$z^0$-Ideals and some special commutative rings

by

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Abstract. In a commutative ring $R$, an ideal $I$ consisting entirely of zero divisors is called a torsion ideal, and an ideal is called a $z^0$-ideal if $I$ is torsion and for each $a \in I$ the intersection of all minimal prime ideals containing $a$ is contained in $I$. We prove that in large classes of rings, say $R$, the following results hold: every $z$-ideal is a $z^0$-ideal if and only if every element of $R$ is either a zero divisor or a unit, if and only if every maximal ideal in $R$ (in general, every prime $z$-ideal) is a $z^0$-ideal, if and only if the sum of any two torsion ideals is either a torsion ideal or $R$. We give a necessary and sufficient condition for every prime $z^0$-ideal to be either minimal or maximal. We show that in a large class of rings, the sum of two $z^0$-ideals is either a $z^0$-ideal or $R$ and we also give equivalent conditions for $R$ to be a $PP$-ring or a Baer ring.

1. Introduction. An ideal $I$ of a commutative ring $R$ is called a $z$-ideal if whenever any two elements of $R$ are contained in the same set of maximal ideals and $I$ contains one of them, then it also contains the other (see [7, 4A.5] for an equivalent definition). These ideals, which are both algebraic and topological objects, were first introduced by Kohls and play a fundamental role in studying the ideal theory of $C(X)$, the ring of continuous real-valued functions on a completely regular Hausdorff space $X$ (see [7]). $z$-Ideals in general commutative rings were studied by Mason [12]. He proved that maximal ideals, minimal prime ideals and some other important ideals in commutative rings are $z$-ideals (see [12, p. 281]).

A special case of $z$-ideals in $C(X)$ consisting entirely of zero divisors are $z^0$-ideals (see [3]–[5]). An ideal $I$ consisting entirely of zero divisors is called a torsion ideal, and an ideal is called a $z^0$-ideal if $I$ is torsion and for each $a \in I$ the intersection of all minimal prime ideals containing $a$ is contained in $I$. $z^0$-Ideals in general commutative rings were also studied by Azarpanah, Karamzadeh and Rezai [4]; these ideals play a fundamental role in studying
the torsion prime ideals. It was proved that the Jacobson radical of a ring 
$R$ is zero if and only if every $z^0$-ideal is a z-ideal (see [4, Proposition 1.10]).

In this paper we will investigate the properties of ideals consisting en-
tirely of zero divisors, such as $z^0$-ideals, torsion prime ideals, prime $z^0$-ideals
and so on.

Throughout, $R$ is a commutative reduced ring with identity. By a reduced
ring we mean a ring without non-zero nilpotent elements. We also say $R$ is
semiprimitive if $\bigcap \text{Max}(R) = (0)$. We denote by Spec($R$), Max($R$) and
Min($R$) the spaces of prime ideals, maximal ideals and minimal ideals of $R$,
respectively. The topology of these spaces is the Zariski topology (see [2]).
The operators cl and int denote the closure and interior. Let

$$
V(a) = \{ P \in \text{Spec}(R) : a \in P \} \quad \text{for all } a \in R,
$$

$$
V(I) = \{ P \in \text{Spec}(R) : I \subseteq P \} \quad \text{for all ideals } I \text{ of } R.
$$

If $S \subseteq \text{Spec}(R)$, we put

$$
V_S(a) = V(a) \cap S, \quad V_S(I) = V(I) \cap S,
$$

$$
M(a) = V(a) \cap \text{Max}(R), \quad M(I) = V(I) \cap \text{Max}(R).
$$

For each $P \in \text{Spec}(R)$, let $O_P = \bigcap_{P' \subseteq P} P'$, where $P'$ ranges over all prime
ideals contained in $P$. It is well known that if $R$ is a reduced ring, then

