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z^0-Ideals and some special commutative rings

by

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Abstract. In a commutative ring R, an ideal I consisting entirely of zero divisors is called a torsion ideal, and an ideal is called a z^0-ideal if I is torsion and for each $a \in I$ the intersection of all minimal prime ideals containing a is contained in I. We prove that in large classes of rings, say R, the following results hold: every z-ideal is a z^0-ideal if and only if every element of R is either a zero divisor or a unit, if and only if every maximal ideal in R (in general, every prime z-ideal) is a z^0-ideal, if and only if every torsion z-ideal is a z^0-ideal and if and only if the sum of any two torsion ideals is either a torsion ideal or R. We give a necessary and sufficient condition for every prime z^0-ideal to be either minimal or maximal. We show that in a large class of rings, the sum of two z^0-ideals is either a z^0-ideal or R and we also give equivalent conditions for R to be a PP-ring or a Baer ring.

1. Introduction. An ideal I of a commutative ring R is called a *z-ideal* if whenever any two elements of R are contained in the same set of maximal ideals and I contains one of them, then it also contains the other (see [7, 4A.5] for an equivalent definition). These ideals, which are both algebraic and topological objects, were first introduced by Kohls and play a fundamental role in studying the ideal theory of $C(X)$, the ring of continuous real-valued functions on a completely regular Hausdorff space X (see [7]). z-Ideals in general commutative rings were studied by Mason [12]. He proved that maximal ideals, minimal prime ideals and some other important ideals in commutative rings are z-ideals (see [12, p. 281]).

A special case of z-ideals in $C(X)$ consisting entirely of zero divisors are z^0-ideals (see [3]–[5]). An ideal I consisting entirely of zero divisors is called a *torsion ideal*, and an ideal is called a *z^0-ideal* if I is torsion and for each $a \in I$ the intersection of all minimal prime ideals containing a is contained in I. z^0-Ideals in general commutative rings were also studied by Azarpanah, Karamzadeh and Rezai [4]; these ideals play a fundamental role in studying

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the torsion prime ideals. It was proved that the Jacobson radical of a ring R is zero if and only if every z^0-ideal is a z-ideal (see [4, Proposition 1.10]).

In this paper we will investigate the properties of ideals consisting entirely of zero divisors, such as z^0-ideals, torsion prime ideals, prime z^0-ideals and so on.

Throughout, R is a commutative reduced ring with identity. By a *reduced ring* we mean a ring without non-zero nilpotent elements. We also say R is *semiprimitive* if $\bigcap \mathrm{Max}(R) = (0)$. We denote by $\mathrm{Spec}(R)$, $\mathrm{Max}(R)$ and $\mathrm{Min}(R)$ the spaces of prime ideals, maximal ideals and minimal ideals of R, respectively. The topology of these spaces is the Zariski topology (see [2]). The operators cl and int denote the closure and interior. Let

$$\begin{aligned} \mathrm{V}(a) &= \{P \in \mathrm{Spec}(R) : a \in P\} \quad \text{for all } a \in R, \\ \mathrm{V}(I) &= \{P \in \mathrm{Spec}(R) : I \subseteq P\} \quad \text{for all ideals } I \text{ of } R. \end{aligned}$$

If $S \subseteq \mathrm{Spec}(R)$, we put

$$\begin{aligned} \mathrm{V}_S(a) &= \mathrm{V}(a) \cap S, \qquad \mathrm{V}_S(I) = \mathrm{V}(I) \cap S, \\ \mathrm{M}(a) &= \mathrm{V}(a) \cap \mathrm{Max}(R), \\ \mathrm{M}(I) &= \mathrm{V}(I) \cap \mathrm{Max}(R). \end{aligned}$$

For each $P \in \mathrm{Spec}(R)$, let $O_P = \bigcap_{P' \subseteq P} P'$, where P' ranges over all prime ideals contained in P. It is well known that if R is a reduced ring, then

$$\begin{aligned} O_P &= \{a \in R : \exists b \in R - P \text{ such that } ab = 0\} \\ &= \{a \in R : P \in \operatorname{int} \mathrm{V}(a)\} \quad \text{(see [18, p. 1482])}. \end{aligned}$$

