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The Boolean space of higher level orderings

by

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Abstract. Let K be an ordered field. The set $X(K)$ of its orderings can be topologized to make it a Boolean space. Moreover, it has been shown by Craven that for any Boolean space Y there exists a field K such that $X(K)$ is homeomorphic to Y. Becker's higher level ordering is a generalization of the usual concept of ordering. In a similar way to the case of ordinary orderings one can define a topology on the space of orderings of fixed exact level. We show that it need not be Boolean. However, our main theorem says that for any n and any Boolean space Y there exists a field, the space of orderings of fixed exact level n of which is homeomorphic to Y.

1. Notation and terminology. In the terminology introduced by Becker, Harman and Rosenberg [2] a *signature* of a formally real field K is a character χ of the multiplicative group $\dot{K}$ with values in the group μ of all complex roots of unity, with additively closed kernel. The *level* $s(\chi)$ of the signature χ, if finite, is defined as $\#\mathrm{Im}(\chi)/2$. The *orderings of higher level* are exactly the kernels of signatures with $s(\chi) < \infty$. If χ is a signature with $s(\chi) = n$, then $P = \ker(\chi)$ is called an *ordering of exact level* n, and an *ordering of level* m for any m such that $n \mid m$. We denote by $s(P)$ the exact level of the ordering P. In general, several signatures have the same kernel. Note that $P = \ker(\chi_1) = \ker(\chi_2)$ if and only if there exists an automorphism κ of μ such that $\chi_1 = \kappa \circ \chi_2$.

For a field K let $\mathrm{eSgn}_n(K)$ be the set of all signatures of K of exact level n and let

$$\mathrm{Sgn}_n(K) = \bigcup \{\mathrm{eSgn}_d(K) : d \mid n\}.$$

Similarly denote by $eX_n(K)$ and $X_n(K)$ the set of all orderings of exact level n and the set of all orderings of level n, respectively. With the standard topology the space $\mathrm{Sgn}_n(K)$ is Boolean (i.e. compact, Hausdorff and totally

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disconnected) [3, Prop. 1.4]. It is known that the set $X_1(K) = eX_1(K)$ of total orders of the field K can be topologized to make it a Boolean space by using as a subbasis the family of *Harrison sets*

$$H(a) := \{P \in X_1(K) : a \in P\}, \quad a \in \dot{K}.$$

Since $H(a)^c := \{P \in X_1(K) : a \notin P\} = H(-a)$, the sets $H(a)$ are clopen. In fact, $\mathrm{Sgn}_1(K)$ and $X_1(K)$ are homeomorphic in the natural way.

In a similar way one can define a topology on $X_n(K)$ by using as a subbasis the family of sets

$$H_n(a) = \{P \in X_n(K) : a \in P\} \quad \text{and} \quad H_n^c(a) = \{P \in X_n(K) : a \notin P\}.$$

This topology makes the space $X_n(K)$ Boolean. Moreover, $X_n(K)$ is homeomorphic to a quotient space $\mathrm{Sgn}_n(K)/\varrho$, where ϱ is the relation

$$\chi_1 \varrho \chi_2 \Leftrightarrow \ker(\chi_1) = \ker(\chi_2).$$

The details can be found in our earlier paper [8, Prop. 1]. The space

$$eX_n(K) = X_n(K) \setminus \bigcup\{X_d(K) : d \,|\, n,\, d < n\}$$

is an open subset of the Boolean space $X_n(K)$. It need not be clopen and hence Boolean. In the last section we give an example of a field for which the subspace of orderings of exact level n is infinite and its topology is discrete, thus not compact.

However, the converse is true, which is our main theorem.

THEOREM 1.1. *Let n be any natural number. Every Boolean space Y is homeomorphic to the space $eX_n(M)$ of orderings of exact level n for some formally real field M.*

In the case $n = 1$ the construction of M was given by Craven in [5]. For any n and Y being the Cantor cube it was given in [8], where it was shown that if F is a real closed field of cardinality $\mathfrak{m}$, then the space $eX_n(K)$ for $K := F(X)(\{\sqrt{(X-a)/X} : a \in \dot{F}\})$ is homeomorphic to the Cantor cube $D_{\mathfrak{m}}$. It was also pointed out that for n odd one could take $K := F(X)(\{\sqrt{X-a} : a \in \dot{F}\})$ [8, Th. 12], which for $n = 1$ was remarked by Craven [5, Remark, p. 230].

The proof of Theorem 1.1 requires considering separately the cases of n even and odd. In the third section, for each even n, we find a field M with $eX_n(M)$ homeomorphic to a given Boolean space; for n odd, this is done in Section 4.

Just as Craven did, we start our construction with a field K for which the space $\mathrm{eSgn}_n(K)$ is homeomorphic to the Cantor cube $D_{\mathfrak{m}}$ containing Y. We get the field M by extending K in such a way as to eliminate unwanted orderings. However, the problem we have to cope with and which does not appear in the case $n = 1$ is controlling the levels of the orderings of K

which extend to M. It turns out that for n odd the field M may be taken the same as in Craven's paper [5] for $n = 1$. The case of n even requires a slightly different approach. When constructing M we have to make use of some results on the space $M(K)$ of real places of K and apply the Separation Criterion.

We shall make use of the concept of *strong approximation property* (*SAP*). Recall that a formally real field K is said to satisfy SAP if the Harrison subbasis consists of all the clopen subsets of $X_1(K)$. This is in fact equivalent to the condition that the Harrison subbasis is a basis for $X_1(K)$ [7, Prop. 17.2].

2. Orderings and their extensions. Let K be a formally real field and let P be a higher level ordering of K. Then P determines the valuation ring

$$A(P) := \{a \in K : \exists_{q \in \mathbb{Q}^+}\ q \pm a \in P\}$$

with the maximal ideal

$$I(P) := \{a \in K : \forall_{q \in \mathbb{Q}^+}\ q \pm a \in P\}$$

and the residue field $k(P)$ such that $\overline{P} := (P \cap \dot{A}(P)) + I(P)$ is an archimedean total order of $k(P)$. Here $\dot{A}(P)$ denotes the set of units of the ring $A(P)$.

Definition 2.1. Let K be a formally real field and let P and Q be orderings of higher level of K. We say that P and Q are *associated* if $A(P) = A(Q)$ and $\overline{P} = \overline{Q}$ on the residue field $k(P)$.

