal torus normalizer for the p-compact group X. Suppose that N may also serve as the maximal torus normalizer for some other p-compact group X' so that we have two monomorphisms

$$X \xleftarrow{\;j\;} N \xrightarrow{\;j'\;} X' \tag{3.6}$$

that are both maximal torus normalizers. The p-compact group X is N-determined if, in this situation, there always exists an isomorphism $f : X \to X'$ under N, i.e. a morphism $f\colon X \to X'$ such that fj and j' are conjugate. A *totally* N-determined p-compact group is an N-determined p-compact group with N-determined automorphisms [66, 7.1].

The following lemma, collecting results from [66, 7.8, 7.10] and (9.6), reduces the problem of determining which p-compact groups are N-determined to the connected and centerless case.

3.7. Lemma. *Let p be an odd prime.*

(1) *The connected p-compact group X is N-determined if its adjoint form PX is.*

(2) *The p-compact group X is N-determined if its identity component X_0 is.*

(3) *The p-compact group X is N-determined if all of its simple factors are.*

Again, in the connected, centerless case we use an inductive procedure.

3.8. Proposition (cf. [66, 7.17]). *In the situation of* (3.6), *suppose that X is connected and centerless and that*

(1) *All objects of $\mathbf{A}(X)$ of rank ≤ 2 have totally N-determined centralizers.*

(2) *For each non-toral rank 2 object (V,ν) of $\mathbf{A}(X)$ there exist a rank 2 object (V,ν') of $\mathbf{A}(X')$ and an isomorphism $f(V,\nu)\colon C_X(V,\nu) \to C_{X'}(V,\nu')$ such that $j'\mu = \nu'$ and*

$$\begin{tikzcd} & C_N(V,\mu) \arrow[dl, "C_j"'] \arrow[dr, "C_{j'}"] & \\ C_X(V,\nu) \arrow[rr, "\cong", "f(V,\nu)"'] & & C_{X'}(V,\nu') \end{tikzcd}$$

commutes for any of the $p+1$ [66, 6.2] *special preferred lifts* (V,μ) *of* (V,ν).

(3) $\lim^2(\mathbf{A}(X);\pi_1(BZC_X)) = 0 = \lim^3(\mathbf{A}(X);\pi_2(BZC_X))$.

Then there exists an isomorphism $f\colon X \to X'$ *under* N.

Proof. For each rank 1 object or *toral* rank 2 object (V,ν) of $\mathbf{A}(X)$, put $\nu' = j'\mu$ where $\mu\colon V \to N$ is *the* preferred lift [67, 4.10], and define $f(V,\nu)\colon C_X(V,\nu) \to C_{X'}(V,\nu')$ to be the unique isomorphism under $C_N(V,\mu)$. Then ν' equals the composition

$$V \xrightarrow{\overline{\nu}} C_X(V,\nu) \xrightarrow{f(V,\nu)} C_{X'}(V,\nu') \xrightarrow{\mathrm{res}} X'$$

and $f(V,\nu)\overline{\nu}$ is the canonical factorization (3.18) $\overline{\nu'}$ of ν'.

Any *non-toral* rank 2 object (V,ν) has $p+1$ special preferred lifts (V,μ) indexed by the set of lines in V [66, 6.2]. By assumption, neither $j'\mu$ nor the isomorphism $f(V,\mu)\colon C_X(V,\nu) \to C_{X'}(V,j'\mu)$ under $C_N(V,\mu)$ depend on the choice of μ. Put $\nu' = j'\mu$ and $f(V,\nu) = f(V,\mu)$ where μ is any of the $p+1$ preferred lifts of ν.

These morphisms $f(V,\nu)$ for $|V| \leq p^2$ respect morphisms in $\mathbf{A}(X)$: Consider for instance a morphism $\alpha\colon (V_1,\nu_1) \to (V_2,\nu_2)$ from a rank 1 object to a rank 2 object. Let $\mu_2\colon V_2 \to N$ be the special preferred lift of ν_2 for which $\mu_1 = \mu_2\alpha$ is the preferred lift of $\nu_1 = \nu_2\alpha$. Since $\nu_1' = j'\mu_1 = j'\mu_2\alpha = \nu_2'\alpha$, the group homomorphism α is an $\mathbf{A}(X')$-morphism $(V_1,\nu_1') \to (V,\nu_2')$. Then

$$\begin{aligned} \nu_2' &= j'\mu_2 = j' \circ \mathrm{res}_N \circ \overline{\mu}_2 = j' \circ \mathrm{res}_N \circ C_N(\alpha)\overline{\mu}_2 \\ &= \mathrm{res}_X \circ C_{j'} \circ C_N(\alpha)\overline{\mu}_2 = \mathrm{res}_X \circ f(V_1,\nu_1) \circ C_X(\alpha)\overline{\nu}_2 \end{aligned}$$

as we see from the commutative diagram

$$\begin{tikzcd} & & C_N(V_1,\mu_1) \arrow[dl, "C_j"'] \arrow[dr, "C_{j'}"] \arrow[rr, "\mathrm{res}_N"] & & N \arrow[d, "j'"] \\ V_2 \arrow[r, "C_X(\alpha)\overline{\nu}_2"'] \arrow[urr, bend left, "C_N(\alpha)\overline{\mu}_2"] & C_X(V_1,\nu_1) \arrow[rr, "\cong", "f(V_1,\nu_1)"'] & & C_{X'}(V_1,\nu_1') \arrow[r, "\mathrm{res}_{X'}"'] & X' \end{tikzcd}$$

and $C_{X'}(\alpha)\overline{\nu_2'} = f(V_1,\nu_1) \circ C_X(\alpha)\overline{\nu}_2$ as we see from the argument of (10.13). Taking centralizers of V_2 we obtain the commutative diagram

$$\begin{tikzcd}
 & C_N(V_2,\mu_2) \arrow[dl, "C_j"'] \arrow[dr, "C_{j'}"] & \\
 & C_{C_N(V_1,\mu_1)}(V_2, C_N(\alpha)\overline{\mu}_2) \arrow[u, "\cong"'] \arrow[ddl] \arrow[ddr] & \\
C_X(V_2,\nu_2) \arrow[rr, "\cong", "f(V_2,\nu_2)"'] & & C_{X'}(V_2,\nu_2') \\
C_{C_X(V_1,\nu_1)}(V_2, C_X(\alpha)\overline{\nu}_2) \arrow[u, "\cong"] \arrow[rr, "\cong", "C_{f(V_1,\nu_1)}"'] \arrow[d] & & C_{C_{X'}(V_1,\nu_1')}(V_2, C_{X'}(\alpha)\overline{\nu_2'}) \arrow[u, "\cong"'] \arrow[d] \\
C_X(V_1,\nu_1) \arrow[rr, "\cong", "f(V_1,\nu_1)"'] & & C_{X'}(V_1,\nu_1')
\end{tikzcd}$$

which shows that the isomorphism $f(V_2,\nu_2)\colon C_X(V_2,\nu_2) \to C_{X'}(V_2,\nu_2')$ under $C_N(V_2,\mu_2)$ is induced from the isomorphism $f(V_1,\nu_1) : C_X(V_1,\nu_1) \to C_{X'}(V_1,\nu_1')$ under $C_N(V_1,\mu_1)$. This implies naturality as we may enlarge the commutative diagram by the morphisms $C_X(V_2,\nu_2) \to C_X(V_1,\nu_1)$ and $C_{X'}(V_2,\nu_2') \to C_{X'}(V_1,\nu_1')$ induced by α (3.20).

Also, if $\alpha \in \mathbf{A}(X)(V,\nu) \subseteq \mathrm{GL}(V)$ is a Quillen automorphism of the rank 2 object (V,ν), and $\mu\colon V \to N$ a special preferred lift of ν, then $\mu\alpha$ is again a special preferred lift of ν and hence $\nu'\alpha = j'\mu\alpha = j'\mu = \nu'$ by assumption. Thus $\mathbf{A}(X)(V,\nu) \subseteq \mathbf{A}(X')(V,\nu')$ and as

$$C_X(V,\nu) \xrightarrow{C_X(\alpha)^{-1}} C_X(V,\nu) \xrightarrow[\cong]{f(V,\nu)} C_{X'}(V,\nu') \xrightarrow{C_X(\alpha)} C_{X'}(V,\nu')$$

is an isomorphism under $C_N(V,\mu\alpha)$, it equals $f(V,\nu)$ by assumption. This is naturality for Quillen automorphisms of (V,ν).

Let now (E,ν) be an object of $\mathbf{A}(X)$ of any rank > 2. Choose a line $L < E$. Define $\nu'\colon E \to X'$ to be the composite monomorphism

$$E \xrightarrow{\overline{\nu}} C_X(E,\nu) \xrightarrow{\mathrm{res}} C_X(L,\nu|L) \xrightarrow[\cong]{f(L,\nu|L)} C_{X'}(L,(\nu|L)') \xrightarrow{\mathrm{res}} X'$$

and define the isomorphism of centralizers $f(E,\nu)\colon C_X(E,\nu) \to C_{X'}(E,\nu')$ to be the isomorphism

$$C_X(E,\nu) \xleftarrow[\cong]{C_{\mathrm{res}}} C_{C_X(L,\nu|L)}(E,\overline{\nu}(L))$$
$$\xrightarrow[\cong]{C_{f(L,\nu|L)}} C_{C_{X'}(L,\nu|L)}(E, f(L,\nu|L)\overline{\nu}(L)) \xrightarrow[\cong]{C_{\mathrm{res}}} C_{X'}(E,\nu')$$

induced by $f(L,\nu|L)$.

To see that this is well-defined, let $L_1 < E$ and $L_2 < E$ be two distinct rank 1 subgroups of E and let $V < E$ be the subgroup generated by them.

Naturality for morphisms from a rank 1 object to a rank 2 object gives a commutative diagram

$$\begin{tikzcd}
 & C_X(L_1,\nu|L_1) \arrow[r, "f(L_1,\nu|L_1)"] & C_{X'}(L_1,(\nu|L_1)') \arrow[dr, "\mathrm{res}"] & \\
E \arrow[ur, "\overline{\nu}(L_1)"] \arrow[r, "\overline{\nu}(V)"] \arrow[dr, "\overline{\nu}(L_2)"'] & C_X(V,\nu|V) \arrow[u] \arrow[d] \arrow[r, "f(V,\nu|V)"] & C_{X'}(V,(\nu|V)') \arrow[u] \arrow[d] \arrow[r, "\mathrm{res}"] & X' \\
 & C_X(L_2,\nu|L_2) \arrow[r, "f(L_2,\nu|L_2)"'] & C_{X'}(L_2,(\nu|L_2)') \arrow[ur, "\mathrm{res}"'] &
\end{tikzcd}$$

showing that neither ν' nor $f(E,\nu)$ depend on the choice of the rank 1 subgroup of E.

To show functoriality of this construction, let $\alpha : (E_1,\nu_1) \to (E_2,\nu_2)$ be a morphism in the category $\mathbf{A}(X)$. Choose a rank one subgroup $L_1 < E_1$ and put $\alpha(L_1) = L_2 < E_2$. Naturality for the rank 1 case gives a commutative diagram

$$\begin{tikzcd}
E_1 \arrow[r] \arrow[d, "\alpha"'] & C_X(L_1,\nu_1|L_1) \arrow[r, "f(L_1,\nu|L_1)"] & C_{X'}(L_1,(\nu_1|L_1)') \arrow[r] & X' \arrow[d, equal] \\
E_2 \arrow[r] & C_X(L_2,\nu_1|L_2) \arrow[u, "C_X(\alpha)"] \arrow[r, "f(L_2,\nu|L_2)"'] & C_{X'}(L_2,(\nu_1|L_2)') \arrow[u, "C_{X'}(\alpha)"'] \arrow[r] & X'
\end{tikzcd}$$

which shows that $\nu_1' = \nu_2'\alpha$, thus

$$\mathbf{A}(X)((E_1,\nu_1),(E_2,\nu_2)) \subseteq \mathbf{A}(X')((E_1,\nu_1'),(E_2,\nu_2')),$$

and implies commutativity of the diagram

$$\begin{tikzcd}
C_X(E_1,\nu_1) \arrow[r, "f(E_1,\nu_1)", "\cong"'] & C_{X'}(E_1,\nu_1') \\
C_X(E_2,\nu_2) \arrow[u, "C_X(\alpha)"] \arrow[r, "\cong", "f(E_2,\nu_2)"'] & C_{X'}(E_2,\nu_2') \arrow[u, "C_X(\alpha)"']
\end{tikzcd}$$

which is naturality.

We have now constructed a collection

$$C_X(E,\nu) \xrightarrow{f(E,\nu)} C_{X'}(E,\nu') \xrightarrow{\mathrm{res}} X', \qquad (E,\nu) \in \mathrm{Ob}(\mathbf{A}(X)),$$

of homotopy $\mathbf{A}(X)$-invariant centric [28] monomorphisms from the centralizers of the homology decomposition of BX [31, 8.1] to BX'. Because the obstruction groups are assumed to vanish, this collection can [100], [66, §2] be realized by a morphism

$$Bf\colon BX \xleftarrow{\simeq} \operatorname{hocolim} BC_X \to BX'$$

such that

$$\begin{tikzcd} C_X(E,\nu) \arrow[r, "f(E,\nu)"] \arrow[d] & C_{X'}(E,\nu') \arrow[d] \\ X \arrow[r, "f"'] & X' \end{tikzcd}$$

commutes for all $(E,\nu) \in \mathrm{Ob}(\mathbf{A}(X))$. In particular, f is a morphism under the maximal torus, for f is a morphism under the maximal rank monomorphisms [31, §4] $X \leftarrow C_N(L,\mu) \to X'$ for some rank 1 object (L,ν) of $\mathbf{A}(X)$. Thus $f\colon X \to X'$ is in fact an isomorphism [32, 5.6], [67, 3.11], and since f is the identity on the maximal torus $T = N_0$, also $\pi_0\mathrm{AM}(f)\colon W \to W$ is the identity map, for W is faithfully represented as a group of operators on T [30, 9.7]. Thus $\pi_*(B\mathrm{AM}(f))$ is the identity automorphism of $\pi_*(BN)$ and $\mathrm{AM}(f)$ is the identity of N [64, 5.2], [5, 3.3]. ∎

Verification of the third assumption reduces to a computation involving Steinberg representations (2.13). For the verification of the second condition we shall use the following lemma which may look rather specialized but in fact applies in all cases considered in this paper.

3.9. Lemma. *Let (V,ν) be a non-toral rank 2 object of $\mathbf{A}(X)$ with special preferred lift $\mu\colon V \to N$ and put $\nu' = j'\mu$. Assume that*

(1) *All rank 2 objects of $\mathbf{A}(X)$, whose centralizers are isomorphic to $C_X(V,\nu)$, are isomorphic to (V,ν).*

(2) $\mathbf{A}(X)(V,\nu) = \mathrm{SL}(V)$. (*Then also* $\mathbf{A}(X')(V,\nu') = \mathrm{SL}(V)$.)

(3) *The isomorphism $f(V,\mu)\colon C_X(V,\nu) \to C_{X'}(V,\nu')$ under $C_N(V,\mu)$ is* $\mathrm{SL}(V)^{\mathrm{op}}$*-equivariant.*

Then $j'\mu_1 = \nu'$ and $f(V,\mu_1) = f(V,\mu)\colon C_X(V,\nu) \to C_{X'}(V,\nu')$ for all special preferred lifts (V,μ_1) of (V,ν).

Proof. The $\mathrm{GL}(V)$-orbit $(V,\nu)\cdot\mathrm{GL}(V)$ contains $p-1$ objects, the $\mathrm{GL}(V)$-orbit $(V,\mu)\cdot\mathrm{GL}(V)$ contains $(p-1)(p+1)$ objects, and the map $j\colon (V,\mu)\cdot\mathrm{GL}(V) \to (V,\nu)\cdot\mathrm{GL}(V)$ is $(p+1)$-to-1 [66, 6.2]. By assumption, the orbit $(V,\mu)\cdot\mathrm{GL}(V)$ contains all special preferred lifts whose centralizers in N are isomorphic to $N(C_X(V,\nu))$. Since X and X' have the same special preferred lifts [66, 7.13], also $j'\colon (V,\mu)\cdot\mathrm{GL}(V) \to (V,j'\mu)\cdot\mathrm{GL}(V)$ is $(p+1)$-to-1. Since the orbit $(V,j'\mu)\cdot\mathrm{GL}(V)$ thus contains $p-1$ objects, the stabilizer subgroup of $(V,j'\mu)$ must be $\mathrm{SL}(V)$ as this is the only subgroup of $\mathrm{GL}(V)$ of that index. Thus the Quillen automorphism group $\mathbf{A}(X')(V,\nu')$ is $\mathrm{SL}(V)$.

Any other special preferred lift of ν has the form $\mu\alpha$ for an α in $\mathrm{SL}(V)$ [66, 6.2], so, clearly, $j'(\mu\alpha) = \nu'\alpha = \nu'$ is independent of the choice of α.

The commutative diagram

$$\begin{tikzcd}
C_N(V,\mu\alpha) \arrow[d, "C_j"'] & & & C_N(V,\mu) \arrow[lll, "C_N(\alpha)"'] \arrow[rrr, "C_N(\alpha)"] \arrow[dl, "C_j"'] \arrow[dr, "C_{j'}"] & & & C_N(V,\mu\alpha) \arrow[d, "C_{j'}"] \\
C_X(V,\nu) & & C_X(V,\nu) \arrow[ll, "C_X(\alpha)"] \arrow[rr, "f(V,\mu)"'] & & C_{X'}(V,\nu') \arrow[rr, "C_{X'}(\alpha)"'] & & C_{X'}(V,\nu')
\end{tikzcd}$$

shows that $f(V,\mu\alpha) = C_{X'}(\alpha)f(V,\mu)C_X(\alpha)^{-1}$, since $C_X(V,\nu)$ has N-determined automorphisms so that $f(V,\mu\alpha) = f(V,\mu)$ by the third assumption. ∎

The canonical factorizations of ν and ν' are $\mathrm{SL}(V)^{\mathrm{op}}$-equivariant (3.19) and they provide a commutative diagram

(3.10)
$$\begin{tikzcd}
& V \arrow[dl, "\overline{\nu}"'] \arrow[dr, "\overline{\nu'}"] & \\
C_X(V,\nu) \arrow[rr, "f(V,\mu)"'] & & C_{X'}(V,\nu')
\end{tikzcd}$$

which shows that the restriction of $f(V,\mu)$ to V is $\mathrm{SL}(V)^{\mathrm{op}}$-equivariant. It is a tautology that $f(V,\mu\alpha) = f(V,\mu)$ for all α in the Borel subgroup stabilizing μ so it is in fact only necessary to check equivariance with respect to one other element (of order p) [91, 3.6.21] of $\mathrm{SL}(V)$.

3.11. *Centers and automorphism groups of p-compact groups.* The following theorem collects some useful facts from various sources that will be applied several times in this paper.

3.12. THEOREM. *Let p be an odd prime and X a connected p-compact group.*

(1) [4, 5] *The semidirect product $\check{N}(X) = \check{T}(X) \rtimes W(X)$ is a discrete approximation* [31, 3.12] *to the maximal torus normalizer $N(X)$.*

(2) [31, §7] *The abelian group $\check{Z}(X)$ given by*

$$\begin{aligned}
&H^0(W(X);\check{T}(X)) \\
&\qquad = (H^0(W(X);L(X)) \otimes \mathbf{Q})/H^0(W(X);L(X)) \times H^1(W(X);L(X))
\end{aligned}$$

is a discrete approximation to the center [69, 31] *of X.*

(3) [66, 7.2] $\mathrm{Aut}(X) \overset{\mathrm{AM}}{\cong} \mathrm{Out}(N(X))$ *provided X is totally N-determined.*

The automorphism group of $N(X)$ sits [64, 5.2] in a short exact sequence

(3.13)
$$\begin{aligned}
0 \to H^1(W(X);\check{T}(X)) \to \mathrm{Aut}(N(X)) \\
\to \mathrm{Aut}(W(X),\check{T}(X),e(X)) \to 1
\end{aligned}$$

where the normal subgroup to the left consists of all automorphisms of $N(X)$ that induce the identity on homotopy groups, and the group to the right consists of all pairs $(\alpha,\theta) \in \mathrm{Aut}(W(X)) \times \mathrm{Aut}(\check{T}(X))$ such that θ is α-linear and the induced automorphism $H^2(\alpha^{-1},\theta)$ [95, 6.7.6] preserves the extension class $e(X)\in H^2(W(X);\check{T}(X))$. The image of $W(X_0)\subseteq W(X)=\pi_0N(X)$ [69, 3.8] in $\mathrm{Aut}(N(X))$ does not intersect the subgroup $H^1(W(X);\check{T}(X))$ (as $W(X_0)$ is represented faithfully in $\mathrm{Aut}(\check{T}(X))$ [30, 9.7]) so there is an induced short exact sequence

$$(3.14)\qquad 0\to H^1(W(X);\check{T}(X))\to \mathrm{Aut}(N(X))/W(X_0) \\ \to \mathrm{Aut}(W(X),\check{T}(X),e(X))/W(X_0)\to 1$$

whose middle term is the target of the Adams–Mahmud homomorphism (3.2).

If X is connected and p is odd, the cohomology group to the left is trivial and $e(X)=0$ [5] so

$$\mathrm{Aut}(N(X))\cong \mathrm{Aut}(W(X),\check{T}(X),0)\cong N_{\mathrm{GL}(L(X))}(W(X))$$

is [66, 3.5], [5, 3.3] the group of self-similarities of the $\mathbf{Z}_p$-reflection group $(W(X),L(X))$ (4.1), and the target of the Adams–Mahmud homomorphism (3.2)

$$(3.15)\qquad \mathrm{Out}(N(X))=\mathrm{Aut}(N(X))/W(X)\cong N_{\mathrm{GL}(L(X))}(W(X))/W(X)$$

is [62, §2] the middle term of an exact sequence

$$(3.16)\quad 1\to \mathrm{Aut}_{\mathbf{Z}_p[W(X)]}(L(X))/Z(W(X))\to \mathrm{Out}(N(X))\to \mathrm{Out}(W(X))$$

of automorphism groups. An automorphism of X is *exotic* if its lift to $N(X)$ [66, 3.7] induces a non-trivial outer automorphism of $W(X)$.

3.17. Remark. Let p and X be as in (3.12).

(1) The formula

$$\pi_j(BZ(X))=H^{2-j}(W(X);L(X)),\quad j=1,\,2,$$

is an alternative version of (3.12.(2)).

(2) The endomorphism monoid of X is given by

$$\mathrm{End}(X)-\{0\} \\ \cong\begin{cases} N_{\mathrm{GL}(L(X))}(W(X))/W(X)=\mathrm{Aut}(X), & p\,|\,|W(X)|,\\ (N_{\mathrm{GL}(L(X)\otimes\mathbf{Q})}(W(X))\cap \mathrm{End}(L(X)))/W(X), & p\nmid |W(X)|,\end{cases}$$

provided X is totally N-determined and simple [65, 5.4]; use [65, 5.6, 5.6] and [64, 5.2] to see this. See [46, 47, 48] for the Lie case.

3.18. *Canonical factorizations* [30, 8.2]. Let $\nu\colon V\to X$ be a monomorphism from an elementary abelian p-group to the p-compact group X. The canonical factorization of ν through its centralizer is the central monomorphism $\overline{\nu}(V)\colon V\to C_X(V,\nu)$ whose adjoint is $V\times V\xrightarrow{+}V\xrightarrow{\nu}X$. The composition $V\xrightarrow{\overline{\nu}}C_X(V,\nu)\xrightarrow{\mathrm{res}}X$ equals ν. If $\alpha\colon(V_1,\nu_1)\to(V_2,\nu_2)$ is a morphism in $\mathbf{A}(X)$ then the canonical factorizations are related by a commutative diagram

$$\begin{tikzcd} V_1 \arrow[r, "\overline{\nu}_1"] \arrow[d, "\alpha"'] & C_X(V_1,\nu_1) \arrow[r, "\mathrm{res}"] & X \arrow[d, equal] \\ V_2 \arrow[r, "\overline{\nu}_2"'] & C_X(V_2,\nu_2) \arrow[r, "\mathrm{res}"'] \arrow[u, "C_X(\alpha)"] & X \end{tikzcd} \tag{3.19}$$

so that $\alpha\colon (V_1,\overline{\nu}_1) \to (V_2, C_X(\alpha)\overline{\nu}_2)$ is a morphism in $\mathbf{A}(C_X(V_1,\nu_1))$. The induced morphisms $C_X(\alpha)$ and $C_{C_X(V_1,\nu_1)}(\alpha)$ can be identified in that the diagram

$$\begin{tikzcd} C_{C_X(V_1,\nu_1)}(V_1,\overline{\nu}_1) \arrow[r, "C_{\mathrm{res}}", "\cong"'] & C_X(V_1,\nu_1) \\ C_{C_X(V_1,\nu_1)}(V_2, C_X(\alpha)\overline{\nu}_2) \arrow[u, "C_{C_X(V_1,\nu_1)}(\alpha)"] \arrow[ur, "\mathrm{res}"] \arrow[r, "C_{\mathrm{res}}", "\cong"'] & C_X(V_2,\nu_2) \arrow[u, "C_X(\alpha)"'] \end{tikzcd} \tag{3.20}$$

commutes up to conjugacy.

4. Cohomologically unique p-compact groups. We shall here discuss to what extent N-determined p-compact groups are determined by their Weyl groups or their mod p cohomology algebras (4.3). The message intended is that cohomological uniqueness [36, 74, 16, 93] is incidental while N-determinism is universal. As the Weyl group of a connected p-compact group is a reflection subgroup of the automorphism group of the lattice we start out by introducing the category of reflection subgroups.

For a commutative domain R, an element g of $\mathrm{GL}(r,R)$ is a *reflection* if the $r\times r$ matrix $I_r - g$ has rank at most 1 where I_r is the $r \times r$ identity matrix. A subgroup W of $\mathrm{GL}(r,R)$ is a *reflection subgroup* if it is generated by the reflections contained in it.

4.1. Definition. For $R = \mathbf{Z}_p, \mathbf{Q}_p, \mathbf{F}_p$, let R-**Refl** be the category with

- objects: pairs (W,L) where L is a finitely generated free R-module and W a finite reflection subgroup of $\mathrm{Aut}_R(L)$, and
- morphisms: pairs $(\alpha,\theta)\colon (W_1,L_1)\to(W_2,L_2)$ where $\alpha\colon W_1\to W_2$ is a group homomorphism and $\theta \in \mathrm{Hom}_R(L_1,L_2)$ an α-linear R-module homomorphism.

A *similarity* is an isomorphism in R-**Refl**. Two objects of $\mathbf{Z}_p$-**Refl** are *R-similar* if they are taken to isomorphic objects of R-**Refl** by the functor $r_R\colon \mathbf{Z}_p$-**Refl** $\to R$-**Refl** induced by $-\otimes_{\mathbf{Z}_p} R$. $G_0(W,L)$ (resp. $G_p(W,L)$) is the set of similarity classes of objects of $\mathbf{Z}_p$-**Refl** that are $\mathbf{Q}_p$-similar (resp. $\mathbf{F}_p$-similar) to the object (W,L). An object (W,L) of $\mathbf{Z}_p$-**Refl** is said to be *simple* if $L\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$ is a simple $\mathbf{Q}_pW$-module.

A similarity class of objects of R-**Refl** amounts to an integer $r \geq 0$ and a conjugacy class (W) of a reflection subgroup of $\mathrm{GL}(r, R)$. The automorphism group $\mathrm{Aut}_{R\text{-}\mathbf{Refl}}(W, L)$ of an R-reflection subgroup is isomorphic to the normalizer $N_{\mathrm{GL}(L\otimes R)}(W)$ of W in $\mathrm{GL}(L \otimes R)$ [62, §2], [61].

In $\mathbf{Z}_p$-**Refl** we shall often write r_0 (resp. r_p) for the functor r_R if $R = \mathbf{Q}_p$ (resp. $R = \mathbf{F}_p$). (Of course, if $R = \mathbf{Z}_p$, then r_R is the identity functor.) By [29, Proof of 5.2], W is a reflection subgroup of $\mathrm{GL}(L)$ if and only if r_pW is a reflection subgroup of $\mathrm{GL}(L \otimes \mathbf{Z}/p)$; also, W and r_pW are abstractly isomorphic groups as the kernel of $\mathrm{GL}(r, \mathbf{Z}_p) \to \mathrm{GL}(r, \mathbf{F}_p)$ contains no nontrivial finite order elements when p is odd [57], [88, 10.7.1]. Two objects, (W_1, L_1) and (W_2, L_2), of $\mathbf{Z}_p$-**Refl** are $\mathbf{Q}_p$-similar iff there exists a morphism $(\alpha, \theta)\colon (W_1, L_1) \to (W_2, L_2)$ in $\mathbf{Z}_p$-**Refl** such that $r_0(\alpha, \theta)$ is an isomorphism in $\mathbf{Q}_p$-**Refl**, and they are $\mathbf{F}_p$-similar iff there exists a group isomorphism $\alpha\colon W_1 \to W_2$ and a $\mathbf{Z}_p$-linear isomorphism $\theta\colon L_1 \to L_2$ such that $(\alpha, \theta \otimes_{\mathbf{Z}_p} \mathbf{F}_p)$ is an isomorphism in $\mathbf{F}_p$-**Refl**. All elements of $G_0(W, L)$, which is a finite set according to the Jordan–Zassenhaus Theorem [24, 24.2], are represented by *centerings* of (W, L), i.e. by objects of the form (W, M) where M is a $\mathbf{Z}_pW$-submodule of L and the index $[L : M]$ is finite. Two centerings, (W, M_1) and (W, M_2), are similar if and only if $A(M_1) = M_2$ for some A in the normalizer $N_{\mathrm{GL}(L\otimes\mathbf{Q})}(W)$ of W in $\mathrm{GL}(L \otimes \mathbf{Q})$ [84, 2.1–2.3].

4.2. Proposition. *Let (W, L) be an object of* $\mathbf{Z}_p$-**Refl**.

(1) $G_0(W, L) = \{(W') < \mathrm{GL}(L) \mid (r_0W') = (r_0W)\}$.
(2) $G_p(W, L) = \{(W') < \mathrm{GL}(L) \mid r_pW' = r_pW\}$.

As usual, (W) *stands for the conjugacy class of the subgroup* W.

Proof. (1) Let $A(W) = \{U \in \mathrm{GL}(L \otimes \mathbf{Q}) \mid U^{-1}WU \subseteq \mathrm{GL}(L)\}$ be the set of automorphisms of $L \otimes \mathbf{Q}$ that conjugate the subgroup $W \subseteq \mathrm{GL}(L)$ to (another) subgroup of $\mathrm{GL}(L)$. We shall define surjections

$$\{(W') < \mathrm{GL}(L) \mid (r_0W') = (r_0W)\} \twoheadleftarrow A(W) \twoheadrightarrow G_0(W, L)$$

and show that the corresponding equivalence relations on $A(W)$ are the same. The left surjection simply takes $U \in A(W)$ to the subgroup conjugacy class $(U^{-1}WU)$. The right map takes $U \in A(W)$ to the similarity class of (W, UL). This is indeed a well-defined surjection because for $U \in \mathrm{GL}(L\otimes\mathbf{Q})$ we have

$$U^{-1}WU \subseteq \mathrm{GL}(L) \Leftrightarrow (U^{-1}WU)(L) = L \Leftrightarrow W(UL) = UL,$$

meaning that UL is a $\mathbf{Z}_pW$-submodule of $L \otimes \mathbf{Q}$ if and only if $U \in A(W)$. The $\mathbf{Z}_p$-**Refl**-objects (W, UL) and (W, VL), $U, V \in A(W)$, are similar if and only if $VAU^{-1}W = WVAU^{-1}$ for some isomorphism of the form

$$UL \xrightarrow[\cong]{U^{-1}} L \xrightarrow[\cong]{A} L \xrightarrow[\cong]{V} VL$$

for an $A \in \mathrm{GL}(L)$. In other words, (W, UL) and (W, VL) are similar if and only if $U^{-1}WU$ and $V^{-1}WV$ are conjugate as subgroups of $\mathrm{GL}(L)$.

(2) The map

$$\{(W') < \mathrm{GL}(L) \mid r_pW' = r_pW\} \to G_p(W, L)$$

taking (W') to (W', L) is clearly well-defined and injective. To see that it is also surjective, let (W_1, L_1) be an object of $\mathbf{Z}_p$-**Refl** that admits a similarity $(\alpha_p, \theta_p)\colon r_p(W_1, L_1) \to r_p(W, L)$ in $\mathbf{F}_p$-**Refl**. Lift the isomorphism θ_p to a $\mathbf{Z}_p$-linear isomorphism $\theta\colon L_1 \to L$. Then (W_1, L_1) and (W', L), $W' = \theta W_1 \theta^{-1}$, are similar and $r_pW' = \theta_p r_p(W_1)\theta_p^{-1} = r_pW$ and thus the subgroup W' is mapped to the element of $G_p(W, L)$ represented by (W_1, L_1). ∎

The Weyl group $W(X)$ of a connected p-compact group X is by birth a finite reflection subgroup of $\mathrm{GL}(L(X))$ [30, 9.7] and $(W(X), L(X))$, $(r_0W(X), L(X) \otimes \mathbf{Q}_p)$, and $(r_pW(X), L(X) \otimes \mathbf{F}_p)$ are objects of R-**Refl** for $R = \mathbf{Z}_p, \mathbf{Q}_p, \mathbf{F}_p$, called the $\mathbf{Z}_p$-*Weyl group* (or just the *Weyl group*), the $\mathbf{Q}_p$-*Weyl group*, and the $\mathbf{F}_p$-*Weyl group* of X, respectively. (As to functoriality we note that any toric morphism [68] between connected p-compact groups determines a morphism between the corresponding reflection subgroups.)

4.3. Definition. Let X be a connected p-compact group.

(1) X is *determined by its R-Weyl group* if any connected p-compact group Y with the same R-Weyl group as X, i.e. with $W(Y)$ R-similar to $W(X)$, is isomorphic to X.

(2) X is a *cohomologically unique p-compact group* if any connected p-compact group Y with $H^*(BY; \mathbf{F}_p)$ isomorphic to $H^*(BX; \mathbf{F}_p)$ as an algebra over the mod p Steenrod algebra, is isomorphic to X.

All p-compact tori are clearly cohomologically unique.

4.4. Corollary. *Let p be an odd prime and X an N-determined connected p-compact group.*

(1) *X is determined by its $\mathbf{Z}_p$-Weyl group $W(X)$.*

(2) *If $G_0(W(X), L(X)) = *$, then X is determined by its $\mathbf{Q}_p$-Weyl group $r_0W(X)$.*

(3) *If $G_p(W(X), L(X)) = *$, then X is determined by its $\mathbf{F}_p$-Weyl group $r_pW(X)$.*

(4) *If X is determined by its $\mathbf{F}_p$-Weyl group, then X is a cohomologically unique p-compact group.*

Proof. At odd primes, the (discrete) maximal torus normalizer of a connected p-compact group, which is a split extension (3.12), is determined, up to isomorphism, by the similarity class of the Weyl group. The next two items are immediate consequences of this, since we are assuming $W(X)$ recoverable from $r_0W(X)$, respectively $r_pW(X)$.

The rational rank $r(X)$ as well as the $\mathbf{F}_p$-Weyl group $r_pW(X)$ can be read off from $H^*(BX,\mathbf{F}_p)$ thanks to Lannes theory [52]. Indeed, $r(X)$ is the maximal $r \geq 0$ for which there exists a monomorphism $(\mathbf{Z}/p)^r \rightarrowtail X$ whose centralizer is a p-compact torus and $r_pW(X)$ is (2.8) the automorphism group in the Quillen category of the object $t(X) \to X$. ∎

4.5. Lemma. *Let W be a finite reflection subgroup of* $\mathrm{GL}(L)$. *Put* $t = L/pL$.

(1) [84, (1) p. 248] *If t is an irreducible $\mathbf{F}_p[W]$-module, then* $G_0(W,L) = *$.

(2) [7, 7.1.2] *If* $H^1(r_pW; \mathrm{Hom}(t,t)) = 0$, *then* $G_p(W,L) = *$.

Proof. For (1), let M be a $\mathbf{Z}_pW$-submodule of L not contained in pL. Since the image of M in $t = L/pL$ is non-trivial, we get $L = M + pL$ by irreducibility and $L = M$ by Nakayama's lemma [86, 9.2]. The H^1-condition of (2) assures that r_pW lifts uniquely to $\mathrm{GL}(r,\mathbf{Z}_p)$. ∎

The sets $G_0(W,L)$ and $G_p(W,L)$ are determined in (11.18, 11.25, 11.26) for (W,L) a simple reflection subgroup and p an odd prime.

For a connected p-compact group X, let SX stand for the universal covering group of X and $PX = X/Z(X)$ the adjoint form of X [31, 69]. Recall that for $(W,L) \in \mathrm{Ob}(\mathbf{Z}_p\text{-}\mathbf{Refl})$ there are associated objects $(SW,SL),(PW,PL) \in \mathrm{Ob}(\mathbf{Z}_p\text{-}\mathbf{Refl})$ [75], (11.1).

