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Equilibrium measures for holomorphic endomorphisms of complex projective spaces

by

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Abstract. Let $f : \mathbb{P} \to \mathbb{P}$ be a holomorphic endomorphism of a complex projective space $\mathbb{P}^k$, $k \geq 1$, and let J be the Julia set of f (the topological support of the unique maximal entropy measure). Then there exists a positive number $\kappa_f > 0$ such that if $\phi : J \to \mathbb{R}$ is a Hölder continuous function with $\sup(\phi) - \inf(\phi) < \kappa_f$, then ϕ admits a unique equilibrium state μ_ϕ on J. This equilibrium state is equivalent to a fixed point of the normalized dual Perron–Frobenius operator. In addition, the dynamical system (f, μ_ϕ) is K-mixing, whence ergodic. Proving almost periodicity of the corresponding Perron–Frobenius operator is the main technical task of the paper. It requires producing sufficiently many "good" inverse branches and controling the distortion of the Birkhoff sums of the potential ϕ. In the case when the Julia set J does not intersect any periodic irreducible algebraic variety contained in the critical set of f, we have $\kappa_f = \log d$, where d is the algebraic degree of f.

1. Introduction. The thermodynamic formalism for holomorphic endomorphisms of the Riemann sphere $\hat{\mathbb{C}}$ and Hölder continuous potentials, with sufficiently small oscillation, was originated in [DU]. The existence and uniqueness of equilibrium states of such potentials was proved there (see also [Pr]). The corresponding Perron–Frobenius operator was shown to be almost periodic and the equilibria were shown to be K-mixing. Later ([DPU], [Ha]) more refined mixing and stochastic properties of these equilibria were established.

The natural question arises about the existence and uniqueness of equilibria in the higher dimensional case, namely, for complex projective spaces of an arbitrary dimension. The existence of equilibria (for all continuous potentials) follows imediately from the fact that for all C^∞ endomorphisms of smooth compact manifolds the entropy function $\mu \mapsto \mathrm{h}_\mu$ ascribing to each

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invariant measure μ its Kolmogorov–Sinai measure-theoretic entropy h_μ is upper semicontinuous (see works of Yomdin [Yo] and Newhouse [Ne]). However, up to our knowledge, so far, for holomorphic endomorphisms of higher dimensional projective spaces, only the case of the potential ϕ identically equal to zero has been treated to address the questions about uniqueness of equilibria, their more direct construction, and finer stochastic properties. The measure of maximal entropy was constructed in [FS1] and [HP] (another construction was presented in [BD]); see [BD] for the proof of its uniqueness. Stochastic properties, in particular upper estimates of exponential speed of convergence of iterates of the corresponding Perron–Frobenius operator, were established in [FS1], [FS2] and [DS3]; for related topics see also [DNS] and [D2]. The expository paper [DS1] contains a complete survey of up to date results.

An interesting class of invariant measures has been constructed and studied in [Du2]. The author applies an approach similar to that in [PUZ, I] and [PUZ, II]. It uses a special coding technique, and the resulting measure is the image, under this coding, of a Gibbs measure for some Hölder continuous potential in the coding space.

Our goal in this paper is to build the thermodynamic formalism for holomorphic endomorphisms $f : \mathbb{P}^k \to \mathbb{P}^k$ and for Hölder continuous potentials $\phi : J \to \mathbb{R}$, with sufficiently small value, depending only on the endomorphisms $f : \mathbb{P}^k \to \mathbb{P}^k$ and denoted by κ_f, of their oscillation $\sup(\phi) - \inf(\phi)$. Here and throughout $J = J(f)$ denotes the *Julia set* of the map $f : \mathbb{P}^k \to \mathbb{P}^k$, i.e. the topological support of the measure of maximal entropy. Note that in the literature our set J is usually denoted by J_k and it may be essentially smaller than the set J_1, which is also frequently called the Julia set and which is defined with the use of the standard normality condition.

Another important backward invariant set is the *exceptional set* E, the largest nontrivial backward invariant algebraic set. It is empty for a generic holomorphic map. We do not know of any example of a map for which the exceptional set intersects the Julia set J (it is easy to construct an example for which E intersects J_1). However, we do not have any general argument showing that $E \cap J$ must be empty. (It seems that components of E with codimension 1 cannot intersect J, but the argument does not extend to higher codimensions.) So from now one we introduce the following.

DEFINITION 1.1. A holomorphic map f is called *regular* if its exceptional set $E = E(f)$ does not intersect the Julia set $J = J(f)$.

Throughout the whole paper we keep the following

ASSUMPTION 1.2. *The map f is regular.*

The regularity hypothesis is essential for the topological exactness of the map $f : J \to J$, which means that for every open set $U \subset \mathbb{P}^k$ intersecting J there exists an integer $n \geq 0$ such that $f^n(U) \supset J$. Indeed, we have the following.

PROPOSITION 1.3. *If $f : \mathbb{P}^k \to \mathbb{P}^k$ is a regular holomorphic endomorphism, then the dynamical system $f : J \to J$ is topologically exact.*

This proposition is instrumental for the whole paper, appearing in the assertions or proofs of the statements such as Proposition 3.6, Proposition 5.8, and Lemma 4.3.

Our class of potentials contains the restrictions to J of Hölder continuous functions $\phi : \mathbb{P}^k \to \mathbb{P}^k$ such that $\sup(\phi) - \inf(\phi) < \kappa_f$. If the Julia set J does not intersect any periodic irreducible algebraic variety contained in the critical set of f, then we can take κ_f as large as possible, namely equal to $\log \deg f$. We observe (see Corollary 3.5) that if k is equal to 2, then this intersection (if nonempty) consists of finitely many critical periodic orbits only, whence κ_f is easier to estimate.

In the proof we cope with estimating the distortion of the Birkhoff sums of the potential ϕ. This task is entirely absent in the case of the measure of maximal entropy, where the distortion is always zero. The bounded distortion for, in a sense, most inverse branches, and the existence of sufficiently many such "good" inverse branches, are the two main tools used to produce upper and lower bounds of iterates of the corresponding Perron–Frobenius operators. In the case of the measure of maximal entropy, this issue actually trivializes; the function identically equal to one is then a fixed point of the Perron–Frobenius operator for free. Another source of serious technical difficulties is the existence of critical periodic varieties intersecting the Julia set J.

A basic notion of ergodic theory is that of metric (Kolmogorov–Sinai) entropy $\mathrm{h}_\mu(f)$ of an f-invariant probability measure μ. The basic notion of thermodynamic formalism is that of topological pressure $\mathrm{P}(\phi) = \mathrm{P}(f, \phi)$ (see [Ru1]). Their alternative definitions and properties can be found for instance in [Wa2] and [PU]. The formula relating these two, seemingly independent, concepts is the celebrated Variational Principle stating that

$$\mathrm{P}(\phi) = \sup\Big\{\mathrm{h}_\mu(f) + \int \phi\, d\mu\Big\}, \tag{1.1}$$

where the supremum is taken over all Borel probability f-invariant measures μ. The measures μ for which $\mathrm{h}_\mu(f) + \int \phi\, d\mu = \mathrm{P}(\phi)$ are called *equilibrium states* for the potential ϕ. We prove the following.

THEOREM 1.4. *For every regular holomorphic endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ of a complex projective space $\mathbb{P}^k$, $k \geq 1$, there exists a positive number*

$\kappa_f > 0$ such that if $\phi : J(f) \to \mathbb{R}$ is a Hölder continuous function with $\sup(\phi) - \inf(\phi) < \kappa_f$, then ϕ admits a unique equilibrium state μ_ϕ on J. This equilibrium state is equivalent to a fixed point of the normalized dual Perron–Frobenius operator. In addition the dynamical system (f, μ_ϕ) is K-mixing, whence ergodic. In the case when the Julia set J does not intersect any periodic irreducible algebraic varieties contained in the critical set of f, we have $\kappa_f = \log\deg f$.

As we have already noted, the existence of equilibria is true for all C^∞ smooth endomorphisms of compact differentiable manifolds. Our proof of existence of equilibria is entirely different; in particular we do not use upper semicontinuity of the entropy function. W prove much more than merely the existence of equilibria. We in fact construct an equilibrium as a fixed point of the normalized dual Perron–Frobenius operator. This gives a piece of a valuable information about the structure of this equilibrium and allows us to deduce the uniqueness of the equilibrium, by showing that the topological pressure function is differentiable.

The K-mixing property is due to almost periodicity of the corresponding Perron–Frobenius operator. We provide a more detailed description of the allowed oscillation κ_f in Section 3. We also provide sufficient conditions for κ_f to be equal to $\log\deg f$, nearly as good as in [DU]. The proof of Theorem 1.4 contains two additional ingredients. Firstly, we prove a form of uniformly subexponentially slow increase of local degrees of iterates of the map $f : \mathbb{P}^k \to \mathbb{P}^k$. This fact is related to some results of Favre ([F1] and [F2]). Secondly, motivated by the argument of M. Gromov [Gr], we prove that the topological pressure is not larger than the logarithm of the eigenvalue of the dual to the Perron–Frobenius operator.

Our paper provides a generalization of corresponding results for the dynamics of rational maps in $\mathbb{P}^1$. The proof required the development of a new approach, and several new major ideas appear in our arguments. Surprisingly, in order to estimate the iterates of the Perron–Frobenius operator, we have to extend our potential ϕ to some neighborhood of J and to estimate the Perron–Frobenius operator in this neighborhood. Furthermore, we have to extend this potential to the entire projective space $\mathbb{P}^k$. We need this extension in order to prove almost periodicity of the Perron–Frobenius operator acting on the whole Banach space $C(\mathbb{P}^k)$ of continuous functions on $\mathbb{P}^k$.

For the proof of the equality $P(\phi) \le \log\lambda$ given in Section 6, we need to know that the iterates of the (normalized) Perron–Frobenius operator are uniformly bounded above everywhere in $\mathbb{P}^k$, and not only in J. The reason is that we follow the idea of Gromov's proof of the equality $\mathrm{h}_{\mathrm{top}}(f) = k\log\deg f$ which does require integrating against the volume measure everywhere in $\mathbb{P}^k$. We stress that we must produce this special extension even if the original potential ϕ was defined everywhere in $\mathbb{P}^k$. The potential ϕ is then modified in such a way as to guarantee a uniform upper bound of the iterates $\hat{\mathcal{L}}^n(\mathbb{1})$ of the Perron–Frobenius operator everywhere in $\mathbb{P}^k$.

Moreover, we have to cope with periodic varieties contained in the critical set which may intersect the Julia set (see Section 3). This phenomenon has no counterpart in dimension 1. We estimate separately the part of the Perron–Frobenius operator acting "along" such "critical periodic varieties". This may cause the maximal allowable oscillation κ_f to be smaller than $\log d$. If no such varieties exist, then the set A_J^*, responsible for all such issues, is empty and calculations become considerably easier. On a first reading the reader may assume that $A_J^* = \emptyset$ and skip the considerations concerning this set.

We now comment on one issue entirely peculiar to the multidimensional case. In higher dimensions there is no obvious canonical way of defining the Julia set. Also, we would not gain more generality if rather than working with the entire properly defined Julia set, we would only assume that we work on some totally invariant closed subset of $\mathbb{P}^k$. Indeed, let T be the Green current for f and denote by $\mathcal{J}_j$ the support of $T^{\wedge j}$. Then

$$\mathcal{J}_1 \supset \cdots \supset \mathcal{J}_k = J.$$

The results of de Thélin (see [dT1], [dT2]) and Dinh (see [D1]) lead to the following corollary (see [DS1, Section 1] for a detailed presentation).

Corollary 1.5. *Let f be an algebraic endomorphism of $\mathbb{P}^k$ of degree d. Let μ be an invariant probability measure. If the* (*topological*) *support of μ does not intersect the set $\mathcal{J}_p$ then*

$$\mathrm{h}_\mu(f) \le (p-1)\log d.$$

However, it can be easily seen that every equilibrium measure ν for the potential ϕ with $\sup(\phi) - \inf(\phi) < \log d$ (the potentials in the current paper do enjoy this property) satisfies $\mathrm{h}_\nu > (k-1)\log d$. This implies that every equilibrium measure for every such potential must charge the Julia set J. Since J is totally invariant, we thus get the following

Corollary 1.6. *The topological support of every ergodic equilibrium state of every potential ϕ, defined in some neighborhood of J, and satisfying $\sup(\phi) - \inf(\phi) < \log d$, is contained in the Julia set J.*

This means that, once we are given such a potential, even defined in the whole $\mathbb{P}^k$, we can always restrict our considerations to the Julia set J. We do this.

