The Boolean space of higher level orderings

by

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Abstract. Let $K$ be an ordered field. The set $X(K)$ of its orderings can be topologized to make it a Boolean space. Moreover, it has been shown by Craven that for any Boolean space $Y$ there exists a field $K$ such that $X(K)$ is homeomorphic to $Y$. Becker’s higher level ordering is a generalization of the usual concept of ordering. In a similar way to the case of ordinary orderings one can define a topology on the space of orderings of fixed exact level. We show that it need not be Boolean. However, our main theorem says that for any $n$ and any Boolean space $Y$ there exists a field, the space of orderings of fixed exact level $n$ of which is homeomorphic to $Y$.

1. Notation and terminology. In the terminology introduced by Becker, Harman and Rosenberg [2] a signature of a formally real field $K$ is a character $\chi$ of the multiplicative group $\hat{K}$ with values in the group $\mu$ of all complex roots of unity, with additively closed kernel. The level $s(\chi)$ of the signature $\chi$, if finite, is defined as $\#\text{Im}(\chi)/2$. The orderings of higher level are exactly the kernels of signatures with $s(\chi) < \infty$. If $\chi$ is a signature with $s(\chi) = n$, then $P = \ker(\chi)$ is called an ordering of exact level $n$, and an ordering of level $m$ for any $m$ such that $n | m$. We denote by $s(P)$ the exact level of the ordering $P$. In general, several signatures have the same kernel. Note that $P = \ker(\chi_1) = \ker(\chi_2)$ if and only if there exists an automorphism $\kappa$ of $\mu$ such that $\chi_1 = \kappa \circ \chi_2$.

For a field $K$ let $\text{eSgn}_n(K)$ be the set of all signatures of $K$ of exact level $n$ and let

$$\text{Sgn}_n(K) = \bigcup\{\text{eSgn}_d(K) : d \mid n\}.$$ 

Similarly denote by $eX_n(K)$ and $X_n(K)$ the set of all orderings of exact level $n$ and the set of all orderings of level $n$, respectively. With the standard topology the space $\text{Sgn}_n(K)$ is Boolean (i.e. compact, Hausdorff and totally

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It is known that the set $X_1(K) = eX_1(K)$ of total orders of the field $K$ can be topologized to make it a Boolean space by using as a subbasis the family of Harrison sets

$$H(a) := \{ P \in X_1(K) : a \in P \}, \quad a \in \hat{K}.$$  

Since $H(a)^c := \{ P \in X_1(K) : a \notin P \} = H(-a)$, the sets $H(a)$ are clopen. In fact, $\text{Sgn}_{1}(K)$ and $X_1(K)$ are homeomorphic in the natural way.

In a similar way one can define a topology on $X_n(K)$ by using as a subbasis the family of sets

$$H_n(a) = \{ P \in X_n(K) : a \in P \} \quad \text{and} \quad H_n^c(a) = \{ P \in X_n(K) : a \notin P \}.$$  

This topology makes the space $X_n(K)$ Boolean. Moreover, $X_n(K)$ is homeomorphic to a quotient space $\text{Sgn}_n(K)/\varrho$, where $\varrho$ is the relation

$$\chi_1\varrho\chi_2 \Leftrightarrow \ker(\chi_1) = \ker(\chi_2).$$

The details can be found in our earlier paper [8, Prop. 1]. The space

$$eX_n(K) = X_n(K) \setminus \bigcup \{ X_d(K) : d \mid n, d < n \}$$

is an open subset of the Boolean space $X_n(K)$. It need not be clopen and hence Boolean. In the last section we give an example of a field for which the subspace of orderings of exact level $n$ is infinite and its topology is discrete, thus not compact.

However, the converse is true, which is our main theorem.

**Theorem 1.1.** Let $n$ be any natural number. Every Boolean space $Y$ is homeomorphic to the space $eX_n(M)$ of orderings of exact level $n$ for some formally real field $M$.

In the case $n = 1$ the construction of $M$ was given by Craven in [5]. For any $n$ and $Y$ being the Cantor cube it was given in [8], where it was shown that if $F$ is a real closed field of cardinality $m$, then the space $eX_n(K)$ for $K := F(X)(\{\sqrt{(X-a)/X} : a \in \hat{F}\})$ is homeomorphic to the Cantor cube $D_m$. It was also pointed out that for $n$ odd one could take $K := F(X)(\{\sqrt{X-a} : a \in \hat{F}\})$ [8, Th. 12], which for $n = 1$ was remarked by Craven [5, Remark, p. 230].

The proof of Theorem 1.1 requires considering separately the cases of $n$ even and odd. In the third section, for each even $n$, we find a field $M$ with $eX_n(M)$ homeomorphic to a given Boolean space; for $n$ odd, this is done in Section 4.

Just as Craven did, we start our construction with a field $K$ for which the space $e\text{Sgn}_n(K)$ is homeomorphic to the Cantor cube $D_m$ containing $Y$. We get the field $M$ by extending $K$ in such a way as to eliminate unwanted orderings. However, the problem we have to cope with and which does not appear in the case $n = 1$ is controlling the levels of the orderings of $K$.
which extend to $M$. It turns out that for $n$ odd the field $M$ may be taken the same as in Craven’s paper [5] for $n = 1$. The case of $n$ even requires a slightly different approach. When constructing $M$ we have to make use of some results on the space $M(K)$ of real places of $K$ and apply the Separation Criterion.

We shall make use of the concept of strong approximation property (SAP). Recall that a formally real field $K$ is said to satisfy SAP if the Harrison subbasis consists of all the clopen subsets of $X_1(K)$. This is in fact equivalent to the condition that the Harrison subbasis is a basis for $X_1(K)$ [7, Prop. 17.2].

2. Orderings and their extensions. Let $K$ be a formally real field and let $P$ be a higher level ordering of $K$. Then $P$ determines the valuation ring

$$A(P) := \{ a \in K : \exists q \in Q^+ \quad q \pm a \in P \}$$

with the maximal ideal

$$I(P) := \{ a \in K : \forall q \in Q^+ \quad q \pm a \in P \}$$

and the residue field $k(P)$ such that $\overline{P} := (P \cap \hat{A}(P)) + I(P)$ is an archimedean total order of $k(P)$. Here $\hat{A}(P)$ denotes the set of units of the ring $A(P)$.

**Definition 2.1.** Let $K$ be a formally real field and let $P$ and $Q$ be orderings of higher level of $K$. We say that $P$ and $Q$ are associated if $A(P) = A(Q)$ and $P = Q$ on the residue field $k(P)$.

