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A connection between multiplication in $C(X)$ and the dimension of X

by

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Abstract. Let X be a compact Hausdorff topological space. We show that multiplication in the algebra $C(X)$ is open iff $\dim X < 1$. On the other hand, the existence of non-empty open sets $U, V \subset C(X)$ satisfying $\mathrm{Int}(U \cdot V) = \emptyset$ is equivalent to $\dim X > 1$. The preimage of every set of the first category in $C(X)$ under the multiplication map is of the first category in $C(X) \times C(X)$ iff $\dim X \le 1$.

Let X be a compact (Hausdorff) topological space. We consider the algebra $C(X)$ of real-valued continuous functions on X with the pointwise addition and multiplication and the norm $\|f\| = \sup_{x\in X} |f(x)|$. Other two natural operations on $C(X)$ are minimum and maximum. All these operations are continuous but, in general, only addition, maximum and minimum are open as mappings from $C(X) \times C(X)$ to $C(X)$ (see [1], [4], [5]).

We recall from [1] the definition of a weakly open map:

Definition. A map of topological spaces is *weakly open* if the image of every non-empty open set has a non-empty interior.

In [1], [4], [5] it is shown that multiplication in $C([0,1])$ is weakly open. We extend this result as follows:

Theorem. *Let X be a compact topological space. The following equivalences hold*:

(1) *multiplication in $C(X)$ is open iff* $\dim X < 1$,
(2) *multiplication in $C(X)$ is weakly open and not open iff* $\dim X = 1$,
(3) *multiplication in $C(X)$ is not weakly open iff* $\dim X > 1$,

where $\dim X$ *denotes the topological* (*covering*) *dimension of* X.

The first of these equivalences was suggested by D. H. Fremlin in January 2004 (oral communication).

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For $A, B \subset C(X)$ set $A \cdot B = \{f \cdot g : f \in A,\ g \in B\}$, and for $f \in C(X)$ and $r > 0$ write $B(f, r) = \{g \in C(X) : \|g - f\| < r\}$.

Proof of Theorem. It is enough to show the implications: (1)$\Leftarrow$, (1)$\Rightarrow$, (3)$\Leftarrow$ and (2)$\Leftarrow$.

(1)$\Leftarrow$: If $\dim X = -1$ (i.e. $X = \emptyset$) then there is nothing to prove. Let $\dim X = 0$. We need to prove that for every pair of open sets $U, V \subset C(X)$ the set $U \cdot V \subset C(X)$ is open. Let $h \in U \cdot V$, i.e. $h = f \cdot g$ for some $f \in U$ and $g \in V$. There exists $\varepsilon > 0$ such that $B(f, \varepsilon) \subset U$ and $B(g, \varepsilon) \subset V$. We will show that $B(h, \varepsilon^2/4) \subset U \cdot V$.

Let $\widehat{h} \in B(h, \varepsilon^2/4)$. We define open subsets of X:

$$F = \{x \in X : |f(x)| > \varepsilon/4\}, \quad G = \{x \in X : |g(x)| > \varepsilon/4\},$$
$$H = \{x \in X : |f(x)| < \varepsilon/3 \text{ and } |g(x)| < \varepsilon/3\}.$$

We have $F \cup G \cup H = X$. Since $\dim X = 0$, there exist clopen, pairwise disjoint sets $\widetilde{F} \subset F$, $\widetilde{G} \subset G$ and $\widetilde{H} \subset H$ such that $\widetilde{F} \cup \widetilde{G} \cup \widetilde{H} = X$. We define $\widehat{f} \in B(f, \varepsilon)$ and $\widehat{g} \in B(g, \varepsilon)$ such that $\widehat{h} = \widehat{f} \cdot \widehat{g}$ as follows:

$$\widehat{f}(x) = f(x), \qquad \widehat{g}(x) = \frac{\widehat{h}(x)}{f(x)} \qquad \text{for } x \in \widetilde{F},$$

$$\widehat{f}(x) = \frac{\widehat{h}(x)}{g(x)}, \qquad \widehat{g}(x) = g(x) \qquad \text{for } x \in \widetilde{G},$$

$$\widehat{f}(x) = \sqrt{|\widehat{h}(x)|}, \quad \widehat{g}(x) = \sqrt{|\widehat{h}(x)|} \cdot \operatorname{sgn}(\widehat{h}(x)) \quad \text{for } x \in \widetilde{H}.$$

It is clear that $\widehat{h} = \widehat{f} \cdot \widehat{g}$. The functions $\widehat{f}$ and $\widehat{g}$ are continuous because they are continuous on each of the clopen sets $\widetilde{F}$, $\widetilde{G}$ and $\widetilde{H}$ covering X.