Our paper is organized as follows. In Section 2, we construct sufficiently many exponentially shrinking inverse branches. In Section 3 we introduce and discuss the maximally allowed oscillation κ_f for the potential ϕ, we introduce the corresponding Perron–Frobenius operator, and we prove the existence of the "geometric" Gibbs state m_ϕ (formula (3.10)). In Section 4, we establish upper and lower uniform bounds of iterates of the Perron–Frobenius operator. In Section 5, we discuss almost periodicity of this operator, and its uniform version, needed for the proof of the uniqueness of the equilibrium state μ_ϕ. Consequently, we produce a continuous fixed point ρ_ϕ of the Perron–Frobenius operator and the f-invariant measure $\mu_\phi = \rho_\phi m_\phi$. Based on almost periodicity of the Perron–Frobenius operator, we establish its spectral properties, and we show that its iterates converge uniformly. Hence we deduce K-mixing of the dynamical system (f, μ_ϕ), whence its ergodicity. We also prove the decay of correlations. In Section 6, developing the idea of Gromov [Gr], we prove the equality of the topological pressure $\mathrm{P}(\phi)$ and the logarithm $\log\lambda$. Section 7 is devoted to proving existence and uniqueness of equilibrium states. Section 8 contains the postponed proof of uniformly subexponentially slow growth of local degrees of iterates of the map. Finally, in Section 9 we give the postponed proof of almost periodicity.

2. Contracting inverse branches. We normalize the Fubini–Study metric ρ on $\mathbb{P}^k$ so that the area A of any ball of radius 1 on a projective line is equal to 1. The following theorem is due to Lelong.

THEOREM 2.1. *There exists a constant $c > 0$ such that if $x \in \mathbb{P}^k$ and $0 < R \le 2\,\mathrm{diam}_\rho(\mathbb{P}^k)$, and if X is a 1-dimensional closed complex variety contained in $B(x, R)$, then*

$$\mathrm{Area}(X \cap B(x, R)) > c^{-1}R^2.$$

Keeping $f : \mathbb{P}^k \to \mathbb{P}^k$ a holomorphic endomorphism, let $\mathrm{Crit}(f)$ be the set of all critical points of f, i.e. points $z \in \mathbb{P}^k$ such that $\deg_z f \ge 2$.

DEFINITION 2.2. Given an integer $n \ge 1$ the *periodic critical set* A_n is the union of orbits of all irreducible varieties that are contained in the critical set and are periodic under the iterate f^l with some $l \le n$. In particular, the orbit of every critical periodic point of period $l \le n$ is in the critical periodic set A_n.

DEFINITION 2.3. Given two integers $1 \le p \le n$ the set E_n^p is defined to consist of all points $x \in \mathbb{P}^k$ for which there exists an integer $0 \le i \le n-1$ such that $f^i(x) \in A_p$. Equivalently,

$$E_n^p = \bigcup_{i=0}^{n-1} f^{-i}(A_p) = f^{-(n-1)}A_p.$$

Obviously, $E_n^p \subset E_{n+1}^p$.

PROPOSITION 2.4. *For every $\beta > 0$ there exist $p = p(\beta)$ and $N = N(\beta)$ such that for every $n \geq N$ and for every $x \notin E_n^p$ we have*

$$\#\{0 \leq j \leq n : f^j(x) \in \operatorname{Crit}(f)\} \leq \beta n.$$

The proof of this proposition is presented in Section 8.

Now, take $\gamma \in (0,1)$. It follows from Proposition 2.4 that there exist two least integers $p_1(\gamma)$ and $p_2(\gamma)$ such that if $z \in \mathbb{P}^k$, $j \geq p_2(\gamma)$, and $f^j(z) \notin A_{p_1(\gamma)}$, then

$$\deg_z(f^j) \leq \gamma^{-j}. \tag{2.1}$$

Let us record the following obvious observation.

REMARK 2.5. Obviously, for $\gamma < d^{-k}$ one can take $p_2(\gamma) = 1$, $p_1(\gamma) = 0$, and $A_{p_1(\gamma)} = \emptyset$. As γ increases, so does the set $A_{p_1(\gamma)}$. However, it is easy to see that for every γ, the set $A_{p_1(\gamma)}$ is a finite union of algebraic varieties. Thus, the three functions $(0,1) \ni \gamma \mapsto p_1(\gamma)$, $p_2(\gamma)$, $A_{p_1(\gamma)}$ are weakly increasing and piecewise constant with a finite number of discontinuities in each interval $[0,t]$, $0 \leq t < 1$.

Put

$$A_\gamma := A_{p_1(\gamma)}.$$

For any two distinct points $a, b \in \mathbb{P}^k$ denote by $\Gamma_{a,b}$ the projective line passing through a and b. The following result holds for a generic (not every) projective line Γ passing through z (see condition (2.6) in the proof below). Below, we shall write "generic projective line" without specifying the precise condition on Γ.

LEMMA 2.6. *For every $\gamma \in (0,1)$, every integer $s \geq p_2(\gamma)$, and every $\eta > 0$ there exists $R(\eta) = R(\gamma, s; \eta) \in (0,1)$ such that for every z in $\mathbb{P}^k \setminus B(A_\gamma, \eta)$, every projective line Γ passing through z, and for all $n \geq 0$, there is a family $W_n(\eta, z, \Gamma)$ of connected components of $f^{-n}(B(z, R(\eta)) \cap \Gamma)$ with the following properties.*

- (a) *For all $0 \leq n \leq s$ the collection $Z_n(\eta, z) = W_n(\eta, z)$ consists of all connected components of $f^{-n}(B(z, R(\eta)) \cap \Gamma)$.*
- (b_n) $\max\{\operatorname{diam}(V) : V \in W_n(\eta, z, \Gamma)\} \leq \gamma^{n/2}$.
- (c_n) *$W_n(\eta, z, \Gamma) \subset Z_n(\eta, z, \Gamma)$ and*

  $$\#(Z_n(\eta, z, \Gamma) \setminus W_n(\eta, z, \Gamma)) \leq \gamma^{-s}(4c\gamma^{-3n} + Gs^{4(k-1)})d^{(k-1)(n+1)},$$

  where $Z_n(\eta, z, \Gamma)$ is the family of all connected components of all sets of the form $f^{-1}(V)$, where $V \in W_{n-1}(\eta, z, \Gamma)$ ($c > 0$ is the constant coming from Lelong's Theorem ([La, Theorem II.3.6] or [McM, Theorem II.3.6]) along with homogeneity of complex projective spaces, and G comes from condition (S''_n) formulated below in the course of the proof; the precise value of G or its definition are not important for further considerations).

(d_n) *For all* $n \geq s+1$ *and* $V \in W_n(\eta, z, \Gamma)$, *we have* $V \cap f(\mathrm{Crit}(f)) = \emptyset$, *and* $f^n|_V$ *is at most* γ^{-s}*-to-*1.

REMARK 2.7. Note that this lemma is correctly stated for all $\gamma \in (0,1)$. However for $\gamma < d^{-1/3}$ it is trivially true (see (c_3) and brings no new information. When applying this lemma, we will always assume that $\gamma \geq d^{-1/6}$.

REMARK 2.8. Although in the proof below we do not explicitly use the geometric distortion lemma from [BD], this lemma has motivated our approach here. We directly use Lelong's inequality instead.

Proof of Lemma 2.6. In virtue of (2.1) there exists $R_1(\eta) > 0$ (depending also on s) so small that if $z \in \mathbb{P}^k \setminus B(A_\gamma, \eta)$ and $x \in f^{-s}(z)$, and if V_x'' is the connected component of $f^{-s}(B(z, R_1(\eta)))$ containing x, then

$$\deg\big(f^s|_{V_x''}\big) \leq \gamma^{-s}. \tag{2.2}$$

We use an approach similar to [Gu, proof of Lemme 3.4]. After noting that for the estimate in [Gu] to hold, $f^l|_{B_i^l \cap V_1}$ need not be 1-to-1, Guedj's construction (in [Gu]) produces a number $0 < R_2(\eta) \leq R_1(\eta)$, and, for every $z \in \mathbb{P}^k \setminus B(A_\gamma, \eta)$, sequences $(W_n''(\eta, z))_{n=0}^\infty$ and $(Z_n''(\eta, z))_{n=0}^\infty$ of connected components of $f^{-n}(B(z, R_2(\eta)))$ with the following properties:

(P'') For all $0 \leq n \leq s$ the collection $Z_n''(\eta, z) = W_n''(\eta, z)$ consists of all connected components of $f^{-n}(B(z, R_2(\eta)))$.

(Q_n'') For all $n \geq s$ the collection $Z_{n+1}''(\eta, z)$ consists of all connected components of the sets $f^{-1}(V)$, where $V \in W_n''(\eta, z)$.

(R_n'') For all $n \geq s+1$,
$$W_n''(\eta, z) = \{V \in Z_n''(\eta, z) : V \cap f(\mathrm{Crit}(f)) = \emptyset\}.$$

(S_n'') For all $n \geq s$,
$$\#(Z_n''(\eta, z) \setminus W_n''(\eta, z)) \leq G s^{4(k-1)} d^{(k-1)(n+1)},$$
with some constant $G \geq 1$ depending only on $f : \mathbb{P}^k \to \mathbb{P}^k$ and η.

Since in $\mathbb{P}^k$ all connected components of all open sets are open, it easily follows from continuity of the map $f : \mathbb{P}^k \to \mathbb{P}^k$ that there exists a radius $r > 0$ such that

$$V_x'' \supset B(x, r) \tag{2.3}$$

for all $z \in \mathbb{P}^k \setminus B(A_\gamma, \eta)$ and all $x \in f^{-s}(z)$, where V_x'' is the connected component of $f^{-s}(B(z, R_1(\eta)))$ containing x. Next, take $0 < R_3(\eta) \leq R_2(\eta)$ so small that

$$V_x' \subset B(x, r) \tag{2.4}$$

for every $z \in \mathbb{P}^k \setminus B(A_\gamma, \eta)$ and every $x \in f^{-s}(z)$, where V'_x is the connected component of $f^{-s}(B(z, R_3(\eta)))$ containing x. Denote the collection of all connected components of $f^{-n}(B(z, R_3(\eta)))$, $n \geq s$, by $W'_n(\eta, z)$. It follows from (2.4) and (2.3) that the map f^{n-s} restricted to each such component is 1-to-1.

Now, for every $0 \leq n \leq s$ define $Z'_n(\eta, z, \Gamma)$ to be the collection of all connected components of the set $f^{-n}(B(z, R_3(\eta)) \cap \Gamma)$, and for all $0 \leq n \leq s$ set $W'_n(\eta, z, \Gamma) = Z'_n(\eta, z, \Gamma)$. Next, we shall construct the collection $W'_n(\eta, z, \Gamma)$, $n \geq s$, recursively such that the conditions (b'_n), (c'_n), and (d_n) are satisfied. The condition (b'_n) is

(b'_n) if $V' \in W'_n(\eta, z, \Gamma)$, then $\mathrm{Area}(V') \leq \gamma^{n+2s}/4c$,

and (c'_n) is the same as (c_n) with γ^{-s} omitted; the constant c comes from Lelong's Theorem ([La, Theorem II.3.6] or [McM, Theorem 2.45]) along with homogeneity of complex projective spaces. The base of our recursion is the set

$$W'_s(\eta, z, \Gamma) = Z'_s(\eta, z, \Gamma).$$

Now assume that for some $n \geq s$ the family $W'_n(\eta, z, \Gamma)$ has been constructed so that conditions (b'_n), (c'_n), and (d_n) are satisfied. The inductive step is to construct the family $W'_{n+1}(\eta, z, \Gamma)$ so that the conditions (b'_{n+1}), (c'_{n+1}), and (d_{n+1}) are satisfied. The family $W'_{n+1}(\eta, z, \Gamma)$ is defined to consist of all connected components V' of all the sets $f^{-1}(G)$ such that $G \in W'_n(\eta, z, \Gamma)$, and V is contained in an element of $W''_{n+1}(\eta, z)$, and

$$\mathrm{Area}(V') \leq (4c)^{-1}\gamma^{n+1+2s}. \tag{2.5}$$

Condition (b'_{n+1}) is then automatically satisfied, as also is the first part of (d_{n+1}), which holds because of (R''_n). Since for every $V' \in W'_{n+1}(\eta, z, \Gamma)$, the map $f^s|_{f^{n+1-s}(V')}$ is at most γ^{-s}-to-1 in view of (2.2), and since by our construction, the map $f^{n+1-s}|_{V'}$ is 1-to-1, condition (d_{n+1}) is thus fully verified.