$$
O_P = \{ a \in R : \exists b \in R - P \text{ such that } ab = 0 \}
$$

$$
= \{ a \in R : P \in \text{int } V(a) \} \quad \text{(see [18, p. 1482]).}
$$

Suppose that $S = \text{Min}(R)$ and $P_a = \bigcap V_S(a)$. It is known that $P_a = \{ b \in R : \text{Ann}(a) \subseteq \text{Ann}(b) \}$, and it is called a basic $z^0$-ideal (see [4]). A proper ideal $I$
is a $z^0$-ideal if $P_a \subseteq I$ for each $a \in I$ (see [3] and [4]). Clearly, minimal prime
ideals, every intersection of $z^0$-ideals, and Ann($a$) and $P_a$ for each $a \in R$
are $z^0$-ideals. It is easy to see that an ideal is a $z^0$-ideal if and only if $a \in I$
and Ann($a$) $\subseteq$ Ann($b$) imply that $b \in I$. Now let $S$ be a dense subspace of
Spec($R$). Since int $V_S(a) = S - V_S(\text{Ann}(a))$, we have: Ann($a$) $\subseteq$ Ann($b$) if
and only if int $V_S(a) \subseteq$ int $V_S(b)$, for each $a, b \in R$. Hence $I$ is a $z^0$-ideal if
and only if $a \in I$ and int $V_S(a) \subseteq$ int $V_S(b)$ imply that $b \in I$. In particular,
in a semiprimitive ring $R$, every $z$-ideal is a $z^0$-ideal (we call $I$ a $z$-ideal if
$a \in I$ and M($a$) $\subseteq$ M($b$) imply that $b \in I$). For other equivalent definitions
of $z^0$-ideals, see [4, Proposition 1.4].

We observe that in a reduced ring $R$, every $z^0$-ideal is a torsion ideal, but
a torsion ideal need not even be a $z$-ideal. Clearly the following questions
concerning torsion ideals, $z$-ideals and $z^0$-ideals are natural:

When is every torsion $z$-ideal a $z^0$-ideal?

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When is every torsion prime $z$-ideal a $z^0$-ideal?

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We are going to answer these questions in Section 2.

A ring is called a Gelfand ring (or a pm ring) if each prime ideal is contained in a unique maximal ideal (see [6]). For a commutative ring $R$, De Marco and Orsatti [6] showed that $R$ is Gelfand if and only if $\text{Max}(R)$ is Hausdorff, if and only if $\text{Spec}(R)$ is normal (in general, not Hausdorff).

A ring $R$ is called a PP-ring if every principal ideal is a projective $R$-module. A ring $R$ is called a Baer ring if the annihilator $\text{Ann}(I)$ of each ideal $I$ in $R$ is generated by an idempotent. A non-zero ideal $I$ in $R$ is said to be essential if it intersects every non-zero ideal non-trivially.

2. Torsion ideals and $z^0$-ideals. The following are Lemmas 2.2 and 2.3 in [17].

**Lemma 2.1.** Let $R$ be a reduced ring and let $S$ be a dense subspace of $\text{Spec}(R)$ containing $\text{Max}(R)$. Then $A$ is a clopen subset of $S$ if and only if there exists an idempotent $e \in R$ such that $A = V_S(e)$.

**Lemma 2.2.** Let $R$ be a reduced ring, $e \in R$ be an idempotent and $S$ be a dense subspace of $\text{Spec}(R)$. Then $(a) \subseteq (e)$ if and only if $V_S(e) \subseteq V_S(a)$, for all $a \in R$.

The following definition is well known, but it is indispensable in studying rings with zero divisors.

**Definition.** A ring $R$ has property $A$ if each finitely generated ideal of $R$ consisting of zero divisors has a non-zero annihilator. $R$ is said to satisfy the annihilator condition, or briefly a.c., if for any $a, b \in R$ there exists an element $c \in R$ with $\text{Ann}(a) \cap \text{Ann}(b) = \text{Ann}(c)$; or equivalently, $\text{int} V(a) \cap \text{int} V(b) = \text{int} V(c)$.

It is well known that most important rings have some of these properties. For example, Noetherian rings [10, p. 56], $C(X)$ [7], rings of Krull dimension zero (each prime ideal is maximal), the polynomial ring $R[x]$ and rings whose classical ring of quotients are regular ([8], [13]) are examples of rings with property $A$. We also observe that $R[x]$, where $R$ is a reduced ring, $C(X)$, Bézout rings (finitely generated ideals are principal) and many other important rings satisfy a.c. ([9], [11], [14]). The reader is referred to [9] for various examples and counterexamples of rings with these properties.

**Definition.** A ring $R$ is called quasi regular if every element of $R$ is either a unit or a zero divisor. Clearly every regular ring is quasi regular, but a quasi regular ring is not necessarily regular (see the next proposition).
Now we give several equivalent conditions for a reduced ring to be quasi regular.

**Proposition 2.3.** Let $R$ be a reduced ring. The following statements are equivalent:

1. $R$ is a quasi regular ring.
2. For every $a \in R$, $V(a) \neq \emptyset$ implies that $\text{int} V(a) \neq \emptyset$.
3. If $a \in R$ is a non-unit, then there exists $b \neq 1$ such that $a = ab$.
4. $\bigcup_{P \in \text{Spec}(R)} O_P$ is the set of all non-unit elements of $R$.
5. Every non-trivial principal ideal of $R$ is non-essential.
6. Every (module) monomorphism from $R$ to $R$ is an isomorphism.

