FUNDAMENTA
MATHEMATICAE
220 (2013)

On special partial types and weak canonical bases in simple theories

by

Ziv Shami (Ariel)

Abstract. We define the notion of a weak canonical base for a partial type in a simple theory. We prove that members of a certain family of partial types, which we call special partial types, admit a weak canonical base; this family properly contains the family of amalgamation bases.

1. Introduction. For a simple theory the notion of the canonical base is essential for the development of parts of the theory such as the theory of analyzability. Given an amalgamation base $p \in S(A)$, the canonical base of p is the minimal hyperimaginary, in the sense of definable closure, $e \in \mathrm{dcl}(A)$ such that p does not fork over e and $p|e$ is an amalgamation base. In this note we define a notion of a weak canonical base for a partial type in a simple theory; it is defined in the same way as the usual canonical base except that it is required to be minimal with respect to bounded closure in the above sense (and there is no requirement on the restriction of the partial type to it). We prove that members of a certain family of partial types (we call them special partial types) have a weak canonical base. This family clearly properly contains the class of amalgamation bases. Our original motivation was to prove Corollary 2.10 so as to obtain a certain definability result that seemed required for the proof of the dichotomy between 1-basedness and supersimplicity proved in [S1]; however, this corollary turned out to be unnecessary for this specific definability result. Nevertheless, it should have other applications to situations where one needs a compactness argument when dealing with families of canonical bases.

The characterization of the class of partial types that admit a weak canonical base appears to be an important problem and it looks reasonable that this class should properly contain the class of special partial types that

2010 *Mathematics Subject Classification*: Primary 03C45.
Key words and phrases: simple theory, forking, weak canonical base, independent.

DOI: 10.4064/fm220-1-1

we deal with in this paper. The class of special partial types is a subclass of the class of partial types obtained by generic composition of a pair of complete types; we say that a partial type $r(x,a)$ over a sufficiently saturated model of T is obtained by *generic composition* of the complete types $p(x,y)$ and $q(y,z)$ (without parameters) if the following condition holds in that model: b realizes r iff there exists c such that $p(b,c)$, $q(c,a)$, and $b \underset{c}{\perp} a$.

Our proof does not seem to extend to general generic composition. The skeleton of the proof of the existence of weak canonical bases is similar to the construction of the usual canonical base.

Throughout this paper, T is assumed to be a first-order simple theory and we work in a monster model $\mathcal{C}$ of T, namely a sufficiently saturated, and sufficiently strongly homogeneous model of T. We will sometime assume, for simplicity, that T is hypersimple, that is, a simple theory with elimination of hyperimaginaries. We only assume basic knowledge of simple theories as in [K], [KP] and [HKP].

2. Weak canonical bases

DEFINITION 2.1. Let Γ_a be a partial type over a tuple a (not necessarily finite). We say that a hyperimaginary e is a *weak canonical base for Γ_a* if

(1) $e \in \mathrm{dcl}(a)$ and Γ_a does not fork over e.
(2) $e \in \mathrm{bdd}(e')$ whenever $e' \in \mathrm{bdd}(a)$ is a hyperimaginary such that Γ_a does not fork over e'.

EXAMPLE 2.2. Let $L = \{E\}$ be a language for a 2-place relation E. Let T be the complete L-theory saying that E is an equivalence relation with infinitely many equivalence classes each of which is infinite. Let $\Gamma(x) \equiv E(x,a) \vee E(x,a')$ for some $a,a' \in \mathcal{C}$ such that $\neg E(a,a')$. Then $\Gamma(x)$ does not have a weak canonical base. To see this, assume otherwise. First, note that clearly $\Gamma(x)$ forks over neither a nor a'. Thus by our assumption and the definition of a weak canonical base, $e \in \mathrm{bdd}(a) \cap \mathrm{bdd}(a')$. So, we get a contradiction since $a \perp a'$ and $\Gamma(x)$ forks over $\emptyset$.