Let us show that (c'_{n+1}) holds. Since $\mathrm{Area}(f^{-(n+1)}(\Gamma)) = d^{(k-1)(n+1)}$, the number of elements from $Z'_{n+1}(\eta, z, \Gamma)$ that fail to satisfy condition (2.5) is bounded above by

$$4c\gamma^{-(n+1+2s)}d^{(k-1)(n+1)} \leq 4c\gamma^{-3(n+1)}d^{(k-1)(n+1)}.$$

Combining this with (S''_n), we conclude that the condition (c'_{n+1}) for W' and Z' is established, and the inductive construction of the family $W'_n(\eta, z, \Gamma)$ satisfying conditions (a'_n), (b'_n), (c'_n), and (d'_n) is complete.

Now, decreasing $R_3(\eta)$ appropriately (the smaller radius will be called $R(\eta)$), we shall check that condition (b_n) also holds. Let $0 < R(\eta) \leq R_3(\eta)$ be sufficiently small as specified later in the course of the proof. For every $n \geq 1$ define $W_n(\eta, z, \Gamma)$ to consist of all connected components V of all elements of $W_n'(\eta, z, \Gamma)$ intersected with $f^{-n}(B(z, R(\eta)) \cap \Gamma)$. Conditions (a_n) and (d_n) immediately follow from (a_n') and (d_n') respectively. Since each element of $W_n'(\eta, z, \Gamma)$ contains at least one and at most γ^{-s} elements of $W_n(\eta, z, \Gamma)$, item (c_n) follows immediately from (c_n').

We now show that (b_n) holds. We shall specify the value of $R(\eta)$ now. First, fix a positive integer

$$M > \#\Big(\Gamma \cap \bigcup_{j=1}^{s} f^j(\mathrm{Crit}(f))\Big). \tag{2.6}$$

This intersection is a finite set of bounded cardinality for a generic line Γ.

Then fix an integer $a > 1$ such that $\gamma^s \log a > 1$, and let $0 < R(\eta) < R_3(\eta)$ be so small that

$$0 < \frac{R(\eta)}{R_3(\eta)} < a^{-(M+1)}.$$

Now, for all $p = 0, 1, \ldots, M$, consider the annuli

$$\Lambda_p = \big(B(z, a^{p+1}R(\eta)) \setminus B(z, a^p R(\eta))\big) \cap \Gamma.$$

By the choice of M, there exists at least one annulus in this collection that does not intersect the set $\bigcup_{j=1}^{s} f^j(\mathrm{Crit}(f))$. Let us keep the notation Λ_p for this specified annulus. Set

$$\begin{aligned} D' &= B(z, R_3(\eta)) \cap \Gamma, & D &= B(z, R(\eta)) \cap \Gamma, \\ D_1 &= B(z, a^p R(\eta)) \cap \Gamma, & D_2 &= B(z, a^{p+1} R(\eta)) \cap \Gamma \end{aligned}$$

(so $D \subset D_1 \subset D_2 \subset D'$). Let $V \in W_n$ be a connected component of $f^{-s}(D)$. Then, let V_1 be the connected component of $f^{-n}(D_1)$ containing V, let V_2 be the connected component of $f^{-n}(D_2)$ containing V_1; further as above, let V' be the connected component of $f^{-n}(D')$ containing V_2 (thus $V \subset V_1 \subset V_2 \subset V'$). Clearly, $V' \cap f^{-n}(\Lambda_p)$ is a union of at most γ^{-s} of the annuli, say, Λ_j', the modulus of each annulus Λ_j' is bounded below by $\gamma^s \log a$, and, after appropriate rearrangement of indices j,

$$V_2 \setminus V_1 = \bigcup_{j=1}^{m} \Lambda_j'$$

with some $m \le \gamma^{-s}$. Since the modulus of every annulus Λ_j' in $V_2 \setminus V_1$ is larger than $\gamma^s \log a > 1$, we have

$$1 > \frac{1}{\mathrm{mod}(\Lambda_j')} = \sup_\rho \frac{\inf_l^2 \mathrm{length}_\rho(l)}{\mathrm{Area}_\rho(\Lambda_j')} \ge \frac{\mathrm{length}^2(l)}{\mathrm{Area}(\Lambda_j')},$$

where the supremum is taken over all measurable Riemannian metrics on Λ_j' and the infimum is taken over all closed piecewise-smooth curves that separate both components of the boundary of Λ_j'. The values length(l) and

$\operatorname{Area}(\Lambda'_j)$ respectively denote the length and the area calculated with respect to the Fubini–Study metric. Thus for every annulus Λ'_j there exists a curve l_j in this family such that

$$\operatorname{length}(l_j) \le \sqrt{\operatorname{Area}(\Lambda'_j)} \le \sqrt{\operatorname{Area}(V_2)} \le \sqrt{\operatorname{Area}(V')}.$$

We claim that this implies

$$\operatorname{diam}(V) < 2\sqrt{c}\sqrt{\operatorname{Area}(V')}\,\gamma^{-s}. \tag{2.7}$$

Indeed, one can enlarge V_1 so that the boundary of this modified domain is exactly the union of the curves $l_1, \ldots, l_m$. Let us keep the notation V_1 for this modified domain. Put $Q = \sqrt{c}\sqrt{\operatorname{Area}(V')}$. Then consider the following two cases. Either

(a) there exists $x \in V_1$ such that $d(x, l_i) > Q$ for all $i = 1, \ldots, m$, or
(b) for every $x \in V_1$ there exists $l_x \in \{l_1, \ldots, l_m\}$ such that $d(x, l_x) < Q$.

In case (a), take $x \in V_1$ with this property, and let

$$U := V_1 \cap B(x, Q) \subset B(x, Q).$$

Then U is a closed algebraic variety in $B(x, Q)$ and, using Lelong's Theorem ([La, Theorem II.3.6] or [McM, Theorem 2.45]), we get

$$\operatorname{Area}(U \cap B(x, Q)) > \frac{1}{c} Q^2.$$

But $\operatorname{Area}(U \cap B(x, Q)) \le \operatorname{Area}(V') = (1/c)Q^2$. This contradiction implies that case (a) never occurs. In case (b) we get

$$V_1 \subset \bigcup_{i=1}^{m} B(l_i, Q) = B(l_1, Q) \cup \bigcup_{i=2}^{m} B(l_i, Q).$$

Since the set V_1 is connected and both sets in the above union are open, they must intersect, say $B(l_2, Q) \cap B(l_1, Q) \ne \emptyset$. Thus, proceeding by induction and permuting the sets $B(l_i, Q)$ if necessary, we can require that

$$B(l_{j+1}, Q) \cap \bigcup_{i=1}^{j} B(l_i, Q) \ne \emptyset.$$

Therefore, if $x \in B(l_1, Q)$ and $y \in B(l_j, Q)$ then

$$\operatorname{dist}(x, y) \le jQ + j \sup_i \operatorname{length}(l_i).$$

This implies that

$$\operatorname{diam}(V_1) \le mQ + m \sup_i \operatorname{length}(l_i) \le 2\sqrt{c}\sqrt{\operatorname{Area}(V')}\,\gamma^{-s}.$$

As $V \subset V_1$, formula (2.7) is thus proved. But, by our condition (b'_n) on the area of V', we can now write

$$\operatorname{diam}(V) \le 2\sqrt{c}\,\frac{\gamma^{n/2}\gamma^{s}}{\sqrt{c}\cdot 2}\gamma^{-s} = \gamma^{n/2}.$$

So, (b_n) is established. ■

The reader is invited to think of elements of $W_n(\eta, z, \varGamma)$ as of good components of $f^{-n}(B(z, R(\eta)) \cap \varGamma)$ and of elements of $Z_n(\eta, z, \varGamma) \setminus W_n(\eta, z, \varGamma)$ as of bad components of $f^{-n}(B(z, R(\eta)) \cap \varGamma)$. Note that if $V \in Z_n(\eta, z, \varGamma) \setminus W_n(\eta, z, \varGamma)$, i.e. if V is a bad component of $f^{-n}(B(z, R(\eta)) \cap \varGamma)$, then all its images $f(V), f^2(V), \dots, f^n(V)$ are good. In Section 4 we will need to deal with an iterate f^q of f rather than with f itself. We will need to estimate the number of inverse images of a given point w, lying in components V_{qn} of $f^{-qn}(B(z, R(\eta)) \cap \varGamma)$ which are bad for the nth iterate of f^q, i.e. $V_{qn} \notin W_{qn}(\eta, z, \varGamma)$ but $f^q(V_{qn}) \in W_{q(n-1)}(\eta, z, \varGamma)$. Precisely, fix an arbitrary integer $q \ge 1$ and let $s = Nq$ be an integral multiple of q. For every $n \ge 0$ and every $w \in B(z, R(\eta)) \cap \varGamma$, set

$$\begin{aligned} &B_n(\eta, z, q, \varGamma; w) \\ &\quad = f^{-qn}(w) \cap \bigcup_{j=0}^{q-1} f^{-j}\Big(\bigcup\{V : V \in Z_{qn-j}(\eta, z, \varGamma) \setminus W_{qn-j}(\eta, z, \varGamma)\}\Big). \end{aligned}$$

Remark 2.9. Note that the radius R depends not only on η but also on N. In the notation, we skip this dependence. N will be fixed in the proof, except for Section 6 (proof of almost periodicity).

A straightforward computation together with the use of Lemma 2.6 leads to the following.

Lemma 2.10. *With the notation and hypotheses of Lemma* 2.6 *assume in addition that $s = qN$ with some integers $q, N \ge 1$. Then, for every $w \in B(z, R(\eta)) \cap \varGamma$, we have*

(a) $B_n(\eta, z, q, \varGamma; w) = \emptyset$ *for all* $0 \le n \le N$.

(b_n) $\#(B_n(\eta, z, q, \varGamma; w))$
$\le q d^{k-2+q}\gamma^{-2qN}(4c\gamma^{-3qn} + G(qN)^{4(k-1)})d^{(k-1)qn}$ *for all* $n \ge 0$.

Proof. Item (a) follows immediately from Lemma 2.6(a). To get (b_n), using Lemma 2.6(c_n), we estimate for all $n \ge N+1$ as follows:

$$\begin{aligned} &\#B_n(\eta, z, q, \varGamma; w) \\ &\quad \le \sum_{j=0}^{q-1} \#\Big(f^{-qn}(w) \cap f^{-j}\Big(\bigcup(Z_{qn-j}(\eta, z, \varGamma; w) \setminus W_{qn-j}(\eta, z, \varGamma; w))\Big)\Big) \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=0}^{q-1} \gamma^{-qN} d^{kj} \#(Z_{qn-j}(\eta,z,\Gamma;w)\setminus W_{qn-j}(\eta,z,\Gamma;w)) \\
&\le \gamma^{-qN} \sum_{j=0}^{q-1} d^{kj}\gamma^{-qN}(4c\gamma^{-3(qn-j)} + G(qN)^{4(k-1)})d^{(k-1)(qn-j+1)} \\
&\le \gamma^{-2qN} \sum_{j=0}^{q-1} (4c\gamma^{-3qn} + G(qN)^{4(k-1)})d^{(k-1)qn} d^{j+k-1} \\
&\le q d^{k-2+q}\gamma^{-2qN}(4c\gamma^{-3qn} + G(qN)^{4(k-1)})d^{(k-1)qn}. \ \blacksquare
\end{aligned}
$$

3. Potentials and conformal measures

3.1. Restrictions on potentials. For every $\gamma \in (0,1)$ let

$$A_{J,\gamma} = A_\gamma \cap J.$$

Since the Julia set J is backward and forward invariant and since the set A_γ fails to be backward invariant, and since the preimages of every point in J are dense in J, there exists a least integer $q = p_3(\gamma) \ge 1$ (an integral multiple of $p_1(\gamma)!$), and a positive Δ such that

$$
\begin{aligned}
(3.1)\qquad &\deg(f^q;\gamma,\Delta) \\
&:= \deg\big(f^q : B(A_{J,\gamma},\Delta)\cap f^{-q}(B(A_{J,\gamma},\Delta)) \to B(A_{J,\gamma},\Delta)\big) \\
&\le d^{qk} - 1.
\end{aligned}
$$

In other words, every point in $B(A_{J,\gamma},\Delta)$ has at least one preimage (under f^q) outside the set $B(A_{J,\gamma},\Delta)$. Note that like p_1 and p_2, the function $p_3 : (0,1) \to \mathbb{N}$ is weakly increasing and is constant throughout the interval $(0,\gamma_*)$. For every $\gamma \in (0,1)$, looking up at (3.1), set

$$(3.2)\qquad g_\gamma := \max\bigg\{\frac{1}{p_3(\gamma)}\log_d(\deg(f^{p_3(\gamma)};\gamma,\Delta)), k-1\bigg\} < k.$$

Now, for every $\kappa \in (0,\log d)$ let

$$(3.3)\qquad \gamma_\kappa = \exp\big(\tfrac{1}{6}(\kappa - \log d)\big).$$

It follows from the definition of the set A_γ that the function $(0,\log d) \ni \kappa \mapsto A_{\gamma_\kappa}$ is weakly increasing and constant on some interval $(0,\kappa_*)$. Consequently, the function $(0,\log d) \ni \kappa \mapsto g_{\gamma_\kappa}$ is weakly increasing and takes on a constant value (in $(0,k)$) throughout some interval $(0,\kappa_*)$. Thus, the function $(0,\log d) \ni \kappa \mapsto k - g_{\gamma_\kappa}$ is weakly decreasing and takes on a constant value (in $(0,k)$) on $(0,\kappa_*)$. Therefore $(k - g_{\gamma_\kappa})\log d > \kappa$ for all $\kappa > 0$ sufficiently close to 0 and we can define the maximal admissible oscillation of our potentials, usually denoted by ϕ, as follows:

$$\kappa_f := \sup\{\kappa \in (0, \log d) : (k - g_{\gamma_\kappa}) \log d > \kappa\} \in (0, \log d], \tag{3.4}$$

and

$$g_f := k - \frac{\kappa_f}{\log d} \in [k-1, k). \tag{3.5}$$

Remark 3.1. We thus fix the value of q according to the chosen value of κ. Having fixed some $\kappa < \kappa_f$, we fix γ_κ according to (3.3). We then fix $p_1(\gamma)$, and finally q, according to (3.1). In particular, if the set $A_{J,\gamma}$ is empty, one can keep $q = 1$.