Suppose that $S = \mathrm{Min}(R)$ and $P_a = \bigcap \mathrm{V}_S(a)$. It is known that $P_a = \{b \in R : \mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)\}$, and it is called a *basic z^0-ideal* (see [4]). A proper ideal I is a z^0-ideal if $P_a \subseteq I$ for each $a \in I$ (see [3] and [4]). Clearly, minimal prime ideals, every intersection of z^0-ideals, and $\mathrm{Ann}(a)$ and P_a for each $a \in R$ are z^0-ideals. It is easy to see that an ideal is a z^0-ideal if and only if $a \in I$ and $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$ imply that $b \in I$. Now let S be a dense subspace of $\mathrm{Spec}(R)$. Since $\operatorname{int} \mathrm{V}_S(a) = S - \mathrm{V}_S(\mathrm{Ann}(a))$, we have: $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$ if and only if $\operatorname{int} \mathrm{V}_S(a) \subseteq \operatorname{int} \mathrm{V}_S(b)$, for each $a, b \in R$. Hence I is a z^0-ideal if and only if $a \in I$ and $\operatorname{int} \mathrm{V}_S(a) \subseteq \operatorname{int} \mathrm{V}_S(b)$ imply that $b \in I$. In particular, in a semiprimitive ring R, every z-ideal is a z^0-ideal (we call I a *z-ideal* if $a \in I$ and $\mathrm{M}(a) \subseteq \mathrm{M}(b)$ imply that $b \in I$). For other equivalent definitions of z^0-ideals, see [4, Proposition 1.4].

We observe that in a reduced ring R, every z^0-ideal is a torsion ideal, but a torsion ideal need not even be a z-ideal. Clearly the following questions concerning torsion ideals, z-ideals and z^0-ideals are natural:

When is every torsion z-ideal a z^0-ideal?

When is every torsion ideal a z^0-ideal?

When is every torsion prime z-ideal a z^0-ideal?
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We are going to answer these questions in Section 2.

A ring is called a *Gelfand ring* (or a *pm ring*) if each prime ideal is contained in a unique maximal ideal (see [6]). For a commutative ring R, De Marco and Orsatti [6] showed that R is Gelfand if and only if $\mathrm{Max}(R)$ is Hausdorff, if and only if $\mathrm{Spec}(R)$ is normal (in general, not Hausdorff).

A ring R is called a *PP-ring* if every principal ideal is a projective R-module. A ring R is called a *Baer ring* if the annihilator $\mathrm{Ann}(I)$ of each ideal I in R is generated by an idempotent. A non-zero ideal I in R is said to be *essential* if it intersects every non-zero ideal non-trivially.

2. Torsion ideals and z^0-ideals. The following are Lemmas 2.2 and 2.3 in [17].

LEMMA 2.1. *Let R be a reduced ring and let S be a dense subspace of* $\mathrm{Spec}(R)$ *containing* $\mathrm{Max}(R)$. *Then A is a clopen subset of S if and only if there exists an idempotent $e \in R$ such that $A = V_S(e)$.*

LEMMA 2.2. *Let R be a reduced ring, $e \in R$ be an idempotent and S be a dense subspace of* $\mathrm{Spec}(R)$. *Then $(a) \subseteq (e)$ if and only if* $\mathrm{V}_S(e) \subseteq \mathrm{V}_S(a)$, *for all $a \in R$.*

The following definition is well known, but it is indispensable in studying rings with zero divisors.

DEFINITION. A ring R has *property A* if each finitely generated ideal of R consisting of zero divisors has a non-zero annihilator. R is said to satisfy the *annihilator condition*, or briefly *a.c.*, if for any $a, b \in R$ there exists an element $c \in R$ with $\mathrm{Ann}(a) \cap \mathrm{Ann}(b) = \mathrm{Ann}(c)$; or equivalently, $\mathrm{int}\,\mathrm{V}(a) \cap \mathrm{int}\,\mathrm{V}(b) = \mathrm{int}\,\mathrm{V}(c)$.

It is well known that most important rings have some of these properties. For example, Noetherian rings [10, p. 56], $C(X)$ [7], rings of Krull dimension zero (each prime ideal is maximal), the polynomial ring $R[x]$ and rings whose classical ring of quotients are regular ([8], [13]) are examples of rings with property A. We also observe that $R[x]$, where R is a reduced ring, $C(X)$, Bézout rings (finitely generated ideals are principal) and many other important rings satisfy a.c. ([9], [11], [14]). The reader is referred to [9] for various examples and counterexamples of rings with these properties.

DEFINITION. A ring R is called *quasi regular* if every element of R is either a unit or a zero divisor. Clearly every regular ring is quasi regular, but a quasi regular ring is not necessarily regular (see the next proposition).

Now we give several equivalent conditions for a reduced ring to be quasi regular.

Proposition 2.3. *Let R be a reduced ring. The following statements are equivalent*:

(1) *R is a quasi regular ring.*
(2) *For every $a \in R$, $\mathrm{V}(a) \neq \emptyset$ implies that $\operatorname{int} \mathrm{V}(a) \neq \emptyset$.*
(3) *If $a \in R$ is a non-unit, then there exists $b \neq 1$ such that $a = ab$.*
(4) *$\bigcup_{P \in \mathrm{Spec}(R)} O_P$ is the set of all non-unit elements of R.*
(5) *Every non-trivial principal ideal of R is non-essential.*
(6) *Every (module) monomorphism from R to R is an isomorphism.*

Proof. (1)⇒(2). For any $a \in R$, $\operatorname{int} \mathrm{V}(a) = \mathrm{Spec}(R) - \mathrm{V}(\mathrm{Ann}(a))$, so $\operatorname{int} \mathrm{V}(a) = \emptyset$ if and only if $\mathrm{Ann}(a) = (0)$.