For every ordering $P$ there exists a total order $P_0$ such that $P$ and $P_0$ are associated. In [2] the authors described the connection between the signature $\chi$ of the ordering $P = \ker(\chi)$ of exact level $n$ and the signature $\chi_0$ of the total order $P_0$ associated with $P$. We have

$$\chi = \chi_0 \cdot \tau \circ v_P,$$

where $v_P$ is the valuation determined by $A(P)$ and $\tau$ is a character of the value group of $v_P$ such that

$$\#\text{Im}(\tau) = \begin{cases} 2n & \text{if } n \text{ is even}, \\ n \text{ or } 2n & \text{if } n \text{ is odd}. \end{cases}$$

This fact allows us to determine all orderings of higher level of any formally real field, if we know the total orders. Moreover, the existence of such a representation for every ordering $P$ implies that if $P$ and $Q$ are associated, then

$$P \cap \hat{A}(P) = Q \cap \hat{A}(Q).$$
Example 2.2. Let $F$ be a real closed field. Consider the function field $F(X)$ with the total order

$$P_0 = \left\{ \frac{f}{g} \in F(X) : \frac{a_s}{b_t} \in \hat{F}^2 \right\},$$

where $a_s, b_t$ are the leading coefficients of the polynomials $f$ and $g$, respectively. Here is a complete list of orderings associated with $P_0$ (cf. [8, Sec. 3]).

For any even $n \in \mathbb{N}$ the set

$$P_n = \left\{ \frac{f}{g} : \left( \frac{a_s}{b_t} \in \hat{F}^2 \land t - s \equiv 0 \pmod{2n} \right) \lor \left( \frac{a_s}{b_t} \in -\hat{F}^2 \land t - s \equiv n \pmod{2n} \right) \right\}$$

is the unique ordering of exact level $n$ associated with $P_0$.

For any odd $n \in \mathbb{N}$ the sets

$$\hat{P}_n = \left\{ \frac{f}{g} : \frac{a_s}{b_t} \in \hat{F}^2 \land t - s \equiv 0 \pmod{n} \right\},$$

$$P_n = \left\{ \frac{f}{g} : (-1)^{t-s} \frac{a_s}{b_t} \in \hat{F}^2 \land t - s \equiv 0 \pmod{n} \right\}$$

are the unique orderings of exact level $n$ associated with $P_0$. Notice that $\hat{P}_1 = P_0$, whereas for $n > 1$ we have $\hat{P}_n \subset P_0$ and $P_n \subset P_1$.

Now we recall some facts on extensions of orderings (cf. [2], [8]).

Let $L/K$ be a field extension and let $P^L$ be an ordering of $L$. Then $P = P^L \cap K$ is an ordering of $K$ and $s(P)$ divides $s(P^L)$. The ordering $P^L$ is called an extension of $P$. If $s(P^L) = s(P)$, then the extension is said to be faithful. If $P^L$ is an extension of $P$, then $A(P^L) \cap K = A(P)$. Notice that if the orderings $P^L$ and $Q^L$ are associated, then so are $P^L \cap K$ and $Q^L \cap K$.

Given two formally real fields $K \subset L$, we obtain the natural mapping

$$\varrho_{L/K} : X_n(L) \to X_n(K)$$

which restricts the orderings of $L$ to the subfield $K$.

Proposition 2.3. The canonical restriction mapping $\varrho_{L/K} : X_n(L) \to X_n(K)$, $\varrho_{L/K}(P^L) = P^L \cap K$, is continuous.

Proof. Let $[H_n(a)]_L$ be a clopen subbasis set of $X_n(K)$. Then

$$\varrho_{L/K}^{-1}([H_n(a)]_K) = [H_n(a)]_L,$$

a clopen subbasis set of $X_n(L)$. $lacksquare$

Now we give a necessary condition for the existence of an extension of a given ordering $P$. 
Proposition 2.4. If an ordering $P$ of $K$ extends to $L$, then there exists a total order $P_0$ which is associated with $P$ and has a faithful extension to $L$.

Proof. Take for $P_0$ the image under $\varrho_{L/K}$ of any total order associated with $P$. ■

The converse need not be true. For example, let $K$ be a field with an ordering $P$ of level $n > 1$ and let $P_0$ be a total order associated with $P$. Consider a real closure $F$ of $(K, P_0)$. Then $F^2$ is an extension of $P_0$ and it is the unique ordering of $F$.

In the case of Galois extensions, we have a simple criterion for the existence of an ordering extension. It is a consequence of [2, Th. 4.4, p. 73] which we now recall in the notation of orderings.

Theorem 2.5. Let $L/K$ be a Galois extension of fields and let $P$ be an ordering of $K$. If $P$ extends to $L$, then either all extensions are faithful or all have level $2s(P)$.

Corollary 2.6. Let $L/K$ be a Galois extension and $P$ be an ordering of $K$. Then $P$ extends to $L$ if and only if there exists a total order $P_0$ associated with $P$ which extends faithfully to $L$.

Proof. Let $P_0$ be a total order of $K$ which is associated with $P$ and extends faithfully to $L$. Let $\chi$ be any signature of $P$ and $\chi_0$ a signature of $P_0$. By [2, Th. 3.4, p. 65], $\chi$ extends to $L$, since $\chi_0$ does. An extension $\chi^L$ of $\chi$ has a finite level, hence $\ker(\chi^L)$ is an ordering and $\ker(\chi^L) \cap K = P$. ■

Corollary 2.7. Let $L/K$ be a Galois extension. Let $P$ be an ordering of $K$ with an extension $P^L$ to $L$ and let $Q$ be an ordering of $K$ associated with $P$. Then there exists an extension $Q^L$ of $Q$ associated with $P^L$.

Proof. Let $\chi, \eta$ be any signatures of $P$ and $Q$, respectively. Let $\chi^L$ be a signature of $P^L$ such that $\chi^L|_K = \chi$. By [2, Th. 3.4, p. 65] there exists an extension $\eta^L$ of $\eta$ such that $A(\ker(\eta^L)) = A(\ker(\chi^L))$ and $\ker(\eta^L) = \ker(\chi^L)$. By [2, Th. 4.4, p. 73] the exact level of $\ker(\eta^L)$ is finite, thus $Q^L := \ker(\eta^L)$ is an extension of $Q$ associated with $P^L$. ■

Let $L/K$ be a Galois extension and let $G(L/K)$ be its topological Galois group. Let $P^L$ be a higher level ordering of $L$. It is a routine matter to check that $\sigma(P^L)$ is a higher level ordering of $L$ for every $\sigma \in G(L/K)$. The next theorem is based on [2, Ths. 4.2 and 4.5] and was proved in [8, Th. 7].