4.6. Lemma. *$SSX = SX = SPX$ and $PPX = PX = PSX$ for any connected p-compact group X.*

Proof. Use [69, 4.7, 5.4, 5.5] and that $BSX = BX\langle 2\rangle$ is the 2-connected cover of BX. ∎

4.7. Proposition. *Let p be an odd prime and X an N-determined connected p-compact group.*

(1) *$H^0(W(X); \check{T}(X)) = \check{Z}(X)$ and $H_0(W(X); L(X)) = \pi_1(X)$.*

(2) *$SL(X) = L(SX)$ and $PL(X) = L(PX)$.*

Proof. The formula for the center of X (3.12(2)) immediately shows that $PL(X) = L(PX)$. By inspection we see that

$$H_0(W(PX); L(PX)) = \pi_2(BPX) \tag{4.8}$$

for any simple p-compact group X. (The formula is known to hold in the Lie case by classical results. The exotic simple p-compact groups are all centerless and polynomial (7.9) so in this case $X = PX$ and $\pi_2(BX) = 0$ because $H^2(BX;\mathbf{F}_p) = 0$. Also $H_0(W(X); L(X)) = 0$ by (11.4.3) for $G_0(W(X)) = *$ (11.18) so that $L(X) = SL(X)$.) Therefore,

$$\begin{aligned} SL(PX) &= \ker(L(PX) \to H_0(W(PX); L(PX))) \\ &= \ker(L(PX) \to \pi_1(PX)) = L(SPX) = L(SX) \end{aligned}$$

for any simple X. For a general connected X, the Splitting Theorem for centerless p-compact groups [32] tells us that $PX = \prod PX_i$ where X_i is simple. Consequently,

$$\begin{aligned} SL(X) = SPL(X) = SL(PX) = \prod SL(PX_i) = \prod L(SX_i) \\ = \prod L(SPX_i) = L(SPX) = L(SX). \end{aligned}$$

From the finite covering $\pi \to SX \times Z(X)_0 \to X$ of [69, 5.4] we obtain a short exact sequence of $\mathbf{Z}_pW(X)$-modules

$$0 \to SL(X) \times H^0(W(X); L(X)) \to L(X) \to \pi \to 0 \tag{4.9}$$

and, using $H_1(W(X); \pi) = 0 = H^0(W(X); SL(X))$ (11.3, 11.4.3), a short exact sequence of $\mathbf{Z}_p$-modules

$$0 \to H^0(W(X); L(X)) \to H_0(W(X); L(X)) \to \pi \to 0$$

identical to the short exact sequence for computing $\pi_1(X)$. ∎

Recall that we write $X_1 \geq X_2$ if there exists an isogeny $X_1 \twoheadrightarrow X_2$ [65, p. 216] in **pcg**, and $(W_1, L_1) \geq (W_2, L_2)$ if there exists an isogeny $(W_1, L_1) \to (W_2, L_2)$ in $\mathbf{Z}_p$-**Refl** (11.23).

4.10. COROLLARY. *Let p be an odd prime and X_1 and X_2 two connected p-compact groups. Assume that PX_2 is totally N-determined.*

(1) *X_1 and X_2 are locally isomorphic* $\Leftrightarrow$ $(W, L)(X_1)$ *and* $(W, L)(X_2)$ *are $\mathbf{Q}_p$-similar.*

(2) $X_1 \geq X_2 \Leftrightarrow (W, L)(X_1) \geq (W, L)(X_2)$.

(3) *The local isomorphism system* [65, 4.7] *of X_2 is poset isomorphic to* $G_0(W(X_2), L(X_2))$.

Proof. Write (W_i, L_i) for $(W(X_i), L(X_i))$, $i = 1, 2$. It is clear from the results of [65, §2–§4] that if X_1 and X_2 are locally isomorphic (and $X_1 \geq X_2$) then (W_1, L_1) and (W_2, L_2) are $\mathbf{Q}_p$-similar (and $(W_1, L_1) \geq (W_2, L_2)$). Conversely, suppose that (W_1, L_1) and (W_2, L_2) are $\mathbf{Q}_p$-similar. Then $(W_1, L_1) \cong (W_2, P_{\check{\alpha}}(SL_2))$ for some diagram $\check{\alpha}\colon \check{\pi}(SL_2) \leftarrowtail \pi(L_1) \xrightarrow{\varphi} \check{T}H^0(W_2; L_2)$ of $\mathbf{Z}_p$-modules (11.20). Since $SL_2 = L(SX_2)$, this means that (W_1, L_1) is similar to $(W(X_2')L(X_2'))$ for the p-compact group

$$X_2' = \frac{SX_2 \times Z(X_2)_0}{(\pi(L_1), \varphi)}$$

locally isomorphic to X_2 [65, 2.8]. But X_2' is totally N-determined if PX_2 is (3.3, 3.7), and therefore X_1 is actually isomorphic to X_2' (4.4). Moreover, if $(W_2, P_{\breve{\alpha}}(SL_2)) \geq (W_2, L_2)$ then (11.21) there is a commutative diagram

$$\begin{tikzcd}
\breve{\pi}(SL_2) \arrow[d, "\cong"'] & \breve{\pi}(L_1) \arrow[l, tail] \arrow[r, "\varphi"] \arrow[d, tail] & \check{T}H^0(W_2; L_2) \arrow[d, two heads] \\
\breve{\pi}(SL_2) & \breve{\pi}(L_2) \arrow[l, tail] \arrow[r] & \check{T}H^0(W_2; L_2)
\end{tikzcd}$$

induced by an automorphism of SL_2 and an epimorphism of $\check{T}H^0(W_2; L_2) = \check{Z}(X_2)_0$ onto itself with finite kernel. But any automorphism of $SL_2 = L(SX_2)$ comes from an automorphism of SX_2 (3.12(3)) and so the above diagram determines [65, 4.3, 4.5] an isogeny $X_1 \twoheadrightarrow X_2$. ∎

4.11. COROLLARY. *Let p be an odd prime. There are fibration sequences*

$$B\pi(L(X)) \to BSX \times B^2H^0(W(X); L(X)) \to BX,$$
$$BX \to B^2LH_0(W(X); \check{T}(X)) \times BPX \to B^2\breve{\pi}(L(X))$$

for any N-determined connected p-compact group X.

Proof. Write (W, L) for the reflection subgroup $(W(X), L(X))$ associated to X. The first of these fibration sequences will follow if we can show that

$$\begin{tikzcd}
B\pi(L) \arrow[r] \arrow[d] & BSX \arrow[d] \\
B^2H^0(W; L) \arrow[r] & BX
\end{tikzcd} \tag{4.12}$$

is a homotopy fiber square. The top horizontal map corresponds to the monomorphism $\pi(L) \rightarrowtail \breve{\pi}(SL) = H^0(W; \check{T}(SL)) = \check{Z}(SX)$ of (11.8.2) and the bottom one corresponds to the monomorphism $\check{T}H^0(W; L) \rightarrowtail H^0(W; \check{T}(L)) = \check{Z}(X)$ of (11.4(1)). There is a fibration

$$BH_0(W; L) \to B\pi(L) \to B^2H^0(W; L)$$

induced from the short exact sequence $0 \to H^0(W; L) \to H_0(W; L) \to \pi(L) \to 0$ of abelian groups. But $H_0(W; L)$ and $\pi_1(X)$ are (4.7) isomorphic abelian groups and thus the left and right vertical maps in (4.12) have identical fibers.

For the second fibration, it is enough to prove that there exists a homotopy fiber square

$$\begin{tikzcd}
BX \arrow[r] \arrow[d] & B^2LH_0(W; \check{T}(L)) \arrow[d] \\
BPX \arrow[r] & B^2\breve{\pi}(L)
\end{tikzcd} \tag{4.13}$$

with an abelian topological group in the lower right corner. The top horizontal map corresponds to the epimorphism $H_0(W;L) \twoheadrightarrow LH_0(W;\check{T}(L))$ of (11.4(4)) and the bottom one to the epimorphism $H_0(W;PL) = \pi(PL) \twoheadrightarrow \check{\pi}(L)$ of (11.8(2)). There is a fibration

$$BH^0(W;\check{T}(L))^\wedge_p \to B^2LH_0(W;\check{T}(L)) \to B^2\check{\pi}(L)$$

obtained by applying the Fiber Lemma [14, II.5.1] to the fibration

$$BH_0(W;\check{T}(L)) \to B^2\check{\pi}(L)$$

with $BH^0(W;\check{T}(L))$ as fiber reflecting the defining short exact sequence for $\check{\pi}(L)$. But $H^0(W;\check{T}(L))$ is (3.12(2)) a discrete approximation to the center of X and thus the left and right vertical maps in (4.13) have identical fibers. ∎

The N-determined connected p-compact group BX is, in other words, the quotient p-compact group of $BSX \times B^2H^0(W(X);L(X))$ corresponding to the subgroup $\pi(L(X))$ (11.8(4)) of the center

$$\check{\pi}(SL(X)) \times \check{T}H^0(W(X);L(X))$$

(4.7, 11.8(1)), or the covering p-compact group [17], [69, 3.3] of

$$B^2LH_0(W(X);\check{T}(X)) \times BPX$$

corresponding to the quotient group $\check{\pi}(L(X))$ (11.8(3)) of the fundamental group $LH_0(W(X);\check{T}(X)) \times \pi(PL(X))$ (4.7, 11.8(1)).

According to [74, 8.1], any "p-convenient and simply connected or pseudo simply connected" compact connected Lie group satisfies (4.5(2)). For our purposes, however, the following corollary will suffice.

4.14. Corollary. *Let p be an odd prime and let X be the p-compact group represented by*

- *the product subgroup* $\mathrm{U}(n_1) \times \ldots \times \mathrm{U}(n_k)$, $n_1 + \ldots + n_k = n$, $n_i \geq 0$, *of* $\mathrm{U}(n)$, *or*
- *the intersection with* $\mathrm{SU}(n)$ *of such a subgroup of* $\mathrm{U}(n)$, *or*
- *the image in* $\mathrm{PU}(n) = \mathrm{U}(n)/\mathrm{U}(1)$ *of such a subgroup of* $\mathrm{U}(n)$.

Then $G_p(W(X), L(X)) = *$.

Proof. Write $t = t(\mathrm{U}(n))$, $t_0 = t(\mathrm{SU}(n))$, and $t_1 = t(\mathrm{PU}(n))$ (the dual to t_0). It suffices (4.5(2)) to show that $H^1(W;-) = 0$ where W is a subgroup of the form $\Sigma_{n_1} \times \ldots \times \Sigma_{n_k}$ of $W(\mathrm{U}(n)) = \Sigma_n$ and the blank is any of the $\mathbf{F}_p\Sigma_n$-modules $\mathrm{Hom}(t,t)$, $\mathrm{Hom}(t_0,t_0)$ or $\mathrm{Hom}(t_1,t_1)$.

Let i be 1 or 2. From the fact that $H^i(\Sigma_n;\mathbf{F}_p) = 0$ for all n when p is odd [50, 12.2.2], we inductively deduce that also $H^i(W;\mathbf{F}_p) = 0$. But then also

$$H^i(W;t_0) \cong H^i(W;t) \cong H^i(W;t_1),$$
$$H^1(W;\mathrm{Hom}(t_0,t_0)) \cong H^1(W;\mathrm{Hom}(t,t_0)),$$
$$H^1(W;\mathrm{Hom}(t_1,t_1)) \cong H^1(W;\mathrm{Hom}(t,t_1)),$$

as we see from the exact sequences in cohomology induced by the short exact sequences

$$0 \to t_0 \to t \xrightarrow{+} \mathbf{F}_p \to 0, \quad 0 \to \mathbf{F}_p \xrightarrow{\Delta} t \to t_1 \to 0,$$
$$0 \to t_0 \to \mathrm{Hom}(t,t_0) \to \mathrm{Hom}(t_0,t_0) \to 0,$$
$$0 \to \mathrm{Hom}(t_1,t_1) \to \mathrm{Hom}(t,t_1) \to t_1 \to 0$$

of $\mathbf{F}_p\Sigma_n$-modules.

Since the representation $t = \mathrm{Ind}_{\Sigma_{n-1}}^{\Sigma_n}(\mathbf{F}_p)$ is induced from the trivial 1-dimensional representation, its restriction to W,

$$\mathrm{res}_W^{\Sigma_n}(t) = \mathrm{res}_W^{\Sigma_n}\mathrm{Ind}_{\Sigma_{n-1}}^{\Sigma_n}(\mathbf{F}_p) = \prod_{x\in W\backslash\Sigma_n/\Sigma_{n-1}} \mathrm{Ind}_{W\cap^x\Sigma_{n-1}}(\mathbf{F}_p),$$

is a product of representations induced from trivial 1-dimensional representations. But $W\cap{}^x\Sigma_{n-1}$, the intersection of W with a conjugate of $\Sigma_{n-1} = \Sigma_1 \times \Sigma_{n-1}$, is again a subgroup of W-type, so it follows that $H^i(W;t) = \prod H^i(W\cap{}^x\Sigma_{n-1};\mathbf{F}_p) = 0$. Furthermore, $\mathrm{Hom}(t,-) = \mathrm{Ind}_{\Sigma_{n-1}}^{\Sigma_n}(-)$ so that, by the same argument, $H^1(W;\mathrm{Hom}(t,-)) = 0$ where the blank can be t, t_0 or t_1. ∎

5. The p-compact group $\mathrm{PGL}(n,\mathbf{C})$. In this section we show N-determinism for the A-family of p-compact groups where p is an odd prime. See Broto and Viruel [16, 15] for an alternative proof and [66, 7.19] for a prototype of Theorem 5.1.

5.1. THEOREM. *The p-compact group $\mathrm{PGL}(n,\mathbf{C})$ is totally N-determined for all $n \geq 1$ and all odd primes p.*

As a consequence (5.3) of this theorem also $\mathrm{GL}(n,\mathbf{C})$, for instance, is totally N-determined so that we may conclude from (3.12(3), 3.16) that

$$\mathrm{Aut}(\mathrm{GL}(n,\mathbf{C})) = \mathrm{Aut}_{\mathbf{Z}_pW(\mathrm{GL}(n,\mathbf{C}))}(L(\mathrm{GL}(n,\mathbf{C}))) = \mathrm{Aut}_{\mathbf{Z}_p[\Sigma_n]}(\mathbf{Z}_p^n)$$

when $n > 2$.

5.2. COROLLARY. *Let X be a p-compact group whose $\mathbf{Q}_p$-Weyl group $r_0W(X)$ is in Clark–Ewing family* 1 *and assume that p is odd. Then*:

(1) *X is totally N-determined.*
(2) *X is determined by its $\mathbf{Z}_p$-Weyl group.*

(3) *For* $n > 2$,

$$\operatorname{End}(X) \cong \begin{cases} \mathbf{Z}_p, & n < p, \\ \mathbf{Z}_p^\times \cup \{0\}, & n \geq p, \end{cases}$$

while $\operatorname{End}(\operatorname{SL}(2,\mathbf{C})) = \mathbf{Z}_p/\mathbf{Z}^\times$.

(4) *If* $\pi_1(X) = 1$, *or* $Z(X) = 1$, *or* $n < p^3$, *then* X *is determined by its* $\mathbf{F}_p$*-Weyl group and* X *is a cohomologically unique p-compact group.*

Proof. This is immediate from (3.3, 3.7) and (3.17(2), 4.4, 11.18). In connection with the application of (3.17(2)), observe that the outer automorphism of the symmetric group Σ_6 [91, 2.2.18, 2.2.20] cannot be lifted to an automorphism of $N(X)$ because all such automorphisms take reflections to reflections. ∎

5.3. Corollary. *Let* X *be the p-compact group represented by*

- *the product subgroup* $\operatorname{GL}(n_1,\mathbf{C}) \times \ldots \times \operatorname{GL}(n_k,\mathbf{C})$, $n_1 + \ldots + n_k = n$, $n_i \geq 0$, *of* $\operatorname{GL}(n,\mathbf{C})$, *or*
- *the intersection of such a subgroup with* $\operatorname{SL}(n,\mathbf{C})$, *or*
- *the image of such a subgroup in* $\operatorname{PGL}(n,\mathbf{C})$.

Then X *is totally* N*-determined*, X *is determined by its* R*-Weyl group for* $R = \mathbf{Z}_p$, $\mathbf{F}_p$, *and* X *is a cohomologically unique p-compact group* (*p odd*).

Proof. That X is totally N-determined follows from (5.1) together with (3.3, 3.7). Apply (4.14, 4.4) for the other properties of X. ∎

We shall prove (5.1) by inductively verifying that $\operatorname{BPGL}(n,\mathbf{C})$ satisfies the sufficient criteria (3.4, 3.8, 3.9) for total N-determinism. For this process, it is crucial (2.12) to have information about the non-toral elementary abelian p-subgroups of $\operatorname{PGL}(n,\mathbf{C}) = \operatorname{GL}(n,\mathbf{C})/\mathbf{C}^\times$ and their centralizers. Thus we shall start out by identifying the non-toral elementary abelian p-subgroups of $\operatorname{PGL}(n,\mathbf{C})$, their Quillen automorphism groups, and their centralizers.

Non-toral elementary abelian p-subgroups of $\operatorname{PGL}(n,\mathbf{C})$ can be constructed from extra-special p-groups contained in $\operatorname{GL}(n,\mathbf{C})$ as follows: Let P be an extra-special p-subgroup (this means $[P,P] = Z(P)$ is of order p [85, 5.3]) and E an elementary abelian p-subgroup of $\operatorname{GL}(n,\mathbf{C})$ such that

$$Z(P) \subseteq Z(\operatorname{GL}(n,\mathbf{C})), \quad [P,E] = \{1\} = P \cap E,$$

where $Z(P)$ is the center of P and $Z(\operatorname{GL}(n,\mathbf{C})) = \mathbf{C}^\times$ the center of $\operatorname{GL}(n,\mathbf{C})$. Then $T = PE$ is a non-abelian subgroup of $\operatorname{GL}(n,\mathbf{C})$ that maps onto a non-trivial non-toral elementary abelian p-subgroup, V, of $\operatorname{PGL}(n,\mathbf{C})$ with kernel $Z(P)$. (V is non-toral because the pre-image of a toral subgroup of $\operatorname{PGL}(n,\mathbf{C})$ is toral in $\operatorname{GL}(n,\mathbf{C})$.)

In fact, all non-trivial non-toral elementary abelian p-subgroups of the Lie group $\mathrm{PGL}(n,\mathbf{C})$ have this form.

5.4. Lemma [41, 3.1]. *Let V be a non-trivial non-toral elementary abelian p-subgroup of* $\mathrm{PGL}(n,\mathbf{C})$*. Then*

- *p divides n, and*
- *there is an inclusion morphism of short exact sequences of groups*

$$\begin{tikzcd} 1 \arrow[r] & Z(P) \arrow[r] \arrow[d] & T \arrow[r] \arrow[d, "\varrho"] & V \arrow[r] \arrow[d, "\varrho"] & 1 \\ 1 \arrow[r] & \mathbf{C}^{\times} \arrow[r] & \mathrm{GL}(n,\mathbf{C}) \arrow[r] & \mathrm{PGL}(n,\mathbf{C}) \arrow[r] & 1 \end{tikzcd}$$

where $T = PE$ is the direct product of an extra-special p-group $P \subseteq \mathrm{GL}(n,\mathbf{C})$ and an elementary abelian p-group $E \subseteq \mathrm{GL}(n,\mathbf{C})$ such that $P \cap E = \{1\} = [P,E]$. The extra-special p-group P can be chosen to have exponent p.

Proof. If n is not divisible by p, then all elementary abelian p-subgroups of $\mathrm{PGL}(n,\mathbf{C})$ are toral. Assume now that p divides n. As $H^2(V;\mathbf{Z}/p)$ maps onto $\mathrm{Hom}(H_2(V);\mathbf{C}^{\times}) = H^2(V;\mathbf{C}^{\times})$, there is a subgroup $R \subseteq \mathrm{GL}(n,\mathbf{C})$ that maps onto V with a kernel that is cyclic of order p and central in $\mathrm{GL}(n,\mathbf{C})$. If R were abelian, then R and V would be toral subgroups.

The commutator subgroup $[R,R]$ is cyclic of order p for it is non-trivial and contained in the kernel of the epimorphism $R \to V$. Thus $V = R/[R,R]$. Let P be a normal subgroup of R such that $P/[R,R]$ is complementary to $Z(R)/[R,R]$. Then $R = PZ(R)$ and P is extra-special as $Z(P) = P \cap Z(R) = [R,R] = [PZ(R),PZ(R)] = [P,P]$.

The commutative diagram

$$\begin{tikzcd} P \times Z(R) \arrow[r] \arrow[d] & \mathrm{GL}(n,\mathbf{C}) \arrow[d] \\ P/[R,R] \times Z(R)/[R,R] \arrow[r] & \mathrm{PGL}(n,\mathbf{C}) \end{tikzcd}$$

has an adjoint diagram

$$\begin{tikzcd} Z(R) \arrow[r] \arrow[d] & C_{\mathrm{GL}(n,\mathbf{C})}(P) \arrow[r] \arrow[d] & \mathrm{GL}(n,\mathbf{C}) \arrow[d] \\ Z(R)/[R,R] \arrow[r] & C_{\mathrm{PGL}(n,\mathbf{C})}(P/[R,R])_0 \arrow[r] & \mathrm{PGL}(n,\mathbf{C}) \end{tikzcd}$$

where the horizontal maps are inclusions and the two rightmost vertical maps are epimorphisms with kernel $\mathbf{C}^{\times}$. The centralizer of P in $\mathrm{GL}(n,\mathbf{C})$ is a product of general linear groups [82, Proposition 4] and $Z(R)$ is included here as an abelian, hence toral, subgroup. Therefore, $Z(R)/[R,R]$ is included as a toral subgroup in (the identity component of) the centralizer of $P/[R,R]$

in $\mathrm{PGL}(n, \mathbf{C})$. It follows that $Z(R)/[R,R]$ is the isomorphic image of an elementary abelian p-group $E \subseteq C_{\mathrm{GL}(n,\mathbf{C})}(P) \subseteq \mathrm{GL}(n, \mathbf{C})$.

By construction, $[P,E] = \{1\} = E \cap Z(P) = E \cap P$, so P and E permute, $T = PE$ is a subgroup of $\mathrm{GL}(n, \mathbf{C})$ that maps onto V with kernel $Z(P)$. For any extra-special p-group $P_- \subseteq \mathrm{GL}(n, \mathbf{C})$ of exponent p^2 with $Z(P_-) = \sqrt[p]{1}$ there is an extra-special p-group $P_+ \subseteq \mathrm{GL}(n, \mathbf{C})$ of exponent p that has the same center, the same centralizer, and the same image in $\mathrm{PGL}(n, \mathbf{C})$ as P_- (5.19). Therefore we can assume $P = P_+$ has exponent p. ■

The commutator subgroup and the center of the covering group $T = PE$ of V are $[T,T] = [P,P] = Z(P) = \sqrt[p]{1}$ and $Z(T) = Z(P)E \supseteq [T,T]$.

By taking commutators in T we get an alternating bilinear form

$$f \colon V \times V \to [T,T] \tag{5.5}$$

on $V = T/[T,T]$, i.e. $f(u_1, u_2) = [\overline{u_1}, \overline{u_2}]$ where $\overline{u} \in T$ is a lift of $u \in V$. This bilinear form may be degenerate in that

$$V^\perp = Z(T)/[T,T] = E$$

and we obtain a non-degenerate alternating bilinear form

$$\bar{f} \colon V/V^\perp \times V/V^\perp \to [T,T] \tag{5.6}$$

by factoring out $V^\perp$ or, equivalently, by restricting to the subspace $P/[P,P] \cong V/V^\perp \cong T/Z(T)$ of V.

Define

$$\mathrm{Isom}(V,f) = \{\alpha \in \mathrm{Aut}(V) \mid f(\alpha(u_1), \alpha(u_2)) = f(u_1,u_2)\},$$
$$\mathrm{Aut}(\bar{f}) = \{(A,a) \in \mathrm{Aut}(V/V^\perp) \times \mathrm{Aut}(Z(T)) \mid \bar{f} \circ (A \times A) = a \circ \bar{f}\}$$

to be the group of all isometries of (V,f) and, respectively, the group of all pairs of automorphisms $(A,a) \in \mathrm{Aut}(V/V^\perp) \times \mathrm{Aut}(Z(T))$ that make

$$\begin{tikzcd}
V/V^\perp \times V/V^\perp \arrow[r, "\bar{f}"] \arrow[d, "A\times A"'] & {[T,T]} \arrow[r, hook] & Z(T) \arrow[d, "a"] \\
V/V^\perp \times V/V^\perp \arrow[r, "\bar{f}"'] & {[T,T]} \arrow[r, hook] & Z(T)
\end{tikzcd}$$

commutative.

Any outer automorphism α of T induces an automorphism $a(\alpha)$ of $Z(T)$ and an automorphism $A(\alpha)$ of $T/Z(T) = V/V^\perp$ such that $(A(\alpha), a(\alpha)) \in \mathrm{Aut}(\bar{f})$.

5.7. Lemma. *For odd p there is a short exact sequence*

$$1 \to \mathrm{Hom}(V/V^\perp, V^\perp) \to \mathrm{Out}(T) \xrightarrow{(A,a)} \mathrm{Aut}(\bar{f}) \to 1$$

for the outer automorphism group of T.

Proof. The 2-cocycle for the extension $Z(T) \to T \to T/Z(T) = V/V^\perp$ is c where

$$\overline{u_1} \cdot \overline{u_2} = c(u_1, u_2)\overline{u_1 u_2}, \quad u_1, u_2 \in T/Z(T),$$

where $\overline{u} \in T$ is a lift of $u \in T/Z(T)$. Since

$$\begin{aligned}\bar{f}(u_1, u_2)c(u_2, u_1) &= [\overline{u_1}, \overline{u_2}](\overline{u_2} \cdot \overline{u_1})(\overline{u_2 u_1})^{-1} = (\overline{u_1} \cdot \overline{u_2})(\overline{u_2 u_1})^{-1} \\ &= (\overline{u_1} \cdot \overline{u_2})(\overline{u_1 u_2})^{-1} = c(u_1, u_2)\end{aligned}$$

for all $u_1, u_2 \in T/Z(T)$, the 2-cochain $\bar{f}$ measures the failure of the 2-cocycle c to be symmetric.

If the pair (A, a) is in $\mathrm{Aut}(\bar{f})$, then the 2-cochain $d = (A^*c)^{-1}(a_*c)$ is symmetric, for

$$\begin{aligned}&\bar{f}(Au_1, Au_2) = a\bar{f}(u_1, u_2) \\ &\qquad \Leftrightarrow \; c(Au_1, Au_2)^{-1}ac(u_1, u_2) = c(Au_2, Au_1)^{-1}ac(u_2, u_1),\end{aligned}$$

and hence $2d = \delta q$ where q is the associated quadratic form, $q(u) = d(u, u)$, viewed as a 1-cocycle. Thus (A, a) can be lifted to an automorphism of T.

The kernel of the map $\alpha \mapsto (A(\alpha), a(\alpha))$ is easily determined as follows. There is a surjection

$$\mathrm{Hom}(T/Z(T), Z(T)) \twoheadrightarrow \ker(\mathrm{Out}(T) \to \mathrm{Aut}(\bar{f}))$$

taking the homomorphism $\varphi\colon T/Z(T) \to Z(T)$ to the automorphism $t \mapsto \varphi(t)t$ of T. This automorphism is inner precisely when $\varphi(t) = [u, t]$ for some $u \in T$. Since any homomorphism $T \to [T, T]$ is of this form, the kernel is isomorphic to

$$\begin{aligned}\mathrm{Hom}(T/Z(T), Z(T))/\mathrm{Hom}(T/Z(T), [T, T]) &\cong \mathrm{Hom}(T/Z(T), Z(T)/[T, T]) \\ &\cong \mathrm{Hom}(V/V^\perp, V^\perp). \quad \blacksquare\end{aligned}$$

For example, if P is an extra-special p-group then $\mathrm{Out}(P)$ is isomorphic to the group $\mathrm{Aut}(\bar{f})$ (when p is odd).

We shall next determine the Quillen automorphism groups of the subgroups $T \subseteq \mathrm{GL}(n, \mathbf{C})$ and $V \subseteq \mathrm{PGL}(n, \mathbf{C})$ of (5.4).

5.8. Definition. For a homomorphism $\varrho\colon H \to G$ of a (finite) group H into a Lie group G, define the *Quillen automorphism group* $\mathbf{A}(G)(H, \varrho)$ as

$$\mathbf{A}(G)(H, \varrho) = \{\alpha \in \mathrm{Out}(H) \mid (\varrho\alpha) = (\varrho)\}$$

where (ϱ) denotes the representation $(\varrho) \in \mathrm{Rep}(H, G) = \mathrm{Hom}(H, G)/G$ represented by the homomorphism ϱ.

If the target of ϱ is $G = \mathrm{GL}(n, \mathbf{C})$, in particular, then

$$\mathbf{A}(G)(H, \varrho) = \{\alpha \in \mathrm{Out}(H) \mid \mathrm{tr}(\varrho\alpha) = \mathrm{tr}(\varrho)\}$$

by complex representation theory.

5.9. LEMMA. *Let $T = PE$ and $V = T/[T,T]$ be as in* (5.4) *and assume that T has exponent p. Then the homomorphism*

$$\mathbf{A}(\mathrm{GL}(n,\mathbf{C}))(T,\varrho) \to \mathbf{A}(\mathrm{PGL}(n,\mathbf{C}))(V,\varrho)$$

is surjective.

Proof. Suppose that $B\mathbf{C}^\times$ normalizes V in $\mathrm{PGL}(n,\mathbf{C})$ for some $B \in \mathrm{GL}(n,\mathbf{C})$. Then $T^B \le T\mathbf{C}^\times$. But if $g \in T$ and $g^B = zh$ for some $z \in \mathbf{C}^\times$ and some $h \in T$, then z must have order p since g and h have order p. Thus z is an element of $\sqrt[p]{1} = [T,T] \le T$. Consequently, $T^B = T$. ∎

In the situation of (5.4), consider first the special case where E is trivial and $T = P$ is an extra-special p-group whose center is central in $\mathrm{GL}(n,\mathbf{C})$. The extra-special p-group P has $|P\colon [P,P]| = p^{2d}$ characters of degree 1 and $p-1$ irreducible characters of degree p^d (described in (5.19)). These irreducible representations of degree p^d are faithful and they are [43, V.16.14] in bijective correspondence with the non-trivial linear forms $\mu\colon Z(P) \to \mathbf{C}^\times$; the representation corresponding to μ is the representation μ^P induced from any extension of μ to a linear form on a maximal normal abelian subgroup of P. Thus the representation ϱ of P has the form

$$\varrho = \sum \mu^P + \sum \chi$$

for some non-trivial linear forms μ on $Z(P)$ and some homomorphisms $\chi\colon V \to \mathbf{C}^\times$. Since ϱ is faithful at least one μ must appear, and since ϱ takes the center of P into the center of $\mathrm{GL}(n,\mathbf{C})$, no χs can occur and exactly one μ occurs. (Observe that for non-identity $g \in Z(P)$, $\mu^P(g) \ne 1 = \chi(g)$.) Thus in fact

$$\varrho = m\mu^P, \quad p^d m = n,$$

for some non-trivial homomorphism $\mu\colon Z(P) \to \mathbf{C}^\times$. From the formula [43, V.16.14], [44, 7.5]

$$\mathrm{tr}\,\varrho(g) = \begin{cases} p^d m\mu(g), & g \in Z(P), \\ 0, & g \notin Z(P), \end{cases}$$

we see that the Quillen automorphism groups

$$\begin{aligned} \mathbf{A}(\mathrm{GL}(n,\mathbf{C}))(P,\varrho) &= \{\alpha \in \mathrm{Out}(P) \mid a(\alpha) = 1\}, \\ \mathbf{A}(\mathrm{PGL}(n,\mathbf{C}))(V,\varrho) &= \mathrm{Sp}(V) \end{aligned}$$

consist of those outer automorphisms of P that restrict to the identity on the center $Z(P)$ and (5.7, 5.9) of all isometries of the non-degenerate space (V,f), respectively. (Note also that $V = P/P \cap \mathbf{C}^\times$ is unique up to isomorphism as an object of $\mathbf{A}(\mathrm{PGL}(n,\mathbf{C}))$.)

In general, $T = PE$ is the direct product of an extra-special p-group with an elementary abelian p-group E. Since the restriction of ϱ to P is of the form $\varrho|P = m\mu^P$, as we have just seen, representation theory for products of groups [43, V.10.3], [44, 8.1] tells us that the representation $\varrho = \mu^P \sharp \chi$ is the outer tensor product of μ^P with a faithful m-dimensional representation χ of $E = V^\perp$. From the formula

$$\operatorname{tr} \varrho(g,e) = \begin{cases} p^d \mu(g)\chi(e), & g \in Z(P), \\ 0, & g \notin Z(P), \end{cases}$$

we see that the Quillen automorphism groups

$$\mathbf{A}(\mathrm{GL}(n,\mathbf{C}))(T,\varrho) = \{\alpha \in \mathrm{Out}(T) \mid a(\alpha) \in \mathbf{A}(\mathrm{GL}(n,\mathbf{C}))(Z(T),\mu\sharp\chi)\},$$
$$\mathbf{A}(\mathrm{PGL}(n,\mathbf{C}))(V,\varrho) = \{\alpha \in \mathrm{Isom}(V,f) \mid \alpha|V^\perp \in \mathbf{A}(\mathrm{PGL}(n,\mathbf{C}))(V^\perp,\chi)\}$$

consist of those outer automorphisms of T that restrict to Quillen automorphisms of the m-dimensional representation $\mu\sharp\chi$ of $Z(T) = Z(P)E$ and of those isometries of the inner product space (V,f) whose restrictions to $V^\perp$ leave the representation χ invariant, respectively.

Define $\mathbf{A}(T) \subseteq \mathrm{Out}(T)$ and $\mathbf{A}(V,f) \subseteq \mathrm{Aut}(V)$ to be the groups

$$\mathbf{A}(T) = \{\alpha \in \mathrm{Out}(T) \mid a(\alpha) = 1\},$$
$$\mathbf{A}(V,f) = \{\alpha \in \mathrm{Isom}(V,f) \mid \alpha \text{ is the identity on } V^\perp\},$$

consisting of those outer automorphisms that restrict to the identity on $Z(T)$, respectively of all isometries of (V,f) that restrict to the identity on $E = V^\perp$. Then $\mathbf{A}(T)$ is a subgroup of the Quillen automorphism group $\mathbf{A}(\mathrm{GL}(n,\mathbf{C}))(T,\varrho)$ (and equal to the latter if T is extra-special). It follows from (5.7) that $\mathbf{A}(T)$, of order $|\mathrm{Sp}(V/V^\perp)|\cdot|\mathrm{Hom}(V/V^\perp,V^\perp)|$, is isomorphic to $\mathbf{A}(V,f)$.

This proves the following lemma.

5.10. Lemma. *The Quillen automorphism group* $\mathbf{A}(\mathrm{GL}(n,\mathbf{C}))(T,\varrho)$ *contains* $\mathbf{A}(T)$ *and the Quillen automorphism group* $\mathbf{A}(\mathrm{PGL}(n,\mathbf{C}))(V,\varrho)$ *contains* $\mathbf{A}(V,f)$. *If* T *is extra-special*, $\mathbf{A}(\mathrm{PGL}(n,\mathbf{C}))(V,\varrho)$ *equals* $\mathrm{Sp}(V)$.

The final step consists in identifying the centralizers and their centers for the subgroups $T \subseteq \mathrm{GL}(n,\mathbf{C})$ and $V \subseteq \mathrm{PGL}(n,\mathbf{C})$ of (5.4). The information we need is obtained in (5.12) as an application of the more general, and elementary, (5.11).

5.11. Lemma. *Let* T *be any subgroup of* $\mathrm{GL}(n,\mathbf{C})$, $\varrho\colon T \to \mathrm{GL}(n,\mathbf{C})$ *the inclusion, and* Z *a central subgroup of* $\mathrm{GL}(n,\mathbf{C})$.

(1) *There is a short exact sequence of Lie groups*

$$1 \to C_{\mathrm{GL}(n,\mathbf{C})}(T)/Z \to C_{\mathrm{GL}(n,\mathbf{C})/Z}(T) \xrightarrow{\partial} \mathrm{Hom}(T,Z)_{(\varrho)} \to 1$$

where the group to the right is the isotropy subgroup for the action of $\mathrm{Hom}(T,Z)$ *on* $(\varrho) \in \mathrm{Rep}(T,\mathrm{GL}(n,\mathbf{C}))$ *and* $\partial(BZ)(g) = [B,g]$.

(2) *The connected component of* $C_{\mathrm{GL}(n,\mathbf{C})/Z}(T)$ *is*

$$C_{\mathrm{GL}(n,\mathbf{C})/Z}(T)_0 = C_{\mathrm{GL}(n,\mathbf{C})}(T)/Z$$

and the group of components $\pi_0(C_{\mathrm{GL}(n,\mathbf{C})/Z}(T))$ *is isomorphic to*

$$\mathrm{Hom}(T,Z)_{(\varrho)} = \{\phi\colon T \to Z \mid \exists B \in \mathrm{GL}(n,\mathbf{C})\ \forall g \in T\colon \phi(g) = [B,g]\}.$$

Proof. The exact sequence of the first point is a consequence of the short exact sequence

$$1 \to C_{\mathrm{GL}(n,\mathbf{C})}(T) \to \{B \in \mathrm{GL}(n,\mathbf{C}) \mid [B,T] \subseteq Z\} \xrightarrow{\partial} \mathrm{Hom}(T,Z)_{(\varrho)} \to 1$$

because $C_{\mathrm{GL}(n,\mathbf{C})/Z}(T)$ is the quotient of the middle group by the central subgroup Z. The second point follows from the first because the centralizer of T in $\mathrm{GL}(n,\mathbf{C})$, a product of general linear groups [82, Proposition 4], is connected. ∎

5.12. Lemma. *Let T and V be as in* (5.4).

(1) *If $T = P$ is extra-special*, *then*

$$C_{\mathrm{PGL}(p^dm,\mathbf{C})}(V) = V \times \mathrm{PGL}(m,\mathbf{C}), \quad Z(C_{\mathrm{PGL}(p^dm,\mathbf{C})}(V)) = V,$$

where the Quillen automorphism $\alpha \in \mathbf{A}(V,f) = \mathrm{Sp}(V)$ acts as $\alpha^{-1} \times 1$ and α, respectively.