(2)⇒(3). Let $ac = 0$. Put $b = 1 + c$, so $a = ab$.

(3)⇒(4). It is sufficient to show that every non-unit element a of R is in O_P for some $P \in \mathrm{Spec}(R)$. By (3), there is $b \in R$ such that $a = ab$. Put $c = 1 - b$, so $c \neq 0$ and $ac = 0$. Hence $\emptyset \neq \mathrm{Spec}(R) - \mathrm{V}(c) \subseteq \mathrm{V}(a)$ implies that $\operatorname{int} \mathrm{V}(a) \neq \emptyset$. This shows that $a \in O_P$ for all $P \in \operatorname{int} \mathrm{V}(a)$.

(4)⇒(5). Let $(0) \neq (a) \neq R$ be a principal ideal in R. Then a is a non-unit element and therefore $a \in O_P$ for some $P \in \mathrm{Spec}(R)$. Thus $\operatorname{int} \mathrm{V}(a) \neq \emptyset$ and this implies that $\mathrm{Ann}(a) \neq (0)$, i.e., (a) is non-essential.

(5)⇒(1) is evident.

(1)⇒(6). Let $\phi : R \to R$ be a monomorphism. Then $\phi(b) = b\phi(1)$ for all $b \in R$. This shows that $\phi(1)$ is a non-zero divisor, hence $\phi(1)$ is unit, i.e., there exists $c \in R$ such that $c\phi(1) = 1$. Thus $\phi(c) = 1$ and $\phi(ac) = a$ for all $a \in R$. So ϕ is an isomorphism.

(6)⇒(1). Let $a \in R$ be a non-zero divisor and define $\phi : R \to R$ by $\phi(b) = ab$ for $b \in R$. Clearly, ϕ is a monomorphism and therefore it is also an epimorphism, hence there exists $b \in R$ such that $\phi(b) = ab = 1$. This completes the proof. ■

Lemma 2.4. *Let R be a semiprimitive Gelfand ring. If A and B are disjoint closed subsets of* $\mathrm{Max}(R)$, *then there exists $a \in R$ such that $A \subseteq \operatorname{int} \mathrm{M}(a)$ and $B \subseteq \operatorname{int} \mathrm{M}(a-1)$.*

Proof. By our hypothesis the space $\mathrm{Max}(R)$ is Hausdorff and compact. Therefore by [7, Theorem 1.15] there are closed sets E and F in $\mathrm{Max}(R)$ such that

$$A \subseteq \operatorname{int} E \subseteq E, \qquad B \subseteq \operatorname{int} F \subseteq F, \qquad E \cap F = \emptyset.$$

Hence there are ideals I and J such that $E = \mathrm{M}(I)$ and $F = \mathrm{M}(J)$. We claim that $I + J = R$. Otherwise there exists $M \in \mathrm{Max}(R)$ such that $I + J \subseteq M$. So $M \in \mathrm{M}(I) \cap \mathrm{M}(J)$, and this is a contradiction. Therefore $a + b = 1$ for some $a \in I$ and $b \in J$. Thus

$$A \subseteq \operatorname{int} \mathrm{M}(I) \subseteq \operatorname{int} \mathrm{M}(a), \quad B \subseteq \operatorname{int} \mathrm{M}(J) \subseteq \operatorname{int} \mathrm{M}(a-1). \blacksquare$$

Next we have the following equivalent conditions in a semiprimitive Gelfand ring with property A.

THEOREM 2.5. *Let R be a semiprimitive Gelfand ring with property A. The following statements are equivalent*:

(1) *R is a quasi regular ring.*
(2) $\mathrm{M}(a) = \mathrm{cl}(\operatorname{int} \mathrm{M}(a))$ *for every $a \in R$.*
(3) *Every z-ideal in R is a z^0-ideal.*
(4) *Every torsion z-ideal is a z^0-ideal.*
(5) *Every maximal ideal in R (in general, every prime z-ideal) is a z^0-ideal.*
(6) *Every maximal ideal in R is a torsion ideal.*
(7) *For each non-unit element $a \in R$, there exists $0 \neq b \in R$ with $P_a \subseteq \mathrm{Ann}(b)$.*
(8) *The sum of any two torsion ideals is either R or a torsion ideal.*

Proof. (1)⇒(2). Let $a \in R$ and $M \in \mathrm{M}(a) - \mathrm{cl}(\operatorname{int} \mathrm{M}(a))$. As $\mathrm{cl}(\operatorname{int} \mathrm{M}(a))$ is closed, by Lemma 2.4 there exists $b \in R$ such that $M \in \operatorname{int} \mathrm{M}(b)$ and $\mathrm{cl}(\operatorname{int} \mathrm{M}(a)) \cap \operatorname{int} \mathrm{M}(b) = \emptyset$. By hypothesis, $\operatorname{int} \mathrm{M}(a) \neq \emptyset$ and since R has property A, it follows that $\mathrm{Ann}(a) \cap \mathrm{Ann}(b) \neq (0)$, i.e., $\operatorname{int} \mathrm{M}(a) \cap \operatorname{int} \mathrm{M}(b) \neq \emptyset$, a contradiction.