Theorem 2.8. Let $L/K$ be a Galois extension and let $P$ be an ordering of $K$. Let $P^L$ be a faithful extension of $P$. Then the map

$$G(L/K) \to \varrho_{L/K}^{-1}(P), \quad \sigma \mapsto \sigma(P^L),$$

is a homeomorphism.
Now we shall answer the question: When does a given ordering $P$ of $K$ extend faithfully to the Galois extension $L$ of $K$?

Let $P$ be an ordering of $K$ of even exact level $n$ and let $P_0$ be any total order associated with $P$. Let $\chi$ be any signature of $P$ and $\chi_0$ a signature of $P_0$. Define

$$P_1 := \ker(\chi_0\chi^n).$$

If $\chi$ has a representation of the form (2.1), then $\chi_0\chi^n = \chi_0 \cdot \tau^n \circ v_P$ and $P_1$ is a total order of $K$ associated with $P_0$ and $P$. Notice that $P_1$ is different from $P_0$.

**Definition 2.9.** If $n$ is even, then the pair $(P_0, P_1)$ defined above is called a pair of total orders associated with $P$.

Now let $P$ be an ordering of $K$ of odd exact level $n$ with a signature $\chi$. Then $\ker(\chi^n)$ is a total order associated with $P$ and $P \subset \ker(\chi^n)$. By [4, Lem. 1.6] such an order is uniquely determined. We denote it by $(P)_0$.

**Proposition 2.10.** Let $K$ be a formally real field and $n$ be odd. Then the map

$$\varphi_K : eX_n(K) \to X_1(K), \quad \varphi_K(P) = (P)_0,$$

is continuous.

*Proof.* It is a routine matter to check that for any $a \in K$ we have $a^n \in P$ iff $a \in (P)_0$. Let $H(a)$ be a Harrison subbasis set. Then

$$\varphi_K^{-1}(H(a)) = \{P \in eX_n(K) : a \in (P)_0\} = H_n(a^n) \cap eX_n(K).$$

The following proposition was proved in [8, Cor. 11].

**Proposition 2.11.** Let $L/K$ be a Galois extension and let $P$ be an ordering of $K$.

1. If $P$ is an ordering of even exact level and there exists a pair $(P_0, P_1)$ of total orders associated with $P$ such that $P_0$ and $P_1$ extend faithfully to $L$, then $P$ also has a faithful extension to $L$.
2. If $P$ is an ordering of odd exact level, then $P$ has a faithful extension to $L$ if and only if $(P)_0$ has a faithful extension to $L$.

For our next result we need the notion of the real holomorphy ring $\mathcal{H}(K)$ of a formally real field $K$. Recall that

$$\mathcal{H}(K) = \bigcap_{P \in X_1(K)} A(P).$$

We denote the group of units of $\mathcal{H}(K)$ by $\mathbb{E}(K)$ (cf. [1]). Notice that if $a \in \mathbb{E}(K)$, then $a$ is a unit of any real valuation of $K$. Therefore, if $a \in \mathbb{E}(K)$, then $a \in P$ or $-a \in P$ for any higher level ordering $P$ of $K$. Moreover, if $a \in P$, then $a \in Q$ for any ordering $Q$ associated with $P$. 
Now we show how to eliminate higher level orderings of a field by extending the base field.

**Lemma 2.12.** Let $K$ be a formally real field, and let $a \in K$ with $\sqrt{a} \notin K$. Let $M := K(\{\sqrt[n]{a} : s = 1, 2, \ldots\})$. Then

1. If $P \in eX_n(K)$ and $a \in \hat{A}(P) \cap P$, then $P$ has a unique extension to $M$ and this extension is faithful.
2. If $a \in \mathcal{E}(K)$, then the map

$$eX_n(M) \to X_n(K), \quad P^M \mapsto P^M \cap K,$$

is a bijection onto $\{P \in eX_n(K) : a \in P\}$.

**Proof.** Let $M_s := K(\sqrt[s]{a})$. Then $M = \bigcup_{s=1}^{\infty} M_s$.

1. By induction we shall show that if $a \in \hat{A}(P) \cap P$, then

- $P$ has exactly two extensions to $M_s$,
- both extensions are faithful,
- only one of them extends to $M_{s+1}$ and this extension is faithful.

First, we deal with the case $s = 1$. Notice that $M_1$ is a Galois extension of $K$. Since $a \in P \cap \hat{A}(P)$ the element $a$ is positive in every total order associated with $P$. By Proposition 2.11 and Theorem 2.8, $P$ has two faithful extensions $P^{M_1}$ and $\sigma(P^{M_1})$, where $\text{id}_{M_1} \neq \sigma \in G(M_1/K)$. Notice that $\sqrt{a} \in \hat{A}(P^{M_1}) \cap \hat{A}(\sigma(P^{M_1}))$, because $a \in \hat{A}(P)$ and the value groups of the valuations determined by $A(P^{M_1})$ and $A(\sigma(P^{M_1}))$ are torsion-free. Thus $\sqrt{a} \in P^{M_1}$ or $-\sqrt{a} \in P^{M_1}$. We may assume that $\sqrt{a} \in P^{M_1}$ and $-\sqrt{a} \in \sigma(P^{M_1})$. Then $\sqrt{a}$ is positive in every total order associated with $P^{M_1}$ and negative in every total order associated with $\sigma(P^{M_1})$. Therefore, by Proposition 2.11, $P^{M_1}$ extends faithfully to $M_2$, and by Proposition 2.4, $\sigma(P^{M_1})$ does not extend to $M_2$.

Now let $P^{M_s} \in eX_n(M_s)$ be the unique extension of $P$ to $M$ which extends to $M_{s+1}$. We have $\sqrt[n]{a} \in \hat{A}(P^{M_s})$, since $a \in \hat{A}(P^{M_s})$ and the value group of the valuation determined by $A(P^{M_s})$ is torsion-free. Moreover, $\sqrt[n]{a} \in P^{M_s}$, since $P^{M_s}$ extends to $M_{s+1}$. To explain the inductive step it suffices to take $M_s$ instead of $K$ and apply the first part of the proof.

In this way we obtain an increasing chain $(P^{M_s})_{s \in \mathbb{N}}$ of orderings of exact level $n$ of the fields $M_s$ such that $P^{M_0} = P$ and $P^{M_s} \cap M_{s-1} = P^{M_{s-1}}$, where $M_0 = K$. It is a routine matter to check that the set $P^M := \bigcup_{s=0}^{\infty} P^{M_s}$ is an ordering of $M$ of exact level $n$. Uniqueness of $P^M$ follows from the uniqueness of $P^{M_s}$.