(2) *If $V^\perp$ has rank one*, *then the component group of $Z(C_{\mathrm{PGL}(p^dm,\mathbf{C})}(V))$ is isomorphic to $V/V^\perp$ or to V.*

(3) *$\pi_1 Z(C_{\mathrm{PGL}(p^dm,\mathbf{C})}(V))$ is a finitely generated free abelian group with trivial $\mathbf{A}(V,f)$-action.*

Proof. In the special case where $T = P$ is extra-special, all elements ϕ of $\mathrm{Hom}(P,\mathbf{C}^\times)$ are of the form $\phi(g) = [h,g]$ for some $h \in P$. Thus all ϕ preserve the representation (ϱ) and it follows from (5.11) that the natural homomorphism

$$V \times \mathrm{PGL}(m,\mathbf{C}) = V \times C_{\mathrm{GL}(p^dm,\mathbf{C})}(P)/\mathbf{C}^\times \to C_{\mathrm{PGL}(p^dm,\mathbf{C})}(V) \tag{5.13}$$

is an isomorphism. Use (5.17) to get the action of the Quillen automorphism group.

For (2), suppose that $T = PE$ where $E = V^\perp$ is one-dimensional. Then

$$\begin{aligned} C_{\mathrm{PGL}(p^dm,\mathbf{C})}(V) &= C_{\mathrm{PGL}(p^dm,\mathbf{C})}(PE) = C_{C_{\mathrm{PGL}(p^dm,\mathbf{C})}(P)}(V^\perp) \\ &= C_{P/[P,P]\times\mathrm{PGL}(m,\mathbf{C})}(V^\perp) = P/[P,P] \times C_{\mathrm{PGL}(m,\mathbf{C})}(V^\perp) \end{aligned}$$

and consequently,

$$ZC_{\mathrm{PGL}(p^dm,\mathbf{C})}(V) = P/[P,P] \times ZC_{\mathrm{PGL}(m,\mathbf{C})}(V^\perp).$$

Here (5.14), the second factor is either connected, in which case

$$\pi_0 C_{\mathrm{PGL}(p^d m,\mathbf{C})}(V) = \pi_0 Z C_{\mathrm{PGL}(m,\mathbf{C})}(V) = P/[P,P] = V/V^{\perp},$$

or disconnected, in which case the center of $Z(C_{\mathrm{PGL}(p^d m,\mathbf{C})}(V))$ is V.

Use (5.17) for (3). ∎

5.14. Lemma. *For any elementary abelian p-group $E \subseteq \mathrm{PGL}(n,\mathbf{C})$ of rank one, either the centralizer $C_{\mathrm{PGL}(n,\mathbf{C})}(E)$ and its center $ZC_{\mathrm{PGL}(n,\mathbf{C})}(E)$ are both connected or $ZC_{\mathrm{PGL}(n,\mathbf{C})}(E) = E$.*

Proof. There is (5.11) an exact sequence

$$1 \to \mathbf{C}^{\times} \to C_{\mathrm{GL}(n,\mathbf{C})}(E) \to C_{\mathrm{PGL}(n,\mathbf{C})}(E) \to \mathrm{Hom}(E,\mathbf{C}^{\times})_{(\chi)} \to 1$$

where $\chi\colon E \to \mathrm{GL}(n,\mathbf{C})$ is a lift. The group to the right is either trivial or cyclic of order p. If it is trivial, then

$$C_{\mathrm{PGL}(n,\mathbf{C})}(E) = C_{\mathrm{GL}(n,\mathbf{C})}(E)/\mathrm{GL}(1,\mathbf{C}),$$
$$Z(C_{\mathrm{PGL}(n,\mathbf{C})}(E)) = Z(C_{\mathrm{GL}(n,\mathbf{C})}(E))/\mathrm{GL}(1,\mathbf{C})$$

are both connected Lie groups [69, 4.6]. Otherwise, $n = rp$ and $\chi = r\varrho$ is a direct sum of a number of copies of the regular representation ϱ of E. Then

$$C_{\mathrm{PGL}(n,\mathbf{C})}(E) = \mathrm{GL}(r,\mathbf{C})^p/\mathrm{GL}(1,\mathbf{C}) \rtimes \langle\sigma\rangle$$

where σ has order p and acts on $\mathrm{GL}(r,\mathbf{C})^p$ by permuting the factors cyclically. Thus the center of the centralizer,

$$Z(C_{\mathrm{PGL}(n,\mathbf{C})}(E)) = (\mathrm{GL}(1,\mathbf{C})^p/\mathrm{GL}(1,\mathbf{C}))^{\langle\sigma\rangle} = H^1(\langle\sigma\rangle\,;\mathrm{GL}(1,\mathbf{C}))$$

is cyclic of order p. ∎

The information collected so far suffices to establish the vanishing of some of the higher limits for the functors $\pi_j(BZC_{\mathrm{PGL}(n,\mathbf{C})})\colon \mathbf{A}(\mathrm{PGL}(n,\mathbf{C})) \to \mathbf{Ab}$ (2.7). We shall make use of the following lemma which, together with its application in the proof of (5.16), is due to J. Grodal.

5.15. Lemma. *Let A be a subgroup and P a parabolic subgroup of $G = \mathrm{GL}(n,\mathbf{F}_p)$ such that $U \subseteq A \subseteq P$ where $P = UL$ is the Levi decomposition* [23, §69A]*. Then*

$$\mathrm{Hom}_{\mathbf{F}_p[A]}(\mathrm{St}(G),M) = \mathrm{Hom}_{\mathbf{F}_p[A/U]}(\mathrm{St}(L),M)$$

for any $\mathbf{F}_p[A]$-module M which is trivial as an $\mathbf{F}_p[U]$-module and finite-dimensional as an $\mathbf{F}_p$-vector space.

Proof. The standard $\mathbf{F}_p[A]$-module isomorphism $\mathrm{Hom}(\mathrm{St}(G),\mathbf{F}_p) \otimes M \xrightarrow{\cong} \mathrm{Hom}(\mathrm{St}(G),M))$ restricts to an isomorphism

$$\mathrm{Hom}(\mathrm{St}(G),\mathbf{F}_p)^U \otimes M \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{F}_p[U]}(\mathrm{St}(G),M)$$

of $\mathbf{F}_p[A/U]$-modules. Since Steinberg modules are self-dual and $\mathrm{St}(G)^U = \mathrm{St}(L)$ [89, 18, 42] we have

$$\mathrm{Hom}(\mathrm{St}(G), \mathbf{F}_p)^U \cong \mathrm{St}(G)^U \cong \mathrm{St}(L) \cong \mathrm{Hom}(\mathrm{St}(L), \mathbf{F}_p)$$

as $\mathbf{F}_p[P/U]$-modules. Thus $\mathrm{Hom}_{\mathbf{F}_p[U]}(\mathrm{St}(G), M) \cong \mathrm{Hom}(\mathrm{St}(L), \mathbf{F}_p) \otimes M \cong \mathrm{Hom}(\mathrm{St}(L), M)$ as $\mathbf{F}_p[A/U]$-modules and consequently

$$\begin{aligned}\mathrm{Hom}_{\mathbf{F}_p[A]}(\mathrm{St}(G), M) &\cong \mathrm{Hom}_{\mathbf{F}_p[U]}(\mathrm{St}(G), M)^{A/U} \cong \mathrm{Hom}(\mathrm{St}(L), M)^{A/U} \\ &= \mathrm{Hom}_{\mathbf{F}_p[A/U]}(\mathrm{St}(L), M)\end{aligned}$$

as vector spaces. ∎

5.16. Lemma. *$\lim^i(\mathbf{A}(\mathrm{PGL}(n, \mathbf{C})), \pi_j(BZC_{\mathrm{PGL}(n,\mathbf{C})})) = 0$ for $j = 1, 2$ and $i = j, j+1$.*

Proof. It suffices (2.13, 5.10, 5.12) to show that the following homomorphism groups are trivial:

- $\mathrm{Hom}_{\mathrm{Sp}(V)}(\mathrm{St}(\mathrm{GL}(V)), V)$ where $\dim_{\mathbf{F}_p} V = 2$,
- $\mathrm{Hom}_{\mathbf{A}(V,f)}(\mathrm{St}(\mathrm{GL}(V)), V)$ and $\mathrm{Hom}_{\mathbf{A}(V,f)}(\mathrm{St}(\mathrm{GL}(V)), V/V^{\perp})$ where $\dim_{\mathbf{F}_p} V = 3$ and f is a non-trivial alternating bilinear form on V,
- $\mathrm{Hom}_{\mathbf{A}(V,f)}(\mathrm{St}(\mathrm{GL}(V)), \mathbf{Z}_p)$ where $\dim_{\mathbf{F}_p} V$ is 3 or 4, f is a non-trivial alternating bilinear form on V, and $\mathbf{Z}_p$ carries the trivial $\mathbf{A}(V, f)$-action.

Note that $\mathbf{Z}_p$ can be replaced by $\mathbf{F}_p$ as target module since the Steinberg module is finitely generated. The first of these groups is clearly trivial as $\mathrm{Sp}(V) = \mathrm{SL}(V)$ contains -1 which acts trivially on the Steinberg module but has no non-trivial fixed points in V. For the remaining cases, we apply (5.15). For us, n is 3 or 4, and the group A consists of the matrices

$$\begin{pmatrix} I_k & * \\ 0 & \mathrm{SL}(2) \end{pmatrix}$$

where I_k is a $k \times k$ identity matrix, $k = 1, 2$. Take P and $U = O_p(P)$ to be the subgroups of $G = \mathrm{GL}(n, \mathbf{F}_p)$ consisting of matrices of the form

$$\begin{pmatrix} \mathrm{GL}(k) & * \\ 0 & \mathrm{GL}(2) \end{pmatrix}, \quad \text{respectively,} \quad \begin{pmatrix} I_k & * \\ 0 & I_2 \end{pmatrix}$$

so that $L = \mathrm{GL}(k) \times \mathrm{GL}(2)$. Then

$$\begin{aligned}\mathrm{Hom}_{\mathbf{F}_p[A]}(\mathrm{St}(G), \mathbf{F}_p) &= \mathrm{Hom}_{\mathbf{F}_p[\mathrm{SL}(2)]}(\mathrm{St}(\mathrm{GL}(k)) \otimes \mathrm{St}(\mathrm{GL}(2)), \mathbf{F}_p) \\ &= \mathrm{Hom}_{\mathbf{F}_p[\mathrm{SL}(2)]}(\mathrm{St}(\mathrm{GL}(2)), \mathrm{Hom}_{\mathbf{F}_p}(\mathrm{St}(\mathrm{GL}(k), \mathbf{F}_p))) \\ &= \bigoplus \mathrm{Hom}_{\mathbf{F}_p[\mathrm{SL}(2)]}(\mathrm{St}(\mathrm{GL}(2)), \mathbf{F}_p)\end{aligned}$$

and, for $n = 3$,

$$\mathrm{Hom}_{\mathbf{F}_p[A]}(\mathrm{St}(G), V/V^{\perp}) = \mathrm{Hom}_{\mathbf{F}_p[\mathrm{SL}(2)]}(\mathrm{St}(\mathrm{GL}(2)), V/V^{\perp}).$$

Using the fact that $\mathrm{St}(\mathrm{GL}(2))$ is an irreducible $\mathbf{F}_p[\mathrm{SL}(2)]$-module we see that both these groups are trivial. Since $V^\perp$ is a trivial $\mathbf{F}_p[A]$-module, also $\mathrm{Hom}_{\mathbf{F}_p[A]}(\mathrm{St}(G), V)$ must be trivial. ■

Proof of Theorem 5.1. $\mathrm{PGL}(n, \mathbf{C})$ is non-modular, hence totally N-determined [66, 3.11, 7.4], for $n < p$. We may therefore, inductively, assume that all elementary abelian p-subgroups of $\mathrm{PGL}(n, \mathbf{C})$ have totally N-determined centralizers (3.3, 3.7, 5.12) [82, Proposition 4]. But then also $\mathrm{PGL}(n, \mathbf{C})$ itself has N-determined automorphisms according to (3.4, 5.16) and is N-determined according to (3.8, 5.16) provided we can verify the conditions of (3.9) when $n = pm$ is divisible by p. It only remains to consider the third condition as the first two have been verified in (5.4, 5.10).

Let $j' \colon N(\mathrm{PGL}(n, \mathbf{C})) \to X$ be the maximal torus normalizer for some p-compact group X. Let (V, ν) denote the non-toral rank 2 object of the category $\mathbf{A}(\mathrm{PGL}(n, \mathbf{C}))$, $\mu \colon V \to N(\mathrm{PGL}(n, \mathbf{C}))$ a preferred lift of $\nu \colon V \to \mathrm{PGL}(n, \mathbf{C})$, and put $\nu' = j'\mu$. The object (V, ν') of $\mathbf{A}(X)$ does not depend on the choice of μ (3.9). We must show that the diagram

$$\begin{tikzcd} C_{\mathrm{PGL}(n,\mathbf{C})}(V,\nu) \arrow[r, "f(V,\mu)"] & C_X(V,\nu') \\ C_{\mathrm{PGL}(n,\mathbf{C})}(V,\nu) \arrow[u, "C_{\mathrm{PGL}(n,\mathbf{C})}(\alpha)"] \arrow[r, "f(V,\mu)"'] & C_X(V,\nu') \arrow[u, "C_X(\alpha)"'] \end{tikzcd}$$

commutes for all $\alpha \in \mathrm{Sp}(V) = \mathrm{SL}(V)$ [43, II.9.12]. This will be the case if application of the identity component functor $(-)_0$ and the component group functor $\pi_0(-)$ gives commutative diagrams [64, 5.3], [62, 3.4, 3.10]. The first of these derived diagrams commutes because $\mathrm{SL}(V)^{\mathrm{op}}$, generated by elements of order p [91, 3.6.21], acts trivially on $C_{\mathrm{PGL}(n,\mathbf{C})}(V,\nu)_0 = \mathrm{PGL}(m, \mathbf{C}) = C_X(V,\nu')_0$ whose automorphism group (5.2) $\mathrm{Aut}(\mathrm{PGL}(m, \mathbf{C})) \cong \mathbf{Z}_p^\times$ (or $\mathbf{Z}_p^\times/\mathbf{Z}^\times$ if $m = 2$) contains no elements of order p. The above diagram also commutes on the level of π_0 for $\pi_0(f(V, \mu))$ is $\mathrm{SL}(V)^{\mathrm{op}}$-equivariant by (3.10, 5.12). ■

5.17. *The action of* $\mathbf{A}(\mathrm{GL}(n, \mathbf{C}))(T, \varrho)$ *on* $C_{\mathrm{GL}(n,\mathbf{C})}(T, \varrho)$. Write $(\varrho) = \mu_1(\varrho_1) + \ldots + \mu_t(\varrho_t)$ as a direct sum of inequivalent irreducible characters $(\varrho_1), \ldots, (\varrho_t)$ with multiplicities $\mu_1, \ldots, \mu_t$, respectively. Then

$$\begin{aligned} C_{\mathrm{GL}(n,\mathbf{C})}(T, \varrho) &= \prod_{\varrho_i \in S(\varrho)} \mathrm{GL}(\mu_i, \mathbf{C}), \\ Z(C_{\mathrm{GL}(n,\mathbf{C})}(T, \varrho)) &= \prod_{\varrho_i \in S(\varrho)} Z(\mathrm{GL}(\mu_i, \mathbf{C})) \end{aligned} \tag{5.18}$$

where $S(\varrho) = \{\varrho_1, \ldots, \varrho_t\}$ is the set of irreducible characters occurring in ϱ.

Let $\Sigma(S(\varrho))$ be the group of all permutations of $S(\varrho)$ and for a given integer-valued function β on $S(\varrho)$ write $\Sigma(S(\varrho))^\beta$ for the subgroup of permutations that preserve β. We shall need the degree function d, recording the degree of ϱ_i, and the multiplicity function $\mu(\varrho)$, recording the multiplicity μ_i of ϱ_i in ϱ. The elements of $N_{\mathrm{GL}(n,\mathbf{C})}(T,\varrho)$ are $\mathbf{C}$-linear automorphisms of $\mathbf{C}^n$ that are α-linear for some automorphism $\alpha \in \mathrm{Aut}(T)$ and there is a homomorphism

$$\begin{aligned}\mathbf{A}(\mathrm{GL}(n,\mathbf{C}))(T,\varrho) &= N_{\mathrm{GL}(n,\mathbf{C})}(T,\varrho)/TC_{\mathrm{GL}(n,\mathbf{C})}(T,\varrho)\\ &\to \Sigma(S(\varrho))^d \cap \Sigma(S(\varrho))^{\mu(\varrho)}\end{aligned}$$

since α permutes the irreducible representations (ϱ_i) in a degree and multiplicity preserving way. Now,

$$\begin{aligned}N_{\mathrm{GL}(n,\mathbf{C})}(T,\varrho) &\subseteq N_{\mathrm{GL}(n,\mathbf{C})}(C_{\mathrm{GL}(n,\mathbf{C})}(T,\varrho)) \subseteq N_{\mathrm{GL}(n,\mathbf{C})}(ZC_{\mathrm{GL}(n,\mathbf{C})}(T,\varrho))\\ &\subseteq \prod \mathrm{GL}(d_i\mu_i,\mathbf{C}) \rtimes \Sigma(S(\varrho))^{d\mu(\varrho)}\end{aligned}$$

and since the first factor of the semidirect product acts trivially on the center $ZC_{\mathrm{GL}(n,\mathbf{C})}(T,\varrho)$, the action homomorphism

$$\mathbf{A}(\mathrm{GL}(n,\mathbf{C}))(T,\varrho) \to \mathrm{Out}(C_{\mathrm{GL}(n,\mathbf{C})}(T,\varrho)) \to \mathrm{Aut}(ZC_{\mathrm{GL}(n,\mathbf{C})}(T,\varrho))$$

factors through $\Sigma(S(\varrho))^d \cap \Sigma(S(\varrho))^{\mu(\varrho)} \subseteq \Sigma(S(\varrho))^{d\mu(\varrho)}$. Note, in particular, that the subgroup $\mathbf{A}(T)$ of $\mathbf{A}(\mathrm{GL}(n,\mathbf{C}))(T,\varrho)$ acts trivially on $C_{\mathrm{GL}(n,\mathbf{C})}(T)$ because any outer automorphism $\alpha \in \mathbf{A}(T)$ restricts to the identity on $Z(T)$ so that it preserves all the irreducible components $\mu^P\sharp\chi_i$ of the representation ϱ.

5.19. *Representations of extra-special p-groups.* We construct explicitly the faithful irreducible representations of the extra-special p-groups.

Let E be an elementary abelian p-group of rank $d \geq 1$ and $\mathbf{C}[E]$ its complex group algebra, or rather, its underlying $|E|$-dimensional complex vector space. Then there is a commutative diagram as in (5.4)

$$\begin{array}{ccc} P & \xhookrightarrow{\varrho} & \mathrm{GL}(\mathbf{C}[E]) \\ \downarrow & & \downarrow \\ V & \xhookrightarrow[\varrho]{} & \mathrm{PGL}(\mathbf{C}[E]) \end{array}$$

where $V = E^\wedge \times E$ is the product of E and its dual $E^\wedge = \mathrm{Hom}(E,\mathbf{C}^\times)$, and $P = P_-, P_+$ is the subgroup of $\mathrm{GL}(\mathbf{C}[E])$,

$$P_- = \langle \omega R_\zeta, T_u\rangle\,, \quad P_+ = \langle R_\zeta, T_u\rangle\,,$$

generated by

$$R_\zeta(v) = \zeta(v)v, \quad T_u(v) = u+v, \quad \zeta \in E^\wedge,\ u,v \in E,$$

where ω is a primitive p^2th root of unity. Since the commutator

$$[\omega R_\zeta, T_u] = [R_\zeta, T_u] = \zeta(u), \quad \zeta \in E^\wedge, u \in V,$$

is scalar multiplication with the complex number $\zeta(u)$, the group P_- (resp. P_+) is extra-special of order p^{1+2d} and exponent p^2 (resp. p). A trace computation reveals that the $p-1$ faithful irreducible representations of P are obtained from the inclusion ϱ by composing with the automorphisms $(\omega)R_\zeta \mapsto (\omega)R_\zeta^i$, $T_u \mapsto T_u$, $0 < i < p$, of P. Observe that P_- and P_+ have the same center, the same centralizer in $\mathrm{GL}(\mathbf{C}[E])$, and the same image in $\mathrm{PGL}(\mathbf{C}[E])$. The same is true for P_- and P_+ considered as subgroups of $\mathrm{GL}(\mathbf{C}[E]^{\oplus m})$ by means of the representation $m\varrho$.

An object (V, ν) of $\mathbf{A}(X)$ is said to be *d-oversize* if

$$\operatorname{codim} \ker(\pi_0(\mu) \colon V \to \pi_0 N(X) = W(X)) \geq d$$

for all preferred lifts $\mu \colon V \to N(X)$ of $\nu \colon V \to X$ and $d \geq 0$ is the greatest such natural number. Thus the 0-oversize objects are the toral objects. It may be worthwhile to note that the A-family provides examples of highly oversized elementary abelian subgroups.

5.20. Proposition. *Let T and V be as in Lemma* 5.4. *If $T = PE$ where P has order p^{1+2d} then (V, ϱ) is a d-oversize object of the Quillen category $\mathbf{A}(\mathrm{PGL}(n, \mathbf{C}))$.*

Proof. We shall first consider the case where $T = P = p_+^{1+2d}$ is extra-special and ϱ is one of the irreducible and faithful representations that we have just considered. Note that P is contained in the maximal torus normalizer $N(\mathrm{GL}(n, \mathbf{C}))$ as

$$N(\mathrm{GL}(n, \mathbf{C})) = \big\langle R_\zeta, T_\sigma \,|\, \zeta \in \operatorname{map}(V, \mathbf{C}^\times),\ \sigma \in \Sigma(V) \big\rangle$$

is generated by all the operators $R_\zeta(v) = \zeta(v)v$, $T_\sigma(v) = \sigma(v)$ for all *functions* ζ from V into $\mathbf{C}^\times$ and all *permutations* σ of the elements of V. Similarly, V is contained in the maximal torus normalizer $N(\mathrm{PGL}(n, \mathbf{C})) = N(\mathrm{GL}(n, \mathbf{C}))/\mathbf{C}^\times$ of $\mathrm{PGL}(n, \mathbf{C})$. The centralizers are

$$C_{N(\mathrm{GL}(n,\mathbf{C}))}(P) = \mathbf{C}^\times, \quad C_{N(\mathrm{PGL}(n,\mathbf{C}))}(V) = V$$

so that the inclusion of V into the maximal torus normalizer is a preferred lift of the inclusion of V into $\mathrm{PGL}(n, \mathbf{C})$. For this preferred lift the intersection of V with the maximal torus has codimension d. But the intersection of V with any maximal torus of $\mathrm{PGL}(n, \mathbf{C})$ is covered by the intersection of P with a maximal torus of $\mathrm{GL}(n, \mathbf{C})$ and such a subgroup has order at most p^{1+d}, which is the order of a maximal normal abelian subgroup of the extra-special p-group P. Thus (V, ϱ) is a d-oversize rank $2d$ object of the Quillen category of $\mathrm{PGL}(n, \mathbf{C})$.

When $\varrho = m\mu^P$,

$$\begin{aligned} C_{N(\mathrm{GL}(n,\mathbf{C}))}(P) &= N(\mathrm{GL}(n,\mathbf{C})) \cap C_{\mathrm{GL}(n,\mathbf{C})}(P) \\ &= N(\mathrm{GL}(n,\mathbf{C})) \cap \mathrm{GL}(m,\mathbf{C}) = N(\mathrm{GL}(m,\mathbf{C})) \end{aligned}$$

so that the centralizer of V in $N(\mathrm{PGL}(n,\mathbf{C}))$ is $V \times N(\mathrm{PGL}(m,\mathbf{C}))$. Again, the inclusion of V into $N(\mathrm{PGL}(n,\mathbf{C}))$ is a preferred lift of the inclusion of V into $\mathrm{PGL}(n,\mathbf{C})$ and we conclude, as above, that (V,ϱ) is a d-oversize rank $2d$ object of $\mathbf{A}(\mathrm{PGL}(n,\mathbf{C}))$.

In general, $T = PE$ is the direct product of an extra-special p-group and an elementary abelian p-group. But still the inclusion of $V = P/[P,P] \times E$ into the maximal torus normalizer is a preferred lift because its adjoint

$$E \to C_{N(\mathrm{PGL}(n,\mathbf{C}))}(P/[P,P]) \to C_{\mathrm{PGL}(n,\mathbf{C})}(P/[P,P])$$

is a preferred lift as E is toral. For this preferred lift, the intersection of V with the maximal torus has codimension d and, as above, this is actually the minimum. Thus (V,ϱ) is a d-oversize rank $> 2d$ object of the Quillen category. ■

6. The 3-compact group F_4. We consider the 3-compact group $(B\mathrm{F}_4)^\wedge_3$ obtained by completing the classifying space $B\mathrm{F}_4$ for the exceptional Lie group F_4 of rank 4.

6.1. Theorem [92]. *The following hold for the* 3*-compact group* F_4:

(1) F_4 *is totally N-determined.*

(2) F_4 *is determined by its R-Weyl group for* $R = \mathbf{Z}_3$, $\mathbf{Q}_3$, $\mathbf{F}_3$.

(3) F_4 *is a cohomologically unique p-compact group.*

(4) $\mathrm{End}(\mathrm{F}_4) - \{0\} = \mathrm{Aut}(\mathrm{F}_4) = N_{\mathrm{GL}(L(\mathrm{F}_4))}(W(\mathrm{F}_4))/W(\mathrm{F}_4)$ *is an abelian group isomorphic to* $\mathbf{Z}_3^\times/\mathbf{Z}^\times \times C_2$ *where the group* C_2 *of order* 2 *is generated by an exotic automorphism.*

Proof. The information provided by Griess [41, 7.4] about elementary abelian p-subgroups of the Lie group F_4 shows that the 3-compact group F_4 satisfies the conditions of (3.8); see (3.9, 3.10) and the remark below (2.13). Combined with (4.4, 11.18, 11.25) this proves the first three items. Direct computation shows that the normalizer

$$N_{\mathrm{GL}(4,\mathbf{Z}_3)}(W(\mathrm{F}_4)) = \langle \mathbf{Z}_3^\times, \varepsilon, W(\mathrm{F}_4)\rangle$$

where

$$\sqrt{-2}\,\varepsilon = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

for the Weyl group of F_4 in $\mathrm{GL}(4,\mathbf{Z}_3)$ as described e.g. in [12]. The final item of the theorem is now a consequence of (3.17(2)). ∎

Note that (6.1) yields a new proof of the existence of Friedlander's exceptional isogeny [39].

7. Polynomial p-compact groups. All connected $\mathbf{F}_p$-local spaces with polynomial mod p cohomology are p-compact groups. We study these polynomial p-compact groups in this section. See also D. Notbohm [72, 76, 79] for further information and for references to the literature about this classical subject.

For any connected p-compact group X, the image of $H^*(BX;\mathbf{F}_p)$ in $H^*(BT(X);\mathbf{F}_p)$ is contained in the invariant ring $H^*(BT(X);\mathbf{F}_p)^{W(X)}$ for the action of the Weyl group on the cohomology of the maximal torus. Much work, summarized in the following lemma, has been done to tell when $H^*(BX;\mathbf{F}_p)$ actually equals this invariant ring.

7.1. Lemma. *Let p be an odd prime and X a p-compact group. The following conditions are equivalent*:

(1) $H^*(BX;\mathbf{F}_p)$ *is a polynomial algebra.*
(2) $H^*(BX;\mathbf{F}_p) = H^*(BT(X);\mathbf{F}_p)^{W(X)}$.
(3) $H^*(BX;\mathbf{F}_p) \subset H^*(BT(X);\mathbf{F}_p)$.
(4) $H^*(BX;\mathbf{F}_p)$ *is concentrated in even degrees.*
(5) $H^*(BX;\mathbf{Z}_p)$ *is concentrated in even degrees and degree-wise free.*
(6) $H^*(BX;\mathbf{Z}_p)$ *is polynomial on even degree generators.*
(7) $H^*(BX;\mathbf{Z}_p) = H^*(BT(X);\mathbf{Z}_p)^{W(X)}$.
(8) $H^*(BX;\mathbf{Z}_p) \subset H^*(BT(X);\mathbf{Z}_p)$.

If X satisfies these equivalent conditions, *then the rational rank r of X* [30, 5.1] *equals the Krull dimension of $H^*(BX;\mathbf{F}_p)$*, *and $|W| = \prod d_i$ where $2d_i$*, *$1 \le i \le r$*, *are the degrees of the polynomial generators* [88, 5.3.5, 5.5.4].

Proof. (1)⇒(2) is [29, 2.11] and (2)⇒(3)⇒(4)⇒(5) is elementary.

(5)⇒(1), (6), (7), (8): As was noted in [70, 4.2], $H^*(\Omega BX;\mathbf{Z}_p)$ is degree-wise free so that Borel's argument [8], [88, 10.7.5] shows that $H^*(BX;\mathbf{F}_p)$ and $H^*(BX;\mathbf{Z}_p)$ are polynomial. But then $H^*(BX;R)$ is the invariant ring for $R = \mathbf{Z}_p,\mathbf{F}_p$ by [29, 2.11] again. The implications (8)⇒(5), (7)⇒(5), (6)⇒(5) are elementary. ∎

7.2. Definition. A p-compact group X is *polynomial* if its cohomology ring $H^*(BX;\mathbf{F}_p)$ is a polynomial $\mathbf{F}_p$-algebra. A $\mathbf{Z}_p$-reflection group $(W,\check{T})$ is *polynomial* if its invariant ring $H^*(\check{T};\mathbf{F}_p)^W$ is a polynomial $\mathbf{F}_p$-algebra.

7.3. Example. A p-compact group X is non-modular if p does not divide the order of $W(X)$. A $\mathbf{Z}_p$-reflection group $(W,\check{T})$ is non-modular if p

does not divide the order of W. Any non-modular p-compact group is connected [69, 3.8] and its Weyl group is a non-modular $\mathbf{Z}_p$-reflection group. The Shephard–Todd theorem [7, 7.2.1] says that any non-modular $\mathbf{Z}_p$-reflection group $(W, \check{T})$ is polynomial, and, clearly, $H^0(W; \check{T}) = 0$ if $(W, \check{T})$ is also simple. Any non-modular p-compact group X is polynomial [66, 3.12], totally N-determined [66, 3.11, 7.7], and determined by its R-Weyl group for $R = \mathbf{Z}_p$, $\mathbf{F}_p$ (4.5, 4.4); if X is also simple, then X is centerless (3.12(2)) and determined by its $\mathbf{Q}_p$-Weyl group (11.18, 4.4).

The Weyl group of any polynomial p-compact group is a polynomial $\mathbf{Z}_p$-reflection group but not all polynomial $\mathbf{Z}_p$-reflection groups are Weyl groups of polynomial p-compact groups (7.4). If the ring of invariants $H^*(\check{T}; \mathbf{F}_p)^W$ for some $\mathbf{Z}_pW$-torus $\check{T}$ is polynomial then W is a $\mathbf{Z}_p$-reflection group [29, Proof of 5.2].

7.4. Remark. Borel [9, 2.5] shows that for a simple compact Lie group G and p an odd prime, the Bousfield $\mathbf{F}_p$-localization $(BG)^\wedge_p$ of BG [13] is a non-polynomial p-compact group $B\hat{G}$ precisely when

- $G = \mathrm{SU}(r+1)/Z$ where Z is a non-trivial central p-subgroup, or
- $G = \mathrm{F}_4$, PE_6, E_6, $\mathrm{E}_7, \mathrm{E}_8$ and $p = 3$, or
- $G = \mathrm{E}_8$ and $p = 5$.

Kemper and Malle [51] show that a simple $\mathbf{Z}_p$-reflection group $(W, \check{T})$ is non-polynomial precisely when it is the Weyl group of one of the Lie p-compact groups on Borel's list—with the exception that $(W, \check{T})(\mathrm{PU}(3))$ at $p = 3$ *is* polynomial because we are in dimension 2 [71, 5.1]. Combining this with (7.27), we see that the invariant ring $H^*(\check{T}; \mathbf{Z}_p)^W$ with $\mathbf{Z}_p$-coefficients is non-polynomial precisely when $(W, \check{T}) = (W, \check{T})(\hat{G})$ is the Weyl group of one of the Lie p-compact groups $\hat{G}$ on Borel's list. Thus the polynomial $\mathbf{Z}_p$-reflection group $(W, \check{T})(\mathrm{PU}(3))$ at $p = 3$ is not the Weyl group of a polynomial p-compact group for then also the invariant ring with $\mathbf{Z}_p$-coefficients would be polynomial (7.1). (Combine the method of (7.24) with the results of [71, §4], [51, §5] to see that $(W, \check{T})(\mathrm{SU}(r+1)/Z)$ is non-polynomial when $p \mid r+1$, $n \geq 3$, and Z is a non-trivial central p-group.)

7.5. Lemma. *Let p be an odd prime. Let $i \colon \check{T} \to X$ be a loop space homomorphism from a $\mathbf{Z}_p$-torus $\check{T}$ to a polynomial p-compact group X. If $H^*(Bi; \mathbf{F}_p)$ induces an isomorphism*

$$H^*(BX; \mathbf{F}_p) \cong H^*(\check{T}; \mathbf{F}_p)^W$$

to the ring of invariants for some finite group W of automorphisms of $\check{T}$, then $i \colon \check{T} \to X$ is a (p-discrete) maximal torus for X and W and $W(X)$ are $\mathbf{F}_p$-similar $\mathbf{Z}_p$-reflection groups.

Proof. As $H^*(Bi; \mathbf{F}_p)$ makes $H^*(\check{T}; \mathbf{F}_p)$ a finitely generated $H^*(BX; \mathbf{F}_p)$-module [88, 2.3.1], $i\colon \check{T} \to X$ is a monomorphism [30, 9.11]. Moreover, $\check{T}$ and $T(X)$ have the same rank, the Krull dimension of $H^*(BX; \mathbf{F}_p)$, so that $i\colon \check{T} \to X$ is indeed a maximal torus. Also, W and $W(X)$ have the same order given by the degrees of the polynomial generators. By Lannes theory [52], the homomorphism $t \to \check{T} \to X$ is W-equivariant up to homotopy because it is so on mod p cohomology. This means that r_pW is contained in the Quillen automorphism group $\mathbf{A}(X)(t)$ of $t \to X$ which is $r_pW(X)$ (2.8). But these two groups have the same order, so they must be identical. ∎

If X is polynomial, then X is connected and, by Lannes theory [52], any monomorphism of a non-trivial elementary abelian p-group into X factors through the maximal torus and hence (2.8) the Quillen category $\mathbf{A}(X)$ is equivalent to $\mathbf{A}(W(X), t(X))$. About the centric [28] functor (2.5)

$$BC_X\colon \mathbf{A}(W(X), t(X))_{\mathrm{op}} \to [\mathbf{pcg}]$$

we know that, for an odd prime p,

(1) $H^*(BC_X; \mathbf{Z}_p) = H^*(\check{T}(X); \mathbf{Z}_p)_0$,
(2) $\pi_j(BZC_X) = L(X)_{2-j}$, $j = 1, 2$.

The formula in item (1) is a consequence of [33, 1.2] and (7.1, 2.10) showing that polynomiality is preserved under taking centralizers of elementary abelian subgroups. The formula in item (2) follows from (2.8). (Recall that $H^*(\check{T}(X); \mathbf{Z}_p)_0$ is (2.2) the functor given by

$$H^*(\check{T}(X); \mathbf{Z}_p)_0(E) = H^*(\check{T}(X); \mathbf{Z}_p)^{W(X)(E)} = H^*(BT(X); \mathbf{Z}_p)^{W(X)(E)}$$

and, similarly (2.3), $L(X)_{2-j}$ is the functor given by

$$L(X)_{2-j}(E) = H^{2-j}(W(X)(E); L(X))$$

for all non-trivial subgroups E of $t(X)$.)

Combined with the acyclicity result of (2.4) this leads to a very simple proof of the homology decomposition for polynomial p-compact groups.

7.6. Proposition [45, 31]. *Let p be an odd prime. For any polynomial p-compact group X, the evaluation map*

$$\operatorname{hocolim}_{\mathbf{A}(W(X),t(X))^{\mathrm{op}}} BC_X \to BX$$

is an $H^*\mathbf{Z}_p$-equivalence.

Alternatively, the full subcategory (2.14) $\mathbf{A}_{\mathcal{C}(p)}(W(X), t(X))^{\mathrm{op}}$ *based on the collection $\mathcal{C}(p)$ of all p-subgroups of W can be used for index category.*

Proof. By (2.4, 2.16) and one of the formulas above, the E_2-page of the Bousfield–Kan spectral sequence for the cohomology of a homotopy colimit collapses onto the vertical axis and therefore the evaluation map is an $H^*\mathbf{Z}_p$-equivalence. ∎

In particular, if X is a polynomial p-compact group and p does not divide the order of the Weyl group (i.e. X is a non-modular p-compact group (7.3)) then BX is $H^*\mathbf{Z}_p$-equivalent to the homotopy colimit of a diagram of the form

$$BT(X) \circlearrowright W(X)^{\mathrm{op}}$$

i.e. to $BN(X)$; this is the case treated by Clark–Ewing [20]. If p divides the order of the Weyl group exactly once, then BX is $H^*\mathbf{Z}_p$-equivalent to the homotopy colimit of a diagram of the form

$$\overline{W}(X)(t^S)^{\mathrm{op}}/W(X)(t^S)^{\mathrm{op}} \circlearrowright BC_X(t^S) \xleftarrow{W(X)(t^S)^{\mathrm{op}}\backslash W(X)^{\mathrm{op}}} BT(X) \circlearrowright W(X)^{\mathrm{op}}$$

with just two nodes; this is the case treated by Aguadé [2]. In general, BX is $H^*\mathbf{Z}_p$-equivalent to the homotopy colimit of a diagram with nodes in one-to-one correspondence with the subgroups of the Sylow p-subgroup of $W(X)$. (The objects t^P, for P a subgroup of $\mathrm{Syl}_p\, W(X)$, generate a skeletal subcategory of $\mathbf{A}_{\mathcal{C}(p)}(W(X), t(X))$.)