For every ordering P there exists a total order P_0 such that P and P_0 are associated. In [2] the authors described the connection between the signature χ of the ordering $P = \ker(\chi)$ of exact level n and the signature χ_0 of the total order P_0 associated with P. We have

$$\chi = \chi_0 \cdot \tau \circ v_P, \tag{2.1}$$

where v_P is the valuation determined by $A(P)$ and τ is a character of the value group of v_P such that

$$\#\mathrm{Im}(\tau) = \begin{cases} 2n & \text{if } n \text{ is even,} \\ n \text{ or } 2n & \text{if } n \text{ is odd.} \end{cases}$$

This fact allows us to determine all orderings of higher level of any formally real field, if we know the total orders. Moreover, the existence of such a representation for every ordering P implies that if P and Q are associated, then

$$P \cap \dot{A}(P) = Q \cap \dot{A}(Q).$$

EXAMPLE 2.2. Let F be a real closed field. Consider the function field $F(X)$ with the total order

$$P_0 = \left\{ \frac{f}{g} \in F(X) : \frac{a_s}{b_t} \in \dot{F}^2 \right\},$$

where a_s, b_t are the leading coefficients of the polynomials f and g, respectively. Here is a complete list of orderings associated with P_0 (cf. [8, Sec. 3]).

For any even $n \in \mathbb{N}$ the set

$$P_n = \left\{ \frac{f}{g} : \left(\frac{a_s}{b_t} \in \dot{F}^2 \wedge t - s \equiv 0 \pmod{2n} \right) \vee \left(\frac{a_s}{b_t} \in -\dot{F}^2 \wedge t - s \equiv n \pmod{2n} \right) \right\}$$

is the unique ordering of exact level n associated with P_0.

For any odd $n \in \mathbb{N}$ the sets

$$\widehat{P}_n = \left\{ \frac{f}{g} : \frac{a_s}{b_t} \in \dot{F}^2 \wedge t - s \equiv 0 \pmod{n} \right\},$$

$$P_n = \left\{ \frac{f}{g} : (-1)^{t-s} \frac{a_s}{b_t} \in \dot{F}^2 \wedge t - s \equiv 0 \pmod{n} \right\}$$

are the unique orderings of exact level n associated with P_0. Notice that $\widehat{P}_1 = P_0$, whereas for $n > 1$ we have $\widehat{P}_n \subset P_0$ and $P_n \subset P_1$.

Now we recall some facts on extensions of orderings (cf. [2], [8]).

Let L/K be a field extension and let P^L be an ordering of L. Then $P = P^L \cap K$ is an ordering of K and $s(P)$ divides $s(P^L)$. The ordering P^L is called an *extension* of P. If $s(P^L) = s(P)$, then the extension is said to be *faithful*. If P^L is an extension of P, then $A(P^L) \cap K = A(P)$. Notice that if the orderings P^L and Q^L are associated, then so are $P^L \cap K$ and $Q^L \cap K$.

Given two formally real fields $K \subset L$, we obtain the natural mapping

$$\varrho_{L/K} : X_n(L) \to X_n(K)$$

which restricts the orderings of L to the subfield K.

PROPOSITION 2.3. *The canonical restriction mapping* $\varrho_{L/K} : X_n(L) \to X_n(K)$, $\varrho_{L/K}(P^L) = P^L \cap K$, *is continuous.*

Proof. Let $[H_n(a)]_K$ be a clopen subbasis set of $X_n(K)$. Then

$$\varrho_{L/K}^{-1}([H_n(a)]_K) = [H_n(a)]_L,$$

a clopen subbasis set of $X_n(L)$. ■

Now we give a necessary condition for the existence of an extension of a given ordering P.

Proposition 2.4. *If an ordering P of K extends to L, then there exists a total order P_0 which is associated with P and has a faithful extension to L.*

Proof. Take for P_0 the image under $\varrho_{L/K}$ of any total order associated with an extension P^L of P. ∎

The converse need not be true. For example, let K be a field with an ordering P of level $n > 1$ and let P_0 be a total order associated with P. Consider a real closure F of (K, P_0). Then $\dot{F}^2$ is an extension of P_0 and it is the unique ordering of F.

In the case of Galois extensions, we have a simple criterion for the existence of an ordering extension. It is a consequence of [2, Th. 4.4, p. 73] which we now recall in the notation of orderings.

Theorem 2.5. *Let L/K be a Galois extension of fields and let P be an ordering of K. If P extends to L, then either all extensions are faithful or all have level $2s(P)$.*

Corollary 2.6. *Let L/K be a Galois extension and P be an ordering of K. Then P extends to L if and only if there exists a total order P_0 associated with P which extends faithfully to L.*

Proof. Let P_0 be a total order of K which is associated with P and extends faithfully to L. Let χ be any signature of P and χ_0 a signature of P_0. By [2, Th. 3.4, p. 65], χ extends to L, since χ_0 does. An extension χ^L of χ has a finite level, hence $\ker(\chi^L)$ is an ordering and $\ker(\chi^L) \cap K = P$. ∎

Corollary 2.7. *Let L/K be a Galois extension. Let P be an ordering of K with an extension P^L to L and let Q be an ordering of K associated with P. Then there exists an extension Q^L of Q associated with P^L.*

Proof. Let χ, η be any signatures of P and Q, respectively. Let χ^L be a signature of P^L such that $\chi^L|_K = \chi$. By [2, Th. 3.4, p. 65] there exists an extension η^L of η such that $A(\ker(\eta^L)) = A(\ker(\chi^L))$ and $\overline{\ker(\eta^L)} = \overline{\ker(\chi^L)}$. By [2, Th. 4.4, p. 73] the exact level of $\ker(\eta^L)$ is finite, thus $Q^L := \ker(\eta^L)$ is an extension of Q associated with P^L. ∎

Let L/K be a Galois extension and let $G(L/K)$ be its topological Galois group. Let P^L be a higher level ordering of L. It is a routine matter to check that $\sigma(P^L)$ is a higher level ordering of L for every $\sigma \in G(L/K)$. The next theorem is based on [2, Ths. 4.2 and 4.5] and was proved in [8, Th. 7].

Theorem 2.8. *Let L/K be a Galois extension and let P be an ordering of K. Let P^L be a faithful extension of P. Then the map*

$$G(L/K) \to \varrho_{L/K}^{-1}(P), \qquad \sigma \mapsto \sigma(P^L),$$

is a homeomorphism.