(2)⇒(3). Let I be a z-ideal, $\operatorname{int} \mathrm{M}(a) = \operatorname{int} \mathrm{M}(b)$ and $a \in I$. By hypothesis we have

$$\mathrm{M}(a) = \mathrm{cl}(\operatorname{int} \mathrm{M}(a)) = \mathrm{cl}(\operatorname{int} \mathrm{M}(b)) = \mathrm{M}(b).$$

Therefore $a \in I$ implies that $b \in I$.

(3)⇒(4) is clear.

(4)⇒(5). Suppose that every torsion z-ideal is a z^0-ideal. We show that every maximal ideal in R is a torsion ideal. First we show that for any zero divisor $a \in R$, $\mathrm{cl}(\operatorname{int} M(a)) = \mathrm{M}(a)$. Suppose not; then there are $M \in \mathrm{M}(a) - \mathrm{cl}(\operatorname{int} \mathrm{M}(a))$ and $b \in R$ such that $\mathrm{cl}(\operatorname{int} \mathrm{M}(a)) \subseteq \mathrm{M}(b)$ and $b \notin M$. On the other hand, $I = \bigcap \mathrm{M}(a)$ is a torsion z-ideal, for if $a' \in I$, then $\mathrm{M}(a) \subseteq \mathrm{M}(a')$, and this implies that $\emptyset \neq \operatorname{int} \mathrm{M}(a) \subseteq \operatorname{int} \mathrm{M}(a')$, i.e., a' is a zero divisor. Hence by hypothesis, I is a z^0-ideal. Thus $\operatorname{int} \mathrm{M}(a) \subseteq \operatorname{int} \mathrm{M}(b)$ implies that $b \in I \subseteq M$, a contradiction.

Now we show that every non-zero divisor is a unit. Suppose that $a \in R$ is a non-zero divisor, so $\mathrm{M}(a) \neq \mathrm{Max}(R)$ and $\operatorname{int} \mathrm{M}(a) = \emptyset$. Hence by Lemma 2.4, there exists $b \in R$ such that $\operatorname{int} \mathrm{M}(b) \neq \emptyset$ and $\mathrm{M}(a) \cap \mathrm{M}(b) = \emptyset$. Thus b

is a zero divisor and we have

$$\begin{aligned}\mathrm{M}(a) \cup \mathrm{M}(b) &= \mathrm{cl}(\mathrm{int}(\mathrm{M}(ab))) = \mathrm{cl}(\mathrm{int}(\mathrm{M}(a) \cup \mathrm{M}(b))) \\ &= \mathrm{cl}(\mathrm{int}(\mathrm{M}(b))) = \mathrm{M}(b).\end{aligned}$$

Therefore $\mathrm{M}(a) \subseteq \mathrm{M}(b)$, which implies that $\mathrm{M}(a) = \emptyset$, i.e., a is a unit. Thus every maximal ideal in R is a torsion ideal.

(5)⇒(6)⇒(7)⇒(8) are evident.

(8)⇒(1). Let $a \in R$ be a non-unit element and $M \neq M' \in \mathrm{Max}(R)$. By Lemma 2.4, there are $b, c \in R$ such that $b \in O_M$ and $c \in O_{M'}$ with $\mathrm{M}(b) \cap \mathrm{M}(c) = \emptyset$. Hence $(b) + (c) = R$ and so $(ab) + (ac) = (a) \neq R$. Since (ab) and (ac) are torsion ideals, a is a zero divisor. ∎

In Theorem 2.5, it is shown that every torsion z-ideal is a z^0-ideal if and only if R is a quasi regular ring. We give a necessary and sufficient condition for every torsion prime z-ideal to be a z^0-ideal. First we give the following definition:

Definition. A ring R is called *weak quasi regular* if for any $a, b \in R$ with $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$, there exists a non-zero divisor $c \in R$ such that $\mathrm{M}(a) \subseteq \mathrm{M}(bc)$. Clearly every quasi regular ring is weak quasi regular.