The decomposition (7.6) is usually only helpful when X is centerless. (Any simple p-compact group X for which $r_0W(X)$ is not in family 1 of the Clark–Ewing list and not equal to $r_0W(\mathrm{E}_6)$ if $p = 3$, is centerless (3.12(2), 11.18).)

Conversely, given a finite group W of automorphisms of a $\mathbf{Z}_p$-torus $\check{T}$ such that $H^0(W;\check{T}) = 0$ and the ring of invariants $H^*(\check{T};\mathbf{F}_p)^W$ is polynomial, does there exist a polynomial p-compact group $X(W)$ with $\mathbf{Z}_p$-reflection subgroup $(W,\check{T})$ and with mod p cohomology isomorphic to this invariant ring? Note that if $X(W)$ exists, then the Quillen category $\mathbf{A}(X(W)) = \mathbf{A}(W,t)$, the maximal torus normalizer $\check{N}(X(W)) = \check{T} \rtimes W$, and the functor $\check{N} \circ C_{X(W)}$, giving the maximal torus normalizers of the centralizers, is the functor $\check{N}\colon \mathbf{A}(W,t)^{\mathrm{op}} \to [\mathbf{Grp}]$ given by

$$\check{N}(E_0 \xrightarrow{wW(E_0)} E_1) = (\check{T} \rtimes W(E_0) \xleftarrow{(w^{-1},c(w^{-1}))} \check{T} \rtimes W(E_1))$$

according to the considerations of the proof of (2.8). This means that if $BX(E)$ denotes the value of $BC_{X(W)}$ on $E \subseteq t$ then there must exist homotopy commutative diagrams

(7.7)
$$\begin{tikzcd} B\check{N}(E_0) \arrow[d, "Bj(E_0)"'] & B\check{N}(E_1) \arrow[l, "\check{N}(wW(E_0))"'] \arrow[d, "Bj(E_1)"] \\ BX(E_0) & BX(E_1) \arrow[l, "X(wW(E_0))"] \end{tikzcd}$$

where the vertical arrows are (discrete) maximal torus normalizers.

7.8. THEOREM (Generalized Clark–Ewing construction). *Let p be an odd prime and $(W,\check{T})$ a polynomial $\mathbf{Z}_p$-reflection group with $H^0(W;\check{T})=0$. Suppose that there exist a centric functor* [28] $BX\colon \mathbf{A}(W,t)^{\mathrm{op}} \to [\mathbf{pcg}]$ *and a natural transformation $Bj\colon B\check{N} \to BX$ such that, for each non-trivial subgroup E of t, $BX(E)$ is a polynomial p-compact group and $Bj(E) : B\check{N}(E) \to BX(E)$ is a p-discrete maximal torus normalizer. Then BX determines an essentially unique functor $BX\colon \mathbf{A}(W,t)^{\mathrm{op}} \to \mathbf{Top}$, and $H^*(BX(W);\mathbf{F}_p) \cong H^*(\check{T};\mathbf{F}_p)^W$ as unstable algebras where*

$$BX(W) = (\operatorname{hocolim}_{\mathbf{A}(W,t)^{\mathrm{op}}} BX)^{\wedge}_p$$

is the $\mathbf{F}_p$-localization of the homotopy colimit. $X(W)$ is a centerless polynomial p-compact group whose Weyl group is $\mathbf{F}_p$-similar to W. If all values of the functor BX are totally N-determined p-compact groups, then also $X(W)$ is totally N-determined.

Alternatively, the full subcategory (2.14) $\mathbf{A}_{\mathcal{C}(p)}(W,t)$ *based on the collection $\mathcal{C}(p)$ of all p-subgroups of W can be used for index category.*

Proof. For any non-trivial subgroup E of t, the p-compact group $BX(E)$ has p-discrete center $\check{Z}(X(E)) = Z(\check{N}(E)) = \check{T}^{W(E)}$ meaning (3.17(1)) that $(\pi_j BZX)(E) = H^{2-j}(W(E);L(\check{T})) = L(\check{T})_{2-j}(E)$ for $j=1,2$. Since these functors are acyclic (2.4), [28, 1.1] tells us that BX lifts, essentially uniquely, to a functor taking values in the category of topological spaces. Let $BX(W)$ be the ($\mathbf{F}_p$-localization of the) homotopy colimit. The polynomial p-compact group $BX(E)$ has cohomology $H^*(BX(E);R) = H^*(\check{T};R)^{W(E)} = H^*(\check{T};R)_0(E)$, $R=\mathbf{F}_p,\mathbf{Z}_p$. Since this functor is acyclic (2.4), the Bousfield–Kan spectral sequence for the cohomology of a homotopy colimit [14, XII.4.5] collapses onto the vertical axis giving the cohomology of $BX(W)$ and so $H^*(BX(W);\mathbf{F}_p) = H^*(\check{T};\mathbf{F}_p)^W$. As this invariant ring is assumed to be polynomial, $X(W)$ is indeed a polynomial p-compact group. The p-compact group morphism $T(\mathrm{GL}(n,\mathbf{C})) = C_{\mathrm{GL}(n,\mathbf{C})}(t) \to X(W)$ is a maximal torus and $r_p(W) = r_pW(X(W))$ by (7.5). According to [66, 4.9] and (2.11, 3.8), $X(W)$ is totally N-determined provided all values of the functor BX are totally N-determined p-compact groups.

We may replace the index category $\mathbf{A}(W,t)$ by any of its full subcategories $\mathbf{I}$ as long as $\lim^{1+j}(\mathbf{I};L(\check{T})_{2-j}) = 0 = \lim^{2+j}(\mathbf{I};L(\check{T})_{2-j})$, $j=1,2$, and $H^*(B\check{T};\mathbf{Z}_p)_0$ is acyclic on $\mathbf{I}$ with $\lim^0$ equal to the invariant ring. For instance, $\mathbf{I} = \mathbf{A}_{\mathcal{C}(p)}(W,t)$, where $\mathcal{C}(p)$ is the collection of all p-subgroups of W is a possibility (2.16). ■

In particular, if p divides the order of W exactly once, we may use the full subcategory $\mathbf{A}(W,t)\{t,t^S\} = \mathbf{I}(W,W(t^S))$ (13.10) generated by the two objects t and t^S where $S = \mathrm{Syl}_p W$ is a Sylow p-subgroup of W.

The $\mathbf{Q}_p$-Weyl group $r_0W(X)$ (4.3) of a connected p-compact group X is a reflection subgroup of $\mathrm{Aut}(L(X)\otimes\mathbf{Q}_p)$ [30, 9.7]. If X is simple in the sense that this Weyl group is an irreducible reflection group then $r_0W(X)$ must occur in the Clark–Ewing classification table [20]. The irreducible reflection groups of this table are divided into four infinite families, denoted 1, 2a, 2b, and 3, and 34 sporadic reflection groups $G_4,\dots,G_{37}$.

7.9. Theorem. *Let p be an odd prime and X a simple p-compact group with Weyl reflection group $(W(X),L(X))$. Assume that*

- $r_0W(X)$ *is not in family* 1,
- *if* $p=3$, *then* $(r_0W(X))\neq(r_0W(\mathrm{F}_4))$, $(r_0W(\mathrm{E}_6))$, $(r_0W(\mathrm{E}_7))$, $(r_0W(\mathrm{E}_8))$, *and*
- *if* $p=5$, *then* $(r_0W(X))\neq(r_0W(\mathrm{E}_8))$.

Then:

(1) *X is a centerless, simply connected, totally N-determined, polynomial p-compact group.*
(2) *X is determined by its R-Weyl group for* $R=\mathbf{Z}_p$, $\mathbf{Q}_p$, $\mathbf{F}_p$.
(3) *X is a cohomologically unique p-compact group.*
(4) $\mathrm{End}(X)$ *is given by* (3.17(2)).

Proof. A glance at the Clark–Ewing classification table [20] (as presented e.g. in [5, Table 1]) reveals that X is either a non-modular p-compact group, which certainly has the stated properties (7.3), or one of the modular p-compact groups treated in (7.10) in which case we apply (7.8, 5.3) together with (4.4, 11.18, 11.25(3)). ∎

7.10. *Construction of modular, centerless, polynomial, simple p-compact groups.* We apply (7.8) to construct polynomial p-compact groups $X(G)$ where $G\subset\mathrm{GL}(r,\mathbf{Q}_p)$ is either

- in family 2a,
- $r_0W(\mathrm{G}_2)$ at $p=3$ from family 2b,
- one of the groups of Aguadé [2, Table 1], or
- $r_0W(\mathrm{E}_6)$ at $p=5$.

There is no ambiguity in pretending that G be a subgroup of $\mathrm{Aut}(\check{T})=\mathrm{GL}(r,\mathbf{Z}_p)$ since $G_0(G)=*$ in each of these cases (11.18). The rings of invariants $H^*(\check{T};R)^G$, $R=\mathbf{F}_p$, $\mathbf{Z}_p$, are polynomial rings (7.4), and from [5, 3.4] we know that $H^0(G;\check{T})=0$. Thus it suffices to find a functor BX that satisfies the conditions of (7.8).

Family 2a (cf. [76]). Let p be an odd prime and $r\geq 1$, $m\geq 2$, $n\geq 2$ natural numbers such that $r\mid m\mid p-1$. Let $C_m\subseteq\mathbf{Z}_p^\times$ be the order m cyclic subgroup of the p-adic units. Define $G(m,r,n)=A(m,r,n)\Sigma_n$ as the subgroup of $\mathrm{GL}(n,\mathbf{Z}_p)=\mathrm{Aut}(\check{T})$, $\check{T}=\check{T}(\mathrm{U}(n))$, generated by the group $W(\mathrm{U}(n))=$

Σ_n of monomial matrices and the abelian group $A(m,r,n)$ of diagonal matrices with entries in C_m and determinant in the index r subgroup of C_m. (For instance, $G(2,1,n) = W(\mathrm{SO}(2n+1))$ and $G(2,2,n) = W(\mathrm{SO}(2n))$.) The subgroup Σ_n normalizes $A(m,r,n)$ and $G(m,r,n) = A(m,r,n) \rtimes \Sigma_n$ is in fact the semidirect product of the two groups. The ring of invariants [71, 2.4], [88, §7.4, Example 1]

$$H^*(\check{T};\mathbf{Z}_p)^{G(m,r,n)} = \mathbf{Z}_p[y_1,\dots,y_{n-1},e], \qquad |y_i| = 2im,\ |e| = 2\frac{m}{r}n,$$

is generated by $e = (x_1 \dots x_n)^{m/r}$ together with the $n-1$ first elementary symmetric polynomials $y_i = \sigma_i(x_1^m,\dots,x_n^m)$, $1 \le i \le n-1$, in the mth powers of the coordinate functions $x_i\colon H^2(\check{T};\mathbf{Z}_p) \to \mathbf{Z}_p$, $1 \le i \le n$, which are considered as having degree 2.

Define $\mathbf{A}_{\mathcal{C}(p)}(G,t)$ where $G = G(m,r,n)$ or $G = \Sigma_n$, $t = t(\mathrm{U}(n))$, to be the full subcategory of $\mathbf{A}(G,t)$ generated by all objects of the form $E = t^P$ for $P \subseteq \mathrm{Syl}_p\,\Sigma_n = \mathrm{Syl}_p\,G(m,r,n)$ a subgroup of a Sylow p-subgroup of Σ_n (which is also a Sylow p-subgroup of $G(m,r,n)$). These two small categories have by definition the same set of objects E, with the same pointwise stabilizer subgroups $G(m,r,n)(E) = \Sigma_n(E)$, and for the morphism sets (2.1) we note that

$$\overline{G}(m,r,n)(E_0,E_1) = A(m,r,n)^{\Sigma_n(E_1)} \times \overline{\Sigma}_n(E_0,E_1),$$

meaning that any morphism $(a,\sigma)\colon E_0 \to E_1$ in $\mathbf{A}_{\mathcal{C}(p)}(G(m,r,n),t)(E_0,E_1)$ factors uniquely as a morphism $\sigma\colon E_0 \to E_1$ in $\mathbf{A}_{\mathcal{C}(p)}(\Sigma_n,t)(E_0,E_1)$ followed by multiplication $a\colon E_1 \to E_1$ by a diagonal matrix $a \in A(m,r,n)^{\Sigma_n(E_1)} = A(m,r,n)_0(E_1)$. To see this, it is convenient to observe that all objects $E = t^P$ of $\mathbf{A}_{\mathcal{C}(p)}(G,t)$ are of the special form

$$E = \{(x_1,\dots,x_n) \in \mathbf{F}_p^n \mid x_i = x_j \text{ iff } i \text{ and } j \text{ are } \mathbf{n}(E)\text{-equivalent}\}$$

for some partition $\mathbf{n}(E)$ of $\mathbf{n} = \{1,\dots,n\}$ into disjoint subsets. (Thus $\mathbf{A}_{\mathcal{C}(p)}(G(m,r,n),t)$ can be viewed as the Grothendieck construction on the functor $A(m,r,n)_0$ from $\mathbf{A}_{\mathcal{C}(p)}(\Sigma_n,t)$ to categories with one object.)

We now define the functor $BX\colon \mathbf{A}_{\mathcal{C}(p)}(G(m,r,n),t)^{\mathrm{op}} \to [\mathbf{pcg}]$ which shall serve as input for the generalized Clark–Ewing construction (7.8). On *objects* $E = t^P \subseteq t = t(\mathrm{U}(n))$ we are forced to put $BX(E) = BC_{\mathrm{U}(n)}(E)$ for the pointwise stabilizer group $G(m,r,n)(E) = \Sigma_n(E) = W(C_{\mathrm{U}(n)}(E))$ and $C_{\mathrm{U}(n)}(E)$, a product of unitary groups [82, Proposition 4], is determined by its $\mathbf{Z}_p$-Weyl group (5.3). For each *morphism* $E_0 \xrightarrow{\sigma} E_1 \xrightarrow{a} E_1$ in $\mathbf{A}_{\mathcal{C}(p)}(G(m,r,n),t)$ we are required (7.7) to fill in the commutative diagram

$$\begin{tikzcd} \check{T} \rtimes \Sigma_n(E_0) \arrow[d] & \check{T} \rtimes \Sigma_n(E_1) \arrow[l, "{(1,c(\sigma^{-1}))}"'] \arrow[d] & \check{T} \rtimes \Sigma_n(E_1) \arrow[l, "{(a^{-1},1)}"'] \arrow[d] \\ C_{\mathrm{U}(n)}(E_0) & C_{\mathrm{U}(n)}(E_1) \arrow[l, dashed] & C_{\mathrm{U}(n)}(E_1) \arrow[l, dashed] \end{tikzcd}$$

of p-compact groups with discrete maximal torus normalizers. To the left we may put the value $BC_{\mathrm{U}(n)}(\sigma)\colon BC_{\mathrm{U}(n)}(E_0) \leftarrow BC_{\mathrm{U}(n)}(E_1)$ on $E_0 \xrightarrow{\sigma} E_1$ of the functor $BC_{\mathrm{U}(n)}\colon \mathbf{A}_{\mathcal{C}(p)}(\Sigma_n, t) \to [\mathbf{pcg}]$. To the right there is just one possibility, denoted $\psi^{a^{-1}}$, for $C_{\mathrm{U}(n)}(E_1)$ have N-determined automorphisms (5.3). This prompts us to declare

$$\begin{aligned} &BX(E_0 \xrightarrow{\sigma} E_1 \xrightarrow[\cong]{a} E_1) \\ &\qquad = BC_{\mathrm{U}(n)}(E_0) \xleftarrow{BC_{\mathrm{U}(n)}(\sigma)} BC_{\mathrm{U}(n)}(E_1) \xleftarrow[\cong]{B\psi^{a^{-1}}} BC_{\mathrm{U}(n)}(E_1) \end{aligned}$$

However, for this to be a valid definition of a functor we need to verify that the relation $\tau \circ a = \tau a \tau^{-1} \circ \tau$, $a \in \Sigma_n(E_0)$, $\tau \in \overline{\Sigma}_n(E_0, E_1)$, which holds in $\mathbf{A}_{\mathcal{C}(p)}(G(m,r,n),t)$, also holds in $[\mathbf{pcg}]$, i.e. that the diagram

$$\begin{tikzcd} C_{\mathrm{U}(n)}(E_0) & C_{\mathrm{U}(n)}(E_0) \arrow[l, "\psi^{a^{-1}}"'] \\ C_{\mathrm{U}(n)}(E_1) \arrow[u, "C_{\mathrm{U}(n)}(\tau)"] & C_{\mathrm{U}(n)}(E_1) \arrow[l, "\psi^{\tau a^{-1}\tau^{-1}}"] \arrow[u, "C_{\mathrm{U}(n)}(\tau)"'] \end{tikzcd}$$

commutes in $[\mathbf{pcg}]$. This is not difficult as $\psi^{\tau a^{-1}\tau^{-1}}$ and the isomorphism induced by $\psi^{a^{-1}}$ on $C_{\mathrm{U}(n)}(E_1)$ have the same effect on the maximal torus normalizer, so are identical. Thus the above definition indeed makes BX into a functor. BX is clearly a centric functor because $BC_{\mathrm{U}(n)}$ is, and we conclude from (7.8) that there exists a centerless, polynomial, totally N-determined p-compact group $XG(m,r,n)$ with reflection subgroup $\mathbf{F}_p$-similar, and hence even $\mathbf{Z}_p$-similar (11.25), to $(G(m,r,n),\check{T})$.

For future reference, we now compute the centralizer of an arbitrary non-trivial subgroup E of $t = t(XG(m,r,n))$. Suppose that E has rank $r > 0$. Choose an $n \times r$ matrix B whose columns form a basis for E. Declare i and j to be equivalent if the ith and jth rows in B are C_m-multiples of each other, $1 \le i,j \le n$. Let $\mathbf{n}(E)$ denote the partition of $\mathbf{n} = \{1,\dots,n\}$ into equivalence classes. If there is a zero-row in B, call the corresponding equivalence class the *null-class*. Suppose that the null-class contains $u_0 \ge 0$ elements and that there are $s \ge 1$ more classes containing $u_1,\dots,u_s$ elements, respectively.

The following lemma, describing the pointwise stabilizer subgroup $G(m,r,n)(E)$, implies that the equivalence relation $\mathbf{n}(E)$ does not depend on the choice of basis.

7.11. Lemma. *The pointwise stabilizer $G(m,r,n)(E)$ of E is isomorphic to the subgroup*

$$G(m,r,u_0)\times\Sigma_{u_1}\times\ldots\times\Sigma_{u_s}$$

where the reflection subgroup $(\Sigma_{u_j},\mathbf{Z}_p^{u_j})$ is similar to $(W,L)(\mathrm{U}(u_j))$, $1\le j\le s$.

Proof. The element $(a,\sigma)\in A(m,r,n)\rtimes\Sigma_n$ stabilizes E pointwise if and only if $a_iB_{\sigma(i)}=B_i$ where B_i, $i=1,\ldots,n$, are the rows of the matrix B. This implies that the permutation σ of the n rows of B must respect the C_m-equivalence classes. Therefore the group homomorphism

$$\begin{gathered}G(m,r,u_0)\times\Sigma_{u_1}\times\ldots\times\Sigma_{u_t}\to G(m,r,n)(E),\\ ((b,\tau),\sigma_1,\ldots,\sigma_s)\mapsto((b,a_1(\sigma_1),\ldots,a_j(\sigma_j)),\tau\sigma_1\ldots\sigma_s),\end{gathered}$$

where $a_j(\sigma_j)_iB_{\sigma_j(i)}=B_i$, $1\le j\le s$, is an isomorphism. Observe in this connection that the product $\prod a_j(\sigma_j)_i=1$; indeed, for fixed j, the product over all $a_j(\sigma_j)_i$, where i runs through the elements of a cycle in the decomposition of σ_j, equals 1. Conjugate this action of Σ_{u_j} by the diagonal matrix consisting of the first non-zero entries in the rows of B to obtain the standard permutation action. ∎

If we take existence for granted, referring to [76], then the above lemma and (3.8) would suffice to show inductively that $XG(m,r,n)$ is totally N-determined.

The 3-*compact group* G_2. Take BX to be the functor on the 2-object category $\mathbf{A}(G,t)\{t^S,t\}^{\mathrm{op}}=\mathbf{I}(G,W(\mathrm{SU}(3)))^{\mathrm{op}}$, $G=W(\mathrm{G}_2)=W(\mathrm{SU}(3))\times Z(G)$, $Z(G)=\{\pm1\}$, indicated by the diagram

$$Z(G)^{\mathrm{op}}\circlearrowright\mathrm{BSU}(3)\xleftarrow{W(\mathrm{SU}(3))^{\mathrm{op}}\backslash G^{\mathrm{op}}}BT\circlearrowleft G^{\mathrm{op}}$$

where $Z(G)$ acts on $\mathrm{BSU}(3)$ via the unstable Adams operations $\psi^{\pm1}$.

The Aguadé groups. These are the reflection groups

$$\begin{gathered}(G_{12},p=3),(G_{29},p=5),(G_{31},p=5),(G_{34},p=7),\\ (G_{36},p=5),(G_{36},p=7),(G_{37},p=7)\end{gathered}$$

where the index refers to their numbering on the Clark–Ewing list [20]. Since p divides the order of the Weyl group only once, it suffices to specify the functor BX on the full subcategory $\mathbf{A}(G,t)\{t^S,t\}^{\mathrm{op}}=\mathbf{I}(G,G(t^S))^{\mathrm{op}}=\mathbf{I}(G,W(\mathrm{SU}(r+1)))^{\mathrm{op}}$ (13.7.6) where r denotes the rank. Take BX to be the functor indicated by the diagram

$$\text{(7.12)}\qquad Z(G)^{\mathrm{op}} \circlearrowright \mathrm{BSU}(r+1) \xleftarrow{W(\mathrm{SU}(r+1))^{\mathrm{op}}\backslash G^{\mathrm{op}}} BT \circlearrowleft G^{\mathrm{op}}$$

where $Z(G)$, which is cyclic of order $2,4,4,6,2,2,2$, acts on $\mathrm{BSU}(r+1)$ via unstable Adams operations. See [4] for more details. (This follows Aguadé's original construction very closely.)

The 5*-compact group* E_6. Take BX to be the functor on $\mathbf{A}(G,t)\{t^S,t\}^{\mathrm{op}} = \mathbf{I}(G,G(t^S))^{\mathrm{op}}$, $G=W(E_6)$, indicated by the diagram

$$C_2^{\mathrm{op}} \circlearrowright \mathrm{BU}(5)\times\mathrm{BU}(1) \xleftarrow{W(\mathrm{U}(5)\times\mathrm{U}(1))^{\mathrm{op}}\backslash G^{\mathrm{op}}} BT \circlearrowleft G^{\mathrm{op}}$$

where C_2 acts on $\mathrm{U}(5)\times\mathrm{U}(1)$ in some way.

7.13. *Automorphisms of* $X(G(m,r,n))$ [21, (2.13)], [76, §7]. Assume first that $A(m,r,n)$ is a characteristic subgroup of $G(m,r,n)$. Then

$$N_{\mathrm{GL}(n,\mathbf{Z}_p)}(G(m,r,n)) = \mathbf{Z}_p^\times G(m,1,n)$$

because this normalizer is contained in the normalizer of $A(m,r,n)$ which equals $\mathbf{Z}_p^\times \wr \Sigma_n$ by the argument of [82, Lemma 3], and, on the other hand, a diagonal matrix $\mathrm{diag}(u_1,\dots,u_n)\in(\mathbf{Z}_p^\times)^n$ normalizes $G(m,r,n)$ if and only if it lies in $\mathbf{Z}_p^\times A(m,1,n)$. Thus (3.12(3)),

$$\mathrm{Out}(X(G(m,r,n))) \cong \mathbf{Z}_p^\times G(m,1,n)/G(m,r,n) \cong \mathbf{Z}_p^\times A(m,1,n)/A(m,r,n)$$

is an abelian group and the exact sequence (3.16) has the form

$$\text{(7.14)}\qquad 1\to \mathbf{Z}_p^\times/ZG(m,r,n) \to \mathbf{Z}_p^\times G(m,1,n)/G(m,r,n) \to C_{(r,n)} \to 1$$

where $ZG(m,r,n)$, the center of $G(m,r,n)$, is cyclic of order $\frac{m}{r}(r,n)$ and $C_{(r,n)}$ denotes a cyclic group of order the greatest common divisor (r,n) of r and n.

Choose a primitive $(p-1)$th root of unity $\zeta\in\mathbf{Z}_p^\times$, choose integers s and t with $(r,n)=sr+tn$, and put $\varepsilon=\mathrm{diag}(\zeta^{(p-1)/m},1,\dots,1)\in\mathbf{Z}_p^\times A(m,1,n)$. Then $\varepsilon A(m,r,n)$ projects onto a generator of the cyclic group $C_{(r,n)}$ and the element

$$\zeta^{\frac{p-1}{m}t}\langle\zeta^{(r,n)},\zeta^{\frac{p-1}{m}\frac{r}{(r,n)}}\rangle \in \langle\zeta\rangle/\langle\zeta^{(r,n)},\zeta^{\frac{p-1}{m}\frac{r}{(r,n)}}\rangle \le H^2(C_{(r,n)};\mathbf{Z}_p^\times/ZG(m,r,n))$$

classifies extension (7.14) because $\zeta^{(p-1)t/m}A(m,r,n) = \varepsilon^{(r,n)}A(m,r,n)$. Consequently,

$$\begin{aligned}\text{(7.14) splits} &\Leftrightarrow \zeta^{\frac{p-1}{m}t}\in\langle\zeta^{(r,n)},\zeta^{\frac{p-1}{m}\frac{r}{(r,n)}}\rangle \Leftrightarrow \frac{p-1}{m}t\in\mathbf{Z}_{((r,n),\frac{p-1}{m}\frac{r}{(r,n)})}\\ &\Leftrightarrow \left((r,n),\frac{p-1}{m}\frac{r}{(r,n)}\right)\Big|\frac{p-1}{m}t \Leftrightarrow (r,n)\Big|\frac{p-1}{m} \text{ in } \mathbf{Z}_{(\frac{r}{(r,n)})}\end{aligned}$$

where at the final stage we observe that $\frac{r}{(r,n)}$ and t are relatively prime since $1 = s\frac{r}{(r,n)} + t\frac{n}{(r,n)}$. For instance, (7.14) splits whenever $(r,n) = (r,n^2)$ for then (r,n) and $\frac{r}{(r,n)}$ are relatively prime so that

$$\left((r,n), \frac{p-1}{m}\frac{r}{(r,n)}\right) = \left((r,n), \frac{p-1}{m}\right)$$

clearly divides $\frac{p-1}{m}t$. More generally,

$$\frac{(p-1)(r,n)}{\max_{x\in\mathbf{Z}}\left\{\left(p-1, \frac{p-1}{m}t + x\left((r,n), \frac{p-1}{m}\frac{r}{(r,n)}\right)\right)\right\}}$$

is the smallest possible order of an exotic automorphism of $X(G(m,r,n))$ projecting onto a generator of $C_{(r,n)}$.

7.15. Lemma [76, §6]. *$A(m,r,n)$ is characteristic in $G(m,r,n)$ if and only if $(m,r,n) \notin \{(2,1,2),(4,2,2),(3,3,3),(2,2,4)\}$.*

Proof. For $n > 4$, $A(m,r,n)$ is the Fitting subgroup of $G(m,r,n)$. (Consult e.g. [85] for general group-theoretic information.) For $2 \le n \le 4$, $\mathrm{Fit}(G(m,r,n)) = A(m,r,n) \rtimes F$ where F is a subgroup of $\mathrm{Fit}(\Sigma_n)$ which is elementary abelian of order n. If $A(m,r,n)$ is not characteristic in $G(m,r,n)$, it is not characteristic in $\mathrm{Fit}(G(m,r,n))$ and then (7.16)

- $n = 2$: $\frac{m}{r}m \mid 2\frac{m}{r}(r,2)$ so that $(m,r) = (2,1),(2,2),(4,2)$ or $(4,4)$,
- $n = 3$: $\frac{m}{r}m^2 \mid 3\frac{m}{r}(r,3)$ so that $(m,r) = (3,3)$,
- $n = 4$: $\frac{m}{r}m^3 \mid 4m\frac{m}{r}(r,2)$ so that $(m,r) = (2,1)$ or $(2,2)$.

Among these options, $(2,2,2)$ is an illegal choice of parameters, $A(4,4,2) \cong C_4$ is the unique cyclic subgroup of order 4 in $G(4,4,2) \cong D_8$, $A(2,1,4)$ is the unique elementary abelian subgroup of order 16 of $G(2,1,4)$, and in the remaining four cases it can be verified that $A(m,r,n)$ is not characteristic in $G(m,r,n)$. ∎

7.16. Lemma. *Let $A \rtimes W$ be the semidirect product for the action of a finite group W on a finite abelian group A. If A is not characteristic in $A \rtimes W$, then $|A|$ divides $|A^\sigma| \cdot |C_W(\sigma)|$ for some non-trivial element $\sigma \in W$.*

Proof. If A is not characteristic, some automorphism of the semidirect product takes an element of A to an element (a,σ) where $\sigma \in W$ is non-trivial. As automorphisms preserve centralizers up to isomorphism, we know that $|A|$ divides $|C_{A\rtimes W}(a,\sigma)|$. The exact sequence

$$1 \to A^\sigma \to C_{A\rtimes W}(a,\sigma) \to C_W(\sigma)$$

shows that $|C_{A\rtimes W}(a,\sigma)|$ divides $|A^\sigma| \cdot |C_W(\sigma)|$. ∎

Since $G(2,1,2)$ is conjugate to $G(4,4,2)$ [21, 2.5], its normalizer was found above. In the remaining three cases, there are exact sequences (3.16)

$$1 \to \mathbf{Z}_p^\times/ZG(4,2,2) \to N_{\mathrm{GL}(2,\mathbf{Z}_p)}(G(4,2,2))/G(4,2,2) \to \Sigma_3 \to 1,$$
$$1 \to \mathbf{Z}_p^\times \to N_{\mathrm{GL}(3,\mathbf{Z}_p)}(G(3,3,3))/G(3,3,3) \to A_4 \to 1,$$
$$1 \to \mathbf{Z}_p^\times/ZG(2,2,4) \to \mathbf{Z}_p^\times W(\mathrm{F}_4)/G(2,2,4) \to \Sigma_3 \to 1,$$

describing the automorphism groups of the groups $XG(4,2,2)$, $XG(3,3,3)$, and $XG(2,2,4)$.

7.17. *Automorphisms of other modular polynomial p-compact groups.* If W is one of the Aguadé reflection groups or $W = W(\mathrm{G}_2)$ and $p = 3$, then (3.17(2))

$$\mathrm{End}(X(W)) - \{0\} = \mathrm{Out}(X(W)) = \mathbf{Z}_p^\times/Z(W)$$

for $N_{\mathrm{GL}(r,\mathbf{Z}_p)}(W) = \mathbf{Z}_p^\times W$ according to [4, 5.7].

The 3-compact group $BX(G_{12})$ is also denoted BDI_2 for, since $G_{12} \subset \mathrm{GL}(2,\mathbf{Z}_3)$ maps isomorphically onto $\mathrm{GL}(2,\mathbf{F}_3)$ [11, p. 272], [88, 10.7.1], the mod 3 cohomology algebra

$$H^*(BX(G_{12});\mathbf{F}_3) = H^*(B\check{T};\mathbf{F}_3)^{\mathrm{GL}(2,3)} = \mathbf{F}_3[x_{12},x_{16}]$$

is the rank 2 mod 3 Dickson algebra [88, 8.1.5]: a polynomial algebra on a generator x_{12} in degree 12 and a generator $x_{16} = P^1x_{12}$ in degree 16. DI_2 has the potential of containing all other connected 3-compact groups of rank 2 as this is certainly true on the level of Weyl groups. Section 10 elaborates on this aspect of DI_2.

7.18. *Structure of polynomial p-compact groups* (cf. [79]). We start by noting that polynomiality of a connected p-compact group is determined by the universal covering p-compact group and the fundamental group.

7.19. Lemma. *Let X be connected p-compact group with universal covering p-compact group SX* [69, 3.3] *and fundamental group $\pi_1(X)$. Then*

(1) *X is polynomial $\Leftrightarrow$ SX is polynomial and $\pi_1(X)$ is a free $\mathbf{Z}_p$-module.*

(2) *If X is polynomial, then $H^*(BX;R) \to H^*(BSX;R)$ is surjective and the kernel is the ideal generated by the degree* 2 *cohomology classes, $R = \mathbf{F}_p$, $\mathbf{Z}_p$, $\mathbf{Q}_p$.*

(3) *If X is polynomial, then*

$$H^*(BX;R) \cong H^*(\pi_2(BX),2;R) \otimes H^*(BSX;R)$$

as graded algebras, $R = \mathbf{F}_p$, $\mathbf{Z}_p$, $\mathbf{Q}_p$.

Proof. If X is polynomial, $H_*(BX;\mathbf{Z}_p)$ is (7.1) concentrated in even degrees and is degreewise free so that, in particular, the second homology group $H_2(BX;\mathbf{Z}_p)=\pi_2(BX)$ is a free $\mathbf{Z}_p$-module. The Serre spectral sequence for the Postnikov fibration $K(\pi_1(X),1)\to BSX\to BX$ collapses at the E_3-page to yield

$$E_3=R\otimes_{H^*(\pi_2(BX),2;R)}H^*(BX;R)=H^*(BSX;R).$$

Conversely, if SX is polynomial and $\pi_1(X)$ is free, then the Serre spectral sequence for the fibration $BSX\to BX\to K(\pi_2(BX),2)$ collapses at the E_2-page for degree reasons and shows that $H^*(BX;\mathbf{F}_p)$ is concentrated in even degrees and is degreewise free. ∎

Let $Y=\prod Y_i$ be a product of finitely many simple, simply connected p-compact groups Y_i, π a (finite) subgroup of the p-discrete center $\check{Z}(Y)=\prod\check{Z}(Y_i)$ of Y, and $\varphi\colon\pi\to\check{S}$ a homomorphism into the discrete approximation $\check{S}$ to a p-compact torus S. Define [65, §2] the p-compact group

$$X=Y\times S/(\pi,\varphi)$$

by the short exact sequence $\pi\xrightarrow{(\mathrm{incl},\varphi)}Y\times S\to X$. Any connected p-compact group has this form with $Y=SX$ and $S=Z(X)_0$ [69, 5.4].

7.20. Corollary. *$X=Y\times S/(\pi,\varphi)$ is polynomial if and only if $\varphi:\pi\to\check{S}$ is a monomorphism and each simple factor Y_i in the universal covering p-compact group $Y=\prod Y_i$ equals* $\mathrm{SU}(n)$ *for some n or is one of the p-compact groups from* (7.9).

Proof. We use criterion (7.19(1)). Elementary homological algebra performed on the short exact sequence $0\to\pi_1(S)\to\pi_1(X)\to\pi\to 0$ shows that $\pi_1(X)$ is a free $\mathbf{Z}_p$-module if and only if $\varphi\colon\pi\to\check{S}$ is injective. (One may also use the functor $\check{T}$ of §11 to see this.) The universal covering p-compact group $Y=\prod Y_i$ is polynomial iff each simple factor Y_i is polynomial (7.28). According to (7.9) and Borel [9, 2.5], Y_i is polynomial iff $Y_i=\mathrm{SU}(n)$ for some n or $r_0W(Y_i)\neq r_0W(\mathrm{F}_4)$, $r_0W(\mathrm{E}_6)$, $r_0W(\mathrm{E}_7)$, $r_0W(\mathrm{E}_8)$ if $p=3$ and $r_0W(Y_i)\neq r_0W(\mathrm{E}_8)$ if $p=5$. ∎

In greater detail, any polynomial p-compact group X is of the form $X=X_1\times X_2$ where $X_1=(\prod\mathrm{SU}(n_i)\times S)/(\pi,\varphi)$ is a polynomial p-compact group whose universal covering is a product of special unitary p-compact groups and X_2 is a product of some of the simple, simply connected, centerless, polynomial p-compact groups of (7.9).

7.21. Corollary. *All polynomial p-compact groups are totally N-determined.*

Proof. Since all simple factors of $PX=PY=\prod PY_i$ are totally N-determined (7.20, 5.2, 7.9), X is totally N-determined (3.3, 3.7). ∎

7.22. Corollary. *Let Y be a simply connected, polynomial p-compact group. Then*

(1) *Y is determined by its $\mathbf{F}_p$-Weyl group, and*
(2) *BY is cohomologically unique among $\mathbf{F}_p$-local spaces.*

Proof. Since Y is totally N-determined (7.21), it suffices (4.4, 4.5) to show that the cohomology group $H^1(W(Y);\mathrm{Hom}(t(Y),t(Y)))$ is trivial. But that is proved by Notbohm in [79, 6.2]: Let $Y=\prod Y_i$ be as in (7.20). Let $S=\prod S_i\subseteq\prod W(Y_i)$ be the product subgroup with factors $S_i=W(Y_i)$ in case $Y_i=\mathrm{SU}(n)$ and $S_i=\mathrm{Syl}_p W(Y_i)$ in case Y_i is one of the p-compact groups from (7.9). Then $|W(Y)\colon S|$ is prime to p. The natural homomorphism $C_Y(t(Y)^S)\to Y$ is a monomorphism of maximal rank [31, 4.3], and, by inspection, $C_Y(t(Y)^S)=\prod C_{Y_i}(t(Y_i)^{S_i})$ is isomorphic to a product of $\mathrm{SU}(n)$s and $\mathrm{U}(n)$s. Thus $H^1(W(t(Y)^S);\mathrm{Hom}(t(Y),t(Y)))=0$ by [74, 8.2]. But then also the cohomology group $H^1(W(Y);\mathrm{Hom}(t(Y),t(Y)))=0$ by a transfer argument because $W(C_Y(t(Y)^S))=W(t(Y)^S)\supseteq S$ has index prime to p in $W(Y)$. ∎

7.23. Corollary. *Any polynomial p-compact group is determined up to local isomorphism by its mod p cohomology algebra considered as an unstable algebra over the Steenrod algebra.*

Proof. The mod p cohomological dimension as well as $H^*(BSX;\mathbf{F}_p)$ (7.19), and hence (7.22) BSX, can be read off from $H^*(BX;\mathbf{F}_p)$ if this is a polynomial algebra. But this information is the local isomorphism class of X [65, 2.6]. ∎

Two locally isomorphic p-compact groups, $X_1=Y\times S/(\pi_1,\varphi_1)$ and $X_2=Y\times S/(\pi_2,\varphi_2)$, are isomorphic iff there exist automorphisms $g\in\mathrm{Out}(Y)$ and $h\in\mathrm{Out}(S)=\mathrm{Aut}(\check{S})$ such that the diagram

$$\begin{tikzcd} \check{Z}(Y) \arrow[d, "\check{Z}(g)"', "\cong"] & \pi_1 \arrow[l, tail] \arrow[r, tail, "\varphi_1"] \arrow[d, "\cong"] & \check{S} \arrow[d, "h", "\cong"'] \\ \check{Z}(Y) & \pi_2 \arrow[l, tail] \arrow[r, tail, "\varphi_2"'] & \check{S} \end{tikzcd}$$

commutes [65, 4.3, 4.5].