It remains to show that $\widehat{f} \in B(f, \varepsilon)$ and $\widehat{g} \in B(g, \varepsilon)$. If $x \in \widetilde{F}$ then

$$|\widehat{f}(x) - f(x)| = 0 < \varepsilon, \quad |\widehat{g}(x) - g(x)| = \left|\frac{\widehat{h}(x) - h(x)}{f(x)}\right| < \frac{\varepsilon^2/4}{\varepsilon/4} = \varepsilon,$$

and similarly for $x \in \widetilde{G}$. Finally, if $x \in \widetilde{H}$ then $|h(x)| = |f(x)|\,|g(x)| < \varepsilon^2/9$. It follows that $|\widehat{h}(x)| \le |h(x)| + \varepsilon^2/4 < \varepsilon^2/9 + \varepsilon^2/4$ and $|\widehat{f}(x)| = |\widehat{g}(x)| < \sqrt{\varepsilon^2/9 + \varepsilon^2/4}$. Then $|\widehat{f}(x) - f(x)| \le |\widehat{f}(x)| + |f(x)| < \sqrt{\varepsilon^2/9 + \varepsilon^2/4} + \varepsilon/3 < \varepsilon$ and similarly $|\widehat{g}(x) - g(x)| < \varepsilon$. Hence $\|\widehat{f} - f\| < \varepsilon$ and $\|\widehat{g} - g\| < \varepsilon$.

(1)$\Rightarrow$: We will show that if $\dim X > 0$ then multiplication is not open. Since $\dim X > 0$, there exists a connected component S of X which has at least two elements, say x_1 and x_2 (cf. [2]). Let $f : X \to \mathbb{R}$ be a continuous function such that $f(x_1) = -1$ and $f(x_2) = 1$. We will prove that $f \cdot f$ is not an interior point of $B(f, 1) \cdot B(f, 1)$. Consider an arbitrary element $\widehat{h} \in B(f, 1) \cdot B(f, 1)$. We have $\widehat{h} = \widehat{f} \cdot \widehat{g}$ for some $\widehat{f}, \widehat{g} \in B(f, 1)$. The function $\widehat{f}$ satisfies $\widehat{f}(x_1) < -1 + 1 = 0$ and $\widehat{f}(x_2) > 1 - 1 = 0$. Thus $\widehat{f}$, and hence

$\widehat{h}$, has a zero in S. It follows that none of the functions $h_n = f \cdot f + 1/n$, $n \in \mathbb{N}$, is in $B(f,1) \cdot B(f,1)$. On the other hand, $\lim_{n\to\infty} h_n = f \cdot f$. Hence $f \cdot f \notin \mathrm{Int}(B(f,1) \cdot B(f,1))$.

(3)$\Leftarrow$: We assume that $\dim X \geq 2$. By the Hemmingsen Lemma (cf. [2]) there exist closed sets $A_1, B_1, A_2, B_2 \subset X$ such that $A_1 \cap B_1 = \emptyset$, $A_2 \cap B_2 = \emptyset$ and if L_1 is a partition between A_1 and B_1, and L_2 is a partition between A_2 and B_2, then $L_1 \cap L_2 \neq \emptyset$. (We recall that a closed set $L \subset X$ is a *partition* between disjoint closed sets $A, B \subset X$ if there exist two disjoint open sets $U, V \subset X$ satisfying $A \subset U$, $B \subset V$ and $U \cup V = X \setminus L$.)

Let $f, g \in C(X)$ be such that $f(x) = 1$ for $x \in A_1$, $f(x) = -1$ for $x \in B_1$, $g(x) = 1$ for $x \in A_2$ and $g(x) = -1$ for $x \in B_2$. We will show that $B(f,1) \cdot B(g,1)$ is nowhere dense.

