A connection between multiplication in $C(X)$ and the dimension of $X$

by

Andrzej Komisarski (Łódź)

Abstract. Let $X$ be a compact Hausdorff topological space. We show that multiplication in the algebra $C(X)$ is open iff $\dim X < 1$. On the other hand, the existence of non-empty open sets $U, V \subset C(X)$ satisfying $\text{Int}(U \cdot V) = \emptyset$ is equivalent to $\dim X > 1$. The preimage of every set of the first category in $C(X)$ under the multiplication map is of the first category in $C(X) \times C(X)$ iff $\dim X \leq 1$.

Let $X$ be a compact (Hausdorff) topological space. We consider the algebra $C(X)$ of real-valued continuous functions on $X$ with the pointwise addition and multiplication and the norm $\|f\| = \sup_{x \in X} |f(x)|$. Other two natural operations on $C(X)$ are minimum and maximum. All these operations are continuous but, in general, only addition, maximum and minimum are open as mappings from $C(X) \times C(X)$ to $C(X)$ (see [1], [4], [5]).

We recall from [1] the definition of a weakly open map:

Definition. A map of topological spaces is weakly open if the image of every non-empty open set has a non-empty interior.

In [1], [4], [5] it is shown that multiplication in $C([0,1])$ is weakly open. We extend this result as follows:

Theorem. Let $X$ be a compact topological space. The following equivalences hold:

1. multiplication in $C(X)$ is open iff $\dim X < 1$,
2. multiplication in $C(X)$ is weakly open and not open iff $\dim X = 1$,
3. multiplication in $C(X)$ is not weakly open iff $\dim X > 1$,

where $\dim X$ denotes the topological (covering) dimension of $X$.

The first of these equivalences was suggested by D. H. Fremlin in January 2004 (oral communication).

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For $A, B \subset C(X)$ set $A \cdot B = \{f \cdot g : f \in A, g \in B\}$, and for $f \in C(X)$ and $r > 0$ write $B(f, r) = \{g \in C(X) : \|g - f\| < r\}$.

**Proof of Theorem.** It is enough to show the implications: $(1) \iff (1) \Rightarrow (3) \iff (2) \iff$.

$(1) \iff$: If $\dim X = -1$ (i.e. $X = \emptyset$) then there is nothing to prove. Let $\dim X = 0$. We need to prove that for every pair of open sets $U, V \subset C(X)$ the set $U \cdot V \subset C(X)$ is open. Let $h \in U \cdot V$, i.e. $h = f \cdot g$ for some $f \in U$ and $g \in V$. There exists $\varepsilon > 0$ such that $B(f, \varepsilon) \subset U$ and $B(g, \varepsilon) \subset V$. We will show that $B(h, \varepsilon^2/4) \subset U \cdot V$.

Let $\hat{h} \in B(h, \varepsilon^2/4)$. We define open subsets of $X$:

$$F = \{x \in X : |f(x)| > \varepsilon/4\}, \quad G = \{x \in X : |g(x)| > \varepsilon/4\},$$
$$H = \{x \in X : |f(x)| < \varepsilon/3 \text{ and } |g(x)| < \varepsilon/3\}.$$  

We have $F \cup G \cup H = X$. Since $\dim X = 0$, there exist clopen, pairwise disjoint sets $\hat{F} \subset F$, $\hat{G} \subset G$ and $\hat{H} \subset H$ such that $\hat{F} \cup \hat{G} \cup \hat{H} = X$. We define $\hat{f} \in B(f, \varepsilon)$ and $\hat{g} \in B(g, \varepsilon)$ such that $\hat{h} = \hat{f} \cdot \hat{g}$ as follows:

$$\hat{f}(x) = f(x), \quad \hat{g}(x) = \frac{\hat{h}(x)}{f(x)} \quad \text{for } x \in \hat{F},$$
$$\hat{f}(x) = \frac{\hat{h}(x)}{g(x)}, \quad \hat{g}(x) = g(x) \quad \text{for } x \in \hat{G},$$
$$\hat{f}(x) = \sqrt{|\hat{h}(x)|}, \quad \hat{g}(x) = \sqrt{|\hat{h}(x)|} \cdot \text{sgn}(\hat{h}(x)) \quad \text{for } x \in \hat{H}.$$

It is clear that $\hat{h} = \hat{f} \cdot \hat{g}$. The functions $\hat{f}$ and $\hat{g}$ are continuous because they are continuous on each of the clopen sets $\hat{F}$, $\hat{G}$ and $\hat{H}$ covering $X$.