The next example shows that there are locally isomorphic but non-isomorphic polynomial p-compact groups with isomorphic mod p cohomology algebras.

7.24. Lemma. *Let $X=Y\times S/(\pi,\varphi)$ be a polynomial p-compact group. If $\pi\subseteq p\check{Z}(Y)$, then $W(X)$ and $W(Y\times S)$ are $\mathbf{F}_p$-similar so that $H^*(BX;\mathbf{F}_p)$ and $H^*(BY\times BS;\mathbf{F}_p)$ are isomorphic unstable polynomial algebras.*

Proof. We may assume that π, and hence S, is non-trivial as otherwise there is nothing to prove. From the short exact sequence $0\to\pi\to\check{T}(Y)\times$

$\check{T}(S) \to \check{T}(X) \to 0$ we get the exact sequence

$$0 \to H^1(\pi;\mathbf{F}_p) \to H^2(\check{T}(X);\mathbf{F}_p) \to H^2(\check{T}(Y)\times\check{T}(S);\mathbf{F}_p) \to H^2(\pi;\mathbf{F}_p) \to 0$$

of $\mathbf{F}_pW(X)$-modules.

The map induced by $\varphi\colon \pi \to \check{S}$,

$$\mathrm{Ext}(\check{T}(S),\mathbf{F}_p) = H^2(\check{T}(S);\mathbf{F}_p) \to H^2(\pi;\mathbf{F}_p) = \mathrm{Ext}(\pi,\mathbf{F}_p),$$

is surjective since φ is injective. This implies that the $\mathbf{F}_pW(X)$-module homomorphism onto $H^2(\pi;\mathbf{F}_p)$ has a right inverse.

We next show that the $\mathbf{F}_pW(X)$-module homomorphism out of $H^1(\pi;\mathbf{F}_p)$ has a left inverse. Write K for the kernel of the map onto $H^2(\pi;\mathbf{F}_p)$ and apply $H_0(W(X);-)$ to the short exact sequence $0 \to H^1(\pi) \to H^2(\check{T}(X)) \to K \to 0$ to get the exact sequence

$$H_1(W(X);H^2(\check{T}(X)))\to H_1(W(X);K) \to H^1(\pi)\to H_0(W(X);H^2(\check{T}(X)))$$

where

$$H_1(W(X);K) \cong H_1(W(X);H^2(\check{T}(Y)\times\check{T}(S))) \cong H_1(W(X);H^2(\check{T}(Y)))$$

since $H_1(W(X);H^2(\pi)) = H_2(W(X);H^2(\pi)) = H_1(W(X);H^2(\check{T}(S))) = 0$ [5, 3.2], [74, 3.1]. It suffices to show that the map into $H_0(W(X);H^2(\check{T}(X)))$ is injective or, by exactness, that the map out of $H_1(W(X);H^2(\check{T}(X)))$ is surjective. The maps $\check{Z}(Y) \to \check{Z}(X) = \check{Z}(Y)\times S/(\pi,\varphi) \to \check{Z}(Y)/\pi$ induce dual maps

$$\mathrm{Hom}(\check{Z}(Y)/\pi,\mathbf{Z}/p^\infty) \to \mathrm{Hom}(\check{Z}(X),\mathbf{Z}/p^\infty) \to \mathrm{Hom}(\check{Z}(Y),\mathbf{Z}/p^\infty)$$

whose composition is surjective under the assumption of the lemma that $\pi \subseteq p\check{Z}(Y)$. Thus the second of these maps, which by (11.17) can be identified with the first map in the above exact sequence, must also be an epimorphism.

We now conclude that

$$\begin{aligned} H^2(\check{T}(X);\mathbf{F}_p) &= H^1(\pi;\mathbf{F}_p) + \mathrm{coker}(H^1(\pi;\mathbf{F}_p) \to H^2(\check{T}(X);\mathbf{F}_p)) \\ &= H^2(\pi;\mathbf{F}_p) + \ker(H^2(\check{T}(Y)\times\check{T}(S);\mathbf{F}_p) \to H^2(\pi;\mathbf{F}_p)) \\ &= H^2(\check{T}(Y)\times\check{T}(S);\mathbf{F}_p) \end{aligned}$$

as $\mathbf{F}_pW(X)$-modules and therefore the two rings of invariants,

$$\begin{aligned} H^*(BX;\mathbf{F}_p) &\cong H^*(\check{T}(X);\mathbf{F}_p)^{W(X)} \cong H^*(\check{T}(Y)\times\check{T}(S);\mathbf{F}_p)^{W(X)} \\ &\cong H^*(BY\times BS;\mathbf{F}_p), \end{aligned}$$

are isomorphic unstable algebras over the mod p Steenrod algebra. ■

7.25. Example [98], [74, 9.6]. The p-compact groups

$$U_i = \mathrm{SU}(p^\nu)\times \mathrm{U}(1)/(\mathbf{Z}/p^i,\mathrm{incl}), \qquad 0 \le i \le \nu,\ \nu \ge 3,$$

from the local isomorphism system of $\mathrm{U}(p^\nu)$ (11.28) are distinct, polynomial (7.19) p-compact groups but (7.24) the Weyl groups $W(U_i)$ are $\mathbf{F}_p$-similar and the unstable algebras $H^*(\mathrm{B}U_i;\mathbf{F}_p)$ are isomorphic for $0 \le i < \nu$ (and distinct from $H^*(\mathrm{BU}(p^\nu);\mathbf{F}_p)$). Thus the polynomial p-compact group $\mathrm{SU}(p^\nu)\times\mathrm{U}(1)$ is not determined by its $\mathbf{F}_p$-Weyl group, not even by its mod p cohomology algebra.

An unstable graded algebra over the mod p Steenrod algebra is

• *polynomial* if its underlying graded algebra over $\mathbf{F}_p$ is polynomial on finitely many generators,

• *topologically realizable* if it is isomorphic to the mod p cohomology of a topological space.

Steenrod's problem [90] asks for the determination of all topologically realizable polynomial algebras over the the mod p Steenrod algebra. A complete solution was found by D. Notbohm [76, 79] but it may still be worthwhile to record also the following form of the answer.

Write $P(H,t) = \sum(\dim_k H^i)t^i$ for the Poincaré series of the graded algebra H over the field k.

7.26. THEOREM. *A polynomial unstable algebra over the mod p Steenrod algebra is topologically realizable if and only if it is isomorphic to* $H^*(BX;\mathbf{F}_p) = H^*(\check{T}(X);\mathbf{F}_p)^{W(X)}$ *for some polynomial p-compact group X* (*as described in* (7.20)). *Moreover*, *the following conditions are equivalent*:

(1) $(W,\check{T})$ *is the Weyl group of a polynomial p-compact group*,
(2) $(W,S\check{T})$ *is polynomial and* $H_0(W;L(\check{T}))$ *is a free* $\mathbf{Z}_p$*-module*,
(3) $(W,\check{T})$ *is polynomial and* $H_0(W;L(\check{T}))$ *is a free* $\mathbf{Z}_p$*-module*,
(4) $(W,\check{T})$ *is polynomial and* $H_1(W;\check{T}) = 0$,
(5) $(W,\check{T})$ *is polynomial and* $P(H^*(\check{T};\mathbf{F}_p)^W,t) = P(H^*(\check{T};\mathbf{Q}_p)^W,t)$,
(6) $H^*(\check{T};\mathbf{Z}_p)^W$ *is polynomial*,

for any $\mathbf{Z}_p$*-reflection group* $(W,\check{T})$.

Proof. If BX is an $\mathbf{F}_p$-local space and $H^*(BX;\mathbf{F}_p)$ is polynomial, then X is a polynomial p-compact group and $H^*(BX;\mathbf{Z}_p) = H^*(\check{T}(X);\mathbf{Z}_p)^{W(X)}$ is a polynomial ring (7.1). This proves (1)⇒(6), and (6)⇔(5) by (7.27); we proceed to show (5)⇒(4)⇒(3)⇒(2)⇒(1).

If (5) holds, then $H^1(W;H^j(\check{T};\mathbf{Z}_p)) = 0$ for all degrees $j \ge 0$ (7.27). For $j=1$, in particular, $H^1(W;L(\check{T})^\vee) = 0$, which, for general reasons (11.11(1), 11.8(8)–(9)), is equivalent to $H_1(W;\check{T}) = 0$ or to $H_0(W;L(\check{T}))$ being a free $\mathbf{Z}_p$-module. From the (split) short exact sequence $0 \to S\check{T} \to \check{T} \to H_0(W;\check{T}) \to 0$ of $\mathbf{Z}_pW$-tori (11.8(10)) we get an epimorphism $H^2(\check{T};\mathbf{F}_p) \twoheadrightarrow H^2(S\check{T};\mathbf{F}_p)$ of $\mathbf{F}_pW$-modules and therefore [71, 4.1] $(W,S\check{T})$ is polynomial. In the splitting (11.15) of $(W,S\check{T})$ into a product of simple $\mathbf{Z}_p$-reflection groups $(W_i, \check{T}_i)$ with $H_0(W_i; L(\check{T}_i)) = 0$, each factor is polynomial (7.28), i.e. not similar to the Weyl groups of F_4, E_{6-8} at $p = 3$ and E_8 at $p = 5$ (7.4). Thus all simple factors of $(W, S\check{T})$ are Weyl groups of simple, simply connected, polynomial p-compact groups, $(W, S\check{T})$ is the Weyl group of the product Y of these, and $(W, \check{T})$, where $\check{T} = (S\check{T} \times \check{S})/(\pi, \varphi)$, is the Weyl group of the p-compact group $X = (Y \times S)/(\pi, \varphi)$, $BS = (B\check{S})^\wedge_p$, which is polynomial by (7.20). ∎

The map

$$\left\{\begin{matrix}\text{Isomorphism classes of}\\ \text{polynomial } p\text{-compact groups}\end{matrix}\right\} \xrightarrow{(W,\check{T})} \left\{\begin{matrix}\text{Similarity classes of polynomial}\\ \mathbf{Z}_p\text{-reflection groups with } H_1 = 0\end{matrix}\right\}$$

is surjective by (7.26(4)) and injective by (7.21).

7.27. Lemma. *Let $(W, \check{T})$ be a $\mathbf{Z}_p$-reflection group. Then $H^*(\check{T}; \mathbf{Z}_p)^W$ is polynomial if and only if $(W, \check{T})$ is polynomial and $P(H^*(\check{T}; \mathbf{F}_p)^W, t) = P(H^*(\check{T}; \mathbf{Q}_p)^W, t)$. If this is the case, then $H^*(\check{T}; \mathbf{Z}_p)^W \otimes \mathbf{Z}/p \cong H^*(\check{T}; \mathbf{F}_p)^W$ and $H^1(W; H^*(\check{T}; \mathbf{Z}_p)) = 0$.*

Proof. The Poincaré series condition ensures that the monomorphism of $H^*(\check{T}; \mathbf{Z}_p)^W \otimes \mathbf{Z}/p$ to $H^*(\check{T}; \mathbf{F}_p)^W$ is an isomorphism. Now, if the Poincaré series condition is satisfied, and $H^*(\check{T}; \mathbf{Z}_p)^W \otimes \mathbf{Z}/p = H^*(\check{T}; \mathbf{F}_p)^W$ is polynomial, then there exist homogeneous elements $x_1, \ldots, x_r \in H^*(\check{T}; \mathbf{Z}_p)^W$, where r is the rank of $\check{T}$, that reduced mod p become polynomial generators for $H^*(\check{T}; \mathbf{F}_p)^W$. Thus the ring homomorphism $\mathbf{Z}_p[x_1, \ldots, x_r] \to H^*(\check{T}; \mathbf{Z}_p)^W$ becomes an isomorphism mod p and hence it is an isomorphism by Nakayama's lemma. Conversely, if $H^*(\check{T}; \mathbf{Z}_p)^W$ is polynomial, then [77, 2.3, 2.4], [78] the polynomiality condition from [11, Ch. 5, §5, Exercice 5], [88, 5.5.4, 5.5.5] can be used to show that $H^*(\check{T}; \mathbf{F}_p)^W$ is polynomial. The last assertion of the lemma follows from the exact sequence

$$\ldots \xrightarrow{\cdot p} H^*(\check{T}; \mathbf{Z}_p)^W \twoheadrightarrow H^*(\check{T}; \mathbf{F}_p)^W \xrightarrow{0} H^1(W; H^*(\check{T}; \mathbf{Z}_p)) \\ \xrightarrow{\cdot p} H^1(W; H^*(\check{T}; \mathbf{Z}_p)) \to \ldots$$

where $H^1(W; H^j(\check{T}; \mathbf{Z}_p))$ is a finite $\mathbf{Z}_p$-module for fixed degree j. ∎

7.28. Lemma. *Let A and B be finitely generated graded algebras over a field k. If $A \otimes_k B$ is a graded polynomial ring over k on homogeneous generators of positive degree, then both factors A and B are polynomial.*

Proof. Since $A \otimes_k B$ is free over A (a k-basis for B provides an A-basis for $A \otimes_k B$) and the global dimension of the polynomial algebra $A \otimes_k B$ is finite by Hilbert's syzygy theorem [7, 4.2.3], the global dimension of A is also finite. Thus A is polynomial by Serre's converse [7, 6.2.3] to Hilbert's theorem. ∎

8. Proofs of Theorem 1.2 and Corollaries 1.3–1.6. This small section contains the proofs of the results stated in the introduction.

Proofs of Theorem 1.2, Corollary 1.3, and Corollary 1.5. The classification of Theorem 1.2 is the content of (5.2, 6.1, 7.9). To obtain Corollaries 1.3 and 1.5, combine this with (4.4, 4.10, 11.18, 11.25) and (3.3, 3.7, 3.12(3)). ∎

Proof of Corollary 1.4. Two connected p-compact groups with similar $\mathbf{Z}_p$-reflection groups have isomorphic maximal torus normalizers (3.12(1)), so are isomorphic (1.2). Thus the map (W, L) is injective.

To prove that the map (W, L) is surjective, let (W, L) be any $\mathbf{Z}_p$-reflection group. Then L sits as the kernel of a short exact sequence (11.5)

$$0 \to L \to LH_0(W; \check{T}) \times PL \to \check{\pi}(L) \to 0$$

of $\mathbf{Z}_pW$-modules. Choose a p-compact torus S and a centerless p-compact group PX such that $S \times PX$ realizes the $\mathbf{Z}_p$-reflection group in the middle. This is possible since PL is a product (11.15) of simple $\mathbf{Z}_p$-reflection groups, each of which is realizable (7.10). The $\mathbf{Z}_p$-reflection group (W, L) is now realized by a covering p-compact group of $S \times PX$ (4.10).

The expression for the automorphism group of X is (3.12(3), 3.15). ∎

Proof of Corollary 1.6. Observe that the maximal torus normalizers [70, 1.3] for X and G become homotopy equivalent after fiberwise completion away from the prime 2. This is because the maximal torus normalizers of the associated p-compact groups split [4] when p is odd; cf. [70, Proof of Proposition 5.5]. Thus there exists a space $(BN)[1/2]$ and rational equivalences

$$(BX)[1/2] \leftarrow (BN)[1/2] \to (BG)[1/2]$$

that p-complete to maximal torus normalizers for the p-compact groups $(BX)^\wedge_p$ and $(BG)^\wedge_p$ at each odd prime p. In this situation, N-determinism of the p-compact group $(BG)^\wedge_p$ and the Arithmetic Square [14, VI.8.1] ensure the existence of a homotopy equivalence $(BG)[1/2] \simeq (BX)[1/2]$ of spaces localized away from 2. ∎

Within the framework of this paper, it can be easily shown (see remark below (2.13)) that also the simple p-compact group $(\mathrm{E}_8, p = 5)$ [94] is totally N-determined, determined by its R-Weyl group for $R = \mathbf{Z}_p, \mathbf{Q}_p, \mathbf{F}_p$, and is a cohomologically unique p-compact group (4.4, 11.18, 11.25). However, more information is needed for the other members of the E-family [6]. If this program goes through we can remove the exceptions from Theorem 1.2, and it will then follow that any finite family (Y_i) of connected, simple, non-abelian, pairwise non-isomorphic p-compact groups is similarity free and any connected p-compact group is completely reducible in the (provisional) sense of [62, 3.4, 3.10] when p is odd. Thus for instance [64, 5.2] will apply to all (non-connected) p-compact groups G and [62, p. 381] will contain a description of the set $\varepsilon_{\mathbf{Q}}(X_1, X_2)$ of rational isomorphisms between any two locally isomorphic p-compact groups, X_1 and X_2, for p odd.

9. N-determinism of product p-compact groups. We show in this section that determinacy behaves well with respect to formation of (finite) products of p-compact groups.

First two lemmas of a general nature. A p-compact group morphism $f\colon X \to Y$ is said to be *trivial* if $Bf\colon BX \to BY$ is null-homotopic.

9.1. Lemma. *Let X and Y be p-compact groups and $X \to Z(Y)$ a p-compact group morphism into the center of Y. If the composite morphism $X \to Z(Y) \to Y$ is trivial, then $X \to Z(Y)$ is trivial.*

Proof. Turn the center $Bz\colon BZ(Y) \to BY$ into a fibration (with fiber $Y/Z(Y)$) and map BX into it to obtain the fibration

$$\mathrm{map}(BX, Y/Z(Y)) \to \mathrm{map}(BX, BZ(Y))_{Bz^{-1}(B0)} \to \mathrm{map}(BX, BY)_{B0}$$

where the total space consists of all maps $BX \to BZ(Y)$ that composed with Bz become null-homotopic. With the help of the Sullivan Conjecture for p-compact groups [31, 9.3], the fiber of this fibration identifies with $Y/Z(Y)$ and the base with BY. Thus the total space identifies with the connected space $BZ(Y) = \mathrm{map}(BX, BZ(Y))_{B0}$. This shows the lemma. ∎

9.2. Lemma. *Let $f\colon X \to Y$ be a p-compact group morphism and $j_p :$ $N_p(X) \to X$ the p-normalizer* [30, 9.8] *of the maximal torus of X. Then*

(1) [32, 5.6] *f is a monomorphism $\Leftrightarrow$ fj_p is a monomorphism*,
(2) [65, 6.6] *f is trivial $\Leftrightarrow$ fj_p is trivial.*

If X is connected, this remains true with the p-normalizer replaced by the maximal torus.

Proof. Suppose the restriction fj_p of f to the p-normalizer is a monomorphism. Then [30, 9.11] $H^*(BN_p(X))$ is a finitely generated $H^*(BY)$-module via $H^*(Bfj_p)$. Since $H^*(BX)$ is an $H^*(BY)$-submodule of $H^*(BN_p(X))$ thanks to the transfer homomorphism [31, 9.13] and $H^*(BY)$ a noetherian graded ring [30, 2.4], $H^*(BX)$ is a finitely generated $H^*(BY)$-module via $H^*(Bf)$. The converse follows from the fact that the composition of two monomorphisms is a monomorphism.

If X is connected, any monomorphism of $\mathbf{Z}/p$ to X factors through the maximal torus monomorphism $i\colon T(X) \to X$ [30, 4.7, 5.6], [31, 3.11]. This implies that if $\check{N}_p(X) \to N_p(X) \to X \to Y$ has a non-trivial kernel, the same is true for $\check{T}(X) \to T(X) \to X \to Y$ [30, §7]; here, $\check{N}_p(X)$ and $\check{T}(X)$ are discrete approximations [30, 6.4]. In other words, if $T(X) \to X \to Y$ is injective, so is $N_p(X) \to X \to Y$ [30, 7.3], [31, 3.5].

The second part of the lemma is [65, 6.6, 6.7]. ∎

We now address N-determinism of automorphisms of product p-compact groups. Let X_1 and X_2 be p-compact groups and

$$X_1 \underset{\iota_1}{\overset{\pi_1}{\leftrightarrows}} X_1 \times X_2 \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} X_2$$

the natural projections and inclusions.

9.3. Lemma. *Let $f\colon X_1 \times X_2 \to X_1$ be a p-compact group morphism such that $f\iota_1\colon X_1 \to X_1$ is an isomorphism and $f\iota_2\colon X_2 \to X_1$ is trivial. Then f is conjugate to $f\iota_1\pi_1$.*

Proof. We want to show that the adjoint of Bf, mapping BX_2 to $\mathrm{map}(BX_1, BX_1)_{B(f\iota_1)}$ which is a space homotopy equivalent [31, 1.3] to $BZ(X_1)$, is null-homotopic. But this follows immediately from (9.1) since composition with the evaluation monomorphism to BX_1 gives the null-homotopic map $Bf \circ B\iota_2$. ∎

9.4. Proposition. *Let X_1 and X_2 be two connected p-compact groups with N-determined automorphisms. Then also the product p-compact group $X_1 \times X_2$ has N-determined automorphisms.*

Proof. Let f be an automorphism of $X_1 \times X_2$ under the product $N_1 \times N_2$ of the two maximal torus normalizers. The morphism $\pi_1 f\iota_1\colon X_1 \to X_1$ is an isomorphism for [69, 3.7], [31, 4.7] it is a rational equivalence [30, 9.7] and a monomorphism (9.2). As also $\pi_1 f\iota_2\colon X_2 \to X_1$ is trivial by (9.2), it follows from (9.3) that $\pi_1 f$ is conjugate to $\pi_1 f\iota_1\pi_1$. Similarly, $\pi_2 f$ is conjugate to $\pi_2 f\iota_2\pi_2$ and thus f is conjugate to the product morphism $f_1 \times f_2$ where $f_1 = \pi_1 f\iota_1$ and $f_2 = \pi_2 f\iota_2$. Thus $N(f) = N_1(f_1) \times N_2(f_2)$ and, since X_1 and X_2 have N-determined automorphisms, it follows that f_1 and f_2 are conjugate to identity morphisms. ∎

Next, we address N-determinism of products. This is based on a slight reformulation of the Splitting Theorem [32, 6.1], [80].

9.5. Theorem. *Assume that p is odd. Let X be a connected p-compact group and $i\colon T \to X$ a maximal torus with normalizer $j\colon N \to X$. For any decomposition $N \xrightarrow{\cong} N_1 \times N_2$ of N into a product of two extended p-compact tori, N_1 and N_2, there exist p-compact groups, X_1 and X_2, and an isomorphism $s\colon X \to X_1 \times X_2$ such that*

$$\begin{tikzcd} N \arrow[r, "\cong"] \arrow[d, "j"'] & N_1 \times N_2 \arrow[d, "j_1 \times j_2"] \\ X \arrow[r, "\cong", "s"'] & X_1 \times X_2 \end{tikzcd}$$

commutes up to conjugacy where $j_1\colon N_1 \to X_1$ and $j_2\colon N_2 \to X_2$ are normalizers of maximal tori.

Proof. Write N_i, $i = 1, 2$, as an extension $T_i \to N_i \to W_i$ of a p-compact torus T_i and a finite group W_i. Then the Weyl group $W = \pi_0(N)$ of X is isomorphic to $W_1 \times W_2$ and W_i acts [32, 6.3] as a reflection group on $\pi_1(T_i) \otimes \mathbf{Q}$. According to [32, 6.1], the splitting $\pi_1(T) \cong \pi_1(T_1) \times \pi_1(T_2)$ as a $W \cong W_1 \times W_2$-module can be realized by a p-compact group splitting $s\colon X \to X_1 \times X_2$. This means that if $N(s)\colon N \to N_1' \times N_2'$, where $j_i'\colon N_i' \to X_i$ is the normalizer of the maximal torus $T_i \to X_i$, $i = 1, 2$, is the lift [65, 5.1] of s, then the discrete approximation [31, 3.12] $\check{N}(s)$ to $N(s)$ determines an isomorphism

$$\begin{tikzcd}
0 \arrow[r] & \check{T} \arrow[r] \arrow[d, "\check{T}(s)"'] & \check{N} \arrow[r] \arrow[d, "\check{N}(s)"] & W \arrow[r] \arrow[d, "W(s)"] & 1 \\
0 \arrow[r] & \check{T}_1 \times \check{T}_2 \arrow[r] & N_1' \times N_2' \arrow[r] & W_1 \times W_2 \arrow[r] & 1
\end{tikzcd}$$

of short exact sequences where $\check{T}(s)$ is the given splitting $\check{T} \cong \check{T}_1 \times \check{T}_2$ and $W(s)$ the given splitting $W \cong W_1 \times W_2$. Relative to the given splitting $\check{N} \cong \check{N}_1 \times \check{N}_2$, the middle isomorphism $\check{N}(s)$ takes $\check{N}_1 \times \check{N}_2$ isomorphically to $\check{N}_1' \times \check{N}_2'$. The composite

$$\check{N}_1 \xrightarrow{\iota_1} \check{N}_1 \times \check{N}_2 \xrightarrow{\check{N}(s)} \check{N}_1' \times \check{N}_2' \xrightarrow{\pi_2'} \check{N}_2',$$

where ι_1 is the injection and π_2' the projection, can be factored through a homomorphism $W_1 \to \check{T}_2$ as it restricts to the trivial morphism $\check{T}_1 \to \check{T}_2$. Since p is assumed to be odd, any such homomorphism is trivial for the reflection group W_1 is generated by elements of order prime to p. This implies that $\check{N}(s)\colon \check{N}_1 \times \check{N}_2 \to \check{N}_1' \times \check{N}_2'$ is the product of two isomorphisms, $\check{N}_1 \to \check{N}_1'$ and $\check{N}_2 \to \check{N}_2'$. Let j_i, $i = 1, 2$, be the composite of this isomorphism $\check{N}_i \to \check{N}_i'$ with $j_i\colon \check{N}_i' \to X_i$. ∎

The assumption that p should be odd is presumably not essential.

9.6. Proposition. *The product of two connected N-determined p-compact groups is N-determined when $p > 2$.*

Proof. This is immediate from the commutative diagram

$$\begin{tikzcd}
 & N_1 \times N_2 \arrow[dl] \arrow[dr, "j'"'] \arrow[drr] & & \\
X_1 \times X_2 & & X' \arrow[r, "\cong", "s"'] & X_1' \times X_2'
\end{tikzcd}$$

where X' is any p-compact group with maximal torus normalizer j' and s the splitting isomorphism from (9.5). For if X_1 and X_2 are N-determined, we get isomorphisms $f_1\colon X_1 \to X_1'$ and $f_2\colon X_2 \to X_2'$ under N_1 and N_2, respectively, and $s^{-1} \circ (f_1 \times f_2)$ is then an isomorphism $X_1 \times X_2 \to X'$ under $N_1 \times N_2$. ∎

The next step is to generalize (9.4) and (9.6) to possibly non-trivial extensions.

9.7. THEOREM. *Let $Y \to G \to X$ be a short exact sequence of connected p-compact groups.*

(1) *If the adjoint forms PX and PY have N-determined automorphisms, so does G.*

(2) *If the adjoint forms PX and PY are N-determined and $p > 2$, so is G.*

Since a connected p-compact group has N-determined automorphisms or is N-determined provided this holds for its adjoint form [66, 4.8, 7.10], the proof of the above theorem is an immediate consequence of (9.4, 9.6) and the lemma below.

9.8. LEMMA. *Let $Y \to G \to X$ be an extension of connected p-compact groups. Then*

(1) *G is locally isomorphic* [65, 2.7] *to $X \times Y$, and*

(2) *the adjoint form PG is isomorphic to $PX \times PY$.*

Proof. Let SX denote the universal covering p-compact group and $S = Z(X)_0$ the identity component of the center of X. Let $Y \to E_1 \to SX \times S$ be the extension obtained by pulling back along the isogeny [69, 5.4] $SX \times S \to X$. Since [64, 3.2, 3.3, 3.4] the projection of $SX \times S$ to S induces a bijection

$$\mathrm{Ext}(S, Y) \xrightarrow{\cong} \mathrm{Ext}(SX \times S, Y)$$

of equivalence classes of extensions, the p-compact group E_1, which is locally isomorphic to G, is isomorphic to $SX \times E_2$ for some extension $Y \to E_2 \to S$ of the p-compact torus S by Y. By [64, 2.6], E_2 is locally isomorphic to $S \times Y$ and hence G is locally isomorphic to $SX \times S \times Y$, which is locally isomorphic to $X \times Y$.

Any connected p-compact group has the same adjoint form as its universal covering p-compact group (4.6). Hence

$$PG \cong P(SX \times SY) \cong PX \times PY. \blacksquare$$

9.9. EXAMPLE. Since [64, 3.3, 3.4]

$$\begin{aligned}\mathrm{Ext}(\mathrm{PU}(p), \mathrm{SU}(p)) &= [\mathrm{BPU}(p), \mathrm{B}^2 Z(\mathrm{SU}(p))] = H^2(\mathrm{BPU}(p); \mathbf{Z}/p) \\ &= \mathrm{Hom}(\mathbf{Z}/p, \mathbf{Z}/p) = \mathbf{Z}/p\end{aligned}$$

there are p equivalence classes of extensions of $\mathrm{PU}(p)$ by $\mathrm{SU}(p)$ in the category of p-compact groups. However, since the local isomorphism system of the p-compact group $\mathrm{SU}(p)\times\mathrm{SU}(p)$ [65, p. 217]

$$\begin{tikzcd}[column sep=small] & \mathrm{SU}(p)\times\mathrm{PU}(p) \arrow[dr] & \\ \mathrm{SU}(p)\times\mathrm{SU}(p) \arrow[ur] \arrow[dr] & & \mathrm{PU}(p)\times\mathrm{PU}(p) \\ & \mathrm{SU}(p)\rtimes\mathrm{PU}(p) \arrow[ur] & \end{tikzcd}$$

consists of very few p-compact groups, we see from (9.10) that the middle p-compact group in any of these extensions must be isomorphic to either the direct product $\mathrm{SU}(p)\times\mathrm{PU}(p)$ or the semidirect product $\mathrm{SU}(p)\rtimes\mathrm{PU}(p)$ for the conjugation action of $\mathrm{PU}(p)$ on $\mathrm{SU}(p)$. All the p-compact groups locally isomorphic to $\mathrm{SU}(p)\times\mathrm{SU}(p)$ are totally N-determined. The automorphism groups are, for instance,

$$\mathrm{Out}(\mathrm{SU}(p)\times\mathrm{PU}(p)) = \mathbf{Z}_p^*\times\mathbf{Z}_p^*,$$
$$\mathrm{Out}(\mathrm{SU}(p)\rtimes\mathrm{PU}(p)) \cong \{(u,v)\in\mathbf{Z}_p^*\times\mathbf{Z}_p^* \mid u\equiv v \bmod p\}\rtimes\mathbf{Z}/2,$$

where $\mathbf{Z}/2$ permutes the coordinates. Formulas like these follow from (5.1) in combination with [65, 4.3] and [62, 3.5].

9.10. Lemma. *For any short exact sequence $Y\xrightarrow{\iota}G\xrightarrow{\pi}X$ of connected p-compact groups there exists a corresponding short exact sequence $Z(Y)\to Z(G)\to Z(X)$ of centers. In particular, Y and G have isomorphic centers if X is centerless.*

Proof. Let $z\colon Z(Y)\to Y$ be the center of Y. In the commutative diagram

$$\begin{tikzcd} \mathrm{map}(BZ(Y),BY)_{Bz} \arrow[r] \arrow[d, "\simeq"'] & \mathrm{map}(BZ(Y),BG)_{B(\iota z)} \arrow[r] \arrow[d] & \mathrm{map}(BZ(Y),BX)_{B0} \arrow[d, "\simeq"] \\ BY \arrow[r] & BG \arrow[r] & BX \end{tikzcd}$$

the horizontal lines are fibration sequences and the vertical arrows are evaluation maps. Note that the middle arrow is a homotopy equivalence since the outer two arrows are homotopy equivalences. This shows that $\iota z\colon Z(Y)\to G$ is central and by naturality we obtain [30, 8.3] a short exact sequence

$$PY\to G/Z(Y)\to X$$

of p-compact groups. This extension is equivalent to the trivial extension [64, 3.4] since PY is centerless [69, 4.7], [31, 6.3]. Thus $G/Z(Y)\cong PY\times X$ and

$$Z(G)/Z(Y)\cong Z(G/Z(Y))\cong Z(PY\times X)\cong Z(X)$$

by [69, 4.6(4)]. ∎

10. Maximal rank subgroups of DI_2. This section contains some general theory for monomorphisms between p-compact groups and it is shown that DI_2 contains essentially unique copies of each of the 3-compact groups $\mathrm{SU}(2)\times\mathrm{SU}(2)$, $\mathrm{U}(2)$, $\mathrm{Spin}(5)$, $\mathrm{SU}(3)$, $\mathrm{PU}(3)$, and G_2.

Recall from the previous section that BDI_2 is the homotopy colimit (at $p=3$) of a diagram of the form

$$(10.1)\qquad (\mathbf{Z}/2)^{\mathrm{op}} \circlearrowright \mathrm{BSU}(3) \xleftarrow{W(\mathrm{SU}(3))^{\mathrm{op}}\backslash W^{\mathrm{op}}} BT(\mathrm{SU}(3)) \circlearrowleft W^{\mathrm{op}}$$

where $\mathbf{Z}/2$ acts on $\mathrm{BSU}(3)$ as $\{\psi^{\pm1}\}$ and W, the Weyl group of DI_2, is the subgroup of $\mathrm{GL}(2,\mathbf{Z}_p)$

$$W = W(\mathrm{DI}_2) = \langle W(\mathrm{SU}(3)), W(\mathrm{PU}(3))\rangle = \langle \sigma,\tau,-E\rangle$$

generated by the Weyl groups of $\mathrm{SU}(3)$ and $\mathrm{PU}(3)$ or, alternatively, by the matrices

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

together with scalar multiplication with -1. The semidirect product

$$\check{N}(\mathrm{DI}_2) = \check{T} \rtimes W$$

where $\check{T} = \check{T}(\mathrm{SU}(3))$ is (3.12) the discrete approximation to the maximal torus normalizer $N(\mathrm{DI}_2)$ for DI_2.

We start our investigation of maximal rank subgroups of DI_2 with some general remarks.

Let X_1 and X_2 be two connected p-compact groups of the same rank. Let $j_1\colon N_1 \to X_1$ and $j_2\colon N_2\to X_2$ be normalizers of maximal tori $i_1\colon T_1\to X_1$ and $i_2\colon T_2\to X_2$.

Consider the map [67, 3.11]

$$(10.2)\qquad N\colon\ \mathrm{Mono}(X_1,X_2)\to \mathrm{Mono}(N_1,N_2)$$

that to any conjugacy class of a monomorphism $f\colon X_1\to X_2$ associates the unique conjugacy class $N(f)\colon N_1\to N_2$ such that

$$\begin{tikzcd} N_1 \arrow[r, "N(f)"] \arrow[d, "j_1"'] & N_2 \arrow[d, "j_2"] \\ X_1 \arrow[r, "f"'] & X_2 \end{tikzcd}$$

commutes up to conjugacy. Here, $\mathrm{Mono}(X_1,X_2)\subset[BX_1,BX_2]$ denotes the set of conjugacy classes of monomorphisms of X_1 into X_2 and $\mathrm{Mono}(N_1,N_2)$ denotes the set of conjugacy classes of maps $BN_1\to BN_2$ inducing monomorphisms on π_1 and isomorphisms on π_2. Note that if $\check{N}_1\to N_1$ and $\check{N}_2\to N_2$ are discrete approximations then

$$[B\check{N}_1, B\check{N}_2] = [BN_1, BN_2]$$

so that

$$\mathrm{Mono}(N_1, N_2) = \mathrm{Mono}(\check{N}_1, \check{N}_2)/\check{N}_2$$

consists of conjugacy classes of monomorphisms of $\check{N}_1$ into $\check{N}_2$. For any monomorphism $f \in \mathrm{Mono}(X_1, X_2)$, we let $\check{N}(f) \in \mathrm{Mono}(\check{N}_1, \check{N}_2)$, determined up to conjugacy, denote any discrete approximation to $N(f)$.

10.3. Definition. The monomorphism $f \in \mathrm{Mono}(X_1, X_2)$ is *N-determined* if the subset $N^{-1}(N(f))$ of $\mathrm{Mono}(X_1, X_2)$ consists of f alone.

Let $W_1 = \pi_0(N_1)$ and $W_2 = \pi_0(N_2)$ denote the Weyl groups.

10.4. Example. If $p \nmid |W_1|$, then all monomorphisms are N-determined. Indeed, it is not difficult to see that (10.2) is bijective in this case.

In case $X_1 = X_2$, the map (10.2) is the homomorphism $N : \mathrm{Out}(X_1) \to \mathrm{Out}(N_1)$ previously encountered. We say that X_1 *has N-determined monomorphisms* if this map is injective; if X_1 is totally N-determined then N is a bijection. Note that (10.2) is equivariant in the sense that there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Mono}(X_1, X_2) \times \mathrm{Out}(X_1) \arrow[r] \arrow[d, "N\times N"'] & \mathrm{Mono}(X_1, X_2) \arrow[d, "N"] \\
\mathrm{Mono}(N_1, N_2) \times \mathrm{Out}(N_1) \arrow[r] & \mathrm{Mono}(N_1, N_2)
\end{tikzcd}$$

relating group actions on sets of monomorphisms.