Aiming at a contradiction assume that $\widehat{h} \in \mathrm{Int}(\overline{B(f,1) \cdot B(g,1)})$. Then $\widehat{h} + \varepsilon \in \overline{B(f,1) \cdot B(g,1)}$ for some $\varepsilon > 0$. Let $\widehat{f} \in B(f,1)$ and $\widehat{g} \in B(g,1)$ be such that $\|\widehat{h} - \widehat{f} \cdot \widehat{g}\| < \varepsilon/2$. If $x \in A_1$ then $\widehat{f}(x) > 1 - 1 = 0$, and if $x \in B_1$ then $\widehat{f}(x) < -1 + 1 = 0$. It follows that $A_1 \subset \{x \in X : \widehat{f}(x) > 0\}$, $B_1 \subset \{x \in X : \widehat{f}(x) < 0\}$ and the set $L_1 = \{x \in X : \widehat{f}(x) = 0\}$ is a partition between A_1 and B_1. Similarly, let $\widetilde{f} \in B(f,1)$ and $\widetilde{g} \in B(g,1)$ satisfy $\|(\widehat{h} + \varepsilon) - \widetilde{f} \cdot \widetilde{g}\| < \varepsilon/2$. We have $A_2 \subset \{x \in X : \widetilde{g}(x) > 0\}$ and $B_2 \subset \{x \in X : \widetilde{g}(x) < 0\}$, so $L_2 = \{x \in X : \widetilde{g}(x) = 0\}$ is a partition between A_2 and B_2. It follows that there exists $x_0 \in L_1 \cap L_2$. By the definition of L_1 one has $\widehat{h}(x_0) + \varepsilon > \widehat{f}(x_0) \cdot \widehat{g}(x_0) - \varepsilon/2 + \varepsilon = \varepsilon/2$. On the other hand, by the definition of L_2 we have $\widehat{h}(x_0) + \varepsilon < \widetilde{f}(x_0) \cdot \widetilde{g}(x_0) + \varepsilon/2 = \varepsilon/2$. This contradiction shows that $B(f,1) \cdot B(g,1)$ is nowhere dense.

(2)$\Leftarrow$: This implication is an immediate consequence of (1)$\Rightarrow$ and of Lemmas 1 and 2 below. ∎

LEMMA 1. *Let X be a compact topological space with* $\dim X \leq 1$ *and let $U, V \subset C(X)$ be non-empty and open. Then there exist $f \in U$ and $g \in V$ such that $f^{-1}(0) \cap g^{-1}(0) = \emptyset$.*

Proof. Let X be any topological space, let (Y, d) be a metric space and let $\varphi : X \to Y$ be continuous. Following Hurewicz and Wallman ([3, Ch. VI]), we say that a point $y \in Y$ is an *unstable value* of φ if for every $\delta > 0$ there exists a continuous mapping $\psi : X \to Y$ satisfying $\forall_{x \in X}\ d(\varphi(x), \psi(x)) < \delta$ and $y \notin \psi[X]$. Theorem VI.1 of [3] states that if $Y = \mathbb{R}^n$ and $\dim X < n$ for some $n \in \mathbb{N}$ then every point of Y is an unstable value of φ.

Under the assumptions of the lemma, we pick $\widetilde{f} \in U$ and $\widetilde{g} \in V$. Then $\varphi = (\widetilde{f}, \widetilde{g})$ maps X into $\mathbb{R}^2$. Since $\dim X < 2$, $(0,0)$ is an unstable value of $(\widetilde{f}, \widetilde{g})$. As U and V are open, there exist $f \in U$ and $g \in V$ satisfying $(0,0) \notin (f,g)[X]$, which completes the proof. ∎

Lemma 2. *Assume that X is a compact topological space, $U, V \subset C(X)$ are non-empty and open and $f \in U$, $g \in V$ satisfy $f^{-1}(0) \cap g^{-1}(0) = \emptyset$. Then there exists $\delta > 0$ such that $B(f \cdot g, \delta) \subset U \cdot V$.*

Proof. There exists $\varepsilon > 0$ satisfying $B(f, \varepsilon) \subset U$ and $B(g, \varepsilon) \subset V$. By normality, there are closed sets $F, G \subset X$ such that $F \subset \{x \in X : f(x) \neq 0\}$, $G \subset \{x \in X : g(x) \neq 0\}$ and $F \cup G = X$. Let a be the smaller of the numbers $\min\{|f(x)| : x \in F\}$ and $\min\{|g(x)| : x \in G\}$). Clearly, $a > 0$. We put

$$\delta = \min\left(a\varepsilon, \frac{a^2}{2}, \frac{a^2\varepsilon}{2a + 2\|f\|}\right).$$

Let $\widehat{h} \in B(f \cdot g, \delta)$. We will find $\widehat{f} \in B(f, \varepsilon) \subset U$ and $\widehat{g} \in B(g, \varepsilon) \subset V$ satisfying $\widehat{f} \cdot \widehat{g} = \widehat{h}$. First, set

$$\widehat{g}(x) = \frac{\widehat{h}(x)}{f(x)} \quad \text{for } x \in F.$$

For $x \in F$ we have

$$|\widehat{g}(x) - g(x)| = \frac{|\widehat{h}(x) - f(x) \cdot g(x)|}{|f(x)|} < \frac{\delta}{a}.$$

Thus the range of $\widehat{g} - g : F \to \mathbb{R}$ is a compact subset of $(-\delta/a, \delta/a)$. Using the Tietze Theorem we extend this function to $\widehat{g} - g : X \to (-\delta/a, \delta/a)$ and put $\widehat{g} = g + (\widehat{g} - g)$. Then clearly $\widehat{g} \in B(g, \varepsilon)$. For $x \in G$ we have

$$|\widehat{g}(x)| \geq |g(x)| - \|\widehat{g} - g\| > a - \frac{\delta}{a} \geq a - \frac{a}{2} = \frac{a}{2}.$$