Let us now record the following obvious observation.

Lemma 3.2. *If $A_p \cap J = \emptyset$ for all $p \geq 1$, then*

$$\kappa_f = \log d. \tag{3.6}$$

Note that the set $A_p \cap J$ may not be empty. The simplest example is provided by some polynomial skew product in $\mathbb{C}^2$ (see Example 9.1 in [J]). In this example, the Julia set contains a "supersaddle point", a fixed point for which one eigenvalue of the derivative is larger than one, while the other one equals zero.

3.2. The case of dimension 2. We now discuss the case $k = 2$. Although the set $A_p \cap J$ does not have to be empty, the task to estimate κ_f reduces to looking at finitely many periodic points only. Indeed, we recall the following lemma from [FS1, Lemma 7.9].

Lemma 3.3. *Suppose that $f : \mathbb{P}^2 \to \mathbb{P}^2$ is a holomorphic map of degree d that maps a compact complex hypersurface Z into itself and such that Z is contained in the critical set of f. Then*

$$\operatorname{dist}(f(z), Z) = o(\operatorname{dist}(z, Z)).$$

Lemma 3.4. *Suppose that $f : \mathbb{P}^2 \to \mathbb{P}^2$ is a holomorphic map of degree d. If $D \subset C$ is an irreducible component of the critical set C, and D is periodic under f ($f^l(D) = D$ for some $l \geq 1$), then D does not intersect the Julia set J.*

Proof. Let $z \in J$ and let U be an arbitrary neighborhood of z. It follows from the construction of the maximal measure that $\bigcup_{n\geq 0} f^n(U) = \mathbb{P}^2 \setminus E$ where E is the exceptional set. Applying Lemma 3.3, we conclude that if D is a periodic irreducible component of the critical set C then there exists a neighborhood of D which is mapped into itself under f^l. Therefore, $D \cap J = \emptyset$. ∎

For every periodic point z of f let $p(z) \geq 1$ be the least integer such that $f^{p(z)}(z) = z$. Denote by $\operatorname{Per}(f)$ the set of all periodic points of f. As a corollary of Lemma 3.4 we get

COROLLARY 3.5. *If $f : \mathbb{P}^2 \to \mathbb{P}^2$ is a regular holomorphic map of degree d, then the set*

$$W := \{z : \deg_z f > d\} = \{z : \deg_z f \geq d+1\}$$

is finite and

$$\kappa_f \geq 2\log d - \max\left\{\log d, \max_{z\in W\cap \mathrm{Per}(f)\cap J}\left\{\frac{1}{p(z)}\log \deg_z(f^{p(z)})\right\}\right\}.$$

In particular, if $W \cap J = \emptyset$, or $W \cap J \cap \mathrm{Per}(f) = \emptyset$ then $\kappa_f = \log d$.

3.3. Conformal measures. As was indicated, our assumption is that $\phi : J \to \mathbb{R}$ is a Hölder continuous function and

$$\sup(\phi) - \inf(\phi) < \kappa_f. \tag{3.7}$$

Let us take the first fruits of this assumption. Fix two positive numbers α and β such that

$$\sup(\phi) - \inf(\phi) < \alpha < \beta < \kappa_f. \tag{3.8}$$

Set

$$\theta = \frac{\beta - \alpha}{2} > 0. \tag{3.9}$$

We consider the dynamical system $f : J \to J$. Let $C(J)$ denote the Banach space of all complex-valued continuous functions on J endowed with the supremum norm. For every $g \in C(J)$ define $\mathcal{L}_\phi g$ by the formula

$$\mathcal{L}_\phi g(z) = \sum_{x\in f^{-1}(z)} e^{\phi(x)} g(x),$$

where the inverse images of critical values of f are counted with multiplicities. Then $\mathcal{L}_\phi g \in C(J)$ and the linear operator $\mathcal{L}_\phi : C(J) \to C(J)$ is bounded. $\mathcal{L}_\phi$ is called the *Perron–Frobenius* (transfer) *operator* associated to the potential ϕ. Consider the dual operator $\mathcal{L}_\phi^* : C^*(J) \to C^*(J)$, $\mathcal{L}_\phi^*\mu(g) = \mu(\mathcal{L}_\phi g)$. Let M_J be the set of all Borel probability measures on J. The map $\mu \mapsto \mathcal{L}_\phi^*\mu/\mathcal{L}_\phi^*\mu(\mathbb{1})$, $\mu \in M_J$, is well-defined and continuous. Since M_J is convex and compact (in the weak-$*$ topology), this map has a fixed point in virtue of the Schauder–Tikhonov Theorem. Denote this fixed point by m_ϕ and set $\lambda = \mathcal{L}_\phi m_\phi(\mathbb{1})$. Then

$$\mathcal{L}_\phi^* m_\phi = \lambda m_\phi. \tag{3.10}$$

Such measures are called *conformal*. Using (3.7), we get

$$\begin{aligned}\lambda = \int \mathcal{L}_\phi \mathbb{1}\, dm_\phi &= \int \sum_{x\in f^{-1}(z)} e^{\phi(x)}\, dm_\phi \\ &\geq \int d^k \exp(\inf(\phi))\, dm_\phi = d^k \exp(\inf(\phi)) \\ &= \exp(k\log d + \inf(\phi)) > \exp(\sup(\phi) - \alpha + k\log d).\end{aligned}$$

Equivalently,

$$\sup(\phi) - \log\lambda < \alpha - k\log d. \tag{3.11}$$

Therefore, because of (3.9) and (3.3), we get

$$\begin{aligned}\log\lambda - \sup(\phi) - (k-1)\log d > \log d - \alpha &= \log d - \beta + (\beta - \alpha) \\ &= -6\log\gamma_\beta + 2\theta.\end{aligned} \tag{3.12}$$

This inequality will be used in future sections. Independently of this, notice that since the Julia set J is completely invariant, iterating (3.10), and making use of the topological exactness of the map $f : J \to J$ (Proposition 1.3), we obtain the following.

PROPOSITION 3.6. *The measure m_ϕ is positive on nonempty open subsets of J; in other words,* $\mathrm{supp}(m_\phi) = J$.

4. Uniform bounds of iterates of the Perron–Frobenius operator. In this section we provide upper and lower uniform bounds on the iterates of the Perron–Frobenius operator. This is naturally done by introducing several auxiliary operators and dealing with them in the following several subsections.

We shall need the following well-known lemma.

LEMMA 4.1. *Suppose that (X,ρ) is a compact metric space and F is a closed subset of X. If $g : F \to \mathbb{R}$ is a Hölder continuous function with an exponent $\alpha \in (0,1)$, then there exists a Hölder continuous function $\tilde{g} : X \to \mathbb{R}$ with the same exponent α and with $\tilde{g}|_F = g$,* $\sup(\tilde{g}) = \sup(g)$ *and* $v_\alpha(\tilde{g}) \le 2v_\alpha(g)$.

4.1. Preliminaries. Consider the number γ_β defined by formula (3.3). There then exists an integer $N_\beta \ge 1$ such that

$$qd^{k-2+q}\gamma_\beta^{-2qN}(4c\gamma_\beta^{-3qn} + G(qN)^{4(k-1)}) \le \gamma_\beta^{-6qn}$$

for all $q \ge 1$, $N \ge N_\beta$ and $n \ge N$. Note that the left-hand side of this inequality is the number appearing in Lemma 2.10. Combining this and (3.12), we get

$$\begin{aligned}qd^{k-2+q}\gamma_\beta^{-2qN}&(4c\gamma_\beta^{-3qn}+G(qN)^{4(k-1)})d^{(k-1)qn}\exp\bigl((\sup(\phi)-\log\lambda)qn\bigr) \\ &= qd^{k-2+q}\gamma_\beta^{-2qN}(4c\gamma_\beta^{-3qn} + G(qN)^{4(k-1)})d^{(k-1)qn} \\ &\quad\cdot \exp\bigl(qn(\sup(\phi) - \log\lambda + (k-1)\log d)\bigr) \\ &\le \gamma_\beta^{-6qn}\exp\bigl(qn(\sup(\phi) - \log\lambda + (k-1)\log d)\bigr) \\ &\le \gamma_\beta^{-6qn}\gamma_\beta^{6qn}e^{-2\theta qn} = e^{-2\theta qn}\end{aligned} \tag{4.1}$$

for all $q \ge 1$, $N \ge N_\beta$ and $n \ge N$. This inequality will allow us to estimate from above the part of the Perron–Frobenius operator corresponding to the inverse images of a given point w lying in bad components, i.e. in $B_n(\eta, z, q, \Gamma; w)$ (see Lemma 2.10).

We also assume N_β to be so large that for all $q \geq 1$ and $N \geq N_\beta$,

$$(1 - e^{-q\theta})^{-1}(1 + (1 - e^{-q\theta})^{-1}\lambda^{-q}e^{q\sup(\phi)}d^{qk})e^{-\theta qN} \leq 1/4. \tag{4.2}$$

This inequality will be used to justify formula (4.26). Now set

$$A_* := A_{\gamma_\beta}, \quad A_J^* = A_{\gamma_\beta} \cap J, \quad g_* = g_{\gamma_\beta}.$$

As mentioned in the Introduction, the calculations below become considerably simpler if $A_J^* = \emptyset$. Apply Lemma 2.6 with $\gamma := \gamma_\beta$. At this point, we also fix the value q, appropriate for $\kappa = \beta$, as Remark 3.1 shows. By (3.2) there exists $\Delta_q^* > 0$ so small that

$$\deg\big(f^q : B(A_J^*, \Delta_q^*) \cap f^{-q}(B(A_J^*, \Delta_q^*)) \to B(A_J^*, \Delta_q^*)\big) \leq d^{g_* q}. \tag{4.3}$$

Now notice that for all closed sets E and F contained in a compact metric space and for all $\varepsilon > 0$ there exists $\delta > 0$ such that $B(E, \delta) \cap B(F, \delta) \subset B(E \cap F, \varepsilon)$. By our choice of $q \geq 1$ as an integral multiple of $(p_1(\gamma_\beta))!$, we have

$$f^q(A_*) \subset A_* \quad \text{and} \quad f^q(A_J^*) \subset A_J^*. \tag{4.4}$$

Since $f^{-q}(A_J^*) \cap A_* \subset J \cap A_* = A_J^*$, we therefore conclude that there exists $\Delta_q^{(1)} \in (0, \Delta_q^*/2)$ such that

$$f^{-q}(B(A_J^*, \Delta_q^{(1)})) \cap B(A_*, \Delta_q^{(1)}) \subset B(f^{-q}(A_J^*) \cap A_*, \Delta_q^*) \subset B(A_J^*, \Delta_q^*). \tag{4.5}$$

It follows from (4.4) and continuity of f^q that there exists $\Delta_q^{(2)} \in (0, \Delta_q^{(1)})$ such that

$$f^q(B(A_J^*, \Delta_q^{(2)})) \subset B(A_J^*, \Delta_q^{(1)}). \tag{4.6}$$

Now, since the sets J and $A_* \setminus B(A_J^*, \Delta_q^{(2)})$ are closed (so compact) and mutually disjoint, there exists $\Delta_q^{(3)} \in (0, \Delta_q^{(2)})$ such that

$$J \cap \overline{B}(A_* \setminus B(A_J^*, \Delta_q^{(2)}), \Delta_q^{(3)}) = \emptyset. \tag{4.7}$$

Also, by (4.6), $f^{-q}(A_* \setminus B(A_J^*, \Delta_q^{(1)})) \cap A_* \subset A_* \setminus B(A_J^*, \Delta_q^{(2)})$, and so there exists $\Delta_q^{(4)} \in (0, \Delta_q^{(3)})$ such that

$$\begin{aligned} f^{-q}\big(B(A_* \setminus B(A_J^*, \Delta_q^{(1)}), \Delta_q^{(4)})\big) \cap B(A_*, \Delta_q^{(4)}) \\ \subset B(A_* \setminus B(A_J^*, \Delta_q^{(2)}), \Delta_q^{(3)}). \end{aligned} \tag{4.8}$$

Since $\{B(A_J^*, \Delta_q^{(1)}), B(A_* \setminus B(A_J^*, \Delta_q^{(1)}), \Delta_q^{(4)})\}$ is an open cover of the compact set A_*, there exists $\Delta_q \in (0, \Delta_q^{(4)}/2)$ such that

$$B(A_*, \Delta_q) \subset B(A_J^*, \Delta_q^{(1)}) \cup B(A_* \setminus B(A_*, \Delta_q^{(1)}), \Delta_q^{(4)}). \tag{4.9}$$

This formula will be however used only in later sections, first in Section 4.3 to estimate the part of the Perron–Frobenius operator taking preimages from the $\varDelta_q$-neighborhood of A_J^*.