**Proof.** $(1) \Rightarrow (2)$. For any $a \in R$, $\text{int} V(a) = \text{Spec}(R) - V(\text{Ann}(a))$, so $\text{int} V(a) = \emptyset$ if and only if $\text{Ann}(a) = (0)$.

$(2) \Rightarrow (3)$. Let $ac = 0$. Put $b = 1 + c$, so $a = ab$.

$(3) \Rightarrow (4)$. It is sufficient to show that every non-unit element $a$ of $R$ is in $O_P$ for some $P \in \text{Spec}(R)$. By $(3)$, there is $b \in R$ such that $a = ab$. Put $c = 1 - b$, so $c \neq 0$ and $ac = 0$. Hence $\emptyset \neq \text{Spec}(R) - V(c) \subseteq V(a)$ implies that $\text{int} V(a) \neq \emptyset$. This shows that $a \in O_P$ for all $P \in \text{int} V(a)$.

$(4) \Rightarrow (5)$. Let $(0) \neq (a) \neq R$ be a principal ideal in $R$. Then $a$ is a non-unit element and therefore $a \in O_P$ for some $P \in \text{Spec}(R)$. Thus $\text{int} V(a) \neq \emptyset$ and this implies that $\text{Ann}(a) \neq (0)$, i.e., $(a)$ is non-essential.

$(5) \Rightarrow (1)$ is evident.

$(1) \Rightarrow (6)$. Let $\phi : R \to R$ be a monomorphism. Then $\phi(b) = b\phi(1)$ for all $b \in R$. This shows that $\phi(1)$ is a non-zero divisor, hence $\phi(1)$ is unit, i.e., there exists $c \in R$ such that $c\phi(1) = 1$. Thus $\phi(c) = 1$ and $\phi(ac) = a$ for all $a \in R$. So $\phi$ is an isomorphism.

$(6) \Rightarrow (1)$. Let $a \in R$ be a non-zero divisor and define $\phi : R \to R$ by $\phi(b) = ab$ for $b \in R$. Clearly, $\phi$ is a monomorphism and therefore it is also an epimorphism, hence there exists $b \in R$ such that $\phi(b) = ab = 1$. This completes the proof.

**Lemma 2.4.** Let $R$ be a semiprimitive Gelfand ring. If $A$ and $B$ are disjoint closed subsets of $\text{Max}(R)$, then there exists $a \in R$ such that $A \subseteq \text{int} M(a)$ and $B \subseteq \text{int} M(a - 1)$.

**Proof.** By our hypothesis the space $\text{Max}(R)$ is Hausdorff and compact. Therefore by [7, Theorem 1.15] there are closed sets $E$ and $F$ in $\text{Max}(R)$ such that

$$A \subseteq \text{int} E \subseteq E, \quad B \subseteq \text{int} F \subseteq F, \quad E \cap F = \emptyset.$$ 

Hence there are ideals $I$ and $J$ such that $E = M(I)$ and $F = M(J)$. We claim that $I + J = R$. Otherwise there exists $M \in \text{Max}(R)$ such that $I + J \subseteq M$. So $M \in M(I) \cap M(J)$, and this is a contradiction. Therefore $a + b = 1$ for
some \( a \in I \) and \( b \in J \). Thus

\[
A \subseteq \text{int } M(I) \subseteq \text{int } M(a), \quad B \subseteq \text{int } M(J) \subseteq \text{int } M(a - 1).
\]

Next we have the following equivalent conditions in a semiprimitive Gelfand ring with property \( A \).

**Theorem 2.5.** Let \( R \) be a semiprimitive Gelfand ring with property \( A \). The following statements are equivalent:

1. \( R \) is a quasi regular ring.
2. \( M(a) = \text{cl}(\text{int } M(a)) \) for every \( a \in R \).
3. Every \( z \)-ideal in \( R \) is a \( z^0 \)-ideal.
4. Every torsion \( z \)-ideal is a \( z^0 \)-ideal.
5. Every maximal ideal in \( R \) (in general, every prime \( z \)-ideal) is a \( z^0 \)-ideal.
6. Every maximal ideal in \( R \) is a torsion ideal.
7. For each non-unit element \( a \in R \), there exists \( 0 \neq b \in R \) with \( P_a \subseteq \text{Ann}(b) \).
8. The sum of any two torsion ideals is either \( R \) or a torsion ideal.