Let G and H be groups. Write $\{G > H\}$ for the set of conjugacy classes of subgroups abstractly isomorphic to H of G.

10.5. Proposition. *Let $i\colon X_1 \to X_2$ be a monomorphism between the two p-compact groups X_1 and X_2 of the same rank. Then the Euler characteristic $\chi(X_2/iX_1) = |W_2 \colon W_1|$ and if*

- *i is N-determined,*
- *X_1 is totally N-determined,*
- *$\{\check{N}_2 > \check{N}_1\}$ is a one-point set,*

then the action $\mathrm{Mono}(X_1, X_2) \times \mathrm{Out}(X_1) \to \mathrm{Mono}(X_1, X_2)$ is transitive and all monomorphisms of X_1 into X_2 are N-determined.

Proof. The first part is [67, 3.11]. For the second part, note first that for any $\alpha \in \mathrm{Out}(X_1)$, $i\alpha$ is an N-determined monomorphism. Suppose namely that $N(f) = N(i\alpha) = N(i)N(\alpha)$ for some monomorphism $f\colon X_1 \to X_2$. Then $N(f\alpha^{-1}) = N(f)N(\alpha)^{-1} = N(i)$, so $f\alpha^{-1} = i$ and therefore $f = i\alpha$.

Let now $f\colon X_1 \to X_2$ be any monomorphism and $\check{N}(f)\colon \check{N}_1 \to \check{N}_2$ a representative for the conjugacy class $N(f)$. Since $\check{N}_2$ contains but a single copy of $\check{N}_1$ up to conjugacy and X_1 is totally N-determined, $\check{N}(f) = \check{N}(i)\check{N}(\alpha)$ for some automorphism α of X_1. Then $\check{N}(f) = \check{N}(i\alpha)$ and $f = i\alpha$. ∎

The third condition is satisfied in case $\check{N}_1 = \check{T}_1 \rtimes W_1$, $\check{N}_2 = \check{T}_2 \rtimes W_2$ are semidirect products and the set $\{W_1 > W_2\}$ is a one-point set.

10.6. Definition. For a monomorphism $f : Y \to X$ of p-compact groups, let $W_X(f)$, or $W_X(Y)$, the *Weyl group of* f, denote the component group of the Weyl space $\mathcal{W}_X(Y)$ [32, 4.1, 4.3].

10.7. Proposition. *Let $f\colon Y \to X$ be a monomorphism of p-compact groups.*

(1) *If the homomorphism $\pi_0(Z(Y)) \to \pi_0(C_X(Y))$ induced by f is surjective, then the Weyl group $W_X(Y)$ is the isotropy subgroup* $\mathrm{Out}(Y)_f$ *for the action of* $\mathrm{Out}(Y)$ *on* $f \in \mathrm{Mono}(Y, X)$.

(2) *If f is centric* [28], *then there is a short exact sequence of loop spaces* [30, 3.2] $Y \to N_X(Y) \to W_X(Y)$ *where $N_X(Y)$ is the normalizer of f* [32, 4.4].

Proof. The monomorphism f determines a fibration

$$\mathcal{W}_X(Y) \to \coprod_{f\circ\alpha\simeq f} \mathrm{map}(BY, BY)_{B\alpha} \xrightarrow{Bf} \mathrm{map}(BY, BX)_{Bf}$$

where the components of the total space are indexed by the isotropy subgroup $\mathrm{Out}(Y)_f$ and the fiber is the Weyl space. The assumptions of the proposition assure that the inclusion of the fiber into the total space is a bijection on π_0. If we make the additional assumption that f be centric, the Weyl space becomes homotopically discrete and the exact sequence of the proposition is the one from [32, 4.6] ∎

10.8. Lemma. *Suppose that $i\colon X_1 \to X_2$ is a monomorphism and let $N(i) \in \mathrm{Mono}(N_1, N_2)$ be the induced monomorphism of normalizers. Then the stabilizer subgroup $\mathrm{Out}(N_1)_{N(i)}$ of $N(i)$ is isomorphic to the quotient group $N_{W_2}(W_1)/W_1$.*

Proof. Note that there is an epimorphism

$$N_{\check{N}_2}(\check{N}_1)/\check{N}_1 \twoheadrightarrow (\mathrm{Aut}(\check{N}_1)/\check{N}_1)_{N(i)} = \mathrm{Out}(N_1)_{N(i)}$$

given by conjugation by elements of $\check{N}_2$ normalizing $\check{N}_1$. This homomorphism is actually also injective, hence bijective, for if conjugation by, say, $n_2 \in N_{\check{N}_2}(\check{N}_1)$ agrees with conjugation by some element $n_1 \in \check{N}_1$, then n_1 and n_2 have the same image in W_2, so that n_2 belongs to $\check{N}_1$. This follows because the Weyl groups of the connected p-compact groups X_1 and X_2 are faithfully represented in their maximal tori. Consequently,

$$\mathrm{Out}(N_1)_{N(i)} \cong N_{\check{N}_2}(\check{N}_1)/\check{N}_1$$

and this last group is isomorphic to the quotient group $N_{W_2}(W_1)/W_1$ by the projection $\check{N}_2 \twoheadrightarrow W_2$. ∎

10.9. Proposition. *Let $i\colon X_1 \to X_2$ be an N-determined monomorphism between the two p-compact groups X_1 and X_2 inducing an epimorphism $\pi_0(Z(X_1)) \to \pi_0(C_{X_2}(X_1))$. Then*

$$W_{X_2}(X_1) = N_{W_2}(W_1)/W_1$$

provided X_1 is totally N-determined.

Proof. The assumptions imply that the Weyl group $W_{X_2}(X_1)$ is isomorphic to the stabilizer subgroup $\mathrm{Out}(X_1)_i$ which again is isomorphic to the stabilizer subgroup $\mathrm{Out}(N_1)_{N(i)}$ for the action

$$\mathrm{Mono}(\check{N}_1, \check{N}_2)/\check{N}_2 \times \mathrm{Aut}(\check{N}_1)/\check{N}_1 \to \mathrm{Mono}(\check{N}_1, \check{N}_2)/\check{N}_2$$

of $\mathrm{Out}(N_1)$ on $N(i) \in \mathrm{Mono}(N_1, N_2)$. Now apply (10.8). ∎

10.10. Example. By (10.4), the inclusion $\check{T} \rtimes \langle -\tau \rangle \rightarrowtail \check{T} \rtimes W$ is realizable by an N-determined monomorphism $i\colon \mathrm{U}(2) \to \mathrm{DI}_2$. The monomorphism i is centric (because $B\mathrm{U}(2) = BT_{h\mathbf{Z}/2}$ and the centralizer $C_{\mathrm{U}(2)}(T) \cong T \cong C_{\mathrm{DI}_2}(T)$) so (10.5, 10.7, 10.9)

$$\chi(\mathrm{DI}_2/\mathrm{U}(2)) = 24 \quad \text{and} \quad W_{\mathrm{DI}_2}(\mathrm{U}(2)) \cong Z(W)$$

and $\mathrm{Out}(\mathrm{U}(2))$ acts transitively on $\mathrm{Mono}(\mathrm{U}(2), \mathrm{DI}_2)$ because $\{\check{T} \rtimes W > \check{T} \rtimes \langle -\tau \rangle\}$ is a one-point set.

10.11. Example. Similarly, $\{W > \mathbf{Z}/2 \times \mathbf{Z}/2\}$ is a one-point set, so there is an essentially unique monomorphism $i\colon \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{DI}_2$ realizing the inclusion of $\check{T} \rtimes (\mathbf{Z}/2 \times \mathbf{Z}/2)$ into $\check{T} \rtimes W$. The monomorphism i is N-determined, centric, and

$$\chi(\mathrm{DI}_2/\mathrm{SU}(2) \times \mathrm{SU}(2)) = 12 \quad \text{and} \quad W_{\mathrm{DI}_2}(\mathrm{SU}(2) \times \mathrm{SU}(2)) \cong Z(W)$$

and $\mathrm{Out}(\mathrm{SU}(2) \times \mathrm{SU}(2))$ acts transitively on $\mathrm{Mono}(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{DI}_2)$.

10.12. Example. Also $\{W > D_8\} = \{D_8\}$, where D_8 is the dihedral group of order 8. It follows that there exists a unique monomorphism $i\colon \mathrm{Spin}(5) \to \mathrm{DI}_2$ realizing the inclusion $\check{T} \rtimes D_8 \rightarrowtail \check{T} \rtimes W$. This monomorphism is centric (because $B\mathrm{Spin}(5)$ and $B(\check{T} \rtimes D_8)$ are $H^*\mathbf{F}_3$-equivalent), so

$$\chi(\mathrm{DI}_2/\mathrm{Spin}(5)) = 6 \quad \text{and} \quad W_{\mathrm{DI}_2}(\mathrm{Spin}(5)) \cong Z(W)$$

and $\mathrm{Out}(\mathrm{Spin}(5))$ acts transitively on $\mathrm{Mono}(\mathrm{Spin}(5), \mathrm{DI}_2)$.

In a situation where a pair of monomorphisms $G \to X_1$ and $G \to X_2$ are given, let us write $\mathrm{map}^{BG}(BX_1, BX_2)$ for the space of maps $BX_1 \to BX_2$ under BG up to homotopy.

10.13. Lemma. *Let $z\colon Z \to X_1$ be a central monomorphism and $i : X_1 \to X_2$ any monomorphism inducing an isomorphism $X_1 \cong C_{X_1}(z) \to$*

$C_{X_2}(f \circ z)$. *Then f induces a homotopy equivalence*

$$\mathrm{map}^{BZ}(BX_1, BX_1) \to \mathrm{map}^{BZ}(BX_1, BX_2)$$

of mapping spaces.

Proof. The spaces $BC_{X_1}(z) = \mathrm{map}(BZ, BX_1)_{Bz}$ and $BC_{X_2}(f \circ z) = \mathrm{map}(BZ, BX_2)_{B(f\circ z)}$ are X_1/Z-spaces and $BC_f(Z)\colon BC_{X_1}(z) \to BC_{X_2}(f\circ z)$ is an X_1/Z-map inducing a map

$$\begin{aligned}\mathrm{map}^{BZ}(BX_1, BX_1) &= BC_{X_1}(z)^{h(X_1/Z)} \\ &\to BC_{X_2}(f \circ z)^{h(X_1/Z)} = \mathrm{map}^{BZ}(BX_1, BX_2)\end{aligned}$$

of homotopy fixed point spaces. If $C_f(z)$ is an isomorphism, then this map is a homotopy equivalence. ∎

This happens for instance for $V \to C_X(V) \to X$ so that

$$\mathrm{map}^{BV}(BC_X(V), BC_X(V)) \simeq \mathrm{map}^{BV}(BC_X(V), BX)$$

for any connected p-compact group X, any elementary abelian p-group V, and any monomorphism $V \to X$.

10.14. Example. Let $i\colon \mathrm{SU}(3) \to \mathrm{DI}_2$ denote the monomorphism arising in the construction (10.1) of BDI_2 as a homotopy colimit. By (10.13), Bi induces a homotopy equivalence

$$\mathrm{map}^{\mathrm{B}\mathbf{Z}/3}(\mathrm{BSU}(3), \mathrm{BSU}(3)) \to \mathrm{map}^{\mathrm{B}\mathbf{Z}/3}(\mathrm{BSU}(3), \mathrm{BDI}_2)$$

where $\mathbf{Z}/3 \to \mathrm{SU}(3)$ is the center, and thus a bijection

$$\mathrm{Out}^+(\mathrm{SU}(3)) \to \mathrm{Mono}(\mathrm{SU}(3), \mathrm{DI}_2),$$

where $\mathrm{Out}^+(\mathrm{SU}(3))$ consists of the unstable Adams operations ψ^u indexed by units $u \in \mathbf{Z}_3^*$ with $u \equiv 1 \bmod 3$. We obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Out}^+(\mathrm{SU}(3)) \arrow[r, "\cong"] \arrow[d, "N"', "\cong"] & \mathrm{Mono}(\mathrm{SU}(3), \mathrm{DI}_2) \arrow[d, tail, "N"] \\
\mathrm{Out}(\check{T} \rtimes W(\mathrm{SU}(3)))/W(\mathrm{SU}(3)) \arrow[r] & \mathrm{Mono}(\check{T} \rtimes W(\mathrm{SU}(3)), \check{T} \rtimes W)/W
\end{tikzcd}$$

and using (10.8) we see that the kernel of the composition going down and then right is trivial. Thus i is N-determined and [28, 4.2] centric. Consequently,

$$\chi(\mathrm{DI}_2/\mathrm{SU}(3)) = 8 \quad \text{and} \quad W_{\mathrm{DI}_2}(\mathrm{SU}(3)) \cong Z(W),$$

$\mathrm{Out}(\mathrm{SU}(3))$ acts transitively on $\mathrm{Mono}(\mathrm{SU}(3), \mathrm{DI}_2)$, and all monomorphisms of $\mathrm{SU}(3)$ into DI_2 are N-determined.

10.15. Example. Similarly, the monomorphism $i\colon \mathrm{SU}(3) \to \mathrm{G}_2$ arising in the construction (7.12) of $B\mathrm{G}_2$ as a homotopy colimit is N-determined and centric. Also, $\mathrm{Out}(\mathrm{SU}(3))$ acts transitively on $\mathrm{Mono}(\mathrm{SU}(3), \mathrm{G}_2)$ with stabilizer subgroup $W_{\mathrm{G}_2}(\mathrm{SU}(3)) = \{\pm E\}$, and $\chi(\mathrm{G}_2/\mathrm{SU}(3)) = 2$.

10.16. Example. The inclusions of the maximal torus and of $\mathrm{SU}(3)$ into DI_2 constitute a homotopy coherent set of maps out of the centralizer diagram (7.10) for BG_2 into BDI_2. Observing that both maps are centric one sees first that the Wojtkowiak obstruction groups vanish according to (13.7) and next that the resulting map $\mathrm{BG}_2 \to \mathrm{BDI}_2$ is a centric monomorphism realizing the inclusion $\check{T} \rtimes W(\mathrm{G}_2) \to \check{T} \rtimes W(\mathrm{G}_2)$ of maximal torus normalizers. As also $\{W > W(\mathrm{G}_2)\} = \{W(\mathrm{G}_2)\}$, we conclude that

$$\chi(\mathrm{DI}_2/\mathrm{G}_2) = 4 \quad \text{and} \quad W_{\mathrm{DI}_2}(\mathrm{G}_2) \cong \{1\}$$

and that $\mathrm{Out}(\mathrm{G}_2)$ acts transitively on $\mathrm{Mono}(\mathrm{G}_2, \mathrm{DI}_2)$.

10.17. Example. $\mathrm{BPU}(3)$ is the homotopy colimit of a diagram of the form

$$\begin{tikzcd} \mathrm{SL}(V)^{\mathrm{op}} \curvearrowright \mathrm{B}V \arrow[r, "S_3^{\mathrm{op}}\backslash \mathrm{SL}(V)^{\mathrm{op}}"'] & BN_3 \arrow[loop below, "\mathbf{Z}/2"] & BT \curvearrowleft W(\mathrm{PU}(3))^{\mathrm{op}} \arrow[l, "S_3^{\mathrm{op}}\backslash W(\mathrm{PU}(3))^{\mathrm{op}}"] \end{tikzcd}$$

where S_3 is a Sylow 3-subgroup of $\mathrm{SL}(V)$ and N_3 is the 3-normalizer of the maximal torus. There is a canonical map $BN_3 \to \mathrm{BDI}_2$ because N_3 is also the 3-normalizer of the maximal torus of DI_2. This map $BN_3 \to \mathrm{BDI}_2$ is centric and it respects the maps of the above diagram up to homotopy. The obstructions to extending $BN_3 \to \mathrm{BDI}_2$ to a map $\mathrm{BPU}(3) \to \mathrm{BDI}_2$ lie in the higher limits of the $\mathbf{A}(\mathrm{PU}(3))$-module

$$\begin{tikzcd} \mathrm{SL}(V) \curvearrowright \pi_*(BT(\mathrm{DI}_2)) & \pi_*(\mathrm{B}\mathbf{Z}3) \arrow[l, "\mathrm{SL}(V)/S_3"'] \arrow[r, "W(\mathrm{PU}(3))/S_3"'] \arrow[loop below, "\mathbf{Z}/2"] & \pi_*(BT) \curvearrowleft W(\mathrm{PU}(3)) \end{tikzcd}$$

which vanish completely (13.7). (We are here implicitly using computations of mapping spaces like $\mathrm{map}(BV, \mathrm{BDI}_2)_{Bi} = BT(\mathrm{DI}_2)$.) Thus there exists a unique homotopy class $Bi\colon \mathrm{BPU}(3) \to \mathrm{BDI}_2$ extending the inclusion of the 3-normalizer. Also, the restriction of i to the 3-normalizer of the maximal torus is a monomorphism, so i itself is a monomorphism (9.2), and i is centric because the Bousfield–Kan spectral sequence [14, XI.7.1] for $\mathrm{map}(\mathrm{BPU}(3), \mathrm{BDI}_2)_{Bi}$ shows that this mapping space is weakly contractible. As also $\{\check{T} \rtimes W > \check{T} \rtimes W(\mathrm{PU}(3))\}$ is a one-point set and $\mathrm{PU}(3)$ is totally N-determined (5.1), (10.5, 10.7, 10.9) show that

$$\chi(\mathrm{DI}_2/\mathrm{PU}(3)) = 8 \quad \text{and} \quad W_{\mathrm{DI}_2}(\mathrm{PU}(3)) = Z(W)$$

and that the group $\mathrm{Out}(\mathrm{PU}(3))$ acts transitively on the set $\mathrm{Mono}(\mathrm{PU}(3), \mathrm{DI}_2)$ of conjugacy classes of monomorphisms.

In view of [10], which says that any connected, closed subgroup of maximal rank of a compact connected Lie group is the normalizer of its center, this example is somewhat surprising.

There is no monomorphism of $\mathrm{PU}(3)$ into G_2 for $\mathbf{A}(\mathrm{PU}(3))(V) = \mathrm{SL}(V)$ (5.10) is too big to be a subgroup of $\mathbf{A}(\mathrm{G}_2)(V) = \Sigma_3 \times \mathbf{Z}/2$ (7.10). Indeed, no non-trivial compact, connected Lie group admits a proper, centerless subgroup of maximal rank [10].

The next example describes the normalizers of the elementary abelian subgroups of DI_2. Strictly speaking, these normalizers are not 3-compact groups, but rather extended 3-compact groups, in that their component groups are not 3-groups.

We start with a general observation.

10.18. Proposition. *Let $\nu\colon V \to X$ be a monomorphism of an elementary abelian p-group V into a p-compact group X.*

(1) *There is a short exact sequence of groups*

$$1 \to \pi_0(C_X(\nu)/V) \to W_X(\nu) \to \mathbf{A}(X)(\nu) \to 1$$

where $C_X(\nu)/V$ is the standard quotient [30, 8.3].

(2) *There is a short exact sequence of loop spaces*

$$C_X(\nu) \to N_X(\nu) \to \mathbf{A}(X)(\nu)$$

where $N_X(\nu)$ is the normalizer of ν [32, 4.4].

Proof. Assuming $B\nu\colon BV \to BX$ to be a fibration, consider the induced fibration

$$\mathcal{W}_X(\nu) \to \coprod_{f\in\mathbf{A}(X)(\nu)} \mathrm{map}(BV, BV)_{Bf} \xrightarrow{B\nu} \mathrm{map}(BV, BX)_{B\nu}$$

where the fibre is the Weyl space [32, 4.1] of ν and the components, each one homotopy equivalent to BV, of the total space are indexed by the automorphism group of ν in the Quillen category. The homotopy exact sequences of this fibration and of its subfibration

$$C_X(\nu)/V \to BV \to BC_X(\nu)$$

give the exact sequence of groups and show that $B(C_X(\nu)/V)$ is the regular covering space of $B\mathcal{W}_X(\nu)$ corresponding to $\pi_0(C_X(\nu)/V) \lhd W_X(\nu)$. Thus there is a pull-back diagram

$$\begin{tikzcd} BC_X(\nu) \arrow[r] \arrow[d] & BN_X(\nu) \arrow[d] \\ B(C_X(\nu)/V) \arrow[r] & B\mathcal{W}_X(\nu) \end{tikzcd}$$

where the horizontal maps are regular covering spaces. ∎

10.19. Example. For any monomorphism $\lambda\colon \mathbf{Z}/3 \to \mathrm{DI}_2$ there is (10.18) a short exact sequence of loop spaces

$$\mathrm{SU}(3) \to N_{\mathrm{DI}_2}(\lambda) \to Z(W)$$

where $Z(W) \cong \mathbf{Z}/2$ acts on SU(3) as $\{\psi^{\pm 1}\}$. Thus

$$N_{\mathrm{DI}_2}(\lambda) = \mathrm{SU}(3) \rtimes Z(W)$$

where $B(\mathrm{SU}(3) \rtimes Z(W))$ denotes the total space of the unique [64, 3.3, 3,7] $B\mathrm{SU}(3)$-fibration over $BZ(W)$ realizing the given monodromy action. (It is not essential in [64, §3] that the component group $\pi_0(X)$ be a p-group.) Since the homotopy fixed point space $BZ(\mathrm{SU}(3))^{hZ(W)}$ is contractible, the inclusion $\check{T} \rtimes W(\mathrm{SU}(3)) \rightarrowtail \mathrm{SU}(3)$ extends uniquely to a short exact sequence morphism

$$\begin{tikzcd} \check{T} \rtimes W(\mathrm{SU}(3)) \arrow[r] \arrow[d, tail] & N_{\check{T}\rtimes W}(\lambda) \arrow[r] \arrow[d] & Z(W) \arrow[d, equal] \\ \mathrm{SU}(3) \arrow[r] & N_{\mathrm{DI}_2}(\lambda) \arrow[r] & Z(W) \end{tikzcd}$$

where $N_{\check{T}\rtimes W}(\lambda) = \check{T} \rtimes \overline{W}(\lambda) = \check{T} \rtimes (W(\mathrm{SU}(3)) \times Z(W))$.

For any monomorphism $\nu\colon (\mathbf{Z}/3)^2 \to \mathrm{DI}_2$ there is a short exact sequence of loop spaces

$$T \to N_{\mathrm{DI}_2}(\nu) \to W$$

so $N_{\mathrm{DI}_2}(\nu)$ is an extended p-compact torus with $\check{T} \rtimes W$ as discrete approximation [31, 3.12].

10.20. Example. The normalizers of the 3-compact subgroups of DI_2 are (10.7(2))

$$N_{\mathrm{DI}_2}(\mathrm{G}_2) = \mathrm{G}_2 \quad \text{and} \quad N_{\mathrm{DI}_2}(X) = X \rtimes Z(W)$$

for $X = \mathrm{U}(2), \mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{Spin}(5), \mathrm{SU}(3), \mathrm{PU}(3)$ where $Z(W)$ acts on X as $\{\psi^{\pm 1}\}$. In each case there is a unique short exact sequence morphism connecting the normalizer in $\check{T} \rtimes W$ of $N_X(T)$ and the normalizer in DI_2 of X. For $X = \mathrm{PU}(3)$, for instance, the picture is

$$\begin{tikzcd} \check{T} \rtimes W(\mathrm{PU}(3)) \arrow[r] \arrow[d, tail] & N_{\check{T}\rtimes W}(\check{T} \rtimes W(\mathrm{PU}(3))) \arrow[r] \arrow[d] & Z(W) \arrow[d, equal] \\ \mathrm{PU}(3) \arrow[r] & N_{\mathrm{DI}_2}(\mathrm{PU}(3)) \arrow[r] & Z(W) \end{tikzcd}$$

where $N_{\check{T}\rtimes W}(\check{T} \rtimes W(\mathrm{PU}(3))) = \check{T} \rtimes (W(\mathrm{PU}(3)) \times Z(W))$. It seems likely that this is another instance of N-determinism.

11. Free $\mathbf{Z}_p$-modules and p-discrete tori. Nearly all material of this section is present, in one form or another, in [75].

A $\mathbf{Z}_p$-module which is isomorphic to $\mathbf{Z}_p^r$ for some finite r will be called a $\mathbf{Z}_p$-*lattice*, and a $\mathbf{Z}_p$-module which is isomorphic to $(\mathbf{Z}/p^\infty)^r = (\mathbf{Q}_p/\mathbf{Z}_p)^r$ for some finite r will be called a $\mathbf{Z}_p$-*torus*.

Let $\check{T}$ and L denote the endo-functors of the category $\mathbf{Ab}$ of abelian groups given by

$$\check{T} = \mathbf{Z}/p^\infty \otimes - \quad \text{and} \quad L = \mathrm{Hom}(\mathbf{Z}/p^\infty, -).$$

Then $\mathrm{Hom}_{\mathbf{Ab}}(\check{T}(A), B) = \mathrm{Hom}_{\mathbf{Ab}}(A, L(B))$ so $(\check{T}, L)$ is a pair of adjoint functors. The left adjoint functor $\check{T}$ is right exact, $\check{T}$ vanishes on finite $\mathbf{Z}_p$-modules, turns $\mathbf{Z}_p$-lattices into $\mathbf{Z}_p$-tori, and its left derived functor $\check{T}_1 = \mathrm{Tor}(\mathbf{Z}/p^\infty, -)$ preserves finite $\mathbf{Z}_p$-modules and vanishes on $\mathbf{Z}_p$-lattices. The right adjoint functor L is left exact, L vanishes on finite $\mathbf{Z}_p$-modules, turns $\mathbf{Z}_p$-tori into $\mathbf{Z}_p$-lattices, and its right derived functor $L_1 = \mathrm{Ext}(\mathbf{Z}/p^\infty, -)$ preserves finite $\mathbf{Z}_p$-modules and vanishes on $\mathbf{Z}_p$-tori. In symbols:

$$\begin{aligned}
&\check{T}(0 \to S \to L \to H \to 0) \\
&\qquad\qquad = (0 \to \check{T}_1(L) \to \check{T}_1(H) \to \check{T}(S) \to \check{T}(L) \to \check{T}(H) \to 0), \\
&L(0 \to H \to \check{T} \to \check{P} \to 0) \\
&\qquad\qquad = (0 \to L(H) \to L(\check{T}) \to L(\check{P}) \to L_1(H) \to L_1(\check{T}) \to 0),
\end{aligned}$$

where S is a $\mathbf{Z}_p$-lattice, $\check{P}$ is a $\mathbf{Z}_p$-torus, and L, H, and $\check{T}$ are $\mathbf{Z}_p$-modules. In fact the pair $(\check{T}, L)$ provides adjoint equivalences [14, p. 181] between the full subcategories of (the underlying abelian groups of) $\mathbf{Z}_p$-lattices and (the underlying abelian groups of) $\mathbf{Z}_p$-tori.

A $\mathbf{Z}_pW$-module whose underlying $\mathbf{Z}_p$-module is a $\mathbf{Z}_p$-lattice will be called a $\mathbf{Z}_pW$-*lattice*, and a $\mathbf{Z}_pW$-module whose underlying $\mathbf{Z}_p$-module is a $\mathbf{Z}_p$-torus will be called a $\mathbf{Z}_pW$-*torus*.

11.1. Definition [75, 1.1.4, 1.1.5]. For a $\mathbf{Z}_pW$-lattice L and a $\mathbf{Z}_pW$-torus $\check{T}$, put

$$\begin{aligned}
SL &= \ker(L \to H_0(W; L)), & PL &= L(P\check{T}(L)), \\
P\check{T} &= \mathrm{coker}(H^0(W; \check{T}) \to \check{T}), & S\check{T} &= \check{T}(SL(\check{T})).
\end{aligned}$$

In plain language, SL is simply the $\mathbf{Z}_pW$-submodule of L generated by the union of the subsets $(1-w)L$, $w \in W$, and $\check{T}(PL)$ is the quotient of $\check{T}(L)$ by the invariants $\check{T}(L)^W$ for the W-action. We have short exact sequences

$$\begin{aligned}
&0 \to SL \to L \to H_0(W; L) \to 0, \\
&0 \to H^0(W; \check{T}(L)) \to \check{T}(L) \to \check{T}(PL) \to 0
\end{aligned} \tag{11.2}$$

defining SL and PL. (SL could perhaps be called the root lattice and PL the weight lattice of L.)

It simplifies matters a great deal to assume that W is generated by elements of order prime to p (as are $\mathbf{Z}_p$-reflection subgroups for odd primes p).

11.3. Lemma. *Suppose that W is generated by elements of order prime to p. Then $H_1(W;H) = 0 = H^1(W;H)$ for any $\mathbf{Z}_p$-module H with trivial W-action.*

Proof. Observe that the abelianization $H_1(W;\mathbf{Z})$ is a finite abelian group generated by elements of order prime to p and apply universal coefficients. ∎

11.4. Lemma. *Let L be a $\mathbf{Z}_pW$-lattice.*

(1) *The $\mathbf{Z}_p$-module homomorphism $H^0(W;L) \to L$ is split injective and the $\mathbf{Z}_p$-module homomorphisms $\check{T}H^0(W;L) \to H^0(W;\check{T}(L))$, $H^0(W;L) \to H_0(W;L)$ are injective.*

(2) $\operatorname{coker}(H^0(W;L) \to H_0(W;L))$ *is finite.*

(3) *$H^0(W;SL) = 0 = H_0(W;\check{T}(SL))$ and $H_0(W;SL)$ is finite. If W is generated by elements of order prime to p, then $H_0(W;SL) = 0$.*

(4) *The $\mathbf{Z}_p$-module homomorphism $\check{T}(L) \to H_0(W;\check{T}(L))$ is split surjective and the $\mathbf{Z}_p$-module homomorphisms $H_0(W;L) \to LH_0(W;\check{T}(L))$, $H^0(W;\check{T}(L)) \to H_0(W;\check{T}(L))$ are surjective.*

(5) $\ker(H^0(W;\check{T}(L)) \to H_0(W;\check{T}(L)))$ *is finite.*

(6) *$H_0(W;\check{T}(PL)) = 0 = H^0(W;PL)$ and $H^0(W;\check{T}(PL))$ is finite. If W is generated by elements of order prime to p, then $H^0(W;\check{T}(PL)) = 0$.*

(7) *$H^0(W;L) \cong LH^0(W;\check{T}(L))$ and $H_0(W;\check{T}(L)) \cong \check{T}H_0(W;L)$.*

(8) *$H^0(W;L) = 0 \Leftrightarrow H_0(W;L)$ is finite $\Leftrightarrow H_0(W;\check{T}(L)) = 0 \Leftrightarrow H^0(W;\check{T}(L))$ is finite.*

(9) $$H_0(W;L) = 0 \Leftrightarrow H_0(W;\check{T}(L)) = 0 = H_1(W;\check{T}(L)) \Leftrightarrow H_0(W;L \otimes_{\mathbf{Z}_p} \mathbf{Z}/p) = 0.$$

(10) $$H^0(W;\check{T}(L)) = 0 \Leftrightarrow H^0(W;L) = 0 = H^1(W;L) \Leftrightarrow H^0(W;\operatorname{Hom}(\mathbf{Z}/p,\check{T}(L))) = 0.$$

Proof. The inclusion $H^0(W;L) \rightarrowtail L$ has a right inverse because its cokernel is a torsion-free, hence free, $\mathbf{Z}_p$-module. Then also $\check{T}H^0(W;L) \to H^0(W;\check{T}(L)) \subseteq \check{T}(L)$ is injective by functoriality. Since the first homology group $H_1(W;L/H^0(W;L))$ is finite, the long exact coefficient sequence in homology shows that $H^0(W;L) \to H_0(W;L)$ is injective. The $\mathbf{Q}_pW$-module $L \otimes \mathbf{Q}_p$ contains $H^0(W;L\otimes\mathbf{Q}_p)$ as a direct summand, so $H^0(W;L\otimes\mathbf{Q}_p) \subseteq H_0(W;L\otimes\mathbf{Q}_p)$, and it contains $H_0(W;L\otimes\mathbf{Q}_p)$ as a direct summand, so $H_0(W;L\otimes\mathbf{Q}_p) \subseteq H^0(W;L\otimes\mathbf{Q}_p)$. Thus the vector spaces $H^0(W;L)\otimes\mathbf{Q}_p \cong H^0(W;L\otimes\mathbf{Q}_p)$ and $H_0(W;L)\otimes\mathbf{Q}_p \cong H_0(W;L\otimes\mathbf{Q}_p)$ have the same dimension. This shows that the cokernel of the monomorphism $H^0(W;L) \rightarrowtail H_0(W;L)$ is finite. Apply the left exact functor $H^0(W;-)$ to (11.2) and, using (1), conclude that $H^0(W;SL) = 0$. Apply the right exact functor to (11.2) and conclude that $H_0(W;SL)$ is finite (and, using (11.3), trivial if W is generated by elements of order prime to p). This proves the first three items and the next three are proved in a dual fashion. For (7), take the short exact sequence $0 \to \mathbf{Z}_p \to \mathbf{Q}_p \to \mathbf{Z}/p^\infty \to 0$ of $\mathbf{Z}_p$-modules. Apply $H_0(W;-) \circ (L \otimes -)$ and $H_0(W;L) \circ -$ to it and compare the results

$$\begin{tikzcd}
H_0(W;L) \arrow[r] & H_0(W;L\otimes \mathbf{Q}_p) \arrow[r] & H_0(W;\check{T}(L)) \arrow[r] & 0 \\
H_0(W;L) \arrow[r] \arrow[u, equal] & H_0(W;L)\otimes \mathbf{Q}_p \arrow[r] \arrow[u, "\cong"] & \check{T}H_0(W;L) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

to see that $\check{T}H_0(W;L) \cong H_0(W;\check{T}(L))$. Dually, compare the values of $H^0(W;-)\circ \mathrm{Hom}(-,\check{T}(L))$ and $\mathrm{Hom}(-,H^0(W;\check{T}(L)))$ applied to the same short exact sequence and conclude that $H^0(W;L)$ and $LH^0(W;\check{T}(L))$ are isomorphic. Combine these isomorphisms with (2) and (5) to obtain (8). To get the formulas of (10) and (11.4), simply apply the right exact functor $H_0(W;-)$ to the short exact sequence $0 \to L \xrightarrow{\cdot p} L \to L\otimes \mathbf{Z}/p \to 0$ and the left exact functor $H^0(W;-)$ to the short exact sequence $0 \to \mathrm{Hom}(\mathbf{Z}/p,\check{T}) \to \check{T} \xrightarrow{\cdot p} \check{T} \to 0$ where $L\otimes \mathbf{Z}/p = \mathrm{Hom}(\mathbf{Z}/p,\check{T})$. ■

From the commutative diagrams with exact rows

$$\begin{tikzcd}
0 \arrow[r] & SL \arrow[r] & L \arrow[r] & H_0(W;L) \arrow[r] & 0 \\
0 \arrow[r] & SL \arrow[r] \arrow[u, equal] & SL\times H^0(W;L) \arrow[r] \arrow[u, tail] & H^0(W;L) \arrow[r] \arrow[u, tail] \arrow[ul, dashed] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & H^0(W;\check{T}(L)) \arrow[r] \arrow[d, two heads] & \check{T}(L) \arrow[r] \arrow[d, two heads] \arrow[dl, dashed] & \check{T}(PL) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H_0(W;\check{T}(L)) \arrow[r] & H_0(W;\check{T}(L))\times \check{T}(PL) \arrow[r] & \check{T}(PL) \arrow[r] & 0
\end{tikzcd}$$

the Snake Lemma produces exact sequences of $\mathbf{Z}_pW$-modules

$$\begin{aligned}
&0 \to SL\times H^0(W;L) \to L \to \pi(L) \to 0,\\
&0 \to \pi(L) \to \check{T}(SL)\times \check{T}H^0(W;L) \to \check{T}(L) \to 0,\\
&0 \to \check{\pi}(L) \to \check{T}(L) \to H_0(W;\check{T}(L))\times \check{T}(PL) \to 0,\\
&0 \to L \to LH_0(W;\check{T}(L))\times PL \to \check{\pi}(L) \to 0,
\end{aligned} \tag{11.5}$$

where $\pi(L)$ and $\check{\pi}(L)$ are the finite groups defined by the short exact sequences

$$0 \to H^0(W;L) \to H_0(W;L) \to \pi(L) \to 0, \tag{11.6}$$

$$0 \to \check{\pi}(L) \to H^0(W;\check{T}(L)) \to H_0(W;\check{T}(L)) \to 0 \tag{11.7}$$

of abelian groups. We have thus constructed functors

$$\check{s} : \mathbf{Z}_pW\text{-}\mathbf{mod} \to (\mathbf{Z}_pW\text{-}\mathbf{mod})^{\bullet\leftarrow\bullet\rightarrow\bullet},$$
$$s : \mathbf{Z}_pW\text{-}\mathbf{mod} \to (\mathbf{Z}_pW\text{-}\mathbf{mod})^{\bullet\rightarrow\bullet\leftarrow\bullet},$$
$$\check{s}(L) = (\check{T}(SL) \leftarrowtail \pi(L) \xrightarrow{\varphi(L)} \check{T}H^0(W;L)),$$
$$s(L) = (PL \twoheadrightarrow \check{\pi}(L) \xleftarrow{\check{\varphi}(L)} LH_0(W;\check{T}(L)))$$

from the category of $\mathbf{Z}_pW$-modules into the category of push-out (pull-back) diagrams of $\mathbf{Z}_pW$-modules. Since we can recover L from the value of these functors in that $\operatorname{colim}\check{s}(L) = \check{T}(L)$ and $L = \lim s(L)$, the classification of $\mathbf{Z}_pW$-modules has been reduced to the classification of $\mathbf{Z}_pW$-modules L with $\pi(L) = 0$ or $\check{\pi}(L) = 0$.