Now we shall answer the question: *When does a given ordering P of K extend faithfully to the Galois extension L of K?*

Let P be an ordering of K of even exact level n and let P_0 be any total order associated with P. Let χ be any signature of P and χ_0 a signature of P_0. Define

$$P_1 := \ker(\chi_0\chi^n).$$

If χ has a representation of the form (2.1), then $\chi_0\chi^n = \chi_0 \cdot \tau^n \circ v_P$ and P_1 is a total order of K associated with P_0 and P. Notice that P_1 is different from P_0.

DEFINITION 2.9. If n is even, then the pair (P_0, P_1) defined above is called a *pair of total orders associated with P*.

Now let P be an ordering of K of odd exact level n with a signature χ. Then $\ker(\chi^n)$ is a total order associated with P and $P \subset \ker(\chi^n)$. By [4, Lem. 1.6] such an order is uniquely determined. We denote it by $(P)_0$.

PROPOSITION 2.10. *Let K be a formally real field and n be odd. Then the map*

$$\varphi_K : eX_n(K) \to X_1(K), \qquad \varphi_K(P) = (P)_0,$$

is continuous.

Proof. It is a routine matter to check that for any $a \in K$ we have $a^n \in P$ iff $a \in (P)_0$. Let $H(a)$ be a Harrison subbasis set. Then

$$\varphi_K^{-1}(H(a)) = \{P \in eX_n(K) : a \in (P)_0\} = H_n(a^n) \cap eX_n(K). \ \blacksquare$$

The following proposition was proved in [8, Cor. 11].

PROPOSITION 2.11. *Let L/K be a Galois extension and let P be an ordering of K.*

(1) *If P is an ordering of even exact level and there exists a pair (P_0, P_1) of total orders associated with P such that P_0 and P_1 extend faithfully to L, then P also has a faithful extension to L.*
(2) *If P is an ordering of odd exact level, then P has a faithful extension to L if and only if $(P)_0$ has a faithful extension to L.*

For our next result we need the notion of the real holomorphy ring $\mathcal{H}(K)$ of a formally real field K. Recall that

$$\mathcal{H}(K) = \bigcap_{P \in X_1(K)} A(P).$$

We denote the group of units of $\mathcal{H}(K)$ by $\mathbb{E}(K)$ (cf. [1]). Notice that if $a \in \mathbb{E}(K)$, then a is a unit of any real valuation of K. Therefore, if $a \in \mathbb{E}(K)$, then $a \in P$ or $-a \in P$ for any higher level ordering P of K. Moreover, if $a \in P$, then $a \in Q$ for any ordering Q associated with P.

Now we show how to eliminate higher level orderings of a field by extending the base field.

LEMMA 2.12. *Let K be a formally real field, and let $a \in K$ with $\sqrt{a} \notin K$. Let $M := K(\{\sqrt[2^s]{a} : s = 1, 2, \ldots\})$. Then*

(1) *If $P \in eX_n(K)$ and $a \in \dot{A}(P) \cap P$, then P has a unique extension to M and this extension is faithful.*
(2) *If $a \in \mathbb{E}(K)$, then the map*

$$eX_n(M) \to X_n(K), \quad P^M \mapsto P^M \cap K,$$

is a bijection onto $\{P \in eX_n(K) : a \in P\}$.

Proof. Let $M_s := K(\sqrt[2^s]{a})$. Then $M = \bigcup_{s=1}^{\infty} M_s$.

(1) By induction we shall show that *if $a \in \dot{A}(P) \cap P$, then*

- *P has exactly two extensions to M_s,*
- *both extensions are faithful,*
- *only one of them extends to M_{s+1} and this extension is faithful.*

First, we deal with the case $s = 1$. Notice that M_1 is a Galois extension of K. Since $a \in P \cap \dot{A}(P)$ the element a is positive in every total order associated with P. By Proposition 2.11 and Theorem 2.8, P has two faithful extensions P^{M_1} and $\sigma(P^{M_1})$, where $\mathrm{id}_{M_1} \neq \sigma \in G(M_1/K)$. Notice that $\sqrt{a} \in \dot{A}(P^{M_1}) \cap \dot{A}(\sigma(P^{M_1}))$, because $a \in \dot{A}(P)$ and the value groups of the valuations determined by $A(P^{M_1})$ and $A(\sigma(P^{M_1}))$ are torsion-free. Thus $\sqrt{a} \in P^{M_1}$ or $-\sqrt{a} \in P^{M_1}$. We may assume that $\sqrt{a} \in P^{M_1}$ and $-\sqrt{a} \in \sigma(P^{M_1})$. Then $\sqrt{a}$ is positive in every total order associated with P^{M_1} and negative in every total order associated with $\sigma(P^{M_1})$. Therefore, by Proposition 2.11, P^{M_1} extends faithfully to M_2, and by Proposition 2.4, $\sigma(P^{M_1})$ does not extend to M_2.

Now let $P^{M_s} \in eX_n(M_s)$ be the unique extension of P to M_s which extends to M_{s+1}. We have $\sqrt[2^s]{a} \in \dot{A}(P^{M_s})$, since $a \in \dot{A}(P^{M_s})$ and the value group of the valuation determined by $A(P^{M_s})$ is torsion-free. Moreover, $\sqrt[2^s]{a} \in P^{M_s}$, since P^{M_s} extends to M_{s+1}. To explain the inductive step it suffices to take M_s instead of K and apply the first part of the proof.

In this way we obtain an increasing chain $(P^{M_s})_{s\in\mathbb{N}}$ of orderings of exact level n of the fields M_s such that $P^{M_0} = P$ and $P^{M_s} \cap M_{s-1} = P^{M_{s-1}}$, where $M_0 = K$. It is a routine matter to check that the set $P^M := \bigcup_{s=0}^{\infty} P^{M_s}$ is an ordering of M of exact level n. Uniqueness of P^M follows from the uniqueness of P^{M_s}.

(2) As pointed out above, if $a \in \mathbb{E}(K)$ and a is negative in an ordering P, then a is negative in any ordering associated with P. Then P does not extend to M. This fact and (1) imply (2). ∎

In the above lemma the assumption on a is very restrictive. In the next lemma we show that for n odd the assumption can be weakened.

LEMMA 2.13. *Let K be a formally real field, and let $a \in K$ with $\sqrt{a} \notin K$. Let $M := K(\{\sqrt[2^s]{a} : s = 1, 2, \ldots\})$. Suppose n is odd.*

(1) *If $P^M \in eX_n(M)$, then $P^M \cap K \in eX_n(K)$.*
(2) *$P \in eX_n(K)$ has a unique faithful extension to M iff $a \in (P)_0$.*
(3) *The map*

$$eX_n(M) \to eX_n(K), \qquad P^M \mapsto P^M \cap K,$$

is a bijection onto $\{P \in eX_n(K) : a \in (P)_0\}$.