**Proof.** (1)⇒(2). Let \( a \in R \) and \( M \in M(a) - \text{cl}(\text{int } M(a)) \). As \( \text{cl}(\text{int } M(a)) \) is closed, by Lemma 2.4 there exists \( b \in R \) such that \( M \in \text{int } M(b) \) and \( \text{cl}(\text{int } M(a)) \cap \text{int } M(b) = \emptyset \). By hypothesis, \( \text{int } M(a) \neq \emptyset \) and since \( R \) has property \( A \), it follows that \( \text{Ann}(a) \cap \text{Ann}(b) \neq (0) \), i.e., \( \text{int } M(a) \cap \text{int } M(b) \neq \emptyset \), a contradiction.

(2)⇒(3). Let \( I \) be a \( z \)-ideal, \( \text{int } M(a) = \text{int } M(b) \) and \( a \in I \). By hypothesis we have

\[
M(a) = \text{cl}(\text{int } M(a)) = \text{cl}(\text{int } M(b)) = M(b).
\]

Therefore \( a \in I \) implies that \( b \in I \).

(3)⇒(4) is clear.

(4)⇒(5). Suppose that every torsion \( z \)-ideal is a \( z^0 \)-ideal. We show that every maximal ideal in \( R \) is a torsion ideal. First we show that for any zero divisor \( a \in R \), \( \text{cl}(\text{int } M(a)) = M(a) \). Suppose not; then there are \( M \in M(a) - \text{cl}(\text{int } M(a)) \) and \( b \in R \) such that \( \text{cl}(\text{int } M(a)) \subseteq M(b) \) and \( b \not\in M \). On the other hand, \( I = \bigcap M(a) \) is a torsion \( z \)-ideal, for if \( a' \in I \), then \( M(a) \subseteq M(a') \), and this implies that \( \emptyset \neq \text{int } M(a) \subseteq \text{int } M(a') \), i.e., \( a' \) is a zero divisor. Hence by hypothesis, \( I \) is a \( z^0 \)-ideal. Thus \( \text{int } M(a) \subseteq \text{int } M(b) \) implies that \( b \in I \subseteq M \), a contradiction.

Now we show that every non-zero divisor is a unit. Suppose that \( a \in R \) is a non-zero divisor, so \( M(a) \neq \text{Max}(R) \) and \( \text{int } M(a) = \emptyset \). Hence by Lemma 2.4, there exists \( b \in R \) such that \( \text{int } M(b) \neq \emptyset \) and \( M(a) \cap M(b) = \emptyset \). Thus \( b \)
is a zero divisor and we have

\[ M(a) \cup M(b) = \text{cl}(\text{int}(M(ab))) = \text{cl}(\text{int}(M(a) \cup M(b))) = \text{cl}(\text{int}(M(b))) = M(b). \]

Therefore \( M(a) \subseteq M(b) \), which implies that \( M(a) = \emptyset \), i.e., \( a \) is a unit. Thus every maximal ideal in \( R \) is a torsion ideal.

(5) \( \Rightarrow \) (6) \( \Rightarrow \) (7) \( \Rightarrow \) (8) are evident.

(8) \( \Rightarrow \) (1). Let \( a \in R \) be a non-unit element and \( M \neq M' \in \text{Max}(R) \). By Lemma 2.4, there are \( b, c \in R \) such that \( b \in O_M \) and \( c \in O_{M'} \) with \( M(b) \cap M(c) = \emptyset \). Hence \( (b) + (c) = R \) and so \( (ab) + (ac) = (a) \neq R \). Since \( (ab) \) and \( (ac) \) are torsion ideals, \( a \) is a zero divisor. \( \blacksquare \)

In Theorem 2.5, it is shown that every torsion \( z \)-ideal is a \( z^0 \)-ideal if and only if \( R \) is a quasi regular ring. We give a necessary and sufficient condition for every torsion prime \( z \)-ideal to be a \( z^0 \)-ideal. First we give the following definition:

**Definition.** A ring \( R \) is called **weak quasi regular** if for any \( a, b \in R \) with \( \text{Ann}(a) \subseteq \text{Ann}(b) \), there exists a non-zero divisor \( c \in R \) such that \( M(a) \subseteq M(bc) \). Clearly every quasi regular ring is weak quasi regular.