Theorem 2.6. *Let R be a semiprimitive ring. Then every torsion prime z-ideal is a z^0-ideal if and only if R is weak quasi regular.*

Proof. First suppose that every torsion z-ideal is a z^0-ideal. To the contrary, suppose $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$ and for every non-zero divisor $c \in R$, $\mathrm{M}(bc)$ does not contain $\mathrm{M}(a)$. So $bc \notin I = \bigcap \mathrm{M}(a)$ for any non-zero divisor $c \in R$. Now we define

$$T = \{b^n c : c \in R \text{ is a non-zero divisor}, n = 0, 1, \ldots\}.$$

Clearly, T is closed under multiplication and $I \cap T = \emptyset$, for $\mathrm{M}(a)$ is a z-ideal and $\mathrm{M}(b^n c) = \mathrm{M}(bc)$ for all $n \in \mathbb{N}$. Now by Theorem 1.1 in [12], there exists a prime z-ideal P such that $I \subseteq P$ and $P \cap S = \emptyset$. So P is a torsion ideal and hence by hypothesis, P must be a z^0-ideal. But $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$, $a \in P$ and $b \notin P$, a contradiction.

Conversely, let P be a torsion z-ideal in R, $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$ and $a \in P$. By hypothesis, there exists a non-zero divisor $c \in R$ with $\mathrm{M}(a) \subseteq \mathrm{M}(bc)$. Since P is a z-ideal, we have $bc \in P$. But $c \notin P$, for c is not a zero divisor, hence $b \in P$, i.e., P is a z^0-ideal. ∎

Next we give some necessary and sufficient conditions, in a reduced ring R with property a.c., for every prime z^0-ideal to be minimal (see [4, Proposition 1.26]).

Proposition 2.7. *Let R be a reduced ring with property a.c. The following statements are equivalent*:

(1) *Every prime z^0-ideal is minimal.*
(2) $\mathrm{Min}(R)$ *is compact.*
(3) *For any $a \in R$, there exists $b \in \mathrm{Ann}(a)$ such that*

$$\mathrm{Ann}(a) \cap \mathrm{Ann}(b) = (0).$$

(4) *For any $a \in R$, there exists $b \in R$ such that*

$$\mathrm{cl}(\mathrm{int}\,\mathrm{V}(a)) = \mathrm{cl}(\mathrm{Spec}(R) - \mathrm{V}(b)).$$

Proof. (1)⇔(2) follows from Proposition 1.26 and Theorem 1.28 in [4].

(2)⇔(3). Let $S = \mathrm{Min}(R)$. By [8, Lemma 3.1(iv) and Theorem 3.4], $\mathrm{Min}(R)$ is compact if and only if for each $a \in R$, there exists $b \in R$ such that $\mathrm{V}_S(b) = \mathrm{V}_S(\mathrm{Ann}(a))$. By hypothesis, there exists $c \in R$ such that $\mathrm{Ann}(a) \cap \mathrm{Ann}(b) = \mathrm{Ann}(c)$. Therefore we have

$$\mathrm{Ann}(a) \cap \mathrm{Ann}(b) = (0) \;\Leftrightarrow\; \mathrm{V}_S(c) = \emptyset \;\Leftrightarrow\; \mathrm{V}_S(b) \subseteq \mathrm{V}_S(\mathrm{Ann}(a)).$$

Also, $b \in \mathrm{Ann}(a) \Leftrightarrow \mathrm{V}_S(\mathrm{Ann}(a)) \subseteq \mathrm{V}_S(b)$, and we are done.

(3)⇔(4). For every $a, b \in R$ we have

$$\begin{aligned}\mathrm{int}\,\mathrm{V}(a) \cap \mathrm{int}\,\mathrm{V}(b) = \emptyset \;\Leftrightarrow\; \mathrm{cl}(\mathrm{int}\,\mathrm{V}(a)) &\subseteq \mathrm{Spec}(R) - \mathrm{int}\,\mathrm{V}(b)\\ &= \mathrm{cl}(\mathrm{Spec}(R) - \mathrm{V}(b)),\end{aligned}$$

and

$$\begin{aligned}b \in \mathrm{Ann}(a) &\Leftrightarrow\; \mathrm{Spec}(R) - \mathrm{V}(b) \subseteq \mathrm{int}\,\mathrm{V}(a)\\ &\Leftrightarrow\; \mathrm{cl}(\mathrm{Spec}(R) - \mathrm{V}(b)) \subseteq \mathrm{cl}(\mathrm{int}\,\mathrm{V}(a)).\end{aligned}$$

This completes the proof. ∎

Corollary 2.8. *Let R be a semiprimitive Gelfand quasi regular ring with property A. The following statements are equivalent*:

(1) *R is a PP-ring.*
(2) *R is a regular ring.*
(3) $\mathrm{Min}(R)$ *is compact.*

Proof. (1)⇒(2). For every $a \in R$, (1) implies that $\mathrm{M}(\mathrm{Ann}(a)) = \mathrm{Max}(R) - \mathrm{int}\,\mathrm{M}(a)$ is clopen. Also by Theorem 2.5(2), $\mathrm{M}(a) = \mathrm{cl}(\mathrm{int}\,\mathrm{M}(a))$, hence $\mathrm{M}(a)$ is clopen, for every $a \in R$. This implies that $O_M = M$ for every $M \in \mathrm{Max}(R)$, i.e., every prime ideal in R is maximal. Thus Proposition 1.3 in [14] implies that R is a regular ring.