2. As pointed out above, if $a \in \mathcal{E}(K)$ and $a$ is negative in an ordering $P$, then $a$ is negative in any ordering associated with $P$. Then $P$ does not extend to $M$. This fact and (1) imply (2). ■
In the above lemma the assumption on \( a \) is very restrictive. In the next lemma we show that for \( n \) odd the assumption can be weakened.

**Lemma 2.13.** Let \( K \) be a formally real field, and let \( a \in K \) with \( \sqrt{a} \notin K \). Let \( M := K(\{ \sqrt[n]{a} : s = 1, 2, \ldots \}) \). Suppose \( n \) is odd.

1. If \( P^M \in eX_n(M) \), then \( P^M \cap K \in eX_n(K) \).
2. \( P \in eX_n(K) \) has a unique faithful extension to \( M \) iff \( a \in (P)_0 \).
3. The map
   \[
eX_n(M) \to eX_n(K), \quad P^M \mapsto P^M \cap K,
   \]
   is a bijection onto \( \{ P \in eX_n(K) : a \in (P)_0 \} \).

**Proof.** As previously, let \( M = \bigcup_{s=1}^{\infty} M_s \), where \( M_s := K(\sqrt[n]{a}) \). Since \( M_s \) is a Galois extension of \( M_{s-1} \) and \( n \) is odd, statement (1) is a consequence of Theorem 2.5.

By induction we show that if \( a \in (P)_0 \), then \( P \) has exactly two faithful extensions to \( M_s \) and only one of them extends faithfully to \( M_{s+1} \).

Notice that if \( P^{M_s} \) is an extension of \( P \) which extends faithfully to \( P^{M_{s+1}} \), then by Proposition 2.11, \( (P^{M_s})_0 \) extends faithfully to \( (P^{M_{s+1}})_0 \), thus \( a \in (P^{M_s})_0 \). Now, it suffices to settle the case \( s = 1 \). If \( a \in (P)_0 \) then by Proposition 2.11, \( P \) extends faithfully to \( M_1 \). Moreover, by Theorem 2.8, there are two faithful extensions \( P^{M_1} \) and \( \sigma(P^{M_1}) \), where \( \id_{M_1} \neq \sigma \in G(M_1/K) \).

We have \( (P^{M_1})_0 \cap K = (P)_0 \) and \( (\sigma(P^{M_1}))_0 \cap K = \sigma((P^{M_1})_0 \cap K = (P)_0 \), since \( (P)_0 \) is uniquely determined. We may assume that \( \sqrt{a} \in (P^{M_1})_0 \). Thus by Proposition 2.11, \( P^{M_1} \) extends faithfully to \( M_2 \). But \( \sigma(P^{M_1}) \) does not extend faithfully to \( M_2 \), since \( -\sqrt{a} \notin (P^{M_1})_0 \).

Let \( P^{M_s} \) be an extension of \( P \) which extends faithfully to \( M_{s+1} \). It is easy to check that \( P^M := \bigcup_{s=1}^{\infty} P^{M_s} \) is a faithful extension of \( P \) to \( M \). Moreover \( P^M \) is uniquely determined, since \( P^{M_s} \) is uniquely determined for any \( s \in \mathbb{N} \).

The converse is obvious, since \( P \) extends faithfully to \( M_1 \) and this implies that \( (P)_0 \) extends faithfully to \( M_1 \) and \( a \in (P)_0 \).

Statement (3) is a simple consequence of (1) and (2).

**Remark 2.14.** In the notation of the previous lemma consider the diagram

\[
eX_n(M) \xrightarrow{\varphi_{M/K}} eX_n(K) \\
\varphi_M \downarrow \quad \varphi_K \\
X_1(M) \xrightarrow{\varphi_{M/K}} X_1(K)
\]

where the vertical maps are as in Proposition 2.10. This diagram commutes. Moreover, if \( \varphi_K \) is a bijection, then so is \( \varphi_M \).
Theorem 2.15. Let $K$ be a formally real field and let $Y \subset eX_n(K)$. Assume that there exists a subset $B \subset \mathbb{E}(K)$ such that $Y = \bigcap \{H_n(\beta) : \beta \in B\}$ \cap $eX_n(K)$. Then there exists an algebraic extension $M$ of $K$ such that the restriction map $\varrho_{M/K}: eX_n(M) \to X_n(K)$ is a bijection onto $Y$. Moreover, if $eX_n(K)$ is compact, then $\varrho_{M/K}$ is a homeomorphism.

Proof. We may assume that $B \cap \hat{K}^2 = \emptyset$ since $\beta \in \hat{K}^2 \cap \mathbb{E}(K)$ implies $H_n(\beta) = X_n(K)$. Define

$$M = K(\{ \sqrt[s]{\beta} : \beta \in B, \ s = 1, 2, \ldots \}).$$

Let $\mathcal{R}$ be the set of pairs $(L, C)$ where $C \subset B$ and $L := K(\{ \sqrt[s]{\beta} : \beta \in C, \ s = 1, 2, \ldots \})$ is a subfield of $M$ such that:

1. $\varrho_{L/K}(eX_n(L)) \subseteq eX_n(K)$,
2. the restriction $\varrho_{L/K}|_{eX_n(L)}$ of $\varrho_{L/K}$ to $eX_n(L)$ is injective,
3. $Y \subseteq \varrho_{L/K}(eX_n(L))$.

Note that $\mathcal{R}$ is nonempty, since $(K, \emptyset) \in \mathcal{R}$, and $\mathcal{R}$ is partially ordered by inclusion on the subsets of $B$. If $(L_1, C_1)$ and $(L_2, C_2)$ are in $\mathcal{R}$ with $C_1 \subset C_2$, then the following diagram commutes:

$$
\begin{array}{ccc}
\vspace{2pt} & \vspace{2pt} eX_n(L_2) & \vspace{2pt} \longrightarrow & \vspace{2pt} eX_n(L_1) & \vspace{2pt} \\
\downarrow & & & & \downarrow \\
\vspace{2pt} & \vspace{2pt} eX_n(K) & \vspace{2pt} \longrightarrow & \vspace{2pt} eX_n(K) & \vspace{2pt}
\end{array}
$$

Let $\{(L_\xi, C_\xi)\}$ be a simply ordered subset of $\mathcal{R}$ and let $L = \bigcup L_\xi$, $C = \bigcup C_\xi$. Then $L = K(\{ \sqrt[s]{\beta} : \beta \in C, \ s = 1, 2, \ldots \})$.