**Theorem 2.6.** Let \( R \) be a semiprimitive ring. Then every torsion prime \( z \)-ideal is a \( z^0 \)-ideal if and only if \( R \) is weak quasi regular.

**Proof.** First suppose that every torsion \( z \)-ideal is a \( z^0 \)-ideal. To the contrary, suppose \( \text{Ann}(a) \subseteq \text{Ann}(b) \) and for every non-zero divisor \( c \in R \), \( M(bc) \) does not contain \( M(a) \). So \( bc \notin I = \bigcap M(a) \) for any non-zero divisor \( c \in R \). Now we define

\[ T = \{ b^n c : c \in R \text{ is a non-zero divisor}, n = 0, 1, \ldots \}. \]

Clearly, \( T \) is closed under multiplication and \( I \cap T = \emptyset \), for \( M(a) \) is a \( z \)-ideal and \( M(b^n c) = M(bc) \) for all \( n \in \mathbb{N} \). Now by Theorem 1.1 in [12], there exists a prime \( z \)-ideal \( P \) such that \( I \subseteq P \) and \( P \cap S = \emptyset \). So \( P \) is a torsion ideal and hence by hypothesis, \( P \) must be a \( z^0 \)-ideal. But \( \text{Ann}(a) \subseteq \text{Ann}(b) \), \( a \in P \) and \( b \notin P \), a contradiction.

Conversely, let \( P \) be a torsion \( z \)-ideal in \( R \), \( \text{Ann}(a) \subseteq \text{Ann}(b) \) and \( a \in P \). By hypothesis, there exists a non-zero divisor \( c \in R \) with \( M(a) \subseteq M(bc) \). Since \( P \) is a \( z \)-ideal, we have \( bc \in P \). But \( c \notin P \), for \( c \) is not a zero divisor, hence \( b \in P \), i.e., \( P \) is a \( z^0 \)-ideal. \( \blacksquare \)

Next we give some necessary and sufficient conditions, in a reduced ring \( R \) with property a.c., for every prime \( z^0 \)-ideal to be minimal (see [4, Proposition 1.26]).

**Proposition 2.7.** Let \( R \) be a reduced ring with property a.c. The following statements are equivalent:
(1) Every prime $z^0$-ideal is minimal.
(2) $\text{Min}(R)$ is compact.
(3) For any $a \in R$, there exists $b \in \text{Ann}(a)$ such that
$$\text{Ann}(a) \cap \text{Ann}(b) = (0).$$
(4) For any $a \in R$, there exists $b \in R$ such that
$$\text{cl}(\text{int} \, V(a)) = \text{cl}(\text{Spec}(R) - V(b)).$$

Proof. (1)$\Leftrightarrow$(2) follows from Proposition 1.26 and Theorem 1.28 in [4].
(2)$\Leftrightarrow$(3). Let $S = \text{Min}(R)$. By [8, Lemma 3.1(iv) and Theorem 3.4], $\text{Min}(R)$ is compact if and only if for each $a \in R$, there exists $b \in R$ such that $V_S(b) = V_S(\text{Ann}(a))$. By hypothesis, there exists $c \in R$ such that $\text{Ann}(a) \cap \text{Ann}(b) = \text{Ann}(c)$. Therefore we have
$$\text{Ann}(a) \cap \text{Ann}(b) = (0) \Leftrightarrow V_S(c) = \emptyset \Leftrightarrow V_S(b) \subseteq V_S(\text{Ann}(a)).$$
Also, $b \in \text{Ann}(a) \Leftrightarrow V_S(\text{Ann}(a)) \subseteq V_S(b)$, and we are done.
(3)$\Leftrightarrow$(4). For every $a, b \in R$ we have
$$\text{int} \, V(a) \cap \text{int} \, V(b) = \emptyset \Leftrightarrow \text{cl}(\text{int} \, V(a)) \subseteq \text{Spec}(R) - \text{int} \, V(b)$$
$$= \text{cl}(\text{Spec}(R) - V(b)),$$
and
$$b \in \text{Ann}(a) \Leftrightarrow \text{Spec}(R) - V(b) \subseteq \text{int} \, V(a)$$
$$\Leftrightarrow \text{cl}(\text{Spec}(R) - V(b)) \subseteq \text{cl}(\text{int} \, V(a)).$$
This completes the proof.

Corollary 2.8. Let $R$ be a semiprimitive Gelfand quasi regular ring with property $A$. The following statements are equivalent:

(1) $R$ is a $PP$-ring.
(2) $R$ is a regular ring.
(3) $\text{Min}(R)$ is compact.