(2)⇒(3) is clear.

(3)⇒(1). Let $a \in R$. By Proposition 2.7(3), there exists $b \in R$ such that $b \in \mathrm{Ann}(a)$ and $\mathrm{Ann}(a) \cap \mathrm{Ann}(b) = (0)$. Hence $\mathrm{V}(a) \cup \mathrm{V}(b) = \mathrm{Spec}(R)$ and $\mathrm{int}(\mathrm{V}(a) \cap \mathrm{V}(b)) = \emptyset$. But since R is a quasi regular ring, we must have $\mathrm{V}(a) \cap \mathrm{V}(b) = \emptyset$, i.e., $\mathrm{V}(a) = \mathrm{int}\,\mathrm{V}(a)$, for all $a \in R$, and this completes the proof. ∎

For an ideal I in R we define $P_I = \bigcap_{a\in I} P_a$. Next we have the following

Lemma 2.9. *Let R be a reduced ring with property a.c. and $a \in R$. Then $\sum_{c\in \mathrm{Ann}(a)} P_{(a,c)} = \bigcup_{c\in \mathrm{Ann}(a)} P_{(a,c)}$ is a z^0-ideal in R.*

Proof. The inclusion $\bigcup_{c\in \mathrm{Ann}(a)} P_{(a,c)} \subseteq \sum_{c\in \mathrm{Ann}(a)} P_{(a,c)}$ is clear. Now we let $b \in \sum_{c\in \mathrm{Ann}(a)} P_{(a,c)}$; then $b = b_1 + \cdots + b_n$, where $b_i \in P_{(a,c_i)}$, $c_i \in \mathrm{Ann}(a)$ and $i = 1, \ldots, n$. By hypothesis, there exists $c \in R$ such that $\bigcap_{i=1}^n \mathrm{Ann}(c_i) = \mathrm{Ann}(c)$, hence $c \in \mathrm{Ann}(a)$. But $\mathrm{Ann}(a) \cap \mathrm{Ann}(c_i) \subseteq \mathrm{Ann}(b_i)$ for all $i = 1, \ldots, n$, so

$$\mathrm{Ann}(a) \cap \mathrm{Ann}(c) = \bigcap_{i=1}^n (\mathrm{Ann}(a) \cap \mathrm{Ann}(c_i)) \subseteq \bigcap_{i=1}^n \mathrm{Ann}(b_i) \subseteq \mathrm{Ann}(b)$$

and this implies $b \in P_{(a,c)}$, showing that $\sum_{c\in \mathrm{Ann}(a)} P_{(a,c)} \subseteq \bigcup_{c\in \mathrm{Ann}(a)} P_{(a,c)}$. Finally, since every $P_{(a,c)}$ is a z^0-ideal, clearly, $\bigcup_{c\in \mathrm{Ann}(a)} P_{(a,c)}$ is also a z^0-ideal. ■

Theorem 2.10. *Let R be a reduced ring with property a.c. and $a \in R$. The following statements are equivalent*:

(1) *Every prime z^0-ideal is minimal or maximal.*
(2) *For any $M \in \mathrm{Max}(R)$ and $a, b \in M$, there are $c \in \mathrm{Ann}(a)$ and $d \notin M$ such that $\mathrm{Ann}(a) \cap \mathrm{Ann}(c) \subseteq \mathrm{Ann}(bd)$.*

Proof. Suppose that every prime z^0-ideal is minimal or maximal but (2) does not hold. Then there are $M \in \mathrm{Max}(R)$ and $a, b \in M$ such that for every $c \in \mathrm{Ann}(a)$ and $d \notin M$, $\mathrm{Ann}(a) \cap \mathrm{Ann}(c) \nsubseteq \mathrm{Ann}(bd)$. Consider

$$T = \{b^n d : d \in R - M,\ n = 1, 2, \ldots\}, \qquad I = \bigcup_{c\in \mathrm{Ann}(a)} P_{(a,c)}.$$

Obviously, T is closed under multiplication. We also have $I \cap T = \emptyset$, for if $b^n d \in P_{(a,c)}$ for some n and $c \in \mathrm{Ann}(a)$, then $\mathrm{Ann}(a) \cap \mathrm{Ann}(c) \subseteq \mathrm{Ann}(bd)$, which is impossible. So there exists a prime ideal P such that $I \subseteq P$ and $P \cap T = \emptyset$. We have already observed in Lemma 2.9 that I is a z^0-ideal and if P is minimal, then P is a z^0-ideal. Now $P \cap T = \emptyset$ and $R - M \subseteq S$ imply that $P \subseteq M$. On the other hand, $\mathrm{Ann}(a) \subseteq P$ and hence P is not minimal, so it must be maximal, i.e., $P = M$. This implies that $b \in M = P$, a contradiction.