Let $P^L \in eX_n(L)$ and let $\chi^L$ be any signature of $P^L$. There exists $\omega \in L$ such that $\chi^L(\omega) = \epsilon_{2n}$, a primitive $2n$th root of unity. But $\omega \in L_\xi$ for some $\xi$, hence $\chi^L|_{L_\xi} \in \text{Sgn}_n(L_\xi)$. This means that $\ker(\chi^L|_{L_\xi}) = P^L \cap L_\xi \in eX_n(L_\xi)$ and $P^L \cap K = P^L \cap L_\xi \cap K \in eX_n(K)$. Thus $(L, C)$ satisfies condition (1). The map $\varrho_{L/K}|_{eX_n(L)}$ is injective since $\varrho_{L_\xi/K}|_{eX_n(L_\xi)}$ is, so $(L, C)$ satisfies (2). Each ordering of $Y$ extends faithfully to each $L_\xi$ and hence to $L = \bigcup L_\xi$, so $(L, C)$ satisfies (3). Therefore $(L, C) \in \mathcal{R}$.

By Zorn’s lemma, $\mathcal{R}$ has a maximal element $(L_0, C_0)$. Suppose $L_0 \neq M$. Then there exists $\beta_0 \in B \setminus C_0$. Since $\beta_0 \in \mathbb{E}(K) \subset \mathbb{E}(L_0)$, by Lemma 2.12 the restriction map

$$eX_n(L_0(\{ \sqrt[s]{\beta_0} : \ s = 1, 2, \ldots \})) \to X_n(L_0)$$

is a bijection onto the set $\{P^{L_0} \in eX_n(L_0) : \beta_0 \in P^{L_0}\}$.

Thus $L_0(\{ \sqrt[s]{\beta_0} : \ s = 1, 2, \ldots \})$ satisfies conditions (1)–(3) and

$$(L_0(\{ \sqrt[s]{\beta_0} : \ s = 1, 2, \ldots \}), C_0 \cup \{\beta_0\}) \in \mathcal{R},$$

contradicting the maximality of $(L_0, C_0)$. Therefore $L_0 = M$. 

Now it suffices to show that \( \varrho_{M/K}(eX_n(M)) \subseteq Y \). Notice that if \( \beta \in \mathcal{B} \), then \( \beta \in \hat{M}^2 \). Let \( P^M \in eX_n(M) \) and let \( P^M_0 \) be a total order associated with \( P^M \). The orderings \( P^M \cap K \) and \( P^M_0 \cap K \) are associated and \( \beta \in P^M_0 \cap \mathbb{E}(K) \). Hence \( \beta \in P^M \cap K \) and \( P^M \in H_n(\beta) \) for every \( \beta \in \mathcal{B} \).

For the next lemma we need the notion of the space \( M(K) \) of \( \mathbb{R} \)-valued places of the field \( K \). Any ordering \( P \) of \( K \) leads to the \( \mathbb{R} \)-valued place \( \lambda_K(P) : K \to \mathbb{R} \cup \{ \infty \} \) attached to a unique order imbedding of the archimedean ordered field \( (k(P), \bar{P}) \) into \( (\mathbb{R}, \mathbb{R}^2) \). Thus we have a map

\[
\lambda_K : \bigcup_{n=1}^{\infty} X_n(K) \to M(K)
\]

which sends an ordering \( P \in X_n(K) \) to \( \lambda_K(P) \), its associated \( \mathbb{R} \)-valued place. Notice that two orderings \( P \) and \( Q \) determine the same \( \mathbb{R} \)-valued place \( \lambda_K(P) = \lambda_K(Q) \) if and only if they are associated.

**Lemma 2.16.** Let \( P \) be an ordering of the field \( F \) and let

\[
K = F(\sqrt{a_1}, \ldots, \sqrt{a_s}), \quad \text{where } a_i \in 1 + I(P), \ i = 1, \ldots, s.
\]

Then the restriction \( \lambda_K \) of

\[
\lambda_K : \bigcup_{n=1}^{\infty} X_n(K) \to M(K)
\]

to the set \( \varrho_{K/F}^{-1}(P) \) is injective.

**Proof.** It suffices to show that the map \( P^K \mapsto A(P^K) \) is injective.

First, we consider the case \( s = 1 \). Let \( K := F(\sqrt{a}), \ P \in eX_n(F) \). Since \( a \in 1 + I(P) \), \( a \) is positive in every total order associated with \( P \). By Proposition 2.11, \( P \) has exactly two extensions \( P^K \) and \( \sigma(P^K) \), where \( \text{id} \neq \sigma \in G(K/F) \), and they are both faithful. We may assume that \( \sqrt{a} \in P^K \), since \( a \in P \). Then \( -\sqrt{a} \in \sigma(P^K) \). Suppose that \( A(P^K) = A(\sigma(P^K)) =: A \) with maximal ideal \( I = I(A) \) and residue field \( k = k(A) \). Then \( \sqrt{a} + I = 1 + I \) or \( -\sqrt{a} + I = 1 + I \), since \( a + I = 1 + I \). Thus \( \sqrt{a} \in P^K \cap \sigma(P^K) \) or \( -\sqrt{a} \in P^K \cap \sigma(P^K) \), a contradiction.

Let now \( K := F(\sqrt{a_1}, \ldots, \sqrt{a_s}) \) and let \( P^K, Q^K \) be different extensions of \( P \) to \( K \). Let \( F_1 := F(\sqrt{a_2}, \ldots, \sqrt{a_s}) \). If \( P^K \cap F_1 \neq Q^K \cap F_1 \), then by the induction assumption \( A(P^K \cap F_1) \neq A(Q^K \cap F_1) \), hence \( A(P^K) \neq A(Q^K) \). If \( P^K \cap F_1 = Q^K \cap F_1 \) then apply the case \( s = 1 \) with \( F = F_1, P = P^K \cap F_1 \) and \( K = F_1(\sqrt{a_1}) \).

Let \( L/K \) be a field extension. The restriction map \( \varrho_{L/K} \) induces the map

\[
\omega_{L/K} : M(L) \to M(K), \quad \omega_{L/K}(\lambda_L(P^L)) = \lambda_K(\varrho_{L/K}(P^L)).
\]
This definition makes sense, because if \( \lambda_L(P_L) = \lambda_L(Q_L) \) (i.e. \( P_L \) and \( Q_L \) are associated), then \( \lambda_K(P_L \cap K) = \lambda_K(Q_L \cap K) \) (i.e. \( P_L \cap K \) and \( Q_L \cap K \) are associated). Moreover, the following diagram commutes:

\[
\begin{array}{ccc}
X(L) & \xrightarrow{\lambda_L} & M(L) \\
\downarrow & & \downarrow \\
X(K) & \xrightarrow{\lambda_K} & M(K)
\end{array}
\]

As an obvious consequence of this fact and Lemma 2.16 we have

**Corollary 2.17.** Let \( P \) be a higher level ordering of the field \( F \) and let \( K = F(\{\sqrt{a} : a \in \mathcal{A}\}) \), where \( \mathcal{A} \subset 1 + I(P) \). Then the restriction \( \lambda_{K,P} \) of \( \lambda : \bigcup_{n=1}^{\infty} X_n(K) \rightarrow M(K) \) to the set \( e_{K/F}^{-1}(P) \) is injective.