Proof. As previously, let $M = \bigcup_{s=1}^{\infty} M_s$, where $M_s := K(\sqrt[2^s]{a})$. Since M_s is a Galois extension of M_{s-1} and n is odd, statement (1) is a consequence of Theorem 2.5.

By induction we show that *if $a \in (P)_0$, then P has exactly two faithful extensions to M_s and only one of them extends faithfully to M_{s+1}.*

Notice that if P^{M_s} is an extension of P which extends faithfully to $P^{M_{s+1}}$, then by Proposition 2.11, $(P^{M_s})_0$ extends faithfully to $(P^{M_{s+1}})_0$, thus $a \in (P^{M_s})_0$. Now, it suffices to settle the case $s = 1$. If $a \in (P)_0$ then by Proposition 2.11, P extends faithfully to M_1. Moreover, by Theorem 2.8, there are two faithful extensions P^{M_1} and $\sigma(P^{M_1})$, where $\mathrm{id}_{M_1} \neq \sigma \in G(M_1/K)$. We have $(P^{M_1})_0 \cap K = (P)_0$ and $(\sigma(P^{M_1}))_0 \cap K = \sigma((P^{M_1})_0) \cap K = (P)_0$, since $(P)_0$ is uniquely determined. We may assume that $\sqrt{a} \in (P^{M_1})_0$. Thus by Proposition 2.11, P^{M_1} extends faithfully to M_2. But $\sigma(P^{M_1})$ does not extend faithfully to M_2, since $-\sqrt{a} \in (\sigma(P^{M_1}))_0$.

Let P^{M_s} be an extension of P which extends faithfully to M_{s+1}. It is easy to check that $P^M := \bigcup_{s=1}^{\infty} P^{M_s}$ is a faithful extension of P to M. Moreover P^M is uniquely determined, since P^{M_s} is uniquely determined for any $s \in \mathbb{N}$.

The converse is obvious, since P extends faithfully to M_1 and this implies that $(P)_0$ extends faithfully to M_1 and $a \in (P)_0$.

Statement (3) is a simple consequence of (1) and (2). ∎

REMARK 2.14. In the notation of the previous lemma consider the diagram

$$\begin{array}{ccc} eX_n(M) & \xrightarrow{\varrho_{M/K}} & eX_n(K) \\ {\scriptstyle \varphi_M}\downarrow & & \downarrow{\scriptstyle \varphi_K} \\ X_1(M) & \xrightarrow{\varrho_{M/K}} & X_1(K) \end{array}$$

where the vertical maps are as in Proposition 2.10. This diagram commutes. Moreover, if φ_K is a bijection, then so is φ_M.

THEOREM 2.15. *Let K be a formally real field and let $Y \subset eX_n(K)$. Assume that there exists a subset $\mathcal{B} \subset \mathbb{E}(K)$ such that $Y = \bigcap\{H_n(\beta) : \beta \in \mathcal{B}\} \cap eX_n(K)$. Then there exists an algebraic extension M of K such that the restriction map $\varrho_{M/K} : eX_n(M) \to X_n(K)$ is a bijection onto Y. Moreover, if $eX_n(K)$ is compact, then $\varrho_{M/K}$ is a homeomorphism.*

Proof. We may assume that $\mathcal{B} \cap \dot{K}^2 = \emptyset$ since $\beta \in \dot{K}^2 \cap \mathbb{E}(K)$ implies $H_n(\beta) = X_n(K)$. Define

$$M = K(\{\sqrt[2^s]{\beta} : \beta \in \mathcal{B},\ s = 1, 2, \ldots\}).$$

Let $\mathcal{R}$ be the set of pairs $(L, \mathcal{C})$ where $\mathcal{C} \subset \mathcal{B}$ and $L := K(\{\sqrt[2^s]{\beta} : \beta \in \mathcal{C}, s = 1, 2, \ldots\})$ is a subfield of M such that:

(1) $\varrho_{L/K}(eX_n(L)) \subseteq eX_n(K)$,
(2) the restriction $\varrho_{L/K}|_{eX_n(L)}$ of $\varrho_{L/K}$ to $eX_n(L)$ is injective,
(3) $Y \subseteq \varrho_{L/K}(eX_n(L))$.

Note that $\mathcal{R}$ is nonempty, since $(K, \emptyset) \in \mathcal{R}$, and $\mathcal{R}$ is partially ordered by inclusion on the subsets of $\mathcal{B}$. If $(L_1, \mathcal{C}_1)$ and $(L_2, \mathcal{C}_2)$ are in $\mathcal{R}$ with $\mathcal{C}_1 \subset \mathcal{C}_2$, then the following diagram commutes:

$$\begin{array}{ccc} eX_n(L_2) & \longrightarrow & eX_n(L_1) \\ \downarrow & & \downarrow \\ eX_n(K) & = & eX_n(K) \end{array}$$

Let $\{(L_\xi, \mathcal{C}_\xi)\}$ be a simply ordered subset of $\mathcal{R}$ and let $L = \bigcup L_\xi$, $\mathcal{C} = \bigcup \mathcal{C}_\xi$. Then $L = K(\{\sqrt[2^s]{\beta} : \beta \in \mathcal{C},\ s = 1, 2, \ldots\})$.

Let $P^L \in eX_n(L)$ and let χ^L be any signature of P^L. There exists $\omega \in L$ such that $\chi^L(\omega) = \epsilon_{2n}$, a primitive $2n$th root of unity. But $\omega \in L_\xi$ for some ξ, hence $\chi^L|_{L_\xi} \in \mathrm{eSgn}_n(L_\xi)$. This means that $\ker(\chi^L|_{L_\xi}) = P^L \cap L_\xi \in eX_n(L_\xi)$ and $P^L \cap K = P^L \cap L_\xi \cap K \in eX_n(K)$. Thus $(L, \mathcal{C})$ satisfies condition (1). The map $\varrho_{L/K}|_{eX_n(L)}$ is injective since $\varrho_{L_\xi/K}|_{eX_n(L_\xi)}$ is, so $(L, \mathcal{C})$ satisfies (2). Each ordering of Y extends faithfully to each L_ξ and hence to $L = \bigcup L_\xi$, so $(L, \mathcal{C})$ satisfies (3). Therefore $(L, \mathcal{C}) \in \mathcal{R}$.