We now extend the function ϕ (more precisely $\sum_{j=0}^{q-1}\phi\circ f^j$) beyond J, in the following way. Fix $\tau\in\mathbb{R}$ so small that $\tau<\sup(\phi)$ and

$$\lambda^{-1}d^k e^{\tau}\le e^{-\theta}. \tag{4.10}$$

Define the function $\phi_q : J\cup\overline{B}(A_*\setminus B(A_J^*,\varDelta_q^{(2)}),\varDelta_q^{(3)})\to\mathbb{R}$ by

$$\phi_q(z)=\begin{cases}\displaystyle\sum_{j=0}^{q-1}\phi(f^j(z)) & \text{if } z\in J,\\ q\tau & \text{if } z\in\overline{B}(A_*\setminus B(A_J^*,\varDelta_q^{(2)}),\varDelta_q^{(3)}).\end{cases}$$

Thus, we require ϕ_q to be negative, with large modulus, on the part of A_* which is away from J. This function is well defined since, by (4.7), the sets and $\overline{B}(A_*\setminus B(A_J^*,\varDelta_q^{(2)}),\varDelta_q^{(3)})$ and J are disjoint. Clearly, this function is Hölder continuous and $\sup(\phi_q)\le q\sup(\phi)$. Let $\tilde{\phi}_q:\mathbb{P}^k\to\mathbb{R}$ be the Hölder continuous extension produced in Lemma 4.1 Remember that $\sup(\tilde{\phi}_q)=\sup(\phi_q)$, and that $\tilde{\phi}_q$ has the same Hölder exponent as ϕ_q and ϕ. Denote this exponent by ω and the ω-variation of $\tilde{\phi}_q$ by H_q, which, by Lemma 4.1, is bounded by the double ω-variation of ϕ_q. For every $g:\mathbb{P}^k\to\mathbb{R}$ let

$$S_n g=\sum_{j=0}^{n-1} g\circ f^{qj}.$$

It follows from Lemma 2.6(b_n) that for every $n\ge 0$ and $z\in\mathbb{P}^k\setminus B(A_*,\eta)$, for a generic projective line $\varGamma$ passing through z, for every connected component $V\in W_{qn}(\eta,z,\varGamma)$, and all $x,y\in V$, we have

$$\begin{aligned}|S_n\tilde{\phi}_q(x)-S_n\tilde{\phi}_q(y)| &\le\sum_{j=0}^{n-1}|\tilde{\phi}_q(f^{qj}(x))-\tilde{\phi}_q(f^{qj}(y))|\\ &\le\sum_{j=0}^{n-1}H_q\rho^{\omega}(f^{qj}(x),f^{qj}(y))\\ &\le H_q\sum_{j=0}^{n-1}\gamma_\beta^{(n-j)q\omega/2}\le H_q\sum_{j=0}^{\infty}\gamma_\beta^{q\omega j/2}\\ &=H_q(1-\gamma_\beta^{q\omega/2})^{-1}.\end{aligned}\tag{4.11}$$

Hence, for all $n\ge 0$,

$$\tilde{C}_q^{-1}\le\frac{\lambda^{-qn}\exp(S_n\tilde{\phi}_q(x))}{\lambda^{-qn}\exp(S_n\tilde{\phi}_q(y))}\le\tilde{C}_q, \tag{4.12}$$

where $\tilde{C}_q=\exp(H_q(1-\gamma_\beta^{\omega/2})^{-1})$.

In this section we will need a few auxiliary Perron–Frobenius operators. First, define $\mathcal{L}_{\tilde{\phi}_q} : C(\mathbb{P}^k) \to C(\mathbb{P}^k)$ by the formula

$$\mathcal{L}_{\tilde{\phi}_q} g(z) = \sum_{x \in f^{-q}(z)} e^{\tilde{\phi}_q(x)} g(x),$$

where the summation is taken over all points of $f^{-q}(z)$ counted with multiplicities. As in Preliminaries, $\mathcal{L}_{\tilde{\phi}_q} : C(\mathbb{P}^k) \to C(\mathbb{P}^k)$ is a bounded linear operator. It is also called the Perron–Frobenius operator associated to the potential $\tilde{\phi}_q$. Define the operators $\hat{\mathcal{L}}_{\tilde{\phi}_q} : C(\mathbb{P}^k) \to C(\mathbb{P}^k)$ and $\hat{\mathcal{L}}_{\phi} : C(J) \to C(J)$ by the formulas

$$\hat{\mathcal{L}}_{\phi} = \lambda^{-1} \mathcal{L}_{\phi} \quad \text{and} \quad \hat{\mathcal{L}}_{\tilde{\phi}_q} = \lambda^{-q} \mathcal{L}_{\tilde{\phi}_q}.$$

Our goal is to prove uniform upper and lower bounds on the iterates $\hat{\mathcal{L}}_{\phi}^n$, $n \geq 0$. This will be done inductively and several auxiliary operators will be involved, labeled with subscripts and superscripts such as $\hat{\mathcal{L}}_{J,q}^n$, $G_{\tilde{\phi}_q,z,\xi}^{(n)}$, $B_{\tilde{\phi}_q,z,\xi}^{(n)}$, and $\hat{\mathcal{L}}_* g$. Fix

$$0 < \eta \leq \Delta_q.$$

For every $n \geq 0$, set

$$\hat{\mathcal{L}}_q^n \mathbb{1} = (\hat{\mathcal{L}}_{\tilde{\phi}_q}^n \mathbb{1})|_{B^c(A_*,\Delta_q)} \quad \text{and} \quad \hat{\mathcal{L}}_{J,q}^n \mathbb{1} = (\hat{\mathcal{L}}_{\phi}^{qn} \mathbb{1})|_{B^c(A_J^*,\Delta_q)}. \tag{4.13}$$

4.2. First estimates away from A_*. Let $R(\eta)$ be the number produced in Lemma 2.6 for $s = Nq$ (recall that, formally, $R(\eta)$ depends also on N, and as before, we skip this dependence in the notation). For every $n \geq 0$ and $z \in \mathbb{P}^k \setminus B(A_*, \eta)$, every projective line Γ passing through z, and every $w \in \Gamma \cap B(z, R_q)$, set

$$\hat{G}_{\tilde{\phi}_q,z,\Gamma}^{(n)}(w) = \sum_{x \in f^{-qn}(w) \cap \bigcup W_{qn}(\eta,z,\Gamma)} \lambda^{-qn} \exp(S_n \tilde{\phi}_q(x)),$$

$$\hat{B}_{\tilde{\phi}_q,z,\Gamma}^{(n)}(w) = \sum_{x \in B_n(\eta,z,q,\Gamma;w)} \lambda^{-qn} \exp(S_n \tilde{\phi}_q(x)).$$

The symbol G is going to indicate that the inverse branches involved in its definition are thought of as good while B stands for branches thought of as bad. It follows from (4.12) that

$$\tilde{C}_q \leq \frac{\hat{G}_{\tilde{\phi}_q,z,\Gamma}^{(n)}(w)}{\hat{G}_{\tilde{\phi}_q,z,\Gamma}^{(n)}(z)} \leq \tilde{C}_q \tag{4.14}$$

for a generic projective line Γ. It also follows from Lemma 2.10 and from formula (4.1) that

$$
\begin{aligned}
(4.15)\qquad \hat{B}^{(n)}_{\tilde{\phi}_q,z,\Gamma}(w) &\le \lambda^{-qn}\exp(\sup(\tilde{\phi}_q)n)\#B_n(\eta,z,q,\Gamma;w)\\
&\le \gamma_\beta^{-2qNq}\exp\bigl(qn(\sup(\phi)-\log\lambda)\bigr)\\
&\quad\cdot qd^{k-2+q}\gamma^{-2qN}(4c\gamma^{-3qn}+G(qN)^{4(k-1)})d^{(k-1)qn}\\
&\le e^{-2\theta qn}.
\end{aligned}
$$

for every $n \ge N+1$. This also holds for $0 \le n \le N$ as then

$$
(4.16)\qquad \hat{B}^{(n)}_{\tilde{\phi}_q,z,\Gamma}(w) = 0.
$$

4.3. Estimates in the neighborhood of A_*. Let $L_\infty(\overline{B}(A_*,\Delta_q))$ be the Banach space of all real-valued bounded functions on $\overline{B}(A_*,\Delta_q)$. For every $h \in L_\infty(\overline{B}(A_*,\Delta_q))$ and every $z \in \overline{B}(A_*,\Delta_q)$, let

$$
(4.17)\qquad \hat{\mathcal{L}}_* h(z) = \sum_{x\in f^{-q}(z)\cap\overline{B}(A_*,\Delta_q))} \lambda^{-q}\exp(\tilde{\phi}_q(x))h(x).
$$

Obviously, $\hat{\mathcal{L}}_* g(z)$ is a linear operator acting on $L_\infty(\overline{B}(A_*,\Delta_q))$. It represents the part of the Perron–Frobenius $\hat{\mathcal{L}}^n_{\tilde{\phi}_q}$ collecting inverse images lying close to A_*. If $z \in \overline{B}(A_J^*,\Delta_q^{(1)})$, then it follows from (4.3), (4.5), (3.4), (3.12), and the fact that $\Delta_q \le \Delta_q^{(4)}/2$ that

$$
\begin{aligned}
(4.18)\qquad \hat{\mathcal{L}}_*\mathbb{1}(z) &\le \lambda^{-q}d^{g_*q}e^{\sup(\tilde{\phi}_q)} \le \exp\bigl(q(\sup(\phi)-\log\lambda+g_*\log d)\bigr)\\
&< \exp\bigl(q(\sup(\phi)-\log\lambda+k\log d-\beta)\bigr) < e^{-2\theta q} < e^{-\theta q}.
\end{aligned}
$$

If, on the other hand, $z \in \overline{B}(A_*\setminus B(A_J^*,\Delta_q^{(1)}),\Delta_q^{(4)})$, then it follows from (4.8), (4.10), and the definitions of ϕ_q and $\tilde{\phi}_q$, that

$$
(4.19)\qquad \hat{\mathcal{L}}_*\mathbb{1}(z) \le \lambda^{-q}d^{kq}e^{q\tau} \le e^{-\theta q}.
$$

Both (4.18) and (4.19) along with (4.9) imply that

$$
\|\hat{\mathcal{L}}_*\| = \|\hat{\mathcal{L}}_*\mathbb{1}\|_\infty \le e^{-\theta q}.
$$