It remains to show that $\hat{f} \in B(f, \varepsilon)$ and $\hat{g} \in B(g, \varepsilon)$. If $x \in \hat{F}$ then

$$|\hat{f}(x) - f(x)| = 0 < \varepsilon, \quad |\hat{g}(x) - g(x)| = \left|\frac{\hat{h}(x) - h(x)}{f(x)}\right| < \frac{\varepsilon^2/4}{\varepsilon/4} = \varepsilon,$$

and similarly for $x \in \hat{G}$. Finally, if $x \in \hat{H}$ then $|\hat{h}(x)| = |f(x)||g(x)| < \varepsilon^2/9$. It follows that $|\hat{h}(x)| < |h(x)| + \varepsilon^2/4 < \varepsilon^2/9 + \varepsilon^2/4$ and $|\hat{f}(x)| = |\hat{g}(x)| < \sqrt{\varepsilon^2/9 + \varepsilon^2/4}$. Then $|\hat{f}(x) - f(x)| = |f(x)| + |f(x)| < \sqrt{\varepsilon^2/9 + \varepsilon^2/4 + \varepsilon/3} < \varepsilon$ and similarly $|\hat{g}(x) - g(x)| < \varepsilon$. Hence $\|\hat{f} - f\| < \varepsilon$ and $\|\hat{g} - g\| < \varepsilon$.

$(1) \Rightarrow$: We will show that if $\dim X > 0$ then multiplication is not open. Since $\dim X > 0$, there exists a connected component $S$ of $X$ which has at least two elements, say $x_1$ and $x_2$ (cf. [2]). Let $f : X \to \mathbb{R}$ be a continuous function such that $f(x_1) = -1$ and $f(x_2) = 1$. We will prove that $f \cdot f$ is not an interior point of $B(f, 1) \cdot B(f, 1)$. Consider an arbitrary element $\tilde{h} \in B(f, 1) \cdot B(f, 1)$. We have $\tilde{h} = \tilde{f} \cdot \tilde{g}$ for some $\tilde{f}, \tilde{g} \in B(f, 1)$. The function $\tilde{f}$ satisfies $\tilde{f}(x_1) < -1 + 1 = 0$ and $\tilde{f}(x_2) > 1 - 1 = 0$. Thus $\tilde{f}$, and hence
\(\widehat{h}\), has a zero in \(S\). It follows that none of the functions \(h_n = f \cdot f + 1/n, n \in \mathbb{N}\), is in \(B(f, 1) \cdot B(f, 1)\). On the other hand, \(\lim_{n \to \infty} h_n = f \cdot f\). Hence \(f \cdot f \not\in \text{Int}(B(f, 1) \cdot B(f, 1))\).

\((3)\Leftarrow\): We assume that \(\dim X \geq 2\). By the Hemmingsen Lemma (cf. [2]) there exist closed sets \(A_1, B_1, A_2, B_2 \subset X\) such that \(A_1 \cap B_1 = \emptyset, A_2 \cap B_2 = \emptyset\) and if \(L_1\) is a partition between \(A_1\) and \(B_1\), and \(L_2\) is a partition between \(A_2\) and \(B_2\), then \(L_1 \cap L_2 \neq \emptyset\). (We recall that a closed set \(L \subset X\) is a partition between disjoint closed sets \(A, B \subset X\) if there exist two disjoint open sets \(U, V \subset X\) satisfying \(A \subset U, B \subset V\) and \(U \cup V = X \setminus L\).)

Let \(f, g \in C(X)\) be such that \(f(x) = 1\) for \(x \in A_1\), \(f(x) = -1\) for \(x \in B_1\), \(g(x) = 1\) for \(x \in A_2\) and \(g(x) = -1\) for \(x \in B_2\). We will show that \(B(f, 1) \cdot B(g, 1)\) is nowhere dense.

Aiming at a contradiction assume that \(\widehat{h} \in \text{Int}(B(f, 1) \cdot B(g, 1))\). Then \(\widehat{h} + \varepsilon \in B(f, 1) \cdot B(g, 1)\) for some \(\varepsilon > 0\). Let \(\widehat{f} \in B(f, 1)\) and \(\widehat{g} \in B(g, 1)\) be such that \(\|\widehat{h} - \widehat{f} \cdot \widehat{g}\| < \varepsilon/2\). If \(x \in A_1\) then \(\widehat{f}(x) > 1 - 1 = 0\), and if \(x \in B_1\) then \(\widehat{f}(x) < -1 + 1 = 0\). It follows that \(A_1 \subset \{x \in X : \widehat{f}(x) > 0\}, B_1 \subset \{x \in X : \widehat{f}(x) < 0\}\) and the set \(L_1 = \{x \in X : \widehat{f}(x) = 0\}\) is a partition between \(A_1\) and \(B_1\). Similarly, let \(\widehat{f} \in B(f, 1)\) and \(\widehat{g} \in B(g, 1)\) satisfy \(\|\widehat{h} + \varepsilon - \widehat{f} \cdot \widehat{g}\| < \varepsilon/2\). We have \(A_2 \subset \{x \in X : \widehat{g}(x) > 0\}\) and \(B_2 \subset \{x \in X : \widehat{g}(x) < 0\}\), so \(L_2 = \{x \in X : \widehat{g}(x) = 0\}\) is a partition between \(A_2\) and \(B_2\). It follows that there exists \(x_0 \in L_1 \cap L_2\). By the definition of \(L_1\) one has \(\widehat{h}(x_0) + \varepsilon > \widehat{f}(x_0) \cdot \widehat{g}(x_0) - \varepsilon/2 + \varepsilon = \varepsilon/2\). On the other hand, by the definition of \(L_2\) we have \(\widehat{h}(x_0) + \varepsilon < \widehat{f}(x_0) \cdot \widehat{g}(x_0) + \varepsilon/2 = \varepsilon/2\). This contradiction shows that \(B(f, 1) \cdot B(g, 1)\) is nowhere dense.