We start by introducing the *special partial types*. First, we will say that a relation $R(x,x')$ is *generically transitive on a partial type* $\pi(x)$ if for all $a',a,a'' \models \pi$, if $a' \underset{a}{\perp} a''$ and $R(a',a) \wedge R(a,a'')$, then $R(a',a'')$.

LEMMA 2.3. *Let $q(y,z), r(z,x) \in S(\emptyset)$ be such that $\exists xyz\ (q(y,z) \wedge r(z,x))$. Assume $q(y,z) \vdash z \in \mathrm{acl}(y)$. Let $p(x) = \exists z\ r(z,x)$ and let $a \models p$. Let Γ_a be defined by*

$$\Gamma_a(y) \equiv \exists z\ (q(y,z) \wedge r(z,a) \wedge y \underset{z}{\perp} a).$$

Then

(1) $\Gamma_a(x)$ *is a partial type.*
(2) *Let* $\bar{R}_\Gamma$ *be the relation defined by* $\bar{R}_\Gamma(c,a,a')$ *iff* $c \models \Gamma_a \wedge \Gamma_{a'}$ *and* $c \underset{a}{\downarrow} a'$ *and* $c \underset{a'}{\downarrow} a$. *Then* $\bar{R}_\Gamma$ *is type-definable and thus so is the relation* $R_\Gamma(x,x') \equiv \exists y\ \bar{R}_\Gamma(y,x,x')$ *on* $p^{\mathcal{C}}$.
(3) *If* $q(y,z) \vdash z \in \operatorname{dcl}(y)$ *and* $\operatorname{tp}(d/a)$ *is an amalgamation base for* $(d,a) \models r$, *then* R_Γ *is generically transitive on* $p^{\mathcal{C}}$.

Proof. (1) is easy since in the definition of Γ_a the complete type of (z,a) is fixed. For (2) note that since $q(y,z) \vdash z \in \operatorname{acl}(y)$, an easy forking computation shows that for all c,a,a' we have $\bar{R}_\Gamma(c,a,a')$ iff there exist d,d' such that

$$q(c,d) \wedge q(c,d') \wedge r(d,a) \wedge r(d',a') \quad \text{and} \quad c \underset{d}{\downarrow} aa' \wedge c \underset{d'}{\downarrow} aa' \wedge d \underset{a}{\downarrow} a' \wedge d' \underset{a'}{\downarrow} a.$$

Again, since q and r are complete we find that $\bar{R}_\Gamma$ is type-definable.

To prove (3), assume $q(y,z) \vdash z \in \operatorname{dcl}(y)$. Let a,a',a'' be such that $a' \underset{a}{\downarrow} a''$ and assume $R_\Gamma(a',a)$ and $R_\Gamma(a,a'')$ (so, clearly $a,a',a'' \models p$). Now, by (2) we know that $\bar{R}_\Gamma(x,a',a)$ is a partial type and clearly by its definition it does not fork over a. Likewise, the partial type $\bar{R}_\Gamma(x,a,a'')$ does not fork over a. It will be sufficient to show the following.

Claim 2.4. *There are* $c' \models \bar{R}_\Gamma(x,a',a)$ *and* $c'' \models \bar{R}_\Gamma(x,a,a'')$ *such that* $\operatorname{Lstp}(c'/a) = \operatorname{Lstp}(c''/a)$.

This is sufficient since $c' \underset{a}{\downarrow} a'$ for all $c' \models \bar{R}_\Gamma(x,a',a)$ and $c'' \underset{a}{\downarrow} a''$ for all $c'' \models \bar{R}_\Gamma(x,a,a'')$, thus by the independence theorem this will imply there exists $c^* \models \bar{R}_\Gamma(x,a',a) \wedge \bar{R}_\Gamma(x,a,a'')$ with $c^* \underset{a}{\downarrow} aa'a''$. In particular, $c^* \underset{aa'}{\downarrow} aa'a''$, and by the definition of $\bar{R}_\Gamma$, $c^* \underset{a'}{\downarrow} aa'a''$. Likewise, $c^* \underset{a''}{\downarrow} aa'a''$. Hence $\bar{R}_\Gamma(a',a'')$.