By Zorn's lemma, $\mathcal{R}$ has a maximal element $(L_0, \mathcal{C}_0)$. Suppose $L_0 \neq M$. Then there exists $\beta_0 \in \mathcal{B} \setminus \mathcal{C}_0$. Since $\beta_0 \in \mathbb{E}(K) \subset \mathbb{E}(L_0)$, by Lemma 2.12 the restriction map

$$eX_n(L_0(\{\sqrt[2^s]{\beta_0} : s = 1, 2, \ldots\})) \to X_n(L_0)$$

is a bijection onto the set $\{P^{L_0} \in eX_n(L_0) : \beta_0 \in P^{L_0}\}$.

Thus $L_0(\{\sqrt[2^s]{\beta_0} : s = 1, 2, \ldots\})$ satisfies conditions (1)–(3) and

$$(L_0(\{\sqrt[2^s]{\beta_0} : s = 1, 2, \ldots\}), \mathcal{C}_0 \cup \{\beta_0\}) \in \mathcal{R},$$

contradicting the maximality of $(L_0, \mathcal{C}_0)$. Therefore $L_0 = M$.

Now it suffices to show that $\varrho_{M/K}(eX_n(M)) \subseteq Y$. Notice that if $\beta \in \mathcal{B}$, then $\beta \in \dot{M}^2$. Let $P^M \in eX_n(M)$ and let P_0^M be a total order associated with P^M. The orderings $P^M \cap K$ and $P_0^M \cap K$ are associated and $\beta \in P_0^M \cap \mathbb{E}(K)$. Hence $\beta \in P^M \cap K$ and $P^M \in H_n(\beta)$ for every $\beta \in \mathcal{B}$. ∎

For the next lemma we need the notion of the space $M(K)$ of $\mathbb{R}$-valued places of the field K. Any ordering P of K leads to the $\mathbb{R}$-valued place $\lambda_K(P) : K \to \mathbb{R} \cup \{\infty\}$ attached to a unique order imbedding of the archimedean ordered field $(k(P), \overline{P})$ into $(\mathbb{R}, \dot{\mathbb{R}}^2)$. Thus we have a map

$$\lambda_K : \bigcup_{n=1}^{\infty} X_n(K) \to M(K)$$

which sends an ordering $P \in X_n(K)$ to $\lambda_K(P)$, its associated $\mathbb{R}$-valued place. Notice that two orderings P and Q determine the same $\mathbb{R}$-valued place $\lambda_K(P) = \lambda_K(Q)$ if and only if they are associated.

LEMMA 2.16. *Let P be an ordering of the field F and let*

$$K = F(\sqrt{a_1}, \ldots, \sqrt{a_s}), \quad \text{where } a_i \in 1 + I(P),\ i = 1, \ldots, s.$$

Then the restriction $\lambda_{K,P}$ of

$$\lambda_K : \bigcup_{n=1}^{\infty} X_n(K) \to M(K)$$

to the set $\varrho_{K/F}^{-1}(P)$ is injective.

Proof. It suffices to show that the map $P^K \mapsto A(P^K)$ is injective.

First, we consider the case $s = 1$. Let $K := F(\sqrt{a})$, $P \in eX_n(F)$. Since $a \in 1 + I(P)$, a is positive in every total order associated with P. By Proposition 2.11, P has exactly two extensions P^K and $\sigma(P^K)$, where $\mathrm{id} \neq \sigma \in G(K/F)$, and they are both faithful. We may assume that $\sqrt{a} \in P^K$, since $a \in P$. Then $-\sqrt{a} \in \sigma(P^K)$. Suppose that $A(P^K) = A(\sigma(P^K)) =: A$ with maximal ideal $I = I(A)$ and residue field $k = k(A)$. Then $\sqrt{a} + I = 1 + I$ or $-\sqrt{a} + I = 1 + I$, since $a + I = 1 + I$. Thus $\sqrt{a} \in P^K \cap \sigma(P^K)$ or $-\sqrt{a} \in P^K \cap \sigma(P^K)$, a contradiction.

Let now $K := F(\sqrt{a_1}, \ldots, \sqrt{a_s})$ and let P^K, Q^K be different extensions of P to K. Let $F_1 := F(\sqrt{a_2}, \ldots, \sqrt{a_s})$. If $P^K \cap F_1 \neq Q^K \cap F_1$, then by the induction assumption $A(P^K \cap F_1) \neq A(Q^K \cap F_1)$, hence $A(P^K) \neq A(Q^K)$. If $P^K \cap F_1 = Q^K \cap F_1$ then apply the case $s = 1$ with $F = F_1$, $P = P^K \cap F_1$ and $K = F_1(\sqrt{a_1})$. ∎

Let L/K be a field extension. The restriction map $\varrho_{L/K}$ induces the map

$$\omega_{L/K} : M(L) \to M(K), \quad \omega_{L/K}(\lambda_L(P^L)) = \lambda_K(\varrho_{L/K}(P^L)).$$

This definition makes sense, because if $\lambda_L(P^L) = \lambda_L(Q^L)$ (i.e. P^L and Q^L are associated), then $\lambda_K(P^L \cap K) = \lambda_K(Q^L \cap K)$ (i.e. $P^L \cap K$ and $Q^L \cap K$ are associated). Moreover, the following diagram commutes:

$$\begin{array}{ccc} X(L) & \xrightarrow{\lambda_L} & M(L) \\ {\scriptstyle \varrho_{L/K}}\big\downarrow & & \big\downarrow{\scriptstyle \omega_{L/K}} \\ X(K) & \xrightarrow{\lambda_K} & M(K) \end{array}$$

As an obvious consequence of this fact and Lemma 2.16 we have

Corollary 2.17. *Let P be a higher level ordering of the field F and let*

$$K = F(\{\sqrt{a} : a \in \mathcal{A}\}),$$

where $\mathcal{A} \subset 1 + I(P)$. Then the restriction $\lambda_{K,P}$ of $\lambda : \bigcup_{n=1}^{\infty} X_n(K) \to M(K)$ to the set $\varrho_{K/F}^{-1}(P)$ is injective.