**Remark 2.18.** In the notation of this corollary suppose that \( Q \) is an ordering of \( F \) associated with \( P \). Consider the following diagram:

\[
\begin{array}{ccc}
e_{K/F}^{-1}(P) & \xrightarrow{\phi_{P,Q}} & e_{K/F}^{-1}(Q) \\
\downarrow & & \downarrow \\
M(K) & \xrightarrow{\lambda_{K,P}} & \lambda_{K,Q} \\
\end{array}
\]

Since \( K/F \) is a Galois extension, by Corollaries 2.7 and 2.17, we can complete the above diagram to

\[
\begin{array}{ccc}
e_{K/F}^{-1}(P) & \xrightarrow{\phi_{P,Q}} & e_{K/F}^{-1}(Q) \\
\downarrow & & \downarrow \\
M(K) & \xrightarrow{\lambda_{K,P}} & \lambda_{K,Q} \\
\end{array}
\]

where \( \phi_{P,Q} \) is bijective. In fact, if \( P^K \) is a fixed extension of \( P \) and \( Q^K \) is an extension of \( Q \) associated with \( P^K \), then \( \phi_{P,Q}(\sigma(P^K)) = \sigma(Q^K) \) for any \( \sigma \in G(K/F) \) and the diagram

\[
\begin{array}{ccc}
e_{K/F}^{-1}(P) & \xrightarrow{\phi_{P,Q}} & e_{K/F}^{-1}(Q) \\
\downarrow & & \downarrow \\
G(K/F) & \xrightarrow{\phi_{P,Q}} & G(K/F)
\end{array}
\]

commutes. By Theorem 2.8, \( \phi_{P,Q} \) is a homeomorphism.
3. Boolean space as a space of orderings of even exact level.

As we have pointed out, every Boolean space is a closed subspace of some Cantor cube. For each infinite cardinal \( \mathfrak{m} \), let \( D_\mathfrak{m} \) denote the Cantor cube of weight \( \mathfrak{m} \). It was shown in [8] that if \( F \) is a real closed field of cardinality \( \mathfrak{m} \) and \( n \) is a fixed natural number, then the space \( eX_n(K) \) for

\[
K := F(X) \left( \left\{ \sqrt{\frac{X - a}{X}} : a \in \hat{F} \right\} \right)
\]

is homeomorphic to \( D_\mathfrak{m} \). Now we briefly recall the explanation of this fact. The reader can find the details in [8, Th. 12].

(1) \( K/F(X) \) is a Galois extension with Galois group homeomorphic to \( D_\mathfrak{m} \).

(2) We have

\[
X_1(K) = H(X) \cup H(-X)
\]

where \( H(X), H(-X) \) are Harrison subbasis sets. Let \( P_0, P_1 \) be the total orders of \( F(X) \) as in Example 2.2. Then

\[
H(X) = g_{K/F(X)}^{-1}(P_0) \quad \text{and} \quad H(-X) = g_{K/F(X)}^{-1}(P_1).
\]

(3) By Corollary 2.6 and Proposition 2.11, every higher level ordering of \( K \) is a faithful extension of some ordering \( P \) of \( F(X) \) associated with \( P_0 \) and \( P_1 \). Therefore if \( n \) is even, then

\[
eX_n(K) = g_{K/F(X)}^{-1}(P_n),
\]

where \( P_n \) is the unique ordering of \( F(X) \) of exact level \( n \) associated with \( P_0 \) and \( P_1 \), and if \( n \) is odd, then

\[
eX_n(K) = g_{K/F(X)}^{-1}(P_n) \cup g_{K/F(X)}^{-1}(\hat{P}_n),
\]

where \( P_n, \hat{P}_n \) are the orderings of exact level \( n \) as in Example 2.2.

(4) If \( P \) is a higher level ordering of \( F(X) \) which extends to \( K \), then by Theorem 2.8, the space \( g_{K/F(X)}^{-1}(P) \) is homeomorphic to \( G(K/F(X)) \), hence to \( D_\mathfrak{m} \).

Now we are able to prove the first part of our main theorem.

**Theorem 3.1.** Let \( n \) be even. Every Boolean space \( Y \) is homeomorphic to the space of orderings of exact level \( n \) for some formally real field \( M \).

**Proof.** Let \( F \) be a real closed field of cardinality \( \mathfrak{m} \) and let

\[
K := F(X) \left( \left\{ \sqrt{\frac{X - a}{X}} : a \in \hat{F} \right\} \right).
\]

Let \( P_0 \) be the total order of \( F(X) \) as in Example 2.2 and let \( P \) be any higher level ordering of \( F(X) \) associated with \( P_0 \) (as yet, we do not assume that
the exact level of $P$ is even). Note that
\[ \frac{X - a}{X} \in 1 + I(P_0) = 1 + I(P) \]
for every $a \in \hat{F}$. By Remark 2.18, we have a homeomorphism
\[ \phi_P : \mathcal{O}_K^{-1}(P) \to \mathcal{O}_K^{-1}(P_0), \]
where $\phi_P(P^K)$ is the unique extension of $P_0$ associated with $P^K$.

If we take as $P$ the total order $P_1$, then we get a bijection which pairs
orders in $H(-X)$ with the associated orders in $H(X)$. Therefore
\[ \bigcap_{P^K \in H(X)} \hat{A}(P^K) = \bigcap_{P^K \in X_1(K)} \hat{A}(P^K) = \mathbb{E}(K). \]

Let $Y$ be a closed subspace of $D_m$. Denote by $Y_P$ the subset of $\mathcal{O}_K^{-1}(P)$
homeomorphic to $Y$. We shall show that there exists a subset $B \subset \mathbb{E}(K)$
such that
\[ Y_P = \bigcap_{\beta \in B} H_n(\beta) \cap \mathcal{O}_K^{-1}(P). \]

The set $\phi_P(Y_P)$ is a closed subspace of $H(X)$, and $\phi_P(Y_P)^c$, the complement of $\phi_P(Y_P)$, is an open subset of $X_1(K)$. Moreover, $\phi_P(Y_P)^c \cap H(X)$ is
open. By [6, Th. 3 and Theorem, p. 346], $K$ satisfies SAP. Therefore,
\[ \phi_P(Y_P)^c \cap H(X) = \bigcup_{\alpha \in \mathcal{A}} H(-\alpha). \]