Proof of Claim 2.4. Since $q(y,z) \vdash z \in \operatorname{dcl}(y)$, by the observation on $\bar{R}_\Gamma$ in the proof of (2), there are d'',c'' such that $q(c'',d'')$, $r(d'',a)$ and $r(d'',a'')$ and $d'' \underset{a}{\downarrow} a''$, and $d'' \underset{a''}{\downarrow} a$ and $c'' \underset{d''}{\downarrow} aa''$. Likewise, there is d' with $d' \underset{a}{\downarrow} a'$ and $d' \underset{a'}{\downarrow} a$ and $r(d',a)$ and $r(d',a')$. Now, since $\operatorname{Lstp}(d'/a) = \operatorname{Lstp}(d''/a)$, there exists c' such that $\operatorname{tp}(c'd'/a) = \operatorname{tp}(c''d''/a)$ and $\operatorname{Lstp}(c'/a) = \operatorname{Lstp}(c''/a)$ and $c' \underset{d'a}{\downarrow} a'$. By the choice of c', $c' \underset{d'}{\downarrow} a$, hence by transitivity $c' \underset{d'}{\downarrow} aa'$. We conclude that $\bar{R}_\Gamma(c'',a,a'')$, $\bar{R}_\Gamma(c',a',a)$ and $\operatorname{Lstp}(c'/a) = \operatorname{Lstp}(c''/a)$, as required.

Definition 2.5. A partial type $\Gamma_a(y)$ as defined in Lemma 2.3 for some $q(y,z), r(z,x) \in S(\emptyset)$ is called a *special partial type* if $q(y,z) \vdash z \in \mathrm{dcl}(y)$ and $\mathrm{tp}(d/a)$ is an amalgamation base for $(d,a) \models r$.

Remark 2.6. Note that, in general, the class of special partial types properly contains the class of amalgamation bases. To see this, let $r(z,x)$ be any complete type over $\emptyset$ such that $\mathrm{tp}(d/a)$ is an amalgamation base for $(d,a) \models r$, and let $q(y,z) = (y = z) \wedge (\exists x\ r(z,x))$. Then, if we apply the definition of Γ_a in Lemma 2.3, we get $\Gamma_a(y) = \mathrm{tp}(d/a)$ for $(d,a) \models r$. To justify properness, we give an example of a special partial type which is not complete. Let $L = \{R\}$ and let T be the L-theory of the random graph. Let $a, b, c \in \mathcal{C}$ be any three distinct elements such that $R(b,c)$ and $R(c,a)$. Then $bc \mathop{\smile\!\!\!\!\!\!\mid}_c a$. Let $q = \mathrm{tp}(bc, c)$, $r = \mathrm{tp}(c,a)$. Let

$$\Gamma_a(y_0y_1) \equiv \exists z\ \Big(q(y_0y_1, z) \wedge r(z,a) \wedge y_0y_1 \mathop{\smile\!\!\!\!\!\!\mid}_z a\Big).$$

Then clearly Γ_a is a special partial type and $\Gamma_a(y_0y_1)$ is equivalent to $(y_0 \neq y_1) \wedge (y_0 \neq a) \wedge (y_1 \neq a) \wedge R(y_0, y_1) \wedge R(y_1, a)$. In particular, Γ_a is not complete.

To prove the theorem we need the following well known fact (used for the construction of the usual canonical base).

Fact 2.7 ([W, Lemma 3.3.1]). *Let $\pi(x)$ be a partial type over $\emptyset$ and let $R(x,x')$ be a type-definable relation over $\emptyset$ that is reflexive, symmetric and generically transitive on $\pi^{\mathcal{C}}$. Let E_R be the transitive closure of R on $\pi^{\mathcal{C}}$. Then E_R is type-definable and for all $a, a' \models \pi$ we have $E_R(a,a')$ iff there exists $b \models \pi$ such that $R(a,b)$ and $R(b,a')$ and $a \mathop{\smile\!\!\!\!\!\!\mid}_{a'} b$ and $b \mathop{\smile\!\!\!\!\!\!\mid}_a a'$.*

Theorem 2.8. *Let Γ_a be a special partial type. Then Γ_a has a weak canonical base.*