Remark 2.18. In the notation of this corollary suppose that Q is an ordering of F associated with P. Consider the following diagram:

$$\begin{array}{ccccc} \varrho_{K/F}^{-1}(P) & & & & \varrho_{K/F}^{-1}(Q) \\ & {\scriptstyle \lambda_{K,P}} \searrow & & \swarrow {\scriptstyle \lambda_{K,Q}} & \\ & & M(K) & & \end{array}$$

Since K/F is a Galois extension, by Corollaries 2.7 and 2.17, we can complete the above diagram to

$$\begin{array}{ccccc} \varrho_{K/F}^{-1}(P) & & \xrightarrow{\phi_{P,Q}} & & \varrho_{K/F}^{-1}(Q) \\ & {\scriptstyle \lambda_{K,P}} \searrow & & \swarrow {\scriptstyle \lambda_{K,Q}} & \\ & & M(K) & & \end{array}$$

where $\phi_{P,Q}$ is bijective. In fact, if P^K is a fixed extension of P and Q^K is an extension of Q associated with P^K, then $\phi_{P,Q}(\sigma(P^K)) = \sigma(Q^K)$ for any $\sigma \in G(K/F)$ and the diagram

$$\begin{array}{ccccc} \varrho_{K/F}^{-1}(P) & & \xrightarrow{\phi_{P,Q}} & & \varrho_{K/F}^{-1}(Q) \\ & \nwarrow & & \nearrow & \\ & & G(K/F) & & \end{array}$$

commutes. By Theorem 2.8, $\phi_{P,Q}$ is a homeomorphism.

3. Boolean space as a space of orderings of even exact level. As we have pointed out, every Boolean space is a closed subspace of some Cantor cube. For each infinite cardinal $\mathfrak{m}$, let $D_{\mathfrak{m}}$ denote the Cantor cube of weight $\mathfrak{m}$. It was shown in [8] that if F is a real closed field of cardinality $\mathfrak{m}$ and n is a fixed natural number, then the space $eX_n(K)$ for

$$K := F(X)\left(\left\{\sqrt{\frac{X-a}{X}} : a \in \dot{F}\right\}\right)$$

is homeomorphic to $D_{\mathfrak{m}}$. Now we briefly recall the explanation of this fact. The reader can find the details in [8, Th. 12].

(1) $K/F(X)$ is a Galois extension with Galois group homeomorphic to $D_{\mathfrak{m}}$.

(2) We have

$$X_1(K) = H(X) \,\dot{\cup}\, H(-X)$$

where $H(X), H(-X)$ are Harrison subbasis sets. Let P_0, P_1 be the total orders of $F(X)$ as in Example 2.2. Then

$$H(X) = \varrho^{-1}_{K/F(X)}(P_0) \quad \text{and} \quad H(-X) = \varrho^{-1}_{K/F(X)}(P_1).$$

(3) By Corollary 2.6 and Proposition 2.11, every higher level ordering of K is a faithful extension of some ordering P of $F(X)$ associated with P_0 and P_1. Therefore if n is even, then

$$eX_n(K) = \varrho^{-1}_{K/F(X)}(P_n),$$

where P_n is the unique ordering of $F(X)$ of exact level n associated with P_0 and P_1, and if n is odd, then

$$eX_n(K) = \varrho^{-1}_{K/F(X)}(P_n) \,\dot{\cup}\, \varrho^{-1}_{K/F(X)}(\widehat{P}_n),$$

where P_n, $\widehat{P}_n$ are the orderings of exact level n as in Example 2.2.

(4) If P is a higher level ordering of $F(X)$ which extends to K, then by Theorem 2.8, the space $\varrho^{-1}_{K/F(X)}(P)$ is homeomorphic to $G(K/F(X))$, hence to $D_{\mathfrak{m}}$.

Now we are able to prove the first part of our main theorem.

THEOREM 3.1. *Let n be even. Every Boolean space Y is homeomorphic to the space of orderings of exact level n for some formally real field M.*

Proof. Let F be a real closed field of cardinality $\mathfrak{m}$ and let

$$K := F(X)\left(\left\{\sqrt{\frac{X-a}{X}} : a \in \dot{F}\right\}\right).$$

Let P_0 be the total order of $F(X)$ as in Example 2.2 and let P be *any higher level ordering* of $F(X)$ associated with P_0 (as yet, we do not assume that the exact level of P is even). Note that

$$\frac{X-a}{X} \in 1 + I(P_0) = 1 + I(P)$$

for every $a \in \dot{F}$. By Remark 2.18, we have a homeomorphism

$$\phi_P : \varrho^{-1}_{K/F(X)}(P) \to \varrho^{-1}_{K/F(X)}(P_0),$$

where $\phi_P(P^K)$ is the unique extension of P_0 associated with P^K.

If we take as P the total order P_1, then we get a bijection which pairs orders in $H(-X)$ with the associated orders in $H(X)$. Therefore

$$\bigcap_{P^K \in H(X)} \dot{A}(P^K) = \bigcap_{P^K \in X_1(K)} \dot{A}(P^K) = \mathbb{E}(K).$$

Let Y be a closed subspace of $D_{\mathfrak{m}}$. Denote by Y_P the subset of $\varrho^{-1}_{K/F(X)}(P)$ homeomorphic to Y. We shall show that there exists a subset $\mathcal{B} \subset \mathbb{E}(K)$ such that

$$Y_P = \bigcap_{\beta \in \mathcal{B}} H_n(\beta) \cap \varrho^{-1}_{K/F(X)}(P).$$

The set $\phi_P(Y_P)$ is a closed subspace of $H(X)$, and $\phi_P(Y_P)^c$, the complement of $\phi_P(Y_P)$, is an open subset of $X_1(K)$. Moreover, $\phi_P(Y_P)^c \cap H(X)$ is open. By [6, Th. 3 and Theorem, p. 346], K satisfies SAP. Therefore,

$$\phi_P(Y_P)^c \cap H(X) = \bigcup_{\alpha \in \mathcal{A}} H(-\alpha).$$

For every $\alpha \in \mathcal{A}$ one observes that $H(\alpha) \cap H(X)$ and $H(-\alpha) \cap H(X)$ are closed and disjoint subsets of $X_1(K)$. By Corollary 2.17, the sets $\lambda(H(\alpha) \cap H(X))$ and $\lambda(H(-\alpha) \cap H(X))$ are disjoint. By the Separation Criterion [7, Prop. 9.13], there exists $\beta \in \bigcap\{\dot{A}(P^K) : P^K \in H(X)\} = \mathbb{E}(K)$ such that $H(\alpha) \cap H(X) \subset H(\beta)$ and $H(-\alpha) \cap H(X) \subset H(-\beta)$. It is not difficult to check that $H(-\alpha) = H(-\beta) \cap H(X)$, since $H(-\alpha) \subset H(X)$. Let $\mathcal{B}$ be the set of β's determined in this way. Then $\phi_P(Y_P) = \bigcap\{H(\beta) : \beta \in \mathcal{B}\} \cap H(X)$ and $Y_P = \bigcap\{H_n(\beta) : \beta \in \mathcal{B}\} \cap \varrho^{-1}_{K/F(X)}(P)$.