For every $\alpha \in \mathcal{A}$ one observes that $H(\alpha) \cap H(X)$ and $H(-\alpha) \cap H(X)$ are
closed and disjoint subsets of $X_1(K)$. By Corollary 2.17, the sets $\lambda(H(\alpha) \cap H(X))$ and $\lambda(H(-\alpha) \cap H(X))$ are disjoint. By the Separation Criterion
[7, Prop. 9.13], there exists $\beta \in \bigcap \{ \hat{A}(P^K) : P^K \in H(X) \} = \mathbb{E}(K)$ such that
$H(\alpha) \cap H(X) \subset H(\beta)$ and $H(-\alpha) \cap H(X) \subset H(-\beta)$. It is not difficult to check that $H(-\alpha) = H(-\beta) \cap H(X)$, since $H(-\alpha) \subset H(X)$. Let $B$ be the
set of $\beta$'s determined in this way. Then $\phi_P(Y_P) = \bigcap \{ H(\beta) : \beta \in B \} \cap H(X)$
and $Y_P = \bigcap \{ H_n(\beta) : \beta \in B \} \cap \mathcal{O}_K^{-1}(P)$.

As we have pointed out, if $n$ is even, then $eX_n(K) = \mathcal{O}_K^{-1}(P_n)$, where
$P_n$ is the unique ordering of exact level $n$ of $F(X)$ associated with $P_0$ and $P_1$.
We use Theorem 2.15 to get a field $M$ with a bijective correspondence between
$eX_n(M)$ and $Y$. Notice that $eX_n(M)$ equals $\mathcal{O}_M^{-1}(P_n) \cap X_n(M)$,
so it is compact. Thus $eX_n(M)$ and $Y$ are homeomorphic.

**Remark 3.2.** Let $n$ be odd and let $K$ be as in the above theorem. Then
$eX_n(K) = \mathcal{O}_K^{-1}(P_n) \cup \mathcal{O}_K^{-1}(\hat{P}_n)$, where $P_n$ and $\hat{P}_n$ are orderings
of exact level $n$ as in Example 2.2. If $\beta \in \mathbb{E}(K)$, then $H_n(\beta)$ contains an ordering
$P_n^K \in \mathcal{O}_K^{-1}(P_n)$ iff $H_n(\beta)$ contains an ordering $\hat{P}_n^K \in \mathcal{O}_K^{-1}(\hat{P}_n)$.
such that $P_n^K$ and $\hat{P}_n^K$ are associated. Let $Y$ be any closed subspace of the Cantor cube $D_m$. In the proof of Theorem 3.1 we have shown that there exists a subset $B \subset \mathbb{R}^n$ such that $Y$ is homeomorphic to $\bigcap \{ H_n(\beta) : \beta \in B \} \cap \varrho^{-1}_{K/F(X)}(P_n)$ and to $\bigcap \{ H_n(\beta) : \beta \in B \} \cap \varrho^{-1}_{K/F(X)}(\hat{P}_n)$. Then $Y \cup Y$ is homeomorphic to $\bigcap \{ H_n(\beta) : \beta \in B \} \cap eX_n(K)$. Let $M$ be as in Theorem 2.15. Then $eX_n(M)$ equals $(\varrho_{M/F(X)}^{-1}(P_n) \cup \varrho_{M/F(X)}^{-1}(\hat{P}_n)) \cap X_n(M)$, so it is compact. Therefore $eX_n(M)$ is homeomorphic to $Y \cup Y \cong D(2) \times Y$, where $D(2)$ is the two-point discrete space.

4. Boolean space as a space of orderings of odd exact level. In this section we prove Theorem 1.1 for odd $n$. The proof is based on the result of Craven in [5]. Let $F$ be a real closed field of cardinality $m$ and let

$$K := F(X)(\{\sqrt[n]{-a} : a \in F\}).$$

By [8, Th. 12], the space $eX_n(K)$ is homeomorphic to the Cantor cube $D_m$ for any odd $n$. In particular, $X_1(K)$ is homeomorphic to $D_m$. Moreover, in the proof of that theorem we have seen that $X_1(K) = \varrho^{-1}_{K/F(X)}(P_0)$ and $eX_n(K) = \varrho^{-1}_{K/F(X)}(\hat{P}_n)$, where $P_0, \hat{P}_n$ are the orderings of $F(X)$ from Example 2.2. Let $Y$ be any closed subset of $X_1(K)$. Since $K$ satisfies SAP, the space $Y^c$ is a union of sets of the Harrison subbasis of $X_1(K)$. Write

$$Y^c = \bigcup_{\alpha \in A} H(-\alpha).$$

As shown by Craven [5, Prop. 2, p. 227], the space $X_1(M)$ is homeomorphic to $Y$ for

$$M := K(\{\sqrt[n]{s\alpha} : \alpha \in A, s = 1, 2, \ldots\}).$$

We shall show that the spaces $eX_n(M)$ and $X_1(M)$ are homeomorphic.

**Theorem 4.1.** Let $n$ be odd. Every Boolean space $Y$ is homeomorphic to the space of orderings of exact level $n$ for some formally real field $M$.

**Proof.** Let $F, K, M$ be the fields defined above. Let $\mathcal{R}$ be the set of pairs $(L, B)$, where $B \subset A$, and let

$$L := K(\{\sqrt[n]{s\alpha} : \alpha \in B, s = 1, 2, \ldots\})$$

be a subfield of $M$ such that

1. $P^L \in eX_n(L) \Rightarrow P^L \cap K \in eX_n(K)$,
2. the map $\varphi_L : eX_n(L) \rightarrow X_1(L)$, $\varphi_L(P^L) = (P^L)_0$, is a bijection.

The set $\mathcal{R}$ is nonempty, since $(K, \emptyset) \in \mathcal{R}$, and it is partially ordered by inclusion on the subsets of $A$. Notice that if $(L_1, B_1)$ and $(L_2, B_2)$ are in $\mathcal{R}$
with $B_1 \subset B_2$, then the following diagram commutes:

$$
\begin{array}{ccc}
e X_n(L_2) & \longrightarrow & e X_n(L_1) \\
\downarrow & & \downarrow \\
X_1(L_2) & \longrightarrow & X_1(L_1)
\end{array}
$$

Let $\{(L_\xi, B_\xi)\}$ be a simply ordered subset of $R$ and set $L = \bigcup L_\xi, B = \bigcup B_\xi$. Then $L := K(\{ \sqrt[n]{\alpha} : \alpha \in B, s = 1, 2, \ldots \})$. Let $P^L \in e X_n(L)$ and let $\chi^L$ be any signature of $P^L$. There exists $\omega \in L$ such that $\chi^L(\omega) = \epsilon_{2n}$, a primitive $2n$th root of unity. But $\omega \in L_\xi$ for some $\xi$, hence $\chi^L|_{L_\xi} \in e \text{Sgn}_n(L_\xi)$. Therefore $\ker(\chi^L|_{L_\xi}) = P^L \cap L_\xi \in e X_n(L_\xi)$ and $P^L \cap K \in e X_n(K)$. Thus $(L, B)$ satisfies condition (1). The map $\varphi_L$ is injective since $\varphi_L$ is. If $P_0^L$ is a fixed order of $L$ then $P^L = \bigcup \varphi^{-1}_{L_\xi}(P_0^L \cap L_\xi)$ is an ordering of exact level $n$ contained in $P_0^L$, hence $(L, B)$ satisfies (2). Thus $(L, B) \in R$.