11.8. Lemma. *Let L be a $\mathbf{Z}_pW$-lattice and assume that W is generated by elements of order prime to p.*

(1) *$\pi(PL) = H_0(W;PL)$, $\check{\pi}(SL) = H^0(W;\check{T}(SL))$, $\check{\pi}(PL) = 0 = \pi(SL)$, and $\pi(PL) \cong \check{\pi}(SL)$.*

(2) *$\pi(L) \to H^0(W;\check{T}(SL)) = \check{\pi}(SL)$ is injective and $\pi(PL) = H_0(W;PL) \to \check{\pi}(L)$ is surjective.*

(3) *$0 \to H_0(W;L) \to LH_0(W;\check{T}(L)) \times \pi(PL) \to \check{\pi}(L) \to 0$ is an exact sequence.*

(4) *$0 \to \pi(L) \to \check{\pi}(SL) \times \check{T}H^0(W;L) \to H^0(W;\check{T}(L)) \to 0$ is an exact sequence.*

(5) *$SSL = SL = SPL$ and $PPL = PL = PSL$.*

(6) *$\pi(L) = 0 \Leftrightarrow SL \times H^0(W;L) = L$ and $\check{\pi}(L) = 0 \Leftrightarrow L = LH_0(W;\check{T}(L)) \times PL$.*

(7) *If $H^0(W;L) = 0$, then $\pi(L) = H_0(W;L)$, $\check{\pi}(L) = H^0(W;\check{T}(L))$, and there is a short exact sequence $0 \to \pi(L) \to \check{\pi}(SL) \to \check{\pi}(L) \to 0$.*

(8) *$\varphi(L) \in \operatorname{Hom}(\pi(L), \check{T}H^0(W;L)) \cong \operatorname{Ext}(\pi(L), H^0(W;L))$ classifies the above abelian extension* (11.6) *and $\check{\varphi}(L) \in \operatorname{Hom}(LH_0(W;\check{T}(L)), \check{\pi}(L)) \cong \operatorname{Ext}(H_0(W;\check{T}(L)), \check{\pi}(L))$ classifies* (11.7)*.*

(9) *$H_0(W;\check{T}(L)) \cong \operatorname{coker}\varphi(L)$, $H_1(W;\check{T}(L)) \cong \ker\varphi(L)$, $H^0(W;L) \cong \ker\check{\varphi}(L)$, $H^1(W;L) \cong \operatorname{coker}\check{\varphi}(L)$.*

(10) *$0 \to H^0(W;L) \to L \to PL \to H^1(W;L) \to 0$ and $0 \to H_1(W;\check{T}(L)) \to \check{T}(SL) \to \check{T}(L) \to H_0(W;\check{T}(L)) \to 0$ are exact sequences.*

Proof. (1) is true because $H^0(W;PL) = 0 = H^0(W;\check{T}(PL))$ by (11.4.6). For (2), note that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi(L) \arrow[r, "(\alpha,\beta)"] \arrow[dr] & \check{T}(SL) \times \check{T}H^0(W;L) \arrow[r] \arrow[d] & \check{T}(L) \arrow[r] & 0 \\
& & \check{T}(L) \times \check{T}(L) \arrow[ur, "+"'] & &
\end{tikzcd}$$

for some homomorphisms α and β. For all $x \in \pi(L)$, $\alpha(x)+\beta(x) = 0$ in $\check{T}(L)$. If $\alpha(x) = 0$ in $\check{T}(SL)$, then also $\alpha(x) = 0$ in $\check{T}(L)$ so $\beta(x) = 0$ in $\check{T}(L)$. But this means that the monomorphism (α, β) takes x to 0, so $x = 0$. Thus α is a monomorphism. Apply the functor $H_0(W; -)$ to a short exact sequence from (11.5) to obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d] & LH_0(W;\check{T}(L)) \times PL \arrow[r] \arrow[d] & \check{\pi}(L) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H_0(W;L) \arrow[r] & LH_0(W;\check{T}(L)) \times H_0(W;PL) \arrow[r] & \check{\pi}(L) \arrow[r] & 0
\end{tikzcd}$$

with a bottom row that is exact according to (11.3). Conclude that $SPL = SL$. Using the exact sequence of (3) and (11.4(7)) we see that there is a short exact sequence

$$0 \to \ker(H_0(W;L) \to L\check{T}H_0(W;L)) \to \pi(PL) \to \check{\pi}(L) \to 0$$

for any $\mathbf{Z}_pW$-module L. Applied to SL, this gives $\pi(PL) \cong \check{\pi}(SL)$.

For (9) and (10) apply the left exact functor $H^0(W;-)$ to one of the short exact sequences from (11.5) and get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] & LH_0(W;\check{T}(L)) \times PL \arrow[r] & \check{\pi}(L) \arrow[r] & 0 \\
0 \arrow[r] & H^0(W;L) \arrow[r] \arrow[u, tail] & LH_0(W;\check{T}(L)) \arrow[r, "\check{\varphi}(L)"] \arrow[u, tail] & \ker(\check{\pi}(L) \to H^1(W;L)) \arrow[r] \arrow[u, tail] & 0
\end{tikzcd}$$

using $H^0(W;PL) = 0 = H^1(W;PL)$ (11.4(10)). Now apply the Snake Lemma. ∎

The group W acts on the dual $\mathbf{Z}_p$-lattice $L^\vee = \mathrm{Hom}(L, \mathbf{Z}_p)$ according to the rule $(w \cdot \varphi)(x) = \varphi(w^{-1}x)$, $w \in W$, $\varphi \in L^\vee$, $x \in L$. The W-equivariant duality pairing

$$\check{T}(L) \times L^\vee \to \mathbf{Z}/p^\infty \tag{11.9}$$

obtained from the identification $L^\vee = \mathrm{Hom}(L, L(\mathbf{Z}/p^\infty)) = \mathrm{Hom}(\check{T}(L), \mathbf{Z}/p^\infty)$ induces pairings

$$\begin{aligned} &H_*(W;\check{T}(L)) \times H^*(W;L^\vee) \to \mathbf{Z}/p^\infty, \\ &H^*(W;\check{T}(L)) \times H_*(W;L^\vee) \to \mathbf{Z}/p^\infty \end{aligned} \tag{11.10}$$

relating homology and cohomology groups. (A *duality pairing* of $\mathbf{Z}_p$-modules is a bilinear map $A \times B \to C$ of $\mathbf{Z}_p$-modules such that the adjoint homomorphisms $A \to \mathrm{Hom}_{\mathbf{Z}_p}(B, \mathbf{C})$ and $B \to \mathrm{Hom}_{\mathbf{Z}_p}(A, \mathbf{C})$ are isomorphisms.)

11.11. Lemma. *Let L be a $\mathbf{Z}_pW$-lattice and $L^\vee$ its dual. Assume that W is generated by elements of order prime to p.*

(1) *The bilinear maps* (11.10) *are duality pairings.*
(2) $S(L^\vee) = (PL)^\vee$.

Proof. It is immediate that

$$H^*(W; L^\vee) = H^*(W; \mathrm{Hom}(\check{T}(L), \mathbf{Z}/p^\infty)) \cong \mathrm{Hom}(H_*(W; \check{T}(L)), \mathbf{Z}/p^\infty)$$

for $\mathrm{Hom}(-, \mathbf{Z}/p^\infty)$ is an exact functor. But then also

$$H_*(W; \check{T}(L)) \cong \mathrm{Hom}(H^*(W; L^\vee), \mathbf{Z}/p^\infty)$$

because $A \cong \mathrm{Hom}(\mathrm{Hom}(A, \mathbf{Z}/p^\infty), \mathbf{Z}/p^\infty)$ for any $\mathbf{Z}_p$-torus, $\mathbf{Z}_p$-lattice, or finite $\mathbf{Z}_p$-module A. Apply the exact functor $\mathrm{Hom}(-, \mathbf{Z}/p^\infty)$ to the short exact sequence $0 \to S(L^\vee) \to L^\vee \to H_0(W; L^\vee) \to 0$ to get the short exact sequence

$$0 \to H^0(W; \check{T}(L)) \to \check{T}(L) \to \check{T}(S(L^\vee)^\vee) \to 0$$

and conclude that $PL = S(L^\vee)^\vee$. ∎

Suppose that the group $W = W_1 \times \ldots \times W_n$ is the direct product of finitely many of its normal subgroups $W_1, \ldots, W_n$. For $j = 1, \ldots, n$, let

$$W_j^\perp = \prod_{i \neq j} W_i$$

denote the product of all these subgroups but W_j. Then $W = W_j \times W_j^\perp$ and $W_j = \bigcap_{i \neq j} W_i^\perp$. Observe that $H^0(W_i^\perp; L)$ is a $\mathbf{Z}_p W_i$-module and also that the direct sum $\coprod H^0(W_i^\perp; L)$ is a $\mathbf{Z}_p W$-module with a natural $\mathbf{Z}_p W$-module homomorphism to L given by addition.

11.12. Lemma [32, 1.5]. *If $H_0(W; \check{T}(L)) = 0 = H^0(W; L)$ for a $\mathbf{Z}_p W$-lattice L, then there is a $\mathbf{Z}_p W$-lattice U and a short exact sequence $0 \to \coprod H^0(W_i^\perp; L) \to L \to U \to 0$ of $\mathbf{Z}_p W$-lattices. Each summand $H^0(W_i^\perp; L)$ is a $\mathbf{Z}_p W_i$-lattice and*

- $H^0(W_i; \check{T} H^0(W_i^\perp; L)) = 0$ *provided* $H^0(W; \check{T}(L)) = 0$,
- $H_0(W_i; H^0(W_i^\perp; L)) = 0$ *provided* $H_0(W; L) = 0$ *and each factor group W_i is generated by elements of order prime to p.*

Proof. This amounts to showing that the addition maps

$$\coprod H^0(W_i^\perp; L) \to L, \qquad \coprod \check{T}(H^0(W_i^\perp; L)) \to \check{T}(L)$$

are injective.

Suppose that (x_i), with $x_i \in H^0(W_i^\perp; L)$, satisfies $\sum x_i = 0$. Then, for an arbitrarily chosen index j, $x_j = -\sum_{i \neq j} x_i$. The left hand side is fixed by $W_j^\perp$ and the right hand side is fixed by $\bigcap_{i \neq j} W_i^\perp = W_j$. Thus x_j is fixed by $W_j^\perp \times W_j = W$, so that $x_j \in H^0(W; L)$. But $H^0(W; L) = 0$ by (11.4(8)). For the other addition map, recall from (11.4(1)) that $\check{T}(H^0(W_i^\perp; L))$ is contained in $H^0(W_i^{\perp};\check{T}(L))$ and proceed as above. The computation

$$\begin{aligned} H^0(W_i;\check{T}(H^0(W_i^{\perp};L))) &\subseteq H^0(W_i;H^0(W_i^{\perp};\check{T}(L))) \\ &= H^0(W_i\times W_i^{\perp};\check{T}(L)) = H^0(W;\check{T}(L)) \end{aligned}$$

shows that $H^0(W_i;\check{T}(H^0(W_i^{\perp};L))) = 0$ if $H^0(W;\check{T}(L)) = 0$. If W_i is generated by elements of order prime to p, then $H_1(W_i;\pi(H^0(W_i^{\perp};L))) = 0$ so that

$$H_0(W_i;H^0(W_i^{\perp};L)) \subseteq H_0(W_i;H_0(W_i^{\perp};L)) = H_0(W_i\times W_i^{\perp};L) = H_0(W;L)$$

proving the final assertion of the lemma. ■

We now specialize to *reflection subgroups.* If $W \subseteq \mathrm{Aut}(L)$ is a group of automorphisms of the $\mathbf{Z}_p$-lattice L, any $w\in W$ restricts to an automorphism Sw of SL and projects to an automorphism Pw of PL. If Sw is the identity on SL, then w is the identity on $\check{T}(L) = \operatorname{colim}\check{s}(L)$ so w is the identity. If w is a reflection on L, then Sw is a reflection on SL because $SL/SL^{\langle\sigma\rangle} \cong L/L^{\langle\sigma\rangle}$. This means that if W is a reflection subgroup of $\mathrm{Aut}(L)$ then also SW (resp. PW) is a reflection subgroup of $\mathrm{Aut}(SL)$ (resp. $\mathrm{Aut}(PL)$). Thus the S-construction and the P-construction (11.1) are endo-functors of the category $\mathbf{Z}_p$-**Refl** of $\mathbf{Z}_p$-reflection subgroups (4.1).

We wish to classify the elements of the category $\mathbf{Z}_p$-**Refl** up to similarity. The preceding general discussion implies the following first reduction of this classification problem.

11.13. Lemma. *Let (W_1,L_1) and (W_2,L_2) be two objects of $\mathbf{Z}_p$-**Refl**. Then the following three statements are equivalent*:

(1) (W_1,L_1) *and* (W_2,L_2) *are similar.*

(2) *The diagram*

$$\begin{tikzcd} \check{T}(SL_1) \arrow[d, "\check{T}(\theta)"', "\cong"] & \pi(L_1) \arrow[l, tail] \arrow[r] \arrow[d, "\cong"'] & \check{T}H^0(W_1;L_1) \arrow[d, "\cong"', "\check{T}(\psi_*)"] \\ \check{T}(SL_2) & \pi(L_2) \arrow[l, tail] \arrow[r] & \check{T}H^0(W_2;L_2) \end{tikzcd}$$

commutes for some similarity $(\alpha,\theta)\colon (SW_1,SL_1)\to(SW_2,SL_2)$, some isomorphism between $\pi(L_1)$ and $\pi(L_2)$, and some isomorphism $\psi : H^0(W_1;L_1) \to H^0(W_2;L_2)$.

(3) *The diagram*

$$\begin{tikzcd} PL_1 \arrow[r, two heads] \arrow[d, "\theta"', "\cong"] & \check{\pi}(L_1) \arrow[d, "\cong"'] & LH_0(W_1;\check{T}(L_1)) \arrow[l] \arrow[d, "\cong"', "(\check{T}\psi)_*"] \\ PL_2 \arrow[r, two heads] & \check{\pi}(L_2) & LH_0(W_2;\check{T}(L_2)) \arrow[l] \end{tikzcd}$$

commutes for some similarity $(\alpha,\theta)\colon (PW_1,PL_1)\to(PW_2,PL_2)$, *some isomorphism between* $\breve{\pi}(L_1)$ *and* $\breve{\pi}(L_2)$, *and an isomorphism* $\psi : H_0(W_1;\check{T}(L_1)) \to H_0(W_2;\check{T}(L_2))$.

The classification of similarity classes of objects (W,L) of $\mathbf{Z}_p$-**Refl** has now been reduced to the case where $\pi(L)=0$ or $\breve{\pi}(L)=0$. Fortunately, this is very easy.

11.14. Theorem [75]. *Let* (W_1,L_1) *and* (W_2,L_2) *be two objects of* $\mathbf{Z}_p$-**Refl** *where* p *is odd. Assume that* $\pi(L_1)=0=\pi(L_2)$ *or* $\breve{\pi}(L_1)=0=\breve{\pi}(L_2)$, $i=1,2$. *Then* (W_1,L_1) *and* (W_2,L_2) *are similar if they are* $\mathbf{Q}_p$-*similar.*

Proof. Assume that (W_1,L_1) and (W_2,L_2) are $\mathbf{Q}_p$-similar objects of $\mathbf{Z}_p$-**Refl** with $\breve{\pi}(L_1)=0=\breve{\pi}(L_2)$. Since $L_i=PL_i\times LH_0(W_i,\check{T}(L_i))$, $i=1,2$ (11.8(6)), it suffices (11.13) to show that (PW_1,PL_1) and (PW_2,PL_2) are similar. As the splitting constructed in (11.15) below depends on rational information only, it suffices to prove the theorem under the additional hypothesis that (W_i,L_i) be simple. This is done in (11.18) below by going through the Clark–Ewing classification table [20]. ■

11.15. Lemma [32, 75]. *Let* (W,L) *be an object of* $\mathbf{Z}_p$-**Refl** *where* p *is odd. If* $H^0(W;\check{T}(L))=0$ (*or* $H_0(W;L)=0$), *then*

$$(W,L)=\prod(W_i,L_i)$$

splits as a product of simple objects of $\mathbf{Z}_p$-**Refl** *with* $H^0(W_i;\check{T}(L_i))=0$ (*or* $H_0(W_i,L_i)=0$) *for all* i.

Proof. We shall only consider the case where $L=P$ is a $\mathbf{Z}_pW$-lattice with $H^0(W;\check{T}(P))=0$. As W is a finite reflection subgroup of $\mathrm{Aut}(P)$ and $H^0(W;P)=0$ (11.4(8)), the $\mathbf{Q}_pW$-module $P\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$ splits as a direct sum $\coprod M_i\cong P\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$ of finitely many irreducible $\mathbf{Q}_pW$-modules $M_1,\dots,M_n$. Each of these irreducible summands occurs with multiplicity one and carries a non-trivial W-action [32, p. 280]. Define W_i to be the subgroup of W that pointwise fixes $\bigoplus_{j\neq i}M_j$ so that the action of W_i is concentrated on the summand M_i. Then $W=\prod W_i$ is the direct product of these normal subgroups [32, 6.3] and, according to (11.12), P is isomorphic to the direct sum of the $\mathbf{Z}_pW$-lattices $H^0(W_i^{\perp};P)$. Observe that each summand $H^0(W_i^{\perp};L)$ is a $\mathbf{Z}_pW_i$-lattice and

- W_i is a reflection subgroup of $\mathrm{Aut}_{\mathbf{Z}_p}(H^0(W_i^{\perp};L))$,
- $(W_i,H^0(W_i^{\perp};L))$ is simple,
- $PH^0(W_i^{\perp};L)=H^0(W_i^{\perp};L)$.

Indeed, the first item is implicit in the proof of [32, 6.3], the second item is clear because the rationalization $H^0(W_i^{\perp};L)\otimes_{\mathbf{Z}_p}\mathbf{Q}_p=H^0(W_i^{\perp};L\otimes_{\mathbf{Z}_p}\mathbf{Q}_p)=M_i$ by construction, and the third item is contained in (11.12). ■

11.16. Lemma. *Let (W, L) be a $\mathbf{Z}_p$-reflection group.*

(1) *(W, L) and $(W, L^\vee)$ are $\mathbf{Q}_p$-similar and $\check{\pi}(L) \cong \pi(L^\vee)$.*
(2) *(W, SL) and $(W, S(L^\vee))$ are $\mathbf{Z}_p$-similar.*
(3) *(W, PL) and $(W, P(L^\vee))$ are $\mathbf{Z}_p$-similar.*
(4) *(W, SL) and $(W, (PL)^\vee)$ are $\mathbf{Z}_p$-similar.*

Proof. For (2), first note that $S(L^\vee) = SP(L^\vee) = S((SL)^\vee)$ by (11.11(2)). But SL is (11.15) a product of simple $\mathbf{Z}_p$-reflection groups (W_i, L_i) with $H_0(W_i; L_i) = 0$. So $(SL)^\vee$ is isomorphic to the product $\prod(W_i, L_i^\vee)$ and $S((SL)^\vee) = \prod(W_i, S(L_i^\vee))$. By inspection (of reflection group family 1 and $W(\mathrm{E}_6)$ at $p = 3$), we see that (W_i, L_i) and $(W_i, S(L_i^\vee))$ are $\mathbf{Z}_p$-similar. Thus SL and $S(L^\vee)$ are $\mathbf{Z}_p$-similar. Moreover, the isomorphisms

$$\begin{aligned} H^0(W; L^\vee) &\cong \mathrm{Hom}(H_0(W; \check{T}(L)), \mathbf{Z}/p^\infty) \cong \mathrm{Hom}(\check{T}H_0(W; L), \mathbf{Z}/p^\infty) \\ &\cong \mathrm{Hom}(H_0(W; L), \mathbf{Z}_p) \end{aligned}$$

from (11.11(1)) show that the lattices $H^0(W; L^\vee)$ and $H^0(W; L)$ have the same rank. Therefore (W, L) and $(W, L^\vee)$ are $\mathbf{Q}_p$-similar (11.5). Finally, $(W, SL) \cong (W, P(L^\vee)^\vee) \cong (W, (PL)^\vee)$ by (11.11(2)) again. ∎

11.17. Lemma. *Let (W, L) be a $\mathbf{Z}_p$-reflection group. Then there are natural group isomorphisms $H_0(W; L^\vee \otimes \mathbf{Z}/p) \cong \mathrm{Ext}(H^0(W; \check{T}(L)), \mathbf{Z}/p)$ and $H_1(W; L^\vee \otimes \mathbf{Z}/p) \cong \mathrm{Hom}(H^0(W; \check{T}(L)), \mathbf{Z}/p)$.*

Proof. Using (11.11), we get $H_0(W; L^\vee \otimes \mathbf{Z}/p) = H_0(W; L^\vee) \otimes \mathbf{Z}/p = \mathrm{Hom}(H^0(W; \check{T}(L)), \mathbf{Z}/p^\infty) \otimes \mathbf{Z}/p = \mathrm{Ext}(H^0(W; \check{T}(L)), \mathbf{Z}/p)$. In the universal coefficient exact sequence

$$0 \to H_1(W; L^\vee) \otimes \mathbf{Z}/p \to H_1(W; L^\vee \otimes \mathbf{Z}/p) \to \mathrm{Tor}(H_0(W; L^\vee), \mathbf{Z}/p) \to 0$$

the term to the right identifies to $\mathrm{Hom}(H_0(W; \check{T}(L)), \mathbf{Z}/p)$ and the term to the left is trivial because $H_1(W; L^\vee) = \mathrm{Hom}(H^1(W; \check{T}(L)), \mathbf{Z}/p^\infty)$ and $H^1(W; \check{T}(L)) = 0$ [5, 3.3]. ∎

Recall that $G_0(W, L)$ stands for the set of similarity classes of reflection subgroups that are $\mathbf{Q}_p$-similar to (W, L) (4.1).

Write $P_{p^i}\mathrm{SU}(r+1)$ for the quotient $\mathrm{SU}(r+1)/C_{p^i}$ of $\mathrm{SU}(r+1)$ by the central subgroup C_{p^i} of order p^i for $0 \le i \le \nu_p(r+1)$ where $\nu_p(r+1)$ is the highest power of p that divides $r+1$.

11.18. Lemma. *Let (W, L) be a simple object of $\mathbf{Z}_p$-**Refl**. Then $G_0(W, L) = *$ except that*

(1) *$G_0(W(\mathrm{SU}(r+1))) = \{W(P_{p^i}\mathrm{SU}(r+1)) \mid 0 \le i \le \nu_p(r+1)\}$ contains $\nu_p(r+1) + 1$ elements.*
(2) *$G_0(W(\mathrm{E}_6)) = \{W(\mathrm{E}_6), W(\mathrm{PE}_6)\}$ contains two elements if $p = 3$.*

Proof. The reflection subgroup $r_p(W,L) = (W, L \otimes \mathbf{Z}/p)$ is irreducible, and hence $G_0(W) = *$ (4.5(1)), unless (W,L) is in Clark–Ewing family 1 or $p=3$ and r_0W is $r_0W(\mathrm{E}_6)$ or $r_0W(\mathrm{G}_2)$ [4, 6.2]. All the Lie cases are covered by G. Maxwell [56, Table I]. (See also [22] or [84, 5.1] for the A-family.) ∎

We learn from (11.18) that two simple objects, (W_1,L_1) and (W_2,L_2), of $\mathbf{Z}_p$-**Refl** are similar if they are $\mathbf{Q}_p$-similar and either $\pi(L_1) \cong \pi(L_2)$ or $\check{\pi}(L_1) \cong \check{\pi}(L_2)$. Combined with the splitting result (11.12), this proves (11.14).

Let (W,L) be an object of $\mathbf{Z}_p$-**Refl**. We shall next describe $G_0(W,L)$ as a partially ordered set. For given diagrams

$$\begin{aligned} &\alpha\colon \pi(PL) = H_0(W;PL) \twoheadrightarrow \check{\pi} \leftarrow LH^0(W;\check{T}(L)), \\ &\check{\alpha}\colon \check{\pi}(SL) = H^0(W;\check{T}(SL)) \leftarrowtail \pi \to \check{T}H^0(W;L) \end{aligned} \tag{11.19}$$

of $\mathbf{Z}_p$-modules, put

$$\begin{aligned} &S_\alpha(PL) = \lim(PL \to \check{\pi} \leftarrow LH^0(W;\check{T}(L))), \\ &\check{T}(P_{\check{\alpha}}(SL)) = \operatorname{colim}(\check{T}(SL) \leftarrow \pi \to \check{T}H^0(W;L)), \end{aligned} \tag{11.20}$$

so that $\check{\pi}(S_\alpha(PL)) = \check{\pi}$ and $\pi(P_{\check{\alpha}}(SL)) = \pi$. There are defining short exact sequences

$$\begin{aligned} &0 \to S_\alpha(PL) \to LH^0(W;\check{T}(L)) \times PL \to \check{\pi} \to 0, \\ &0 \to SL \times H^0(W;L) \to P_{\check{\alpha}}(SL) \to \pi \to 0 \end{aligned}$$

of $\mathbf{Z}_pW$-modules. We have previously (11.5) seen that

$$S_{\pi(PL)\twoheadrightarrow\check{\pi}(L)\leftarrow H^0(W;L)}(PL) = L = P_{\check{\pi}(SL)\leftarrowtail\pi(L)\to\check{T}H^0(W;L)}(SL).$$

By naturality, W is a reflection subgroup of $\operatorname{Aut}(S_\alpha(PL))$ and of $\operatorname{Aut}(P_{\check{\alpha}}(SL))$. Also by naturality, there are morphisms

$$\begin{aligned} &S_\alpha(PL) \to S_{\pi(PL)\twoheadrightarrow 0\leftarrow H^0(W;L)}(PL) = PL \times H^0(W;L), \\ &H^0(W;L) \times SL = P_{\check{\pi}(SL)\leftarrowtail 0\to\check{T}H^0(W;L)}(SL) \to P_{\check{\alpha}}(SL) \end{aligned}$$

showing that $(W, S_\alpha(PL))$ and $(W, P_{\check{\alpha}}(SL))$ are $\mathbf{Q}_p$-similar to (W,L). Conversely, any element of $G_0(W,L)$ will have this form because if (W_1,L_1) and (W_2,L_2) are $\mathbf{Q}_p$-similar then (SW,SL_1) and (SW,SL_2) ((PW,PL_1) and (PW,PL_2)) are $\mathbf{Z}_p$-similar by (11.14) and clearly $H^0(W_1;L_1)$ and $H^0(W_2;L_2)$ are isomorphic $\mathbf{Z}_p$-lattices.

Declare two diagrams of the form considered in (11.19) to be *equivalent* if they can be connected by an automorphism in $\operatorname{Aut}_{\mathbf{Z}_p\text{-}\mathbf{Refl}}(W,PL)$ (or $\operatorname{Aut}_{\mathbf{Z}_p\text{-}\mathbf{Refl}}(W,SL)$) (4.1) and an automorphism in $\operatorname{Aut}(H^0(W;L))$ as in (11.13).

11.21. Lemma. *For any object* (W, L) *of* $\mathbf{Z}_p$-**Refl**, p *odd*, *there is a bijection between the following three sets*:

(1) $G_0(W, L)$.

(2) *Equivalence classes of diagrams*

$$\pi(PL) \twoheadrightarrow \check{\pi} \leftarrow LH^0(W; \check{T}(L))$$

of $\mathbf{Z}_p$*-modules.*

(3) *Equivalence classes of diagrams*

$$\check{\pi}(SL) \leftarrowtail \pi \rightarrow \check{T}H^0(W; L)$$

of $\mathbf{Z}_p$*-modules.*

Since $\check{\pi}(SL) \cong \pi(PL)$ is a finite group (11.4(2)), $G_0(W, L)$ is a finite set. Our next aim is to introduce an ordering relation on $G_0(W, L)$.

11.22. Lemma. *For a* $\mathbf{Z}_p$-**Refl** *morphism* $(\alpha, \theta)\colon (W_1, L_1) \to (W_2, L_2)$ *the following three statements are equivalent*:

(1) $r_0(\alpha, \theta)\colon r_0(W_1, L_1) \to r_0(W_2, L_2)$ *is a similarity in* $\mathbf{Q}_p$-**Refl** *and* W_2 *acts trivially on* $\operatorname{coker}\theta$.

(2) $S(\alpha, \theta)\colon S(W_1, L_1) \to S(W_2, L_2)$ *is a similarity in* $\mathbf{Z}_p$-**Refl** *and the induced morphism of* $\mathbf{Z}_p$*-tori* $\check{T}((\alpha, \theta)_*)\colon \check{T}H^0(W_1; L_1) \to \check{T}H^0(W_2; L_2)$ *is an epimorphism with finite cokernel.*

(3) $P(\alpha, \theta)\colon P(W_1, L_1) \to P(W_2, L_2)$ *is a similarity in* $\mathbf{Z}_p$-**Refl** *and the induced morphism of* $\mathbf{Z}_p$*-lattices* $(\alpha, \theta)_*\colon H^0(W_1; L_1) \to H^0(W_2; L_2)$ *is a monomorphism with finite kernel.*

Proof. Assume that $L_1 \to L_2$ is injective with finite cokernel H. Then there is a short exact sequence

$$0 \to \ker(H_0(W_1; L_1) \to H_0(W_2; L_2)) \to \operatorname{coker}(SL_1 \to SL_2) \\ \to \ker(H \to H_0(W_2; H)) \to 0$$

provided by the Snake Lemma. If the middle term is trivial, then $H = H_0(W_2; H)$. If W_2 acts trivially on H, then the kernel to the left is trivial because $H_1(W_2; H) = 0$ by (11.3), and the kernel to the right is trivial because $H = H_0(W_2; H)$. The proof for $PL_1 \to PL_2$ is completely dual. ∎

11.23. Definition. An *isogeny* is a $\mathbf{Z}_p$-**Refl** morphism $(\alpha, \theta) : (W_1, L_1) \to (W_2, L_2)$ that satisfies one of the three equivalent conditions of (11.22).

Write $(W_1, L_1) \geq (W_2, L_2)$ if there exists an isogeny $(W_1, L_1) \to (W_2, L_2)$.

11.24. Lemma. *If* $(W_1, L_1) \geq (W_2, L_2) \geq (W_1, L_1)$ *then* (W_1, L_1) *and* (W_2, L_2) *are similar objects of* $\mathbf{Z}_p$-**Refl**.

Proof. An isogeny $(W_1, L_1) \to (W_2, L_2)$ induces a commutative diagram

$$\begin{tikzcd}
\check{T}(SL_1) \arrow[d, "\cong"'] & \pi(L_1) \arrow[l, tail] \arrow[r] \arrow[d, "\cong"] & \check{T}H^0(W_1; L_1) \arrow[d, dashed] \\
\check{T}(SL_2) & \pi(L_2) \arrow[l, tail] \arrow[r] & \check{T}H^0(W_2; L_2)
\end{tikzcd}$$

which can be completed [59] by a vertical isomorphism to the right. ∎

Thus the relation $\geq$ induces a partial ordering relation on the set of similarity classes of objects of $\mathbf{Z}_p$-**Refl**; in particular on the set $G_0(W, L)$. For any object (W, L),

$$(W, SL \times H^0(W; L)) \geq (W, L) \geq (W, LH_0(W; \check{T}(L)) \times PL)$$

by (11.5) and actually

$$\begin{aligned}
G_0(W, L) &= \{(W', L') \mid (W', L') \geq (W, LH_0(W; \check{T}(L)) \times PL)\} \\
&= \{(W', L') \mid (W, SL \times H^0(W; L)) \geq (W', L')\}
\end{aligned}$$

is the set of similarity classes of objects above $LH_0(W; \check{T}(L)) \times PL$ or below $SL \times H^0(W; L)$.

I close this section with a few remarks about the set $G_p(W, L)$ (4.1).

11.25. Lemma. *Let (W, L) be an object of* $\mathbf{Z}_p$-**Refl**.

(1) *If (W, L) is simple, then $G_p(W, L) \subseteq G_0(W, L)$.*

(2) $G_0(W, L) \cap G_p(PW, PL) = * = G_0(W, L) \cap G_p(SW, SL)$ *if* $H^0(W; L) = 0$.

(3) *If (W, L) is simple, then $G_p(W, L) = *$ unless (W, L) is similar to $(W(X), L(X))$ for $X = P_{p^i}\mathrm{SU}(r + 1))$, $0 < i < \nu_p(r + 1)$.*

Proof. $G_p(W) \subseteq G_0(W)$ when W is simple because any two abstractly isomorphic groups from the Clark–Ewing list happen to have the same rank r and to be conjugate as subgroups of $\mathrm{GL}(r, \mathbf{Q}_p)$ [4, 2.6]. When $H^0(W; L) = 0$, PL is the unique object of $G_0(W, L)$ with $\check{\pi} = 0$ (11.21); this condition can (11.8(6)) be read off from $L \otimes \mathbf{Z}/p$. ∎

11.26. Example. Put $(W, L) = (W, L)(\mathrm{PU}(r+1))$ so that $\pi(L)$ is cyclic of order p^ν where this is the highest power of p that divides $r + 1$. The $\nu + 1$ elements of $G_0(W, L)$ are represented by the centerings (W, L_i), $0 \leq i \leq \nu$, where $L_i \subseteq L$ is the inverse image of the order p^i subgroup of $\pi(L)$ (11.21), [84, 5.1]. Thus $\pi(L_i)$ is cyclic of order p^i and $L = L_\nu$. Assume now that $0 < i < \nu$ so that both $\pi(L_i)$ and $\check{\pi}(L_i)$ are non-trivial cyclic p-groups. As pointed out to me by D. Notbohm, tensoring the commutative diagram of $\mathbf{Z}_pW$-modules with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L_0 \arrow[r] \arrow[d, equal] & L_i \arrow[r] \arrow[d] & \pi(L_i) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L_0 \arrow[r] & L_\nu \arrow[r] \arrow[d] & \pi(L_\nu) \arrow[r] \arrow[d] & 0 \\
 & & \breve{\pi}(L_i) \arrow[r, equal] \arrow[d] & \breve{\pi}(L_i) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

with $\mathbf{Z}/p$ results in the commutative diagram

$$\begin{tikzcd}
 & & & \operatorname{Tor}(\breve{\pi}(L_i), \mathbf{Z}/p) \arrow[r, equal] \arrow[d] & \operatorname{Tor}(\breve{\pi}(L_i), \mathbf{Z}/p) \arrow[d, "\cong"] & \\
0 \arrow[r, tail] & \operatorname{Tor}(\pi(L_i), \mathbf{Z}/p) \arrow[r, tail] & L_0 \otimes \mathbf{Z}/p \arrow[r] & L_i \otimes \mathbf{Z}/p \arrow[r] & \pi(L_i) \otimes \mathbf{Z}/p \arrow[r] & 0
\end{tikzcd}$$

with a split epimorphism to the right. We conclude that

$$L_i \otimes \mathbf{Z}/p \cong \operatorname{coker}(H^0(W; L_0 \otimes \mathbf{Z}/p) \to L_0 \otimes \mathbf{Z}/p) \oplus H_0(W; L_i \otimes \mathbf{Z}/p),$$

$0 < i < \nu$, as $\mathbf{F}_pW$-modules. (These modules are irreducible [38] and it is no coincidence [84, 3.3] that $L_i \otimes \mathbf{Z}/p$ have the same irreducible constituents, namely

$$\operatorname{coker}(H^0(W; L_0 \otimes \mathbf{Z}/p) \to L_0 \otimes \mathbf{Z}/p) \cong \ker(L_\nu \otimes \mathbf{Z}/p \to H_0(W; L_\nu \otimes \mathbf{Z}/p))$$

and $\mathbf{Z}/p$, for all i.) This shows that $G_p(W(P_{p^i}\mathrm{SU}(r+1)))$ consists of $\nu - 2$ elements for $0 < i < \nu$.

11.27. Example (cf. (9.9)). For

$$(W, L) = (W(X), L(X)), \qquad X = \mathrm{SU}(p) \times \mathrm{SU}(p),$$

the set $G_0(W, L)$ consists of four elements corresponding to the four subgroup-orbits under the action of the automorphism group $\mathrm{Aut}_{\mathbf{Z}_p\textbf{-Refl}}(W, PL)$ $= (\mathbf{Z}_p^\times \times \mathbf{Z}_p^\times) \rtimes \mathbf{Z}/2$ on $H_0(W; PL) = \mathbf{Z}/p \times \mathbf{Z}/p$.

11.28. Example. $G_0(W(X), L(X))$ for $X = \mathrm{U}(p^\nu)$ is the poset $\{(i, j) \in \mathbf{Z} \times \mathbf{Z} \mid 0 \leq j \leq i \leq \nu\}$ with lexicographic ordering. The point (i, j) corresponds to the diagram

$$\mathbf{Z}/p^\nu \supseteq \mathbf{Z}/p^i \xrightarrow{\cdot p^j} \mathbf{Z}/p^\infty$$

where $\mathbf{Z}/p^i$ is the subgroup of order p^i of $H^0(W(X); \check{T}(SL(X))) = \mathbf{Z}/p^\nu \subseteq \mathbf{Z}/p^\infty$. $\mathrm{U}(p^\nu)$ corresponds to $(\nu, 0)$ in this formalism. If $i_1 \leq i_2$ and $j_1 \leq j_2$, then the commutative diagram

$$\begin{tikzcd}
\mathbf{Z}/p^{\nu} \arrow[d, equal] & \mathbf{Z}/p^{i_1} \arrow[l, two heads] \arrow[r, "\cdot p^{j_1}"] \arrow[d, tail] & \mathbf{Z}/p^{\infty} \arrow[d, "\cdot p^{j_2-j_1}"] \\
\mathbf{Z}/p^{\nu} & \mathbf{Z}/p^{i_2} \arrow[l, two heads] \arrow[r, "\cdot p^{j_2}"] & \mathbf{Z}/p^{\infty}
\end{tikzcd}$$

shows that $(i_1, j_1) \leq (i_2, j_2)$.