\((2)\Leftarrow\): This implication is an immediate consequence of \((1)\Rightarrow\) and of Lemmas 1 and 2 below.

**Lemma 1.** Let \(X\) be a compact topological space with \(\dim X \leq 1\) and let \(U, V \subset C(X)\) be non-empty and open. Then there exist \(f \in U\) and \(g \in V\) such that \(f^{-1}(0) \cap g^{-1}(0) = \emptyset\).

**Proof.** Let \(X\) be any topological space, let \((Y, d)\) be a metric space and let \(\varphi : X \to Y\) be continuous. Following Hurewicz and Wallman ([3, Ch. VII]), we say that a point \(y \in Y\) is an unstable value of \(\varphi\) if for every \(\delta > 0\) there exists a continuous mapping \(\psi : X \to Y\) satisfying \(\forall x \in X\ d(\varphi(x), \psi(x)) < \delta\) and \(y \not\in \psi[X]\). Theorem VI.1 of [3] states that if \(Y = \mathbb{R}^n\) and \(\dim X < n\) for some \(n \in \mathbb{N}\) then every point of \(Y\) is an unstable value of \(\varphi\).

Under the assumptions of the lemma, we pick \(\tilde{f} \in U\) and \(\tilde{g} \in V\). Then \(\varphi = (\tilde{f}, \tilde{g})\) maps \(X\) into \(\mathbb{R}^2\). Since \(\dim X < 2\), \((0, 0)\) is an unstable value of \((\tilde{f}, \tilde{g})\). As \(U\) and \(V\) are open, there exist \(f \in U\) and \(g \in V\) satisfying \((0, 0) \not\in (f, g)[X]\), which completes the proof.
Lemma 2. Assume that $X$ is a compact topological space, $U, V \subset C(X)$ are non-empty and open and $f \in U$, $g \in V$ satisfy $f^{-1}(0) \cap g^{-1}(0) = \emptyset$. Then there exists $\delta > 0$ such that $B(f \cdot g, \delta) \subset U \cdot V$.

Proof. There exists $\varepsilon > 0$ satisfying $B(f, \varepsilon) \subset U$ and $B(g, \varepsilon) \subset V$. By normality, there are closed sets $F, G \subset X$ such that $F \subset \{ x \in X : f(x) \neq 0 \}$, $G \subset \{ x \in X : g(x) \neq 0 \}$ and $F \cup G = X$. Let $a$ be the smaller of the numbers $\min \{|f(x)| : x \in F\}$ and $\min \{|g(x)| : x \in G\}$. Clearly, $a > 0$. We put

$$\delta = \min \left( a\varepsilon, \frac{a^2\varepsilon}{2}, \frac{a^2\varepsilon}{2a + 2\|f\|} \right).$$

Let $\hat{h} \in B(f \cdot g, \delta)$. We will find $\hat{f} \in B(f, \varepsilon) \subset U$ and $\hat{g} \in B(g, \varepsilon) \subset V$ satisfying $\hat{f} \cdot \hat{g} = \hat{h}$. First, set

$$\hat{g}(x) = \frac{\hat{h}(x)}{f(x)} \quad \text{for} \ x \in F.$$

For $x \in F$ we have

$$|\hat{g}(x) - g(x)| = \left| \frac{\hat{h}(x) - f(x) \cdot g(x)}{|f(x)|} \right| < \frac{\delta}{a}.$$

Thus the range of $\hat{g} - g : F \to \mathbb{R}$ is a compact subset of $(-\delta/a, \delta/a)$. Using the Tietze Theorem we extend this function to $\hat{g} - g : X \to (-\delta/a, \delta/a)$ and put $\hat{g} = g + (\hat{g} - g)$. Then clearly $\hat{g} \in B(g, \varepsilon)$. For $x \in G$ we have

$$|\hat{g}(x)| \geq |g(x)| - \|\hat{g} - g\| > a - \frac{\delta}{a} \geq a - \frac{a}{2} = \frac{a}{2}.$$