Proof. (1)$\Rightarrow$(2). For every $a \in R$, (1) implies that $\text{M}(\text{Ann}(a)) = \text{Max}(R) - \text{int} \, M(a)$ is clopen. Also by Theorem 2.5(2), $M(a) = \text{cl}(\text{int} \, M(a))$, hence $M(a)$ is clopen, for every $a \in R$. This implies that $O_M = M$ for every $M \in \text{Max}(R)$, i.e., every prime ideal in $R$ is maximal. Thus Proposition 1.3 in [14] implies that $R$ is a regular ring.
(2)$\Rightarrow$(3) is clear.
(3)$\Rightarrow$(1). Let $a \in R$. By Proposition 2.7(3), there exists $b \in R$ such that $b \in \text{Ann}(a)$ and $\text{Ann}(a) \cap \text{Ann}(b) = (0)$. Hence $V(a) \cup V(b) = \text{Spec}(R)$ and $\text{int}(V(a) \cap V(b)) = \emptyset$. But since $R$ is a quasi regular ring, we must have $V(a) \cap V(b) = \emptyset$, i.e., $V(a) = \text{int} \, V(a)$, for all $a \in R$, and this completes the proof.
For an ideal $I$ in $R$ we define $P_I = \bigcap_{a \in I} P_a$. Next we have the following

**Lemma 2.9.** Let $R$ be a reduced ring with property a.c. and $a \in R$. Then $\sum_{c \in \text{Ann}(a)} P_{(a,c)} = \bigcup_{c \in \text{Ann}(a)} P_{(a,c)}$ is a $z^0$-ideal in $R$.

**Proof.** The inclusion $\bigcup_{c \in \text{Ann}(a)} P_{(a,c)} \subseteq \sum_{c \in \text{Ann}(a)} P_{(a,c)}$ is clear. Now we let $b \in \sum_{c \in \text{Ann}(a)} P_{(a,c)}$; then $b = b_1 + \cdots + b_n$, where $b_i \in P_{(a,c_i)}$, $c_i \in \text{Ann}(a)$ and $i = 1, \ldots, n$. By hypothesis, there exists $c \in R$ such that $\bigcap_{i=1}^n \text{Ann}(c_i) = \text{Ann}(c)$, hence $c \in \text{Ann}(a)$. But $\text{Ann}(a) \cap \text{Ann}(c) \subseteq \text{Ann}(b_i)$ for all $i = 1, \ldots, n$, so

$$
\text{Ann}(a) \cap \text{Ann}(c) = \bigcap_{i=1}^n (\text{Ann}(a) \cap \text{Ann}(c_i)) \subseteq \bigcap_{i=1}^n \text{Ann}(b_i) \subseteq \text{Ann}(b)
$$

and this implies $b \in P_{(a,c)}$, showing that $\sum_{c \in \text{Ann}(a)} P_{(a,c)} \subseteq \bigcup_{c \in \text{Ann}(a)} P_{(a,c)}$. Finally, since every $P_{(a,c)}$ is a $z^0$-ideal, clearly, $\bigcup_{c \in \text{Ann}(a)} P_{(a,c)}$ is also a $z^0$-ideal.

**Theorem 2.10.** Let $R$ be a reduced ring with property a.c. and $a \in R$. The following statements are equivalent:

1. Every prime $z^0$-ideal is minimal or maximal.
2. For any $M \in \text{Max}(R)$ and $a, b \in M$, there are $c \in \text{Ann}(a)$ and $d \not\in M$ such that $\text{Ann}(a) \cap \text{Ann}(c) \subseteq \text{Ann}(bd)$.

**Proof.** Suppose that every prime $z^0$-ideal is minimal or maximal but (2) does not hold. Then there are $M \in \text{Max}(R)$ and $a, b \in M$ such that for every $c \in \text{Ann}(a)$ and $d \not\in M$, $\text{Ann}(a) \cap \text{Ann}(c) \not\subseteq \text{Ann}(bd)$. Consider

$$
T = \{b^n d : d \in R - M, n = 1, 2, \ldots \}, \quad I = \bigcup_{c \in \text{Ann}(a)} P_{(a,c)}.
$$

Obviously, $T$ is closed under multiplication. We also have $I \cap T = \emptyset$, for if $b^n d \in P_{(a,c)}$ for some $n$ and $c \in \text{Ann}(a)$, then $\text{Ann}(a) \cap \text{Ann}(c) \subseteq \text{Ann}(bd)$, which is impossible. So there exists a prime ideal $P$ such that $I \subseteq P$ and $P \cap T = \emptyset$. We have already observed in Lemma 2.9 that $I$ is a $z^0$-ideal and if $P$ is minimal, then $P$ is a $z^0$-ideal. Now $P \cap T = \emptyset$ and $R - M \subseteq S$ imply that $P \subseteq M$. On the other hand, $\text{Ann}(a) \subseteq P$ and hence $P$ is not minimal, so it must be maximal, i.e., $P = M$. This implies that $b \in M = P$, a contradiction.