Conversely, suppose that (2) holds and $P \subseteq M$ is a prime z^0-ideal, for some $M \in \mathrm{Max}(R)$. Suppose P is neither maximal nor minimal. Then there are $a \in P$ and $b \in M - P$ such that $\mathrm{Ann}(a) \subseteq P$. Now by (2) there are $c \in \mathrm{Ann}(a)$ and $d \notin M$ such that $\mathrm{Ann}(a) \cap \mathrm{Ann}(c) \subseteq \mathrm{Ann}(bd)$. Since $(a, c) \subseteq P$ and P is a z^0-ideal, hence $bd \in P_{(a,c)} \subseteq P$. But $d \notin P$, for $d \notin M$, hence $b \in P$, a contradiction. ■

3. PP**-rings and** z^0**-ideals.** In this section we investigate the relation between z^0-ideals and PP-rings.

REMARK. We note that $\operatorname{int} \mathrm{V}_S(I) = S - \mathrm{V}_S(\operatorname{Ann}(I))$ for every dense subspace S of $\operatorname{Spec}(R)$ and every ideal I in R. Therefore if R is a reduced ring, then R is a PP-ring if and only if there exists a dense subspace S of $\operatorname{Spec}(R)$ containing $\operatorname{Max}(R)$ such that for each $a \in R$, $\operatorname{int} \mathrm{V}_S(a)$ is clopen in S. Also R is a Baer ring if and only if there exists a dense subspace S of $\operatorname{Spec}(R)$ containing $\operatorname{Max}(R)$ such that every open set in S has closure open in S (i.e., S is an extremally disconnected space; see [1] and [16]).

The sum of two z^0-ideals in a reduced ring (even in $C(X)$) may be a proper ideal which is not a z^0-ideal (see [3, p. 20]). If $C(X)$ is a PP-ring, then the sum of any two z^0-ideals in $C(X)$ is either a z^0-ideal or $C(X)$ (see [3, Proposition 2.13] and [15, Theorem 1.1]). Next we generalize this fact.

THEOREM 3.1. *Let R be a reduced PP-ring with property a.c. The sum of any two z^0-ideals in R is either a z^0-ideal or R.*

Proof. Let I and J be two z^0-ideals in R and suppose that $I + J \neq R$. Let $a \in I + J$ and $\operatorname{int} \mathrm{V}(a) = \operatorname{int} \mathrm{V}(b)$ for some $b \in R$. We will show that $b \in I + J$. We have $a = c + d$, where $c \in I$ and $d \in J$. We may assume that $c \neq 0 \neq d$, for otherwise we clearly have $b \in I + J$. Now by the Remark, $\operatorname{int} \mathrm{V}(c)$ and $\operatorname{int} \mathrm{V}(d)$ are clopen sets, and since I and J are z^0-ideals, we have $\operatorname{int} \mathrm{V}(c) \neq \emptyset \neq \operatorname{int} \mathrm{V}(d)$. Then by Lemma 2.1, there are idempotents $e, e' \in R$ such that $\operatorname{int} \mathrm{V}(c) = \mathrm{V}(e)$ and $\operatorname{int} \mathrm{V}(d) = \mathrm{V}(e')$. Since I and J are z^0-ideals, we infer that $e \in I$ and $e' \in J$. Now by our hypothesis, there exists $e'' \in R$ such that

$$\mathrm{V}(e'') = \mathrm{V}(e) \cap \mathrm{V}(e') \subseteq \operatorname{int} \mathrm{V}(a) = \operatorname{int} \mathrm{V}(b).$$

Thus Lemma 2.2 implies that $b \in (e'')$, i.e., $b \in I + J$. ∎

DEFINITION. A ring R has *property p.z.* if every principal z-ideal in R is generated by an idempotent.

EXAMPLE. Suppose R is a semireal-closed F-ring, i.e., for each $a \succ 0$ there exists $b \in R$ with $a = b^2$ (see [12, p. 288]). We show that R has property $p.z.$ To see this suppose $I = (a)$ is a non-zero z-ideal in R. Then $(|a|) = (a)$ and so there exists $b \in R$ such that $|a| = b^2$. Hence $\mathrm{M}(|a|) = \mathrm{M}(b)$ implies that $b \in (|a|)$. Therefore there is $c \in R$ such that $b = b^2c$, i.e., $\mathrm{M}(b) \cup \mathrm{M}(1 - bc) = \operatorname{Max}(R)$. Since $\mathrm{M}(a) = \mathrm{M}(|a|) = \mathrm{M}(b)$, $\mathrm{M}(a)$ is clopen in $\operatorname{Max}(R)$. By Lemma 2.1, there is an idempotent $e \in R$ such that $\mathrm{M}(a) = \mathrm{M}(e)$. Since (a) is a z-ideal, we have $(e) \subseteq (a)$. Also by Lemma 2.2, $(a) \subseteq (e)$, i.e., $(a) = (e)$. Hence every principal z-ideal in R is generated by an idempotent. In particular, $C(X)$ has property $p.z.$

The following theorems are generalizations of Theorem 2.10 in [3].