Consequently, for all $n \ge 0$ and all $z \in \overline{B}(A_*,\Delta_q)$,

$$
(4.20)\qquad \hat{\mathcal{L}}^n_*\mathbb{1}(z) \le e^{-\theta qn}.
$$

4.4. Estimates away from A_*: the inductive step. Now, using the estimates from Sections 4.2 and 4.3 we obtain an inductive formula for the bound on the iterates of the Perron–Frobenius operator evaluated at points away from A_*. Keep $0 < \eta \le \Delta_q$ fixed. Fix $z \in \mathbb{P}^k \setminus B(A_*,\eta)$, a projective line Γ passing through z, and a point $w \in \Gamma \cap B(z,R(\eta))$. Set

$$
(4.21)\qquad B_j(w) = B^{(j)}_{\tilde{\phi}_q,z,\Gamma}(w).
$$

Then

$$
\begin{aligned}
(4.22)\qquad & \hat{\mathcal{L}}^n_{\tilde{\phi}_q}\mathbb{1}(w)\\
&= \sum_{j=1}^n \sum_{x\in B^c(A_*,\Delta_q)\cap B_j(w)} \lambda^{-qj}\exp(S_j\tilde{\phi}_q(x))\hat{\mathcal{L}}_q^{n-j}\mathbb{1}(x)\\
&\quad + \sum_{j=1}^n \sum_{x_1\in \Lambda_1^j(w)} \lambda^{-qj}\exp(S_j\tilde{\phi}_q(x_1)) \sum_{i=0}^{n-j} \sum_{x_2\in \Lambda_2^i(x_1)} \lambda^{-qi}\exp(S_i\tilde{\phi}_q(x_2))\\
&\qquad \cdot \sum_{x_3\in \Lambda_3(x_2)} \lambda^{-q} e^{\tilde{\phi}_q(x_3)}\hat{\mathcal{L}}_q^{n-(j+i+1)}\mathbb{1}(x_3) + \hat{G}^{(n)}_{\tilde{\phi}_q,z,\Gamma}(w),
\end{aligned}
$$

where

$$
\begin{aligned}
&\Lambda_1^j(w) = B(A_*,\Delta_q)\cap B_j(w),\\
(4.23)\qquad &\Lambda_2^i(x_1) = f^{-qi}(x_1)\cap \bigcap_{l=0}^{i} f^{-ql}(B(A_*,\Delta_q)),\\
&\Lambda_3(x_2) = f^{-q}(x_2)\cap B^c(A_*,\Delta_q).
\end{aligned}
$$

and the operator $\hat{\mathcal{L}}_q$ was defined in (4.13). In other words, backward trajectories starting from w are divided into groups according to the number of consecutive steps at which the trajectory stays close to A_*.

Denote the first summand in (4.22) by $\Sigma_1^{(n)}(w)$ and the second by $\Sigma_2^{(n)}(w)$. We will estimate each of them separately. Set, for all $l\ge 1$,

$$
\begin{aligned}
M_l^*(\tilde{\phi}_q) &= \max\{\|\hat{\mathcal{L}}_q^j\mathbb{1}\|_\infty : 1\le j\le l\},\\
M_l^*(\phi) &= \max\{\|\hat{\mathcal{L}}_\phi^{qj}\mathbb{1}|_{J\cap B^c(A_*,\Delta_q)}\|_\infty : 1\le j\le l\}.
\end{aligned}
$$

We start with $\Sigma_1^{(n)}(w)$. Because of (4.15) and (4.16), we have

$$
\begin{aligned}
(4.24)\qquad \Sigma_1^{(n)}(w) &\le \sum_{j=1}^n \sum_{x\in B_j(w)} \lambda^{-qj}\exp(S_j\tilde{\phi}_q(x))M^*_{n-1}(\tilde{\phi}_q)\\
&\le M^*_{n-1}(\tilde{\phi}_q)\sum_{j=1}^n \hat{B}^{(j)}_{\tilde{\phi}_q,z,\Gamma}(w)\\
&= M^*_{n-1}(\tilde{\phi}_q)\sum_{j=N}^n \hat{B}^{(j)}_{\tilde{\phi}_q,z,\Gamma}(w) \le M^*_{n-1}(\tilde{\phi}_q)\sum_{j=N}^n e^{-\theta qj}\\
&\le (1-e^{-q\theta})^{-1}e^{-\theta qN}M^*_{n-1}(\tilde{\phi}_q).
\end{aligned}
$$

Now we turn to $\Sigma_2^{(n)}$. The calculation below is long but straightforward. Because of (4.15), (4.16), and (4.20) (see also the definition of $\hat{\mathcal{L}}_*$ (4.17)) we have

$$
\begin{aligned}
(4.25)\quad & \Sigma_2^{(n)}(w) \\
&\le \sum_{j=1}^{n}\sum_{x_1\in\Lambda_1^j(w)}\lambda^{-qj}\exp(S_j\tilde{\phi}_q(x_1))\sum_{i=0}^{n-j}\sum_{x_2\in\Lambda_2^i(x_1)}\lambda^{-qi}\exp(S_i\tilde{\phi}_q(x_2)) \\
&\quad\cdot\sum_{x_3\in\Lambda_3(x_2)}\lambda^{-q}e^{\tilde{\phi}_q(x_3)}M_{n-1}^*(\tilde{\phi}_q) \\
&\le M_{n-1}^*(\tilde{\phi}_q)\sum_{j=1}^{n}\sum_{x_1\in\Lambda_1^j(w)}\lambda^{-qj}\exp(S_j\tilde{\phi}_q(x_1)) \\
&\quad\cdot\sum_{i=0}^{n-j}\sum_{x_2\in\Lambda_2^i(x_1)}\lambda^{-qi}\exp(S_i\tilde{\phi}_q(x_2))\hat{\mathcal{L}}_{\tilde{\phi}_q}\mathbb{1}(x_2) \\
&\le M_{n-1}^*(\tilde{\phi}_q)\|\hat{\mathcal{L}}_{\tilde{\phi}_q}\|_\infty\sum_{j=1}^{n}\sum_{x_1\in\Lambda_1^j(w)}\lambda^{-qj}\exp(S_j\tilde{\phi}_q(x_1)) \\
&\quad\cdot\sum_{i=0}^{n-j}\sum_{x_2\in\Lambda_2^i(x_1)}\lambda^{-qi}\exp(S_i\tilde{\phi}_q(x_2)) \\
&\le M_{n-1}^*(\tilde{\phi}_q)\|\hat{\mathcal{L}}_{\tilde{\phi}_q}\|_\infty\sum_{j=1}^{n}\sum_{x_1\in\Lambda_1(w)}\lambda^{-qj}\exp(S_j\tilde{\phi}_q(x_1))\sum_{i=0}^{n-j}\tilde{\mathcal{L}}_*^i\mathbb{1}(x_1) \\
&\le M_{n-1}^*(\tilde{\phi}_q)\|\hat{\mathcal{L}}_{\tilde{\phi}_q}\|_\infty\sum_{j=1}^{n}\sum_{x_1\in\Lambda_1^j(w)}\lambda^{-qj}\exp(S_j\tilde{\phi}_q(x_1))\sum_{i=0}^{n-j}e^{-\theta qi} \\
&\le M_{n-1}^*(\tilde{\phi}_q)\|\hat{\mathcal{L}}_{\tilde{\phi}_q}\|_\infty\sum_{j=1}^{n}\sum_{x_1\in\Lambda_1^j(w)}\lambda^{-qj}\exp(S_j\tilde{\phi}_q(x_1))(1-e^{-q\theta})^{-1} \\
&\le M_{n-1}^*(\tilde{\phi}_q)\|\hat{\mathcal{L}}_{\tilde{\phi}_q}\|_\infty(1-e^{-q\theta})^{-1}\sum_{j=1}^{n}\hat{B}^{(j)}_{\tilde{\phi}_q,z,\Gamma}(w) \\
&= M_{n-1}^*(\tilde{\phi}_q)\|\hat{\mathcal{L}}_{\tilde{\phi}_q}\|_\infty(1-e^{-q\theta})^{-1}\sum_{j=N}^{n}\hat{B}^{(j)}_{\tilde{\phi}_q,z,\Gamma}(w) \\
&\le M_{n-1}^*(\tilde{\phi}_q)\|\hat{\mathcal{L}}_{\tilde{\phi}_q}\|_\infty(1-e^{-q\theta})^{-1}\sum_{j=N}^{\infty}e^{-\theta qj} \\
&\le (1-e^{-q\theta})^{-2}\|\hat{\mathcal{L}}_{\tilde{\phi}_q}\|_\infty e^{-\theta qN}M_{n-1}^*(\tilde{\phi}_q) \\
&\le \lambda^{-q}e^{\sup(\phi_q)}d^{qk}(1-e^{-q\theta})^{-2}e^{-\theta qN}M_{n-1}^*(\tilde{\phi}_q) \\
&\le (\lambda^{-1}e^{\sup(\phi)}d^k)^q(1-e^{-q\theta})^{-2}e^{-\theta qN}M_{n-1}^*(\tilde{\phi}_q).
\end{aligned}
$$

Combining (4.22), (4.24) and (4.25) together, and making use of (4.2), we get for all $z \in \mathbb{P}^k \setminus B(A_*, \eta)$, for a generic projective line Γ passing through z, and for every $w \in \Gamma \cap B(z, R(\eta))$,

$$
\begin{aligned}
(4.26)\qquad \hat{\mathcal{L}}^n_{\tilde\phi_q}(\mathbb{1})(w) &\le (1-e^{-q\theta})^{-1}e^{-\theta qN}M^*_{n-1}(\tilde\phi_q)\\
&\quad + (\lambda^{-1}e^{\sup(\phi)}d^k)^q(1-e^{-q\theta})^{-2}e^{-\theta qN}M^*_{n-1}(\tilde\phi_q) + \hat{G}^{(n)}_{\tilde\phi_q,z,\Gamma}(w)\\
&\le (1-e^{-q\theta})^{-1}(1+(1-e^{-q\theta})^{-1}\lambda^{-q}e^{q\sup(\phi)}d^{qk})e^{-\theta qN}M^*_{n-1}(\tilde\phi_q)\\
&\quad + \hat{G}^{(n)}_{\tilde\phi_q,z,\Gamma}(w)\\
&\le \hat{G}^{(n)}_{\tilde\phi_q,z,\Gamma}(w) + \tfrac14 M^*_{n-1}(\tilde\phi_q).
\end{aligned}
$$

4.5. Conclusion: upper estimates for the operator $\hat{\mathcal{L}}_\phi$ on the Julia set. In particular, we can write the above estimate for $w = z$, $z \in J \setminus B(A_*, \eta)$. Note that, for $z \in J$ we have $\hat{\mathcal{L}}^{qn}_\phi \mathbb{1}(z) = \hat{\mathcal{L}}\tilde\phi^n_q(z)$ and in estimate (4.26), $M^*_{n-1}(\tilde\phi_q)$ can be replaced by $M^*_{n-1}(\phi)$. We thus get

$$(4.27)\qquad \hat{\mathcal{L}}^{qn}_\phi \mathbb{1}(z) \le \hat{G}^{(n)}_{\tilde\phi_q,z,\Gamma}(z) + \tfrac14 M^*_{n-1}(\phi)$$

for all $n \ge 1$ and generic projective lines Γ passing through z. So, we have to estimate the good term $\hat{G}^{(n)}_{\tilde\phi_q,z,\Gamma}(z)$. The idea is to observe that the ratios of $\hat{G}$ evaluated at various points are uniformly bounded, and then to note that at some point the value of $\hat{G}$ is uniformly bounded above. Precisely, it follows from (4.14) that

$$(4.28)\qquad \tilde{C}_q^{-1} \le \hat{G}^{(n)}_{\tilde\phi_q,z,\Gamma}(w)/\hat{G}^{(n)}_{\tilde\phi_q,z,\Gamma}(z) \le \tilde{C}_q$$

for all $w \in \Gamma \cap B(z, R(\eta))$. In view of (3.10), for every $z \in J \setminus B(A_*, \eta)$ we get

$$1 = \int \hat{\mathcal{L}}^q_\phi \mathbb{1}\, dm_\phi \ge \int_{B(z,R_q)} \hat{\mathcal{L}}^q_\phi \mathbb{1}\, dm_\phi \ge \hat{C}_q^{-1}\hat{\mathcal{L}}^q_\phi \mathbb{1}(w)$$

with some $w \in J \cap B(z, R_q)$, where $\hat{C}_q^{-1} = \inf\{m_\phi(B(y, R_q)) : y \in J\}$ is positive in virtue of Proposition 3.6. Hence $\hat{\mathcal{L}}^q_\phi \mathbb{1}(w) \le \hat{C}_q$. Since the function $\hat{\mathcal{L}}^n_{\tilde\phi_q}\mathbb{1} : \mathbb{P}^k \to \mathbb{R}$ is continuous, we may assume that $w \ne z$, the line $\Gamma_{z,w}$ is generic and $\hat{\mathcal{L}}^n_{\tilde\phi_q}\mathbb{1}(w) \le 2\hat{C}_q$. So, $\hat{G}^{(n)}_{\tilde\phi_q,z,\Gamma_{z,w}}(w) \le 2\hat{C}_q$. Along with (4.28), this implies that

$$\hat{G}^{(n)}_{\tilde\phi_q,z,\Gamma_{z,w}}(z) \le \tilde{C}_q \hat{G}^{(n)}_{\tilde\phi_q,z,\Gamma_{z,w}}(w) \le C_q := 2\hat{C}_q\tilde{C}_q.$$

Inserting this into (4.27), we get

$$(4.29)\qquad \hat{\mathcal{L}}^{qn}_\phi \mathbb{1}(z) \le C_q + \tfrac14 M^*_{n-1}(\phi)$$

for all $n \ge 1$ and all $z \in J \setminus B(A_*, \eta)$. Since

$$M_l^*(\phi) = \max\{\|\hat{\mathcal{L}}_\phi^{qj} 1\!\!1|_{J\cap B^c(A_*,\Delta_q)}\|_\infty : 1 \le j \le l\}$$

and since $\eta \le \Delta_q$, formula (4.29) obviously yields

$$M_n^*(\phi) \le C_q + \tfrac{1}{4} M_{n-1}^*(\phi). \tag{4.30}$$

We can now prove, by induction, the following.