Conversely, suppose that (2) holds and $P \subseteq M$ is a prime $z^0$-ideal, for some $M \in \text{Max}(R)$. Suppose $P$ is neither maximal nor minimal. Then there are $a \in P$ and $b \in M - P$ such that $\text{Ann}(a) \subseteq P$. Now by (2) there are $c \in \text{Ann}(a)$ and $d \not\in M$ such that $\text{Ann}(a) \cap \text{Ann}(c) \subseteq \text{Ann}(bd)$. Since $(a, c) \subseteq P$ and $P$ is a $z^0$-ideal, hence $bd \in P_{(a,c)} \subseteq P$. But $d \not\in P$, for $d \not\in M$, hence $b \in P$, a contradiction.
3. PP-rings and $z^0$-ideals. In this section we investigate the relation between $z^0$-ideals and PP-rings.

Remark. We note that $\text{int } V_S(I) = S - V_S(\text{Ann}(I))$ for every dense subspace $S$ of $\text{Spec}(R)$ and every ideal $I$ in $R$. Therefore if $R$ is a reduced ring, then $R$ is a PP-ring if and only if there exists a dense subspace $S$ of $\text{Spec}(R)$ containing $\text{Max}(R)$ such that for each $a \in R$, $\text{int } V_S(a)$ is clopen in $S$. Also $R$ is a Baer ring if and only if there exists a dense subspace $S$ of $\text{Spec}(R)$ containing $\text{Max}(R)$ such that every open set in $S$ has closure open in $S$ (i.e., $S$ is an extremally disconnected space; see [1] and [16]).

The sum of two $z^0$-ideals in a reduced ring (even in $C(X)$) may be a proper ideal which is not a $z^0$-ideal (see [3, p. 20]). If $C(X)$ is a PP-ring, then the sum of any two $z^0$-ideals in $C(X)$ is either a $z^0$-ideal or $C(X)$ (see [3, Proposition 2.13] and [15, Theorem 1.1]). Next we generalize this fact.

Theorem 3.1. Let $R$ be a reduced PP-ring with property a.c. The sum of any two $z^0$-ideals in $R$ is either a $z^0$-ideal or $R$.

Proof. Let $I$ and $J$ be two $z^0$-ideals in $R$ and suppose that $I + J \neq R$. Let $a \in I + J$ and $\text{int } V(a) = \text{int } V(b)$ for some $b \in R$. We will show that $b \in I + J$. We have $a = c + d$, where $c \in I$ and $d \in J$. We may assume that $c \neq 0 \neq d$, for otherwise we clearly have $b \in I + J$. Now by the Remark, $\text{int } V(c)$ and $\text{int } V(d)$ are clopen sets, and since $I$ and $J$ are $z^0$-ideals, we have $\text{int } V(c) \neq \emptyset \neq \text{int } V(d)$. Then by Lemma 2.1, there are idempotents $e, e' \in R$ such that $\text{int } V(c) = V(e)$ and $\text{int } V(d) = V(e')$. Since $I$ and $J$ are $z^0$-ideals, we infer that $e \in I$ and $e' \in J$. Now by our hypothesis, there exists $e'' \in R$ such that

$$V(e'') = V(e) \cap V(e') \subseteq \text{int } V(a) = \text{int } V(b).$$

Thus Lemma 2.2 implies that $b \in (e'')$, i.e., $b \in I + J$. ■

Definition. A ring $R$ has property p.z. if every principal $z$-ideal in $R$ is generated by an idempotent.