Theorem 3.2. *Let R be a reduced ring with property p.z. Every basic z^0-ideal in R is principal if and only if R is a PP-ring.*

Proof. Suppose every basic z^0-ideal is principal. We will show that for each $a \in R$, $\operatorname{int} \mathrm{V}(a)$ is clopen in $\operatorname{Spec}(R)$. It suffices to prove this for $a \in R$ which is a zero divisor, for if $\operatorname{Ann}(a) = (0)$, then $\operatorname{int} \mathrm{V}(a) = \emptyset$. Now let $P_a = (b)$ and $\operatorname{Ann}(a) \neq (0)$. Then by hypothesis, $P_a = (e)$, where $e^2 = e$. Hence $a \in (e)$ implies that $\mathrm{V}(e) \subseteq \mathrm{V}(a)$, and $e \in P_a$ implies that $\operatorname{int} \mathrm{V}(a) \subseteq \operatorname{int} \mathrm{V}(e) = \mathrm{V}(e)$. Hence $\mathrm{V}(e) = \operatorname{int} \mathrm{V}(a)$ is clopen.

Conversely, let R be a PP-ring and $a \in R$ with $\operatorname{Ann}(a) \neq (0)$. Then $\operatorname{int} \mathrm{V}(a) \neq \emptyset$ is a clopen set and hence by Lemma 2.1, there exists an idempotent $e \in R$ with $\operatorname{int} \mathrm{V}(a) = \mathrm{V}(e)$. Since $P_a = \{b \in R : \operatorname{int} \mathrm{V}(a) \subseteq \operatorname{int} \mathrm{V}(b)\}$, Lemma 2.2 implies that $P_a = (e)$. ∎

Theorem 3.3. *Let R be a semiprimitive Gelfand ring with property p.z. Every intersection of basic z^0-ideals in R is principal if and only if R is a Baer ring.*

Proof. Suppose every intersection of basic z^0-ideals is principal and G is an open set in $\operatorname{Max}(R)$. Then by Lemma 2.4, there is $T \subseteq R$ such that $G = \bigcup_{a\in T} \operatorname{int} \mathrm{M}(a)$. By hypothesis, there is an idempotent $e \in R$ such that $\bigcap_{a\in T} P_a = (e)$. We now claim that $\operatorname{cl} G = \mathrm{M}(e)$ and this implies that $\operatorname{cl} G$ is clopen. To see this, we note that $e \in P_a$ for all $a \in T$, i.e., $\operatorname{int} \mathrm{M}(a) \subseteq \operatorname{int} \mathrm{M}(e)$ for all $a \in T$. Hence $G \subseteq \mathrm{M}(e)$ implies that $\operatorname{cl} G \subseteq \mathrm{M}(e)$. Now suppose for contradiction that $M \in \mathrm{M}(e) - \operatorname{cl} G$. Then by Lemma 2.4, there exists $b \in R$ such that $\operatorname{cl} G \subseteq \mathrm{M}(b)$ and $M \in \mathrm{M}(b-1)$, so $\operatorname{int} \mathrm{M}(a) \subseteq \mathrm{M}(b)$ for all $a \in T$, hence $b \in P_a$ for all $a \in T$. This shows that $b \in \bigcap_{a\in T} P_a = (e)$. But $M \in \mathrm{M}(e) \cap \mathrm{M}(b-1)$, so $\mathrm{M}(e) \not\subseteq \mathrm{M}(b)$, i.e., $b \notin (e)$, a contradiction.

Conversely, let R be a Baer ring and let $I = \bigcap_{a\in T} P_a$ for some $T \subseteq R$. By hypothesis, $G = \operatorname{cl}(\bigcup_{a\in T} \operatorname{int} \mathrm{M}(a))$ is an open set, so Lemma 2.1 implies that $G = \mathrm{M}(e)$ for some idempotent $e \in R$. Clearly, $\operatorname{int} \mathrm{M}(a) \subseteq \operatorname{int} \mathrm{M}(e)$ for all $a \in T$, which means that $e \in P_a$ for all $a \in T$. Hence $(e) \subseteq I$. Now let $b \in I$. Then $\operatorname{int} \mathrm{M}(a) \subseteq \operatorname{int} \mathrm{M}(b)$ for all $a \in T$, which means that $G \subseteq \mathrm{M}(b)$. Thus by Lemma 2.2, $b \in (e)$, i.e., $I \subseteq (e)$ and therefore $I = (e)$. ∎

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