Now we define

$$\widehat{f}(x) = \begin{cases} f(x) & \text{for } x \in F, \\ \widehat{h}(x)/\widehat{g}(x) & \text{for } x \in G. \end{cases}$$

The definition is correct, since by definition of $\widehat{g}$, for $x \in F \cap G$ we have $f(x) = \widehat{h}(x)/\widehat{g}(x)$, and if $x \in G$ then $|\widehat{g}(x)| \geq a/2 > 0$. Clearly $\widehat{f} \cdot \widehat{g} = \widehat{h}$. The function $\widehat{f}$ is continuous, because $\widehat{f}|_F$ and $\widehat{f}|_G$ are continuous.

We still have to prove that $\|\widehat{f} - f\| < \varepsilon$. For $x \in F$ one has $\widehat{f}(x) = f(x)$ while if $x \in G$ then

$$\begin{aligned}
|\widehat{f}(x) - f(x)| &= \left|\frac{\widehat{h}(x)}{\widehat{g}(x)} - f(x)\right| \leq \left|\frac{\widehat{h}(x)}{\widehat{g}(x)} - \frac{f(x) \cdot g(x)}{\widehat{g}(x)}\right| + \left|\frac{f(x) \cdot g(x)}{\widehat{g}(x)} - f(x)\right| \\
&= \frac{|\widehat{h}(x) - f(x) \cdot g(x)|}{|\widehat{g}(x)|} + |f(x)| \frac{|g(x) - \widehat{g}(x)|}{|\widehat{g}(x)|} < \frac{\delta}{a/2} + \|f\| \frac{\delta/a}{a/2} \\
&= \delta \frac{2a + 2\|f\|}{a^2} \leq \varepsilon. \quad \blacksquare
\end{aligned}$$

For a Tikhonov ($\mathrm{T}3\frac{1}{2}$) topological space X we denote by $C_{\mathrm{b}}(X)$ the algebra of real-valued continuous bounded functions defined on X. Let βX denote the Čech–Stone compactification of X. Since for every Tikhonov space X the space βX is a compact Hausdorff space and the algebras $C_{\mathrm{b}}(X)$ and $C(\beta X)$ are isometrically isomorphic, we have:

COROLLARY 1. *Let X be a Tikhonov topological space. The following equivalences hold*:

(1) *multiplication in $C_{\mathrm{b}}(X)$ is open iff* $\dim \beta X < 1$,
(2) *multiplication in $C_{\mathrm{b}}(X)$ is weakly open and not open iff* $\dim \beta X = 1$,
(3) *multiplication in $C_{\mathrm{b}}(X)$ is not weakly open iff* $\dim \beta X > 1$.

Another corollary concerns sets of the first category:

COROLLARY 2. *Let X be a compact topological space. The following conditions are equivalent*:

(i) $\dim X \leq 1$.
(ii) *For any set $A \subset C(X)$ of the first category its preimage under the multiplication map is a set of the first category in $C(X) \times C(X)$.*

Corollary 2 extends the results obtained by Balcerzak, Wachowicz and Wilczyński, who showed that the preimage of a residual subset of $C([0,1])$ under the multiplication map is a residual subset of $C([0,1]) \times C([0,1])$. As a matter of fact, the proof of the implication (i)⇒(ii) is identical to the argument in [1], [4] and [5].

Proof of Corollary 2. Denote by B' the preimage of a set $B \subset C(X)$ under the multplication map. Let $\dim X \leq 1$ and let $A \subset C(X)$ be of the first category. Clearly, $A' = \bigcup_{n=1}^{\infty} A'_n$, where the $A_n \subset C(X)$ are nowhere dense. It remains to show that the sets A'_n are nowhere dense. Assume the contrary: for some n there exists a non-empty open set $U \subset C(X) \times C(X)$ satisfying $U \subset \overline{A'_n}$. It follows that $\cdot\,[U] \subset \cdot\,[\overline{A'_n}] \subset \overline{A}_n$, which is not the case because A_n is nowhere dense and $\operatorname{Int}(\cdot\,[U]) \neq \emptyset$ (cf. Theorem, (1)⇐ and (2)⇐).

Conversely, let $\dim X > 1$. There exist non-empty open open sets $U, V \subset C(X)$ such that $W = U \cdot V$ is nowhere dense (cf. proof of Theorem, (3)⇐). On the other hand, $W' \supset U \times V$ is not of the first category in $C(X) \times C(X)$. ∎

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