Lemma 4.2. *There exists a constant $Q_q^+ > 0$ such that $\|\hat{\mathcal{L}}_\phi^{qn}\|_\infty \le Q_q^+$ for all $n \ge 0$.*

Proof. Put $Q_q^* = \max\{\frac{4}{3}C_q, M_0^*(\phi)\}$. It easily follows by induction that $M_n^*(\phi) \le Q_q^*$ for every $n \ge 0$. The case $n = 0$ is obvious. So, suppose that $n \ge 1$ and $M_{n-1}^*(\phi) \le Q_q^*$. We then get, by (4.30),

$$M_n^*(\phi) \le C_q + \tfrac{1}{4} Q_q^* \le \tfrac{3}{4} Q_q^* + \tfrac{1}{4} Q_q^* = Q_q^*. \tag{4.31}$$

The inductive proof is complete. Now, we estimate the iterates of $\hat{\mathcal{L}}_\phi^q$ on the remaining part of J (i.e. on $J \cap B(A_*, \Delta_q)$). Fix $n \ge 0$ and $w \in B(A_*, \Delta_q)$. Keeping the same Λ's as defined in (4.23), it then follows directly from (4.20) that

$$\begin{aligned}
&\hat{\mathcal{L}}_{\tilde\phi_q}^n 1\!\!1(w) \\
&= \sum_{j=0}^{n} \sum_{x\in\Lambda_2^j(w)} \lambda^{-qj} \exp(S_j\tilde\phi_q(x)) \sum_{y\in\Lambda_3(w)} \lambda^{-q} e^{\tilde\phi_q(y)} \hat{\mathcal{L}}_q^{n-j-1} 1\!\!1(y) \\
&\le M_{n-1}^*(\tilde\phi_q) \sum_{j=0}^{n} \sum_{x\in\Lambda_2^j(w)} \lambda^{-qj} \exp(S_j\tilde\phi_q(x)) \max\Big\{1, \sum_{y\in\Lambda_3(x)} \lambda^{-q} e^{\tilde\phi_q(y)}\Big\} \\
&\le M_{n-1}^*(\tilde\phi_q) \max\{1, (\lambda^{-1} d^k e^{\sup(\phi)})^q\} \sum_{j=0}^{n} \hat{\mathcal{L}}_*^j 1\!\!1(w) \\
&\le M_{n-1}^*(\tilde\phi_q) \max\{1, (\lambda^{-1} d^k e^{\sup(\phi)})^q\} \sum_{j=0}^{n} e^{-\theta q j} \\
&\le \max\{1, (\lambda^{-1} d^k e^{\sup(\phi)})^q\}(1 - e^{-q\theta})^{-1} M_{n-1}^*(\tilde\phi_q).
\end{aligned} \tag{4.32}$$

Moreover, if $w \in B(A_*, \Delta_q) \cap J$, then $M_{n-1}^*(\tilde\phi_q)$ can be replaced by $M_{n-1}^*(\phi)$, and then along with (4.31), this estimate gives

$$\hat{\mathcal{L}}_\phi^{qn} 1\!\!1(w) \le \max\{1, (\lambda^{-1} d^k e^{\sup(\phi)})^q\}(1 - e^{-q\theta})^{-1} Q_q^*.$$

4.6. Estimates from below. Somewhat surprisingly, from this upper bound, actually from its proof (formula (4.32)), we also get a uniform lower bound on the iterates of the Perron–Frobenius operator acting on $C(J)$.

Lemma 4.3. *There exists a constant $Q_q^- > 0$ such that $\hat{\mathcal{L}}_\phi^{qn}\mathbb{1}(z) \ge Q_q^-$ for all $n \ge 0$ and $z \in J$.*

Proof. Fix $n \ge 0$. Take $z_n \in J \setminus B(A_*, \Delta_q)$ such that $\hat{\mathcal{L}}_\phi^{qn}\mathbb{1}(z_n) = M_n^*(\phi)$. It then follows from (4.27) that

$$M_n^*(\phi) \le \hat{G}^{(n)}_{\tilde{\phi}_q, z_n, \Gamma}(z_n) + \tfrac{1}{4}M_{n-1}^*(\phi) \le \hat{G}^{(n)}_{\tilde{\phi}_q, z_n, \Gamma}(z_n) + \tfrac{1}{4}M_n^*(\phi)$$

for a generic projective line Γ passing through z_n. Therefore,

$$\hat{G}^{(n)}_{\tilde{\phi}_q, z_n, \Gamma}(z_n) \ge \tfrac{3}{4}M_n^*(\phi). \tag{4.33}$$

But $\int \hat{\mathcal{L}}_\phi^{qn}\mathbb{1}\, dm_\phi = \int \mathbb{1}\, dm_\phi = 1$, and so there exists a point $y_n \in J$ such that $\hat{\mathcal{L}}_\phi^{qn}\mathbb{1}(y_n) \ge 1$. If $y_n \in J \setminus B(A_*, \Delta_q)$, then we get

$$M_n^*(\phi) \ge \hat{\mathcal{L}}_\phi^{qn}\mathbb{1}(y_n) \ge 1.$$

Otherwise, that is, if $y_n \in B(A_*, \Delta_q)$, it follows from (4.32) that

$$\begin{aligned} 1 \le \hat{\mathcal{L}}_\phi^{qn}\mathbb{1}(y_n) &\le (\lambda^{-1}d^k e^{\sup(\phi)})^q(1 - e^{-q\theta})^{-1}M_{n-1}^*(\tilde{\phi}_q) \\ &\le (\lambda^{-1}d^k e^{\sup(\phi)})^q(1 - e^{-q\theta})^{-1}M_n^*(\tilde{\phi}_q). \end{aligned}$$

Thus, $M_n^*(\phi) \ge (\lambda d^{-k}e^{-\sup(\phi)})^q(1 - e^{-q\theta})$. In either case,

$$M_n^*(\phi) \ge M := \min\{1, (\lambda d^{-k}e^{-\sup(\phi)})^q(1 - e^{-q\theta})\}.$$

Hence, by (4.33),

$$\hat{G}^{(n)}_{\tilde{\phi}_q, z_n, \Gamma}(z_n) \ge \tfrac{3}{4}M.$$

Thus, using (4.14) we obtain, for every $w \in B(z_n, R(\eta)) \cap \Gamma$,

$$\hat{\mathcal{L}}^n_{\tilde{\phi}_q}\mathbb{1}(w) \ge \hat{G}^{(n)}_{\tilde{\phi}_q, z_n, \Gamma}(w) \ge 3(4\hat{C}_q)^{-1}M.$$

This holds for a generic line Γ. Since $\hat{\mathcal{L}}^n_{\tilde{\phi}_q}\mathbb{1}$ is continuous, this inequality extends to all $w \in B(z_n, R(\eta))$. Since, by Proposition 1.3, the map $f^q : J \to J$ is topologically exact, there exists $l \ge 1$ such that $f^{ql}(B(z_n, R(\eta)) \cap J) = J$. Hence, for every $x \in J$ and $n \ge l$, there exists $\xi \in B(z_n, R_q) \cap J$ such that $f^{ql}(\xi) = x$. Therefore,

$$\hat{\mathcal{L}}_\phi^{qn}\mathbb{1}(x) \ge \lambda^{-ql}\exp(ql\inf(\phi))\hat{\mathcal{L}}_\phi^{q(n-l)}\mathbb{1}(\xi) \ge 3(4\hat{C}_q)^{-1}M\lambda^{-ql}\exp(ql\inf(\phi)). \ \blacksquare$$

As a consequence of Lemmas 4.2 and 4.3, we get the following.

Lemma 4.4. *There exist constants $Q_+ > 0$ and $Q_- > 0$ such that $\|\hat{\mathcal{L}}_\phi^n\|_\infty \le Q_+$ for all $n \ge 0$, and $\hat{\mathcal{L}}_\phi^n\mathbb{1}(z) \ge Q_-$ for all $n \ge 0$ and $z \in J$.*

5. Almost periodicity of the Perron–Frobenius operator. In this section we establish almost periodicity of our Perron–Frobenius operator.

5.1. A modification of the potential away from the Julia set. We shall need the Perron–Frobenius operator to act on $C(\mathbb{P}^k)$ and all its iterates, as operators on $C(\mathbb{P}^k)$, to be uniformly bounded above. We construct yet another extension of the potential ϕ, as a matter of fact, of ϕ_q. First, as an immediate consequence of (4.28) and Lemma 4.2, we get

$$\hat{G}^{(n)}_{\tilde{\phi}_q,z,\Gamma}(w) \le \tilde{C}_q Q_q^+ \tag{5.1}$$

for all $z \in J \setminus B(A_*, \eta)$, for generic projective lines Γ passing through z and for all $w \in B(z, R(\eta)) \cap \Gamma$.

LEMMA 5.1. *There exists a Hölder continuous function $\hat{\phi}_q : \mathbb{P}^k \to \mathbb{R}$ with the following properties:*

(a) *There exists a neighborhood $U \subset \mathbb{P}^k$ of J such that $\hat{\phi}_q|_U = \tilde{\phi}_q$. In particular, $\hat{\phi}_q|_J = \sum_{j=0}^{q-1} \phi \circ f^j$.*
(b) *$\hat{\phi}_q \le \tilde{\phi}_q$ throughout $\mathbb{P}^k$.*
(c) *$\hat{Q}_q := \sup_{n\ge 0}\{\|\hat{\mathcal{L}}^n_{\hat{\phi}_q} \mathbb{1}\|_\infty\} < \infty$.*

Proof. Assume without loss of generality that

$$R(\eta) \le \tfrac{1}{2}\Delta_q.$$

Set

$$B_q := B(J \setminus B(A_*, \Delta_q/2), R(\eta)), \qquad F_q := (\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q).$$

Since $f^q(J) = J$, and $F_q \cap J = \emptyset$, there exists $\varepsilon_q \in (0, R(\eta)/2)$ such that

$$\overline{B}(J, \varepsilon_q) \cap (F_q \cup f^{-q}(F_q)) = \emptyset.$$

Fix $t > 0$ so large that

$$e^{-t}(\lambda^{-2} d^{2k} e^{\sup(\phi)})^q (1 + e^{-q\theta}(1 - e^{-q\theta})^{-1}) \le 1/4 \tag{5.2}$$

and

$$-t \le \inf\{\tilde{\phi}_q(z) : z \in F_q \cup f^{-q}(F_q)\}.$$

So, by Lemma 4.1 there exists a Hölder continuous function $\hat{\phi}_q : \mathbb{P}^k \to \mathbb{R}$ such that $\hat{\phi}_q|_{\overline{B}(J,\varepsilon_q)} = \tilde{\phi}_q$, $\hat{\phi}_q$ restricted to $F_q \cup f^{-q}(F_q)$ is equal to $-t$ and $\sup(\hat{\phi}_q) \le \max\{\sup(\tilde{\phi}_q|_{\overline{B}(J,\varepsilon_q)}), -t\} \le \sup(\tilde{\phi}_q)$. Conditions (a) and (b) are satisfied by the very definition of $\hat{\phi}$ with $U = B(J, \varepsilon_q)$. We shall check (c). Put

$$M_n^{(1)} = \sup\{\hat{\mathcal{L}}^n_{\hat{\phi}_q} \mathbb{1}(z) : z \in B_q\}, \qquad M_n^{(2)} = \sup\{\hat{\mathcal{L}}^n_{\hat{\phi}_q} \mathbb{1}(z) : z \in F_q\},$$
$$M_n^*(\hat{\phi}_q) = \sup\{\hat{\mathcal{L}}^n_{\hat{\phi}_q} \mathbb{1}(z) : z \in \mathbb{P}^k \setminus B(A_*, \Delta_q)\} = \max\{M_n^{(1)}, M_n^{(2)}\}.$$