By Zorn’s lemma, $R$ has a maximal element $(L_0, B_0)$. Suppose $L_0 \neq M$. Then there exists $\alpha_0 \in A \setminus B_0$. By Lemma 2.13 and Remark 2.14, the field $L_0(\{ \sqrt[n]{\alpha_0} : s = 1, 2, \ldots \})$ satisfies conditions (1), (2), so

$$(L_0(\{ \sqrt[n]{\alpha_0} : s = 1, 2, \ldots \}), B_0 \cup \{ \alpha_0 \}) \in R,$$

contradicting the maximality of $(L_0, B_0)$. Therefore $L_0 = M$.

It suffices to show that the bijection $\varphi_M$ is a homeomorphism. As pointed out above, $e X_n(K) = g^{-1}_{K/F(X)}(\hat{P}_n)$. Therefore $e X_n(M)$ equals $g^{-1}_{M/F(X)}(\hat{P}_n)$ and $X_n(M)$ and is compact. By Proposition 2.10, the map $\varphi_M$ is continuous. A continuous bijection of a compact space onto a Hausdorff space is a homeomorphism. ■

5. The space of orderings of exact level $n$ of $R(X)$. All total orders of $R(X)$ were described in [9, Example 1.1.4]. They are as follows:

$$
P = \left\{ \frac{f}{g} \in R(X) : \frac{a_s}{b_t} \in \mathbb{R}^2 \right\}, \quad Q = \left\{ \frac{f}{g} \in R(X) : (-1)^{t-s} \frac{a_s}{b_t} \in \mathbb{R}^2 \right\},
$$

where $a_s, b_t$ are the leading coefficients of the polynomials $f$ and $g$, respectively, and for any $a \in R$,

$$
P^a = \left\{ (X - a)^k \frac{f}{g} \in R(X) : \frac{f(a)}{g(a)} \in \mathbb{R}^2 \right\},
$$

$$
Q^a = \left\{ (X - a)^k \frac{f}{g} \in R(X) : (-1)^k \frac{f(a)}{g(a)} \in \mathbb{R}^2 \right\},
$$

where $f(a) \neq 0$ and $g(a) \neq 0$. The orderings $P$ and $Q$ are associated, as also are $P^a$ and $Q^a$ for any $a \in R$. 
Let \( n \) be even. Then
\[
P_n = \left\{ \frac{f}{g} : \left( \frac{a_s}{b_t} \in \mathbb{R}^2 \land t - s \equiv 0 \pmod{2n} \right) \right. \\
\left. \lor \left( \frac{a_s}{b_t} \in -\mathbb{R}^2 \land t - s \equiv n \pmod{2n} \right) \right\}
\]
is the unique ordering of \( \mathbb{R}(X) \) of exact level \( n \) associated with \( P \) and \( Q \), and
\[
P^a_n = \left\{ (X - a)^k \frac{f}{g} : \left( \frac{f(a)}{g(a)} \in \mathbb{R}^2 \land k \equiv 0 \pmod{2n} \right) \right. \\
\left. \lor \left( \frac{f(a)}{g(a)} \in -\mathbb{R}^2 \land k \equiv n \pmod{2n} \right) \right\}
\]
is the unique ordering of \( \mathbb{R}(X) \) of exact level \( n \) associated with \( P^a \) and \( Q^a \).

Let \( n \) be odd. Then
\[
P_n = \left\{ \frac{f}{g} : \frac{a_s}{b_t} \in \mathbb{R}^2 \land t - s \equiv 0 \pmod{n} \right\},
\]
\[
Q_n = \left\{ \frac{f}{g} : (-1)^{t-s}\frac{a_s}{b_t} \in \mathbb{R}^2 \land t - s \equiv 0 \pmod{n} \right\}
\]
are the unique orderings of \( \mathbb{R}(X) \) of exact level \( n \) associated with \( P \) and \( Q \), and
\[
P^a_n = \left\{ (X - a)^k \frac{f}{g} : \frac{f(a)}{g(a)} \in \mathbb{R}^2 \land k \equiv 0 \pmod{n} \right\},
\]
\[
Q^a_n = \left\{ (X - a)^k \frac{f}{g} : (-1)^k \frac{f(a)}{g(a)} \in \mathbb{R}^2 \land k \equiv 0 \pmod{n} \right\}
\]
are the unique orderings of \( \mathbb{R}(X) \) of exact level \( n \) associated with \( P^a \) and \( Q^a \). It is readily verified that for \( n \) even we have
\[
\{P_n\} = \bigcap \{ H_n^c(X^{2d}) : d \mid n, d < n \},
\]
and for \( a \in \mathbb{R} \) we have
\[
\{P^a_n\} = H_n(X^n) \cap \bigcap \{ H_n^c((X - a)^{2d}) : d \mid n, d < n \}.
\]
Thus all one-point sets are open and the topology induced on \( e \mathbb{R}_n(\mathbb{R}(X)) \) from \( X_n(\mathbb{R}(X)) \) is discrete.

Similarly, one checks that if \( n > 1 \) is odd, then
\[
\{P_n\} = H_n(X^n) \cap \bigcap \{ H_n^c(X^{2d}) : d \mid n, d < n \},
\]
\[
\{Q_n\} = H_n(-X^n) \cap \bigcap \{ H_n^c(X^{2d}) : d \mid n, d < n \},
\]
and for $a \in \mathbb{R}$,
\[
\{P_n^a\} = H_n(X^2) \cap H_n((X - a)^n) \cap \bigcap \{H^c_n((X - a)^{2d}) : d \mid n, d < n\},
\]
\[
\{Q_n^a\} = H_n(X^2) \cap H_n(-(X - a)^n) \cap \bigcap \{H^c_n((X - a)^{2d}) : d \mid n, d < n\},
\]
which proves that the topological space $eX_n(\mathbb{R}(X))$ is discrete. Since it is infinite, it cannot be compact.

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