As we have pointed out, if n is even, then $eX_n(K) = \varrho^{-1}_{K/F(X)}(P_n)$, where P_n is the unique ordering of exact level n of $F(X)$ associated with P_0 and P_1. We use Theorem 2.15 to get a field M with a bijective correspondence between $eX_n(M)$ and Y. Notice that $eX_n(M)$ equals $\varrho^{-1}_{M/F(X)}(P_n) \cap X_n(M)$, so it is compact. Thus $eX_n(M)$ and Y are homeomorphic. ∎

Remark 3.2. Let n be odd and let K be as in the above theorem. Then $eX_n(K) = \varrho^{-1}_{K/F(X)}(P_n) \,\dot\cup\, \varrho^{-1}_{K/F(X)}(\widehat{P}_n)$, where P_n and $\widehat{P}_n$ are orderings of exact level n as in Example 2.2. If $\beta \in \mathbb{E}(K)$, then $H_n(\beta)$ contains an ordering $P_n^K \in \varrho^{-1}_{K/F(X)}(P_n)$ iff $H_n(\beta)$ contains an ordering $\widehat{P}_n^K \in \varrho^{-1}_{K/F(X)}(\widehat{P}_n)$

such that P_n^K and $\widehat{P}_n^K$ are associated. Let Y be any closed subspace of the Cantor cube $D_{\mathfrak{m}}$. In the proof of Theorem 3.1 we have shown that there exists a subset $\mathcal{B} \subset \mathbb{E}(K)$ such that Y is homeomorphic to $\bigcap\{H_n(\beta) : \beta \in \mathcal{B}\} \cap \varrho_{K/F(X)}^{-1}(P_n)$ and to $\bigcap\{H_n(\beta) : \beta \in \mathcal{B}\} \cap \varrho_{K/F(X)}^{-1}(\widehat{P}_n)$. Then $Y \dot\cup Y$ is homeomorphic to $\bigcap\{H_n(\beta) : \beta \in \mathcal{B}\} \cap eX_n(K)$. Let M be as in Theorem 2.15. Then $eX_n(M)$ equals $(\varrho_{M/F(X)}^{-1}(P_n) \dot\cup \varrho_{M/F(X)}^{-1}(\widehat{P}_n)) \cap X_n(M)$, so it is compact. Therefore $eX_n(M)$ is homeomorphic to $Y \dot\cup Y \cong D(2) \times Y$, where $D(2)$ is the two-point discrete space.

4. Boolean space as a space of orderings of odd exact level. In this section we prove Theorem 1.1 for odd n. The proof is based on the result of Craven in [5]. Let F be a real closed field of cardinality $\mathfrak{m}$ and let

$$K := F(X)(\{\sqrt{X-a} : a \in \dot{F}\}).$$

By [8, Th. 12], the space $eX_n(K)$ is homeomorphic to the Cantor cube $D_{\mathfrak{m}}$ for any odd n. In particular, $X_1(K)$ is homeomorphic to $D_{\mathfrak{m}}$. Moreover, in the proof of that theorem we have seen that $X_1(K) = \varrho_{K/F(X)}^{-1}(P_0)$ and $eX_n(K) = \varrho_{K/F(X)}^{-1}(\widehat{P}_n)$, where P_0, $\widehat{P}_n$ are the orderings of $F(X)$ from Example 2.2. Let Y be any closed subset of $X_1(K)$. Since K satisfies SAP, the space Y^c is a union of sets of the Harrison subbasis of $X_1(K)$. Write

$$Y^c = \bigcup_{\alpha \in \mathcal{A}} H(-\alpha).$$

As shown by Craven [5, Prop. 2, p. 227], the space $X_1(M)$ is homeomorphic to Y for

$$M := K(\{\sqrt[2^s]{\alpha} : \alpha \in \mathcal{A},\ s = 1, 2, \ldots\}).$$

We shall show that the spaces $eX_n(M)$ and $X_1(M)$ are homeomorphic.

THEOREM 4.1. *Let n be odd. Every Boolean space Y is homeomorphic to the space of orderings of exact level n for some formally real field M.*

Proof. Let F, K, M be the fields defined above. Let $\mathcal{R}$ be the set of pairs $(L, \mathcal{B})$, where $\mathcal{B} \subset \mathcal{A}$, and let

$$L := K(\{\sqrt[2^s]{\alpha} : \alpha \in \mathcal{B},\ s = 1, 2, \ldots\})$$

be a subfield of M such that

(1) $P^L \in eX_n(L) \Rightarrow P^L \cap K \in eX_n(K)$,
(2) the map $\varphi_L : eX_n(L) \to X_1(L)$, $\varphi_L(P^L) = (P^L)_0$, is a bijection.

The set $\mathcal{R}$ is nonempty, since $(K, \emptyset) \in \mathcal{R}$, and it is partially ordered by inclusion on the subsets of $\mathcal{A}$. Notice that if $(L_1, \mathcal{B}_1)$ and $(L_2, \mathcal{B}_2)$ are in $\mathcal{R}$

with $\mathcal{B}_1 \subset \mathcal{B}_2$, then the following diagram commutes:

$$\begin{array}{ccc} eX_n(L_2) & \longrightarrow & eX_n(L_1) \\ \big\downarrow & & \big\downarrow \\ X_1(L_2) & \longrightarrow & X_1(L_1) \end{array}$$

Let $\{(L_\xi, \mathcal{B}_\xi)\}$ be a simply ordered subset of $\mathcal{R}$ and set $L = \bigcup L_\xi$, $\mathcal{B} = \bigcup \mathcal{B}_\xi$. Then $L := K(\{\sqrt[2^s]{\alpha} : \alpha \in \mathcal{B},\ s = 1, 2, \ldots\})$. Let $P^L \in eX_n(L)$ and let χ^L be any signature of P^L. There exists $\omega \in L$ such that $\chi^L(\omega) = \epsilon_{2n}$, a primitive $2n$th root of unity. But $\omega \in L_\xi$ for some ξ, hence $\chi^L|_{L_\xi} \in \mathrm{eSgn}_n(L_\xi)$. Therefore $\ker(\chi^L|_{L_\xi}) = P^L \cap L_\xi \in eX_n(L_\xi)$ and $P^L \cap K \in eX_n(K)$. Thus $(L, \mathcal{B})$ satisfies condition (1). The map φ_L is injective since φ_{L_ξ} is. If P_0^L is a fixed order of L then $P^L = \bigcup \varphi_{L_\xi}^{-1}(P_0^L \cap L_\xi)$ is an ordering of exact level n contained in P_0^L, hence $(L, \mathcal{B})$ satisfies (2). Thus $(L, \mathcal{B}) \in \mathcal{R}$.