12. Shapiro's lemma. The main purpose of this section is to introduce some notation to be used in Section 13.

For any set S and any abelian group M we put

$$M[S] = \mathbf{Z}[S] \otimes_{\mathbf{Z}} M, \qquad M\langle S\rangle = \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[S], M)$$

where $\mathbf{Z}[S]$ stands for the free abelian group with basis S. $M[-]$ is a covariant and $M\langle -\rangle$ a contravariant functor from the category of sets to the category **Ab** of abelian groups. ($M[-]$ (resp. $M\langle -\rangle$) is the left (resp. right) adjoint of the forgetful functor from abelian groups to sets.) $M\langle S\rangle$ can also be considered as the abelian group of all functions $u\colon S \to M$. In case S is a left G-set and M a left G-module for some group G, the rules

$$g(s \otimes m) = gs \otimes gm, \quad (gu)(s) = gu(g^{-1}s), \qquad g \in G,\ s \in S,\ m \in M,\ u\colon S \to M,$$

make $M[S]$ and $M\langle S\rangle$ into left G-modules. A special case occurs when S is the left G-set G/H of left cosets of a subgroup H of G.

12.1. Lemma. *$M[G/H]$ is isomorphic to the induced module $\mathrm{Ind}_H^G(M)$ and $M\langle G/H\rangle$ is isomorphic to the coinduced module $\mathrm{Coind}_H^G(M)$.*

Proof. Let T be a set of left coset representatives for G/H.

The set T is a basis for the free right $\mathbf{Z}H$-module $\mathbf{Z}G$. The induced module $\mathrm{Ind}_H^G(M) = \mathbf{Z}G \otimes_{\mathbf{Z}H} M$ is [95, 6.3.4] the sum over $|T|$ copies $t \otimes M$ of M with G-action $g(t \otimes m) = s \otimes hm$ where $gt = sh$, $s \in T$, $h \in H$. The module $M[G/H] = \mathbf{Z}[G/H] \otimes_{\mathbf{Z}} M$ is the sum over $|T|$ copies $t \otimes M$ of M with G-action $g(t \otimes m) = s \otimes gm$. The $\mathbf{Z}$-linear isomorphism $M[G/H] \to \mathrm{Ind}_H^G(M)$ that takes $t \otimes m$ to $t \otimes t^{-1}m$ is G-linear as it takes $g(t \otimes m) = y \otimes gm$ to $y \otimes y^{-1}gm = y \otimes ht^{-1}m = g(t \otimes t^{-1}m)$.

The set $T^{-1} = \{t^{-1} \mid t \in T\}$ is a basis for the free left $\mathbf{Z}H$-module $\mathbf{Z}G$. The coinduced module $\mathrm{Coind}_H^G(M)$ is [95, 6.3.4] the product over $|T|$ copies $\pi_t M$ of M, where $\pi_t m\colon \mathbf{Z}G \to M$ is the H-map sending t^{-1} to $m \in M$ and z^{-1} to 0 for all $z \neq t$ in T. The G-action is given by $g(\pi_t m) = \pi_y(hm)$. The module $M\langle G/H\rangle$ is the product over $|T|$ of copies $\varrho_t M$ of M, where $\varrho_t m\colon G/H \to M$ is the set map sending tH to m and zH to 0 for all $z \neq t$ in T. The G-action is given by $g(\varrho_t m) = \varrho_y(gm)$. The $\mathbf{Z}$-linear isomorphism

$\mathrm{Coind}_H^G(M) \to M\langle G/H\rangle$ that takes $\pi_t m$ to $\varrho_t(tm)$ is G-linear as it takes $g(\pi_t m) = \pi_y(hm)$ to $\varrho_y(yhm) = \varrho_y(gtm) = g\varrho_t(tm)$. ■

Let S_{hG} denote the homotopy colimit of S viewed as a functor from the category G to the category of sets. (S_{hG} is the nerve of the small groupoid that has S for object set and $\{g \in G \mid gs_1 = s_2\}$ as the set of morphisms $s_1 \to s_2$.) The next lemma is just a reformulation of Shapiro's lemma.

12.2. Lemma. *There are natural isomorphisms*

$$H_*(G; M[S]) \cong H_*(S_{hG}; M), \qquad H^*(G; M\langle S\rangle) \cong H^*(S_{hG}; M).$$

Proof. Let X be a set of representatives for the G-orbits in S and $G(x)$ the isotropy subgroup at $x \in X$. Then there are a homotopy equivalence

$$\coprod_{x\in X} BG(x) \to S_{hG}$$

and isomorphisms of G-modules

$$M[S] \cong \coprod M[G/G(x)] \cong \coprod \mathrm{Ind}_{G(x)}^G(M),$$
$$M\langle S\rangle \cong \prod M\langle G/G(x)\rangle \cong \prod \mathrm{Coind}_{G(x)}^G(M),$$

induced by the isomorphism $S \cong \coprod G/G(x)$ of G-sets. These isomorphisms combine, with the help of Shapiro's lemma, to the isomorphisms of the lemma. ■

In other words,

$$\begin{aligned} H_*(G; M[S]) &\cong \coprod H_*(G(x); M), \\ H^*(G; M\langle S\rangle) &\cong \prod H^*(G(x); M), \end{aligned} \tag{12.3}$$

where $x \in S$ runs through a set of representatives for the orbit set S/G.

13. Cellular cohomology of small categories. The following is a general discussion of the derived functors of the inverse limit.

Let $\mathbf{I}$ be a small category such that

- $\mathbf{I}$ has only finitely many objects,
- any endomorphism is an isomorphism,
- any isomorphism is an automorphism,

meaning that $\mathbf{I}$ is a special kind of very small ordered category [81] or EI-category [54]. $\mathbf{I}$ could for instance be a skeletal subcategory of the Quillen category of a p-compact group.

Write $S(i,j)$ for the set of morphisms from the object i to the object j and $\mathbf{I}(i)$ for the group of morphisms $i \to i$. Under the above assumptions, the set $\mathrm{Ob}(\mathbf{I})$ of objects of $\mathbf{I}$ has the structure of a partially ordered set (poset), where $i \leq j$ if there is a morphism from i to j. Let $K(\mathbf{I}) = \mathrm{Cx}(\mathrm{Ob}(\mathbf{I}))$

denote the ordered simplicial complex associated to $\mathrm{Ob}(\mathbf{I})$. The vertex set of $K(\mathbf{I})$ is the poset $\mathrm{Ob}(\mathbf{I})$ and the p-simplices, $p > 0$, is the set of all strictly increasing sequences $(i_0 \dots i_p)$ of elements of $\mathrm{Ob}(\mathbf{I})$ (where $i < j$ if $i \leq j$ and $i \neq j$). The ordered simplicial complex $K(\mathbf{I})$ is *d-dimensional* if there exists a string $i_0 \to \dots \to i_d$ of d morphisms between distinct objects but no such string of $d+1$ morphisms. $K(\mathbf{I})$ is again a poset with ordering given by inclusion.

For any p-simplex $(i_0 \dots i_p) \in K(\mathbf{I})_p$, put

$$\mathbf{I}(i_0 \dots i_p) = \mathbf{I}(i_p) \times \dots \times \mathbf{I}(i_0) \quad \text{and} \quad S(i_0 \dots i_p) = S(i_{p-1}, i_p) \times \dots \times S(i_0, i_1)$$

with the convention that for $p = 0$, $S(i_0)$ is understood to be a point. Form the homotopy orbit space

$$S_{h\mathbf{I}}(i_0 \dots i_p) = S(i_0 \dots i_p)_{h\mathbf{I}(i_0 \dots i_p)}$$

for the action of the group $\mathbf{I}(i_0 \dots i_p)$ on the set $S(i_0 \dots i_p)$ given by

$$(a_p, \dots, a_0) \cdot (a_{p-1,p}, \dots, a_{01}) = (a_p a_{p-1,p} a_{p-1}^{-1}, \dots, a_1 a_{01} a_0^{-1})$$

for all $a_j \in \mathbf{I}(i_j)$ and $a_{j-1,j} \in \mathbf{I}(i_{j-1}, i_j)$. This homotopy orbit space construction provides a functor

$$S_{h\mathbf{I}} \colon K(\mathbf{I})^{\mathrm{op}} \to \mathbf{Sp}$$

from the opposite poset of $K(\mathbf{I})$ to the category $\mathbf{Sp}$ of simplicial sets. For any inclusion $\sigma \leq \sigma'$ of simplices, the map $S_{h\mathbf{I}}(\sigma) \leftarrow S_{h\mathbf{I}}(\sigma')$ is induced by the obvious projection $\mathbf{I}(\sigma) \leftarrow \mathbf{I}(\sigma')$ and the map $S(\sigma) \leftarrow S(\sigma')$ given by composition or omission of morphisms in the usual way.

Let now $M \colon \mathbf{I} \to \mathbf{Ab}$ be a functor. Consider the functor $H^q M \colon K(\mathbf{I}) \to \mathbf{Ab}$ that takes the p-simplex $(i_0 \dots i_p) \in K(\mathbf{I})_p$ to the abelian group

$$H^q(S_{h\mathbf{I}}(i_0 \dots i_p); M(i_p)) = H^q(\mathbf{I}(i_0 \dots i_p); M(i_p) \langle S(i_0 \dots i_p) \rangle).$$

Define cohomology of $K(\mathbf{I})$ with coefficients in $H^q M$, $H^*(K(\mathbf{I}); H^q M)$, as the cohomology of the cochain complex $(C(K(\mathbf{I}); H^q M), \delta)$:

$$\text{(13.1)} \qquad \dots \to \prod_{(i_0 \dots i_{p-1}) \in K(\mathbf{I})_{p-1}} H^q M(i_0 \dots i_{p-1}) \xrightarrow{\delta^{p-1}} \prod_{(i_0 \dots i_p) \in K(\mathbf{I})_p} H^q M(i_0 \dots i_p) \to \dots$$

with differential

$$\delta^{p-1}(U)(i_0 \dots i_p) = \sum_{j=0}^{p} (-1)^j \phi_*^j \varrho_j^* U(i_0 \dots \widehat{i_j} \dots i_p)$$

for all cochains $U \in C^{p-1}(K(\mathbf{I}); H^qM)$ and all p-simplices $(i_0 \dots i_p)$ of the simplicial complex $K(\mathbf{I})$. Here,

$$\varrho_j \colon \mathbf{I}(i_p) \times \dots \times \mathbf{I}(i_0) \to \mathbf{I}(i_p) \times \dots \times \widehat{\mathbf{I}(i_j)} \times \dots \times \mathbf{I}(i_0)$$

is the projection and the homomorphisms

$$M(i_p)\,\langle \mathbf{I}(i_0, \dots, i_p)\rangle \xleftarrow{\phi^j} \begin{cases} M(i_p)\,\langle \mathbf{I}(i_1, \dots, i_p)\rangle & \text{if } j = 0, \\ M(i_p)\langle \mathbf{I}(i_0, \dots, \widehat{i_j}, \dots, i_p)\rangle & \text{if } 0 < j < p, \\ M(i_{p-1})\,\langle \mathbf{I}(i_0, \dots, i_{p-1})\rangle & \text{if } j = 0, \end{cases}$$

are given by

$$\phi^j(u)(a_{p-1,p}, \dots, a_{01}) = \begin{cases} u(a_{p-1,p}, \dots, a_{12}) & \text{if } j = 0, \\ u(a_{p-1,p}, \dots, a_{j,j+1}a_{j-1,j}, \dots, a_{01}) & \text{if } 0 < j < p, \\ M(a_{p-1,p})u(a_{p-2,p-1}, \dots, a_{01}) & \text{if } j = p. \end{cases}$$

It will become clear later that $\delta\delta = 0$, i.e. that $(C(K(\mathbf{I}); H^qM), \delta)$ is indeed a cochain complex.

Let $\lim^*(\mathbf{I}; -)$ denote the right derived functors of the inverse limit functor $\lim \colon \mathbf{Ab}^{\mathbf{I}} \to \mathbf{Ab}$.

13.2. Theorem [54], [87]. *There is a first quadrant cohomological spectral sequence E_r^{pq} with*

$$E_1^{pq} = C^p(K(\mathbf{I}); H^qM) \quad \textit{and} \quad E_2^{pq} = H^p(K(\mathbf{I}); H^qM)$$

converging to $\lim^{p+q}(\mathbf{I}; M)$.

This spectral sequence is associated to a descending filtration on the cochain complex $C(\mathbf{I}; M)$ that has $\lim^*(\mathbf{I}; M)$ for cohomology groups:

Let $\mathbf{\Delta}$ be the category of totally ordered finite sets and weakly order preserving maps. The cosimplicial replacement functor [14, XI.5]

$$\prod\nolimits^* \colon \mathbf{Ab}^{\mathbf{I}} \to \mathbf{Ab}^{\mathbf{\Delta}}$$

takes the abelian $\mathbf{I}$-group M to the cosimplicial abelian group $\prod^* M$ that in codegree n is the abelian group of twisted n-cochains of $\mathbf{I}$ with coefficients in M, i.e.

$$\Big(\prod\nolimits^* M\Big)^n = \prod_{i_0 \to \dots \to i_n \in N(\mathbf{I})_n} M(i_n)$$

consists of all functions U from $N(\mathbf{I})_n$ with values $U(i_0 \to i_1 \to \dots \to i_n)$ in $M(i_n)$. (As usual, the nerve, $N(\mathbf{I})$, of $\mathbf{I}$ is the singular set of I: The simplicial set that in degree 0 is the set of objects of $\mathbf{I}$ and in degree $n > 0$ is the set of all sequences $i_0 \to i_1 \to \dots \to i_n$ of n composable morphisms in $\mathbf{I}$.) The coface maps

$$d^j(U)(i_0 \to \dots \to i_{n+1}) = U(i_0 \to \dots \to \widehat{i_j} \to \dots \to i_{n+1}), \quad 0 \le j \le n,$$
$$d^{n+1}(U)(i_0 \to \dots \to i_{n+1}) = M(i_n \to i_{n+1})U(i_0 \to \dots \to i_n)$$

are the obvious ones. Define $C(\mathbf{I}; M)$ to be the underlying cochain complex whose differential is the alternating sum $\sum(-1)^i d^i$. The ith cohomotopy group of the cosimplicial abelian group $\prod^* M$,

$$\pi^i\Big(\prod{}^* M\Big) = H^i(C(\mathbf{I}; M)), \quad i \geq 0,$$

is defined [14, X.7.1] as the ith cohomology group of its underlying cochain complex $C(\mathbf{I}; M)$.

13.3. Lemma [14, XI.6.2], [81, Lemma 2]. *The functors*

$$\mathbf{Ab}^{\mathbf{I}} \xrightarrow{\prod^*} \mathbf{Ab}^{\boldsymbol{\Delta}} \xrightarrow{\pi^i} \mathbf{Ab}, \quad i \geq 0,$$

form a universal cohomological δ-functor [95, 2.1.1] *with* lim *in degree* 0.

In other words, $\lim^* = \pi^* \circ \prod^*$ and $\lim^i(\mathbf{I}; M) = H^i(C(\mathbf{I}; M))$ is the ith cohomology group of the cochain complex $C(\mathbf{I}; M)$ of $\mathbf{I}$ with (twisted) coefficients M.

Define l to be the function on $N(\mathbf{I})$ that is 0 on $N(\mathbf{I})_0$; on $N(\mathbf{I})_1$, $l(i \to i) = 0$ while $l(i \to j) = 1$ if i and j are non-isomorphic; and in general

$$l(i_0 \to i_1 \to \ldots \to i_n) = \sum_{i=0}^{n-1} l(i_i \to i_{i+1}),$$

the function l counts the number of strict inequalities in the string $i_0 \leq i_1 \leq \ldots \leq i_n$. This makes the nerve into a filtered simplicial set

$$\emptyset = F_0 N(\mathbf{I}) \subseteq F_1 N(\mathbf{I}) \subseteq \ldots \subseteq F_p N(\mathbf{I}) \subseteq F_{p+1} N(\mathbf{I}) \subseteq \ldots \subseteq N(\mathbf{I})$$

where

$$F_p N(\mathbf{I}) = \{i_0 \to i_1 \to \ldots \to i_n \in N(\mathbf{I}) \mid l(i_0 \to i_1 \to \ldots \to i_n) < p\}$$

is the set of all strings of composable morphisms where less than p of the morphisms have non-isomorphic domain and codomain. Since $\mathbf{I}$ has only finitely many equivalence classes of objects, the filtration is finite: $F_{d+1} N(\mathbf{I}) = N(\mathbf{I})$ if $K(\mathbf{I})$ has dimension d.

The filtration we are going to use is the induced descending filtration on the cochain complex $C(\mathbf{I}; M)$,

$$\begin{aligned} C(\mathbf{I}; M) = F_0 C(\mathbf{I}; M) \supseteq F_1 C(\mathbf{I}; M) \supseteq \ldots \\ \supseteq F_p C(\mathbf{I}; M) \supseteq F_{p+1} C(\mathbf{I}; M) \supseteq \ldots \supseteq \{0\} \end{aligned} \tag{13.4}$$

where

$$F_p C(\mathbf{I}; M) = \{U \in C(\mathbf{I}; M) \mid U F_p(N(\mathbf{I})) = 0\}$$

consists of all cochains that vanish on $F_p N(\mathbf{I})$. This filtration is finite: $F_{d+1} C(\mathbf{I}; M) = \{0\}$ if $K(\mathbf{I})$ is d-dimensional.

Proof of Theorem 13.2. Suppose that $K(\mathbf{I})$ is d-dimensional. Then the E_1-page of the spectral sequence associated [95, 5.4.1] to the filtration (13.4) satisfies $E_1^{pq} = 0$ whenever $p > d$ and $E_1^{d*} = H^*(F_dC(\mathbf{I}; M)[-d])$ where $F_dC(\mathbf{I}; M)[-d]$ is the translated cochain complex [95, 1.2.8] that in degree n equals $F_dC(\mathbf{I}; M)^{d+n}$. Note that

$$F_dC(\mathbf{I}; M)[-d] = \bigoplus_{(i_0 \dots i_d) \in K(\mathbf{I})_d} C(i_0 \dots i_d; M)$$

splits as a direct product over the d-simplices in $K(\mathbf{I})$ of the cochain complexes $C(i_0 \dots i_d; M)$ given by

$$\begin{aligned} &C(i_0 \dots i_d; M)^n \\ &\quad = M(i_d)\Big\langle \coprod_{r_0+\dots+r_d=n} \mathbf{I}(i_d)^{r_d} \times \mathbf{I}(i_{d-1}, i_d) \times \dots \times \mathbf{I}(i_0, i_1) \times \mathbf{I}(i_0)^{r_0} \Big\rangle \end{aligned}$$

with a differential that is the restriction of the differential on $C(\mathbf{I}; M)$. The claim is that the cohomology of $C(i_0 \dots i_d; M)$ equals $H^*M(i_0 \dots i_d)$ as defined above (13.1). The standard cochain complex for computing this cohomology group is

$$\begin{aligned} (13.5) \qquad &\mathrm{Hom}_{\mathbf{I}(i_d)\times\dots\times\mathbf{I}(i_0)}(B_*(\mathbf{I}(i_d)) \otimes \dots \otimes B_*(\mathbf{I}(i_0)), \\ &\qquad\qquad M(i_d)\,\langle \mathbf{I}(i_{d-1}, i_d) \times \dots \times \mathbf{I}(i_0, i_1)\rangle) \end{aligned}$$

where $B_*(\mathbf{I}(i_d)) \otimes \dots \otimes B_*(\mathbf{I}(i_0))$ as the tensor product of unnormalized bar resolutions has

$$\begin{aligned} \partial(a_{d\bullet} \otimes \dots \otimes a_{0\bullet}) = a_{d\bullet} &\otimes \dots \otimes \partial a_{0\bullet} \\ &+ (-1)^{r_0} a_{d\bullet} \otimes \dots \otimes \partial a_{1\bullet} \otimes a_{0\bullet} + \dots \\ &+ (-1)^{r_0+\dots+r_{d-1}} \partial a_{d\bullet} \otimes a_{d-1\bullet} \otimes \dots \otimes a_{0\bullet} \end{aligned}$$

as its differential. Here, $a_{j\bullet} = a_{jr_j} \otimes \dots \otimes a_{j1}$ where $a_{jk} \in \mathbf{I}(a_j)$ and

$$\partial a_{j\bullet} = a_{jr_j} \otimes \dots \otimes a_{j2} + \sum_{i_j=1}^{r_j-1} \dots a_{j,i_j+1} a_{ji_j} \otimes \dots + (-1)^{r_j} a_{jr_j} a_{j,r_j-1} \otimes \dots \otimes a_{j1}$$

as usual [95, 6.5.1]. In fact, there is an isomorphism, σ, of cochain complexes from the standard cochain complex (13.5) to $C(i_0 \dots i_d; M)$ given by

$$\begin{aligned} &\sigma(U)(a_{d\bullet}, a_{d-1,d}, \dots, a_{1\bullet}, a_{01}, a_{0\bullet}) \\ &\qquad = (-1)^{r_1+r_3+\dots} U(a_{d\bullet}, a_{d\bullet}a_{d-1,d}, \dots, a_{1\bullet}, a_{1\bullet}a_{01}, a_{0\bullet}) \end{aligned}$$

where $a_{j\bullet}a_{j-1,j} = a_{jr_j} \dots a_{j1}a_{j-1,j}$ and the sign is (-1) raised to the power that is the sum over all *odd* j of $r_j = |a_{j\bullet}|$. I leave it to the reader to check that this isomorphism σ indeed commutes with the differentials. The conclusion is that

$$E_1^{dq} = H^q(F_dC(\mathbf{I}; M)[-d]) \cong \prod_{(i_0 \dots i_d) \in K(\mathbf{I})_d} H^qM(i_0 \dots i_d)$$

is isomorphic to the degree d group of the cochain complex $C(K(\mathbf{I}); H^q M)$ (13.1).

This same pattern repeats itself at all stages of the filtration as

$$F_p C(\mathbf{I}; M)^{p+q} = F_{p+1} C(\mathbf{I}; M)^{p+q} \oplus \prod_{(i_0 \dots i_p) \in K(\mathbf{I})_p} C(i_0 \dots i_p; M)^{p+q}$$

and, in fact, there is an isomorphism

$$F_p C(\mathbf{I}; M)[-p] / F_{p+1} C(\mathbf{I}; M)[-p] \cong \prod_{(i_0 \dots i_p) \in K(\mathbf{I})_p} C(i_0 \dots i_p; M)$$

of cochain complexes. So, by the above computation,

$$E_1^{pq} \cong \prod_{(i_0 \dots i_p) \in K(\mathbf{I})_p} H^q M(i_0 \dots i_p)$$

is isomorphic to $C^p(K(\mathbf{I}); H^q M)$.

It remains to compute the d_1-differential. Again, it will be sufficient to consider the differential $d_1^{d-1,q} \colon E_1^{d-1,q} \to E_1^{dq}$ as similar arguments apply in general. Consider a cohomology class $[U(i_0 \dots i_{d-1})]$ in $H^q(i_{d-1} \dots i_0; M)$ represented by the q-cocycle

$$U(i_0 \dots i_{d-1}) \colon$$
$$\prod_{r_{d-1} + \dots r_0 = q} \mathbf{I}(i_{d-1})^{r_{d-1}} \times \dots \times \mathbf{I}(i_0)^{r_0} \to M(i_{d-1}) \left\langle \mathbf{I}(i_{d-2}, i_{d-1}) \times \dots \times \mathbf{I}(i_0, i_1) \right\rangle$$

and extend this to an element of $F_{d-1} C(\mathbf{I}; M)^{q+d-1}$ by mapping the other $(q+d-1)$-simplices of $N(\mathbf{I})$ to 0. The image $d_1^{d-1,q}[U(i_0 \dots i_{d-1})]$ is represented by $(\sigma^{-1}\delta\sigma)U(i_0 \dots i_{d-1})$ where δ is the zigzag-homomorphism of the short exact sequence

$$0 \to F_d C(\mathbf{I}; M) \to F_{d-1} C(\mathbf{I}; M) \to F_d C(\mathbf{I}; M)/F_{d-1} C(\mathbf{I}; M) \to 0$$

of cochain complexes. This means that $d_1^{d-1,q}[U(i_0 \dots i_{d-1})]$ vanishes on all $(q+d)$-simplices of $N(\mathbf{I})$ except on the ones of the form

$$i_0 \to \dots \to i_{j-1} \xrightarrow{a_{i_{j-1} i'_j}} i'_j \xrightarrow{a_{i'_j i_j}} i_j \to \dots \to i_{d-1} \tag{13.6}$$

for some object i'_j of $\mathbf{I}$, where it has the value

$$(-1)^{q+j} U(a_{d-1\bullet}, a_{d-2,d-1}, \dots, a_{j\bullet}, a_{i_{j-1} i'_j} a_{i'_j i_j}, a_{j-1\bullet}, \dots, a_{01}, a_{0\bullet})$$

assuming, for simplicity, that $0 < j < d-1$. We must compare this to the homomorphism

$$\begin{aligned}
&B_*(\mathbf{I}(i_{d-1})) \otimes \ldots \otimes B_*(\mathbf{I}(i_j)) \otimes B_*(\mathbf{I}(i'_j)) \otimes B_*(\mathbf{I}(i_{j-1})) \otimes \ldots \otimes B_*(\mathbf{I}(i_0)) \\
&\qquad \to B_*(\mathbf{I}(i_{d-1})) \otimes \ldots \otimes B_*(\mathbf{I}(i_0)) \\
&\qquad \xrightarrow{U(i_0 \ldots i_{d-1})} M(i_{d-1}) \langle \mathbf{I}(i_{d-2}, i_{d-1}) \times \ldots \times \mathbf{I}(i_0, i_1) \rangle \\
&\qquad \xrightarrow{\phi^j} M(i_{d-1}) \left\langle \mathbf{I}(i_{d-2}, i_{d-1}) \times \ldots \times \mathbf{I}(i'_j, i_j) \times \mathbf{I}(i_{j-1}, i'_j) \times \ldots \times \mathbf{I}(i_0, i_1) \right\rangle
\end{aligned}$$

where the first homomorphism takes $a_{d-1\bullet} \otimes \ldots \otimes a_{j\bullet} \otimes a'_{j\bullet} \otimes a_{j-1\bullet} \otimes \ldots \otimes a_{0\bullet}$ to $a_{d-1\bullet} \otimes \ldots \otimes a_{j\bullet} \otimes 1_\bullet \otimes a_{j-1\bullet} \otimes \ldots \otimes a_{0\bullet}$. Assuming $U(i_0 \ldots i_{d-1})$ to be normalized [95, 6.5.5], this agrees with the value of $(\sigma^{-1}\delta\sigma)U(i_0 \ldots i_{d-1})$ on the $(q+d)$-simplex (13.6) except that the sign is missing. ∎

There is also a dual spectral sequence

$$E^2_{pq} = H_p(K(\mathbf{I}); H_q M) \Rightarrow \mathrm{colim}_{p+q}(\mathbf{I}; M)$$

where $H_q M(i_0 \ldots i_p) = H_q(S_{h\mathbf{I}}(i_0 \ldots i_p); M(i_0))$.

13.7. Example. 1. [95, 3.5.12] If the category $\mathbf{I}$ is a poset S, the spectral sequence (13.2) for a functor $M\colon S \to \mathbf{Ab}$ degenerates to a cochain complex

$$\ldots \to \prod_{(s_0 \ldots s_{p-1}) \in K(S)_{p-1}} M(s_{p-1}) \to \prod_{(s_0 \ldots s_p) \in K(S)_p} M(s_p) \to \ldots$$

with cohomology $\lim^*(S; M)$.

2. If the category $\mathbf{I}$ is a group G and $M\colon G^{\mathrm{op}} \to \mathbf{Ab}$ a G-module, then the spectral sequence 13.2 collapses onto the vertical axis in the sense that $E_1^{0j} = H^j(G; M)$ and $E_1^{ij} = 0$ for $i > 0$.

3. Suppose that $\mathbf{I}$ is a category

$$0 \xrightarrow{S(0,1)} 1$$

with two objects and no non-identity automorphisms. Then $\lim^n(\mathbf{I}; M) = 0$ for $n > 1$ and there is an exact sequence

$$0 \to \lim{}^0(\mathbf{I}; M) \to M(0) \times M(1) \xrightarrow{\delta} M(1)\langle S(0,1) \rangle \to \lim{}^1(\mathbf{I}; M) \to 0$$

where $\delta(m_0, m_1)(a) = m_1 - M(a)(m_0)$ for all morphisms $a \in S(0,1)$.

4. For a category $\mathbf{I}$ with two objects, 0 and 1, there is a long exact sequence

$$\begin{aligned}
&\ldots \to H^j(\mathbf{I}(0); M(0)) \oplus H^j(\mathbf{I}(1); M(1)) \\
&\qquad\qquad \xrightarrow{d_1} H^j(E(0,1); M(1)) \to \lim{}^{j+1}(\mathbf{I}; M) \to \ldots
\end{aligned}$$

where we are assuming that $\mathbf{I}(1) \times \mathbf{I}(0)$ acts transitively on $S(0,1)$ with stabilizer subgroup $E(0,1)$.

5. With $\mathbf{I} = \mathbf{A}(W,t)\{E,t\}$, the full subcategory of $\mathbf{A}(W,t)$ containing t and one of its non-trivial subspaces $E \neq t$, we get a long exact sequence

$$\ldots \to H^j(\overline{W}(E)/W(E); M(E)) \times H^j(W; M(t)) \xrightarrow{d_1} H^j(\overline{W}(E); M(t)) \\ \to \lim^{j+1}(\mathbf{I}; M) \to \ldots$$

where the homomorphism d_1 is induced from $M(E) \to M(t)$ and from the inclusion $\overline{W}(E) \subset W$.

In case $E = t^S \neq t$ is the fixed-point space for the action of the Sylow p-subgroup S of W and $M(t^S)$ and $M(t)$ are $\mathbf{Z}_{(p)}$-modules, we conclude that there is an exact sequence

$$0 \to \lim^0(\mathbf{I}; M) \to M(t^S)^{\overline{W}(t^S)/W(t^S)} \times M(t)^W \\ \to M(t)^{\overline{W}(t^S)} \to \lim^1(\mathbf{I}; M) \to 0 \tag{13.8}$$

and

$$\lim^{j+1}(\mathbf{I}; M) = \frac{H^j(\overline{W}(t^S); M(t))}{H^j(W; M(t))}, \quad j \geq 1. \tag{13.9}$$

This quotient group vanishes if S has order p for $N_{\overline{W}(t^S)}(S) = N_W(S)$ (2.15) and both cohomology groups equal $H^j(S; M(t))^{N_W(S)}$ as these are the stable elements [19, 9.1, 10.1] in this case. Assuming, furthermore, that $M(t^S) = M^{W(t^S)}$ and $M(t) = M$ for some $\mathbf{Z}_{(p)}[W]$-module M, we recover the formula

$$\lim^j(\mathbf{I}; M) = \begin{cases} M^W & \text{if } j = 0, \\ 0 & \text{if } j > 0, \end{cases}$$

from [2].

6. Let H be a subgroup of the group G and

$$\mathbf{I}(G,H) = \mathcal{O}(G)^{\mathrm{op}}\{G/H, G/\{e\}\}$$

the full subcategory

$$\begin{tikzcd} G/H \arrow[loop left, "N_G(H)/H"] \arrow[r, "G/H"] & G/\{e\} \arrow[loop right, "G"] \end{tikzcd} \tag{13.10}$$

of $\mathcal{O}(G)^{\mathrm{op}}$ containing the two objects $G/\{e\}$ and G/H. The limits of any functor $M\colon \mathbf{I}(G,H) \to \mathbf{Ab}$ fit into a long exact sequence

$$\ldots \to H^j(N_G(H)/H; M(G/H)) \oplus H^j(G; M(G/\{e\})) \\ \xrightarrow{d_1} H^j(N_G(H); M(G/\{e\})) \to \lim^{j+1}(\mathbf{I}(G,H); M) \to \ldots$$

where the homomorphism d_1 is induced from $M(H) : M(G/H) \to M(G/\{e\})$ and from the inclusion of $N_G(H)$ into G.

7. Let $\mathbf{I}$ be a category with three objects, 0, 1_1, and 1_2, of the shape

$$\begin{tikzcd} & 1_1 \arrow[loop right] \\ 0 \arrow[loop left] \arrow[ur, "\mathbf{I}_1"] \arrow[dr, "\mathbf{I}_2"'] & \\ & 1_2 \arrow[loop right] \end{tikzcd}$$

and let M be an $\mathbf{I}$-module with $M(0) = 0$, $M(1_1) = M_1$, and $M(1_2) = M_2$. Then

$$\lim{}^*(\mathbf{I}; M) = \lim{}^*(\mathbf{I}_1; M_1) \times \lim{}^*(\mathbf{I}_2; M_2)$$

where $\mathbf{I}_1$ is the full subcategory generated by the objects 0 and 1_1, $\mathbf{I}_2$ the full subcategory generated by the objects 0 and 1_2, and the $\mathbf{I}(1_1)$-module M_1 is considered as an $\mathbf{I}_1$-module and the $\mathbf{I}(1_2)$-module M_2 as an $\mathbf{I}_2$-module. Of course, this extends to an arbitrary star shaped finite category with outward pointing arrows.

8. Let $\mathbf{I}$ be a category with three objects, 0_1, 0_2, and 1, of the shape

$$\begin{tikzcd} 0_1 \arrow[loop left] \arrow[dr, "\mathbf{I}_1"] & \\ & 1 \arrow[loop right] \\ 0_2 \arrow[loop left] \arrow[ur, "\mathbf{I}_2"'] & \end{tikzcd}$$

and let M be an $\mathbf{I}$-module with $M(0_1) = 0 = M(0_2)$ and $M(1) = M$. Then there is a Mayer–Vietoris sequence

$$\begin{aligned} \ldots \to H^j(\mathbf{I}(1); M) \to \lim{}^j(\mathbf{I}_1; M_1) \times \lim{}^j(\mathbf{I}_2; M_2) \\ \to \lim{}^j(\mathbf{I}; M) \to H^{j+1}(\mathbf{I}(1); M) \to \ldots \end{aligned}$$

where $\mathbf{I}_1$ is the full subcategory generated by the objects 0_1 and 1 and $\mathbf{I}_2$ the full subcategory generated by the objects 0_2 and 1.

In the next lemma, $R(\mathbf{A})$ means the full subcategory containing all objects of the form Ra for $a \in \mathrm{Ob}(\mathbf{A})$.

13.11. Lemma. *Let*

$$\mathbf{I} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{J}$$

be an adjunction between two small categories, $\mathbf{I}(i, Rj) = \mathbf{J}(Li, j)$ *for all* $i \in \mathrm{Ob}(\mathbf{I}), j \in \mathrm{Ob}(\mathbf{J})$, *and* $\mathbf{A}$ *a full subcategory of* $\mathbf{J}$. *Then*

(1) $\lim^*(\mathbf{J}; L^*M) \cong \lim^*(\mathbf{I}; M)$,

(2) $\lim^*(\mathbf{A}; M) \cong \lim^*(R(\mathbf{A}); L^*M) \cong \lim^*(LR(\mathbf{A}); M)$,

for any functor $M \colon \mathbf{J} \to \mathbf{Ab}$.

Proof. Since any left adjoint functor is left cofinal, the first assertion is a consequence of the Cofinality Theorem [14, XI.9.2, XI.7.2].

For the proof of the second assertion, where we may assume that $\mathrm{Ob}(\mathbf{J}) = \mathrm{Ob}(\mathbf{A}) \cup \mathrm{Ob}(LR\mathbf{A})$, we consider the inclusion functors

$$\mathbf{A} \hookrightarrow \mathbf{J} \hookleftarrow LR\mathbf{A}.$$

The inclusion of $LR\mathbf{A}$ into $\mathbf{J}$ is left cofinal for the universal arrow $LRa \to a$ is a terminal object in the over category $LR\mathbf{A} \downarrow a$ for all $a \in \mathrm{Ob}(\mathbf{J})$. For the other inclusion, consider the Grothendieck spectral sequence

$$\lim^p(\mathbf{J}; \lim^q(a \downarrow \mathbf{A}; M)) \Rightarrow \lim^{p+q}(\mathbf{A}; M)$$

If a is an object of $\mathbf{A}$, the identity of a is an initial object in the under category $a \downarrow \mathbf{A}$. Otherwise, note that the restrictions

$$Ra \downarrow R\mathbf{A} \underset{R}{\overset{L}{\rightleftarrows}} LRa \downarrow \mathbf{A}$$

are adjoint functors so that

$$\lim^q(LRa \downarrow \mathbf{A}; M) \cong \lim^q(Ra \downarrow R\mathbf{A}; L^*M) = \begin{cases} MLRa, & q = 0, \\ 0, & q > 0, \end{cases}$$

because the identity of Ra is an initial object in the under category $Ra \downarrow R\mathbf{A}$. We conclude that $\lim^*(\mathbf{A}; M) \cong \lim^*(\mathbf{J}; M) \cong \lim^*(LR\mathbf{A}; M)$. Finally, observe that there is an induced adjointness between $R\mathbf{A}$ and $LR\mathbf{A}$ so that also the two groups $\lim^*(R\mathbf{A}; L^*M)$ and $\lim^*(LR\mathbf{A}; M)$ are isomorphic. ∎

13.12. PROPOSITION. *Let* $\mathbf{J}$ *be a full subcategory of* $\mathbf{I}$. *If* M *vanishes on all objects outside* $\mathbf{J}$ *and if any object of* $\mathbf{I}$ *with a morphism to an object of* $\mathbf{J}$ *is in* $\mathbf{J}$, *then* $\lim^*(\mathbf{I}; M) \cong \lim^*(\mathbf{J}; M)$.

Proof. The cochain projection map

$$\prod_{i_0 \to \ldots \to i_n \in N(\mathbf{I})_n} M(i_n) \to \prod_{j_0 \to \ldots \to j_n \in N(\mathbf{J})_n} M(j_n)$$

is an isomorphism. ∎

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