Proof. By Lemma 2.3 and Fact 2.7 we know that the transitive closure of R_Γ (as defined in Lemma 2.3), which we denote by E_Γ, is type-definable and for all $a' \models \mathrm{tp}(a)$ we have $E_\Gamma(a,a')$ iff there exists $b \models \mathrm{tp}(a)$ such that $R_\Gamma(a,b)$ and $R_\Gamma(b,a')$ and $a \mathop{\smile\!\!\!\!\!\!\mid}_{a'} b$ and $b \mathop{\smile\!\!\!\!\!\!\mid}_a a'$.

Let $e = a_{E_\Gamma}$. Clearly $e \in \mathrm{dcl}(a)$. First we show Γ_a does not fork over e. Pick $a' \models \mathrm{tp}(a)$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_{a_{E_\Gamma}} a'$ and $\mathrm{tp}(a'/a_{E_\Gamma}) = \mathrm{tp}(a/a_{E_\Gamma})$. In particular, $E_\Gamma(a,a')$. Let b be as above. Then easily $a \mathop{\smile\!\!\!\!\!\!\mid}_{a_{E_\Gamma}} b$ and thus if $c \models \Gamma_a \wedge \Gamma_b$ with $a \mathop{\smile\!\!\!\!\!\!\mid}_b c$, then since $a_{E_\Gamma} \in \mathrm{dcl}(b)$ we conclude that $a \mathop{\smile\!\!\!\!\!\!\mid}_{a_{E_\Gamma}} c$. To prove (2) of 2.1, assume $\Gamma_a(x)$ does not fork over some $e' \in \mathrm{bdd}(a)$. Let $\sigma \in \mathrm{Aut}(\mathcal{C}/\mathrm{bdd}(e'))$

and let $a' = \sigma(a)$. Pick a^* such that $\operatorname{tp}(a^*/\operatorname{bdd}(e')) = \operatorname{tp}(a/\operatorname{bdd}(e'))$ and $a^* \mathop{\smile\!\!\!\!\!\!|}_{e'} aa'$. By the independence theorem, neither $\Gamma_a \wedge \Gamma_{a^*}$ nor $\Gamma_{a^*} \wedge \Gamma_{a'}$ forks over e'. Since $e' \in \operatorname{bdd}(a^*) \cap \operatorname{bdd}(a) \cap \operatorname{bdd}(a')$, we have $E_\Gamma(a, a')$. Thus $e \in \operatorname{bdd}(e')$.

Remark 2.9. Definition 2.5 of special partial types can be applied in the more general context of hyperimaginaries. It is not hard to check that Theorem 2.8 remains true in this context; the main properties we need for that are the following.

First, for the proof of Fact 2.7 we only need two properties besides standard forking computations: the first one is that $a \mathop{\smile\!\!\!\!\!\!|}_{c} b$ if and only if for every $\phi = \phi(x, y) \in L$ and $k < \omega$ we have $D(\operatorname{tp}(a/c), \phi, k) = D(\operatorname{tp}(a/bc), \phi, k)$, and the second property is the type-definability of the $D(-, \phi, k)$-rank in the following sense: for any, possibly infinite, tuples of the sorts s_0, s_1 and for $\phi = \phi(x, y) \in L$ and $k, n < \omega$ the set $\{(a_0, a_1) \in \mathcal{C}^{s_0} \times \mathcal{C}^{s_1} \mid D(\operatorname{tp}(a_1/a_0), \phi, k) \geq n\}$ is type-definable. These properties remain true in the hyperimaginary context and thus so does Fact 2.7. For the proofs in this paper (and even for knowing that special partial types are in fact types) we only need, in addition, the following property: if b_0, c_0 are hyperimaginaries, then for any fixed hyperimaginary sort S_E (where E is a type-definable equivalence relation over $\emptyset$), the set $\{(a, b, c) \mid a \in S_E^{\mathcal{C}},\ a \mathop{\smile\!\!\!\!\!\!|}_{c} b,\ \operatorname{tp}(b, c) = \operatorname{tp}(b_0, c_0)\}$ is type-definable (i.e. the union of the classes of members of this set is type-definable).