Now we define

$$\hat{f}(x) = \begin{cases} f(x) & \text{for} \ x \in F, \\ \frac{\hat{h}(x)}{\hat{g}(x)} & \text{for} \ x \in G. \end{cases}$$

The definition is correct, since by definition of $\hat{g}$, for $x \in F \cap G$ we have $f(x) = \frac{\hat{h}(x)}{\hat{g}(x)}$, and if $x \in G$ then $|\hat{g}(x)| \geq a/2 > 0$. Clearly $\hat{f} \cdot \hat{g} = \hat{h}$. The function $\hat{f}$ is continuous, because $f|_F$ and $\hat{f}|_G$ are continuous.

We still have to prove that $\|\hat{f} - f\| < \varepsilon$. For $x \in F$ one has $\hat{f}(x) = f(x)$ while if $x \in G$ then

$$|\hat{f}(x) - f(x)| = \left| \frac{\hat{h}(x)}{\hat{g}(x)} - f(x) \right| \leq \left| \frac{\hat{h}(x) - f(x) \cdot g(x)}{\hat{g}(x)} \right| + \left| \frac{f(x) \cdot g(x)}{\hat{g}(x)} - f(x) \right| = \frac{|\hat{h}(x) - f(x) \cdot g(x)|}{|\hat{g}(x)|} + |f(x)| \frac{|g(x) - \hat{g}(x)|}{|\hat{g}(x)|} < \frac{\delta}{a/2} + \|f\| \frac{\delta/a}{a/2} = \frac{\delta}{a} + \frac{\delta}{a/2} = \frac{\delta}{a/2} \leq \varepsilon. \quad \blacksquare$$
For a Tikhonov $(T3\frac{1}{2})$ topological space $X$ we denote by $C_b(X)$ the algebra of real-valued continuous bounded functions defined on $X$. Let $\beta X$ denote the Čech–Stone compactification of $X$. Since for every Tikhonov space $X$ the space $\beta X$ is a compact Hausdorff space and the algebras $C_b(X)$ and $C(\beta X)$ are isometrically isomorphic, we have:

**Corollary 1.** Let $X$ be a Tikhonov topological space. The following equivalences hold:

1. multiplication in $C_b(X)$ is open iff $\dim \beta X < 1$,
2. multiplication in $C_b(X)$ is weakly open and not open iff $\dim \beta X = 1$,
3. multiplication in $C_b(X)$ is not weakly open iff $\dim \beta X > 1$.

Another corollary concerns sets of the first category:

**Corollary 2.** Let $X$ be a compact topological space. The following conditions are equivalent:

(i) $\dim X \leq 1$.
(ii) For any set $A \subset C(X)$ of the first category its preimage under the multiplication map is a set of the first category in $C(X) \times C(X)$.

Corollary 2 extends the results obtained by Balcerzak, Wachowicz and Wilczyński, who showed that the preimage of a residual subset of $C([0, 1])$ under the multiplication map is a residual subset of $C([0, 1]) \times C([0, 1])$. As a matter of fact, the proof of the implication (i)$\Rightarrow$(ii) is identical to the argument in [1], [4] and [5].

**Proof of Corollary 2.** Denote by $B'$ the preimage of a set $B \subset C(X)$ under the multiplication map. Let $\dim X \leq 1$ and let $A \subset C(X)$ be of the first category. Clearly, $A' = \bigcup_{n=1}^{\infty} A'_n$, where the $A_n \subset C(X)$ are nowhere dense. It remains to show that the sets $A'_n$ are nowhere dense. Assume the contrary: for some $n$ there exists a non-empty open set $U \subset C(X) \times C(X)$ satisfying $U \subset \overline{A'_n}$. It follows that $\cdot [U] \subset \cdot [\overline{A'_n}] \subset \overline{A_n}$, which is not the case because $A_n$ is nowhere dense and $\text{Int}(\cdot [U]) \neq \emptyset$ (cf. Theorem, (1)$\Leftrightarrow$ and (2)$\Leftarrow$).

Conversely, let $\dim X > 1$. There exist non-empty open open sets $U, V \subset C(X)$ such that $W = U \cdot V$ is nowhere dense (cf. proof of Theorem, (3)$\Leftarrow$). On the other hand, $W' \supset U \times V$ is not of the first category in $C(X) \times C(X)$. □

**References**


Department of Probability Theory and Statistics
Faculty of Mathematics
University of Łódź
Banacha 22
90-238 Łódź, Poland
E-mail: andkom@math.uni.lodz.pl
URL: http://www.math.uni.lodz.pl/~andkom/

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