By Zorn's lemma, $\mathcal{R}$ has a maximal element $(L_0, \mathcal{B}_0)$. Suppose $L_0 \neq M$. Then there exists $\alpha_0 \in \mathcal{A} \setminus \mathcal{B}_0$. By Lemma 2.13 and Remark 2.14, the field

$$L_0(\{\sqrt[2^s]{\alpha_0} : s = 1, 2, \ldots\})$$

satisfies conditions (1), (2), so

$$(L_0(\{\sqrt[2^s]{\alpha_0} : s = 1, 2, \ldots\}), \mathcal{B}_0 \cup \{\alpha_0\}) \in \mathcal{R},$$

contradicting the maximality of $(L_0, \mathcal{B}_0)$. Therefore $L_0 = M$.

It suffices to show that the bijection φ_M is a homeomorphism. As pointed out above, $eX_n(K) = \varrho_{K/F(X)}^{-1}(\widehat{P}_n)$. Therefore $eX_n(M)$ equals $\varrho_{M/F(X)}^{-1}(\widehat{P}_n)$ $\cap\, X_n(M)$ and is compact. By Proposition 2.10, the map φ_M is continuous. A continuous bijection of a compact space onto a Hausdorff space is a homeomorphism. ■

5. The space of orderings of exact level n of $\mathbb{R}(X)$. All total orders of $\mathbb{R}(X)$ were described in [9, Example 1.1.4]. They are as follows:

$$P = \left\{ \frac{f}{g} \in \mathbb{R}(X) : \frac{a_s}{b_t} \in \dot{\mathbb{R}}^2 \right\}, \qquad Q = \left\{ \frac{f}{g} \in \mathbb{R}(X) : (-1)^{t-s}\,\frac{a_s}{b_t} \in \dot{\mathbb{R}}^2 \right\},$$

where a_s, b_t are the leading coefficients of the polynomials f and g, respectively, and for any $a \in \mathbb{R}$,

$$P^a = \left\{ (X-a)^k \frac{f}{g} \in \mathbb{R}(X) : \frac{f(a)}{g(a)} \in \dot{\mathbb{R}}^2 \right\},$$
$$Q^a = \left\{ (X-a)^k \frac{f}{g} \in \mathbb{R}(X) : (-1)^k \frac{f(a)}{g(a)} \in \dot{\mathbb{R}}^2 \right\},$$

where $f(a) \neq 0$ and $g(a) \neq 0$. The orderings P and Q are associated, as also are P^a and Q^a for any $a \in \mathbb{R}$.

Let n be even. Then

$$P_n = \left\{ \frac{f}{g} : \left(\frac{a_s}{b_t} \in \dot{\mathbb{R}}^2 \wedge t - s \equiv 0 \pmod{2n} \right) \vee \left(\frac{a_s}{b_t} \in -\dot{\mathbb{R}}^2 \wedge t - s \equiv n \pmod{2n} \right) \right\}$$

is the unique ordering of $\mathbb{R}(X)$ of exact level n associated with P and Q, and

$$P_n^a = \left\{ (X-a)^k \, \frac{f}{g} : \left(\frac{f(a)}{g(a)} \in \dot{\mathbb{R}}^2 \wedge k \equiv 0 \pmod{2n} \right) \vee \left(\frac{f(a)}{g(a)} \in -\dot{\mathbb{R}}^2 \wedge k \equiv n \pmod{2n} \right) \right\}$$

is the unique ordering of $\mathbb{R}(X)$ of exact level n associated with P^a and Q^a. Let n be odd. Then

$$P_n = \left\{ \frac{f}{g} : \frac{a_s}{b_t} \in \dot{\mathbb{R}}^2 \wedge t - s \equiv 0 \pmod{n} \right\},$$

$$Q_n = \left\{ \frac{f}{g} : (-1)^{t-s} \, \frac{a_s}{b_t} \in \dot{\mathbb{R}}^2 \wedge t - s \equiv 0 \pmod{n}) \right\}$$

are the unique orderings of $\mathbb{R}(X)$ of exact level n associated with P and Q, and

$$P_n^a = \left\{ (X-a)^k \, \frac{f}{g} : \frac{f(a)}{g(a)} \in \dot{\mathbb{R}}^2 \wedge k \equiv 0 \pmod{n} \right\},$$

$$Q_n^a = \left\{ (X-a)^k \, \frac{f}{g} : (-1)^k \, \frac{f(a)}{g(a)} \in \dot{\mathbb{R}}^2 \wedge k \equiv 0 \pmod{n} \right\}$$

are the unique orderings of $\mathbb{R}(X)$ of exact level n associated with P^a and Q^a. It is readily verified that for n even we have

$$\{P_n\} = \bigcap \{H_n^c(X^{2d}) : d \,|\, n,\, d < n\},$$

and for $a \in \mathbb{R}$ we have

$$\{P_n^a\} = H_n(X^n) \cap \bigcap \{H_n^c((X-a)^{2d}) : d \,|\, n,\, d < n\}.$$

Thus all one-point sets are open and the topology induced on $eX_n(\mathbb{R}(X))$ from $X_n(\mathbb{R}(X))$ is discrete.

Similarly, one checks that if $n > 1$ is odd, then

$$\{P_n\} = H_n(X^n) \cap \bigcap \{H_n^c(X^{2d}) : d \,|\, n,\, d < n\},$$

$$\{Q_n\} = H_n(-X^n) \cap \bigcap \{H_n^c(X^{2d}) : d \,|\, n,\, d < n\},$$

and for $a \in \mathbb{R}$,

$$\{P_n^a\} = H_n(X^2) \cap H_n((X-a)^n) \cap \bigcap\{H_n^c((X-a)^{2d}) : d \,|\, n,\, d < n\},$$

$$\{Q_n^a\} = H_n(X^2) \cap H_n(-(X-a)^n) \cap \bigcap\{H_n^c((X-a)^{2d}) : d \,|\, n,\, d < n\},$$

which proves that the topological space $eX_n(\mathbb{R}(X))$ is discrete. Since it is infinite, it cannot be compact.

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