Example. Suppose $R$ is a semireal-closed $F$-ring, i.e., for each $a \succ 0$ there exists $b \in R$ with $a = b^2$ (see [12, p. 288]). We show that $R$ has property p.z. To see this suppose $I = (a)$ is a non-zero $z$-ideal in $R$. Then $(|a|) = (a)$ and so there exists $b \in R$ such that $|a| = b^2$. Hence $M(|a|) = M(b)$ implies that $b \in (|a|)$. Therefore there is $c \in R$ such that $b = b^2c$, i.e., $M(b) \cup M(1 - bc) = \text{Max}(R)$. Since $M(a) = M(|a|) = M(b)$, $M(a)$ is clopen in $\text{Max}(R)$. By Lemma 2.1, there is an idempotent $e \in R$ such that $M(a) = M(e)$. Since $(a)$ is a $z$-ideal, we have $(e) \subseteq (a)$. Also by Lemma 2.2, $(a) \subseteq (e)$, i.e., $(a) = (e)$. Hence every principal $z$-ideal in $R$ is generated by an idempotent. In particular, $C(X)$ has property p.z.

The following theorems are generalizations of Theorem 2.10 in [3].
**Theorem 3.2.** Let $R$ be a reduced ring with property $p.z$. Every basic $z^0$-ideal in $R$ is principal if and only if $R$ is a $PP$-ring.

**Proof.** Suppose every basic $z^0$-ideal is principal. We will show that for each $a \in R$, $\text{int} \ V(a)$ is clopen in $\text{Spec}(R)$. It suffices to prove this for $a \in R$ which is a zero divisor, for if $\text{Ann}(a) = (0)$, then $\text{int} \ V(a) = \emptyset$. Now let $P_a = (b)$ and $\text{Ann}(a) \neq (0)$. Then by hypothesis, $P_a = (e)$, where $e^2 = e$. Hence $a \in (e)$ implies that $V(e) \subseteq V(a)$, and $e \in P_a$ implies that $\text{int} \ V(a) \subseteq \text{int} \ V(e) = V(e)$. Hence $V(e) = \text{int} \ V(a)$ is clopen.

Conversely, let $R$ be a $PP$-ring and $a \in R$ with $\text{Ann}(a) \neq (0)$. Then $\text{int} \ V(a) \neq \emptyset$ is a clopen set and hence by Lemma 2.1, there exists an idempotent $e \in R$ with $\text{int} \ V(a) = V(e)$. Since $P_a = \{b \in R : \text{int} \ V(a) \subseteq \text{int} \ V(b)\}$, Lemma 2.2 implies that $P_a = (e)$. ■

**Theorem 3.3.** Let $R$ be a semiprimitive Gelfand ring with property $p.z$. Every intersection of basic $z^0$-ideals in $R$ is principal if and only if $R$ is a Baer ring.

**Proof.** Suppose every intersection of basic $z^0$-ideals is principal and $G$ is an open set in $\text{Max}(R)$. Then by Lemma 2.4, there is $T \subseteq R$ such that $G = \bigcup_{a \in T} \text{int} \ M(a)$. By hypothesis, there is an idempotent $e \in R$ such that $\bigcap_{a \in T} P_a = (e)$. We now claim that $\text{cl} G = M(e)$ and this implies that $\text{cl} G$ is clopen. To see this, we note that $e \in P_a$ for all $a \in T$, i.e., $\text{int} \ M(a) \subseteq \text{int} \ M(e)$ for all $a \in T$. Hence $G \subseteq M(e)$ implies that $\text{cl} G \subseteq M(e)$. Now suppose for contradiction that $M \in M(e) - \text{cl} G$. Then by Lemma 2.4, there exists $b \in R$ such that $\text{cl} G \subseteq M(b)$ and $M \in M(b - 1)$, so $\text{int} \ M(a) \subseteq M(b)$ for all $a \in T$, hence $b \in P_a$ for all $a \in T$. This shows that $b \in \bigcap_{a \in T} P_a = (e)$. But $M \in M(e) \cap M(b - 1)$, so $M(e) \not\subseteq M(b)$, i.e., $b \not\in (e)$, a contradiction.

Conversely, let $R$ be a Baer ring and let $I = \bigcap_{a \in T} P_a$ for some $T \subseteq R$. By hypothesis, $G = \text{cl} (\bigcup_{a \in T} \text{int} \ M(a))$ is an open set, so Lemma 2.1 implies that $G = M(e)$ for some idempotent $e \in R$. Clearly, $\text{int} \ M(a) \subseteq \text{int} \ M(e)$ for all $a \in T$, which means that $e \in P_a$ for all $a \in T$. Hence $(e) \subseteq I$. Now let $b \in I$. Then $\text{int} \ M(a) \subseteq \text{int} M(b)$ for all $a \in T$, which means that $G \subseteq M(b)$. Thus by Lemma 2.2, $b \in (e)$, i.e., $I \subseteq (e)$ and therefore $I = (e)$. ■

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**References**


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