Here is a corollary of our main theorem. For simplicity we assume that T is hypersimple (rather than just simple). In the following, when we write $\operatorname{Cb}(a/b)$, we mean the usual canonical base of (the amalgamation base) $\operatorname{tp}(a/\operatorname{bdd}(b))$ (where $\operatorname{bdd}(b)$ denotes the set of hyperimaginaries of countable length whose type over b is bounded). The assumption that T is hypersimple implies that such a canonical base exists as a set of imaginary elements and a type over an algebraically closed set in $\mathcal{C}^{\mathrm{eq}}$ is an amalgamation base (since $\operatorname{bdd}(A)$ is interdefinable with $\operatorname{acl}^{\mathrm{eq}}(A)$ for every set $A \subseteq \mathcal{C}^{\mathrm{eq}}$).

Corollary 2.10. *Let T be a simple theory with elimination of hyperimaginaries and work in $\mathcal{C}^{\mathrm{eq}}$. Let d, a be some tuples* (*possibly infinite*) *and let $p \in S(d)$ be such that for $c \models p$, $d \in \operatorname{dcl}(c)$. Let*

$$S = \{c \in p^{\mathcal{C}} \mid c \mathop{\smile\!\!\!\!\!\!|}_{d} a\}.$$

Then there exists $c^* \in S$ such that $\bigcap_{c \in S} \operatorname{acl}(\operatorname{Cb}(c/a)) = \operatorname{acl}(\operatorname{Cb}(c^*/a))$.

Proof. Let $\tilde{a} = \operatorname{acl}(a)$. Let $\Gamma_{\tilde{a}}$ be the special partial type over $\tilde{a}$ defined by the types $r = \operatorname{tp}(d, \tilde{a})$ and $q = \operatorname{tp}(c, d)$ for some $c \models p$. Note that

$\{\mathrm{tp}(c/\tilde{a}) \mid c \in S\} = \{\mathrm{tp}(c/\tilde{a}) \mid c \models \Gamma_{\tilde{a}}\}$, and clearly $S \subseteq \Gamma_{\tilde{a}}^{\mathcal{C}}$. By Theorem 2.8, there is a weak canonical base of $\Gamma_{\tilde{a}}$, call it e. Let $c' \models \Gamma_{\tilde{a}}$ be such that $c' \underset{e}{\downarrow} a$, and let $e^* = \mathrm{Cb}(c'/a)$. Then by the definition of the usual canonical base, $e^* \in \mathrm{bdd}(e)$. By the observation above, there exists $c^* \in S$ such that $e^* = \mathrm{Cb}(c^*/a)$. Now, to finish the proof it will be sufficient to show that $e \in \mathrm{bdd}(\mathrm{Cb}(c/a))$ for every $c \in S$. Indeed, let $c \in S$, then $c \models \Gamma_{\tilde{a}}$. Let $e_c = \mathrm{Cb}(c/a)$. Then $c \underset{e_c}{\downarrow} a$, and since e is a weak canonical base of $\Gamma_{\tilde{a}}$, we conclude $e \in \mathrm{bdd}(e_c)$.

References

[HKP] B. Hart, B. Kim and A. Pillay, *Coordinatization and canonical bases in simple theories*, J. Symbolic Logic 65 (2000), 293–309.

[K] B. Kim, *Forking in simple unstable theories*, J. London Math. Soc. 57 (1998), 257–267.

[KP] B. Kim and A. Pillay, *Simple theories*, Ann. Pure Appl. Logic 88 (1997), 149–164.

[W] F. O. Wagner, *Simple Theories*, Kluwer, Dordrecht, 2000.

[S1] Z. Shami, *Countable hypersimple unidimensional theories*, J. London Math. Soc. 83 (2011), 309–332.

Ziv Shami
Department of Mathematics and Computer Sciences
The Ariel University Center of Samaria
44837 Ariel, Israel
E-mail: zivshami@gmail.com

Received 3 February 2009;
in revised form 23 October 2012