Fix now $z \in J \setminus B(A_*, \Delta_q/2)$, a projective line Γ passing through z, and a point $w \in \Gamma \cap B(z, R(\eta))$. Since $\hat{\phi}_q \le \tilde{\phi}_q$, it then follows from (4.26), applied with $\eta = \Delta_q/2$, that

$$\hat{\mathcal{L}}^n_{\hat{\phi}_q}\mathbb{1}(w) \le \hat{G}^{(n)}_{\tilde{\phi}_q,z,\Gamma}(w) + \tfrac{1}{4}M^*_{n-1}(\hat{\phi}_q).$$

Applying (5.1) we thus further get

$$\hat{\mathcal{L}}^n_{\hat{\phi}_q}\mathbb{1}(w) \le \tilde{C}_q Q^+_q + \tfrac{1}{4}M^*_{n-1}(\hat{\phi}_q).$$

This means that

$$M^{(1)}_n \le \tilde{C}_q Q^+_q + \tfrac{1}{4}M^*_{n-1}(\hat{\phi}_q). \tag{5.3}$$

We shall now estimate the iterates $\hat{\mathcal{L}}^n_{\hat{\phi}_q}\mathbb{1}$ in exactly the same manner as for $\hat{\mathcal{L}}^n_{\tilde{\phi}_q}\mathbb{1}$ in (4.25) and (4.32). The improvement is that now the estimate is valid everywhere in $\mathbb{P}^k$. So, again, for every $j \ge 0$ put

$$\Lambda_j = \bigcap_{i=0}^{j} f^{-qi}(B(A_*,\Delta_q)).$$

Using the definition of $\hat{\phi}_q$ and (4.20), and the same straightforward stratification as in (4.25), we deduce for all $w \in \mathbb{P}^k$ that

$$\begin{aligned}
(5.4)\quad & \hat{\mathcal{L}}^n_{\hat{\phi}_q}\mathbb{1}(w) \\
&= \sum_{x\in f^{-q}(w)\cap B^c(A_*,\Delta_q)} \lambda^{-q} e^{\hat{\phi}_q(x)} \hat{\mathcal{L}}^{n-1}_{\hat{\phi}_q}\mathbb{1}(x) \\
&\quad + \sum_{j=1}^{n} \sum_{y\in f^{-qj}(w)\cap\Lambda_j} \lambda^{-qj}\exp(S_j\hat{\phi}_q(y)) \sum_{x\in f^{-q}(y)\cap B^c(A_*,\Delta_q)} \lambda^{-q} e^{\hat{\phi}_q(x)} \hat{\mathcal{L}}^{n-(j+1)}_{\hat{\phi}_q}\mathbb{1}(x) \\
&\quad + \sum_{x\in f^{-n}(w)\cap\Lambda_{n-1}} \lambda^{-qn}\exp(S_n\hat{\phi}_q(x)) \\
&\le \sum_{x\in f^{-q}(w)\cap B^c(A_*,\Delta_q)} \lambda^{-q} e^{\sup(\hat{\phi}_q)} \hat{\mathcal{L}}^{n-1}_{\hat{\phi}_q}\mathbb{1}(x) \\
&\quad + M^*_{n-1}(\hat{\phi}_q) \sum_{j=1}^{n} \sum_{y\in f^{-qj}(w)\cap\Lambda_j} \lambda^{-qj}\exp(S_j\hat{\phi}_q(y)) \sum_{x\in f^{-q}(y)\cap B^c(A_*,\Delta_q)} \lambda^{-q} e^{\hat{\phi}_q(x)} \\
&\quad + \sum_{x\in f^{-q}(w)} \lambda^{-q} e^{\hat{\phi}_q(x)} \hat{\mathcal{L}}^{n-1}_{*}\mathbb{1}(x) \\
&\le (\lambda^{-1}d^k e^{\sup(\phi)})^q M^*_{n-1}(\hat{\phi}_q) \\
&\quad + M^*_{n-1}(\hat{\phi}_q) \sum_{j=1}^{n} \sum_{y\in f^{-qj}(w)\cap\Lambda_j} \lambda^{-qj}\exp(S_j\hat{\phi}_q(y))(\lambda^{-1}d^k e^{\sup(\phi)})^q \\
&\quad + (\lambda^{-1}d^k e^{\sup(\phi)})^q e^{-\theta q(n-1)}
\end{aligned}$$

$$
\begin{aligned}
&\le (\lambda^{-1}d^k e^{\sup(\phi)})^q M_{n-1}^*(\hat{\phi}_q) + M_{n-1}^*(\hat{\phi}_q)(\lambda^{-1}d^k e^{\sup(\phi)})^q \sum_{j=1}^{n} \hat{\mathcal{L}}_*^j \mathbb{1}(w) \\
&\quad + (\lambda^{-1}d^k e^{\sup(\phi)})^q e^{-\theta q(n-1)} \\
&\le (\lambda^{-1}d^k e^{\sup(\phi)})^q M_{n-1}^*(\hat{\phi}_q) + M_{n-1}^*(\hat{\phi}_q)(\lambda^{-1}d^k e^{\sup(\phi)})^q \sum_{j=1}^{n} e^{-\theta qj} \\
&\quad + (\lambda^{-1}d^k e^{\sup(\phi)})^q e^{-\theta q(n-1)} \\
&\le (\lambda^{-1}d^k e^{\sup(\phi)})^q M_{n-1}^*(\hat{\phi}_q) + M_{n-1}^*(\hat{\phi}_q)(\lambda^{-1}d^k e^{\sup(\phi)})^q e^{-q\theta}(1-e^{-q\theta})^{-1} \\
&\quad + (\lambda^{-1}d^k e^{\sup(\phi)})^q \\
&\le TM_{n-1}^*(\hat{\phi}_q) + (\lambda^{-1}d^k e^{\sup(\phi)})^q,
\end{aligned}
$$

where $T = (\lambda^{-1}d^k e^{\sup(\phi)})^q(1+e^{-q\theta}(1-e^{-q\theta})^{-1})$. Therefore, using the definition of $\hat{\phi}_q$ and the definition of t in (5.2), for every $w \in F_q$ we get

$$
\begin{aligned}
\hat{\mathcal{L}}_{\hat{\phi}_q}^n \mathbb{1}(w) &= \sum_{x\in f^{-q}(w)} \lambda^{-q} e^{\hat{\phi}_q(x)} \hat{\mathcal{L}}_{\hat{\phi}_q}^{n-1} \mathbb{1}(x) \\
&\le \sum_{x\in f^{-q}(w)} \lambda^{-q} e^{\hat{\phi}_q(x)} (TM_{n-1}^*(\hat{\phi}_q) + (\lambda^{-1}d^k e^{\sup(\phi)})^q) \\
&= (TM_{n-1}^*(\hat{\phi}_q) + (\lambda^{-1}d^k e^{\sup(\phi)})^q) \sum_{x\in f^{-q}(w)} \lambda^{-q} e^{-t} \\
&= (\lambda^{-1}d^k)^q e^{-t}(TM_{n-1}^*(\hat{\phi}_q) + (\lambda^{-1}d^k e^{\sup(\phi)})^q) \le \tfrac{1}{4} M_{n-1}^*(\hat{\phi}_q) + \tfrac{1}{4}.
\end{aligned}
$$

Thus,

$$M_n^{(2)} \le \tfrac{1}{4} + \tfrac{1}{4} M_{n-1}^*(\hat{\phi}_q).$$

Together with (5.3), this gives

$$M_n^*(\hat{\phi}_q) \le \max\{\tfrac{1}{4}, \tilde{C}_q Q_q^+\} + \tfrac{1}{4} M_{n-1}^*(\hat{\phi}_q).$$

Now we can prove in the same standard way as in the proof of Lemma 4.2 that $M^* := \sup_{n\ge 1} M_n^*(\hat{\phi}_q) < \infty$. So, applying (5.4) for every $n \ge 1$ we get

$$\|\hat{\mathcal{L}}_{\hat{\phi}_q}^n \mathbb{1}\|_\infty \le TM^* + (\lambda^{-1}d^k e^{\sup(\phi)})^q. \ \blacksquare$$

5.2. Bounded distortion. We will need the following strengthening of the distortion property (4.11).

Lemma 5.2. *For every $\varepsilon > 0$ and $\eta \in (0, \Delta_q]$ there exists $\delta_1 > 0$ such that*

$$|S_n\hat{\phi}_q(x) - S_n\hat{\phi}_q(y)| \le \varepsilon$$

for all $n \ge 1$, $z \in \mathbb{P}^k \backslash B(A_*, \eta)$, a generic projective line Γ passing through z, *all connected components $V \in W_{qn}(\eta, z, \Gamma)$ and all $x, y \in V$ with*

$$\rho(f^{qn}(x), f^{qn}(y)) \leq \delta_1.$$

Proof. Let $\hat{H} > 0$ be the Hölder constant of the Hölder continuous function $\hat{\phi}_q : \mathbb{P}^k \to \mathbb{R}$ produced in Lemma 5.1. Fix an integer $k \geq 1$ so large that $\hat{H} \sum_{j=k+1}^{\infty} \gamma^{\alpha qj/2} \leq \varepsilon/2$. Since all the functions $S_j \hat{\phi}_q$, $j = 1, \ldots, k$, are continuous (and there are only finitely many of them) it suffices to prove the lemma for all $n \geq k+1$. Take $\delta_2 > 0$ so small that $|\hat{\phi}_q(b) - \hat{\phi}_q(a)| \leq \varepsilon/2k$ whenever $\rho(a, b) \leq \delta_2$. By Lemma 2.6(b_n) there exists $\delta_1 > 0$ so small that for all $n \geq k+1$ and every $n - k \leq j \leq n-1$, we have $\rho(f^{qj}(x), f^{qj}(y)) \leq \delta_2$ whenever z, ξ, V, x, y are as in the hypothesis of the lemma. Applying Lemma 2.6(b_n) again, with n, z, ξ, V, x, y as in the hypothesis, we get

$$\begin{aligned}
&|S_n \hat{\phi}_q(x) - S_n \hat{\phi}_q(y)| \\
&\quad \leq \sum_{j=0}^{n-(k+1)} |\hat{\phi}_q(f^{qj}(x)) - \hat{\phi}_q(f^{qj}(y))| + \sum_{j=n-k}^{n-1} |\hat{\phi}_q(f^{qj}(x)) - \hat{\phi}_q(f^{qj}(y))| \\
&\quad \leq \sum_{j=0}^{n-(k+1)} \hat{H} \gamma^{q(n-j)\alpha/2} + \sum_{j=n-k}^{n-1} \frac{\varepsilon}{2k} \leq \varepsilon. \ \blacksquare
\end{aligned}$$

5.3. Compactness. Recall that a bounded linear operator $L : B \to B$ acting on a Banach space B is called *almost periodic* if for every $x \in B$, the closure $\overline{\{L^n(x) : n \geq 0\}}$ is compact in B.

PROPOSITION 5.3. *The Perron–Frobenius operator $\hat{\mathcal{L}}_{\hat{\phi}_q} : C(\mathbb{P}^k) \to C(\mathbb{P}^k)$ is almost periodic.*

The proof of Proposition 5.3 is technically involved. It is postponed to Section 9.

As a consequence of this proposition and the fact that every $g \in C(J)$ extends continuously to $\mathbb{P}^k$ with the same supremum norm, we obtain

PROPOSITION 5.4. *The Perron–Frobenius operator $\hat{\mathcal{L}}_{\phi_q} : C(J) \to C(J)$ is almost periodic, and consequently, so is the operator $\hat{\mathcal{L}}_{\phi} : C(J) \to C(J)$.*

It follows that the sequence $(n^{-1} \sum_{j=0}^{n-1} \hat{\mathcal{L}}_{\phi}^{j} \mathbb{1})_{n=0}^{\infty}$ is pre-compact, and it is easy to see that any of its limit points ρ_ϕ is a fixed point of the operator $\hat{\mathcal{L}}_\phi$ and its integral against the measure m_ϕ is equal to 1. Therefore:

PROPOSITION 5.5. *There exists a continuous function $\rho_\phi : J \to [0, \infty)$ with the following properties: $\hat{\mathcal{L}}_\phi \rho_\phi = \rho_\phi$, $\int \rho_\phi \, dm_\phi = 1$, $Q_- \leq \inf(\rho_\phi) \leq \sup(\rho_\phi) \leq Q_+$. In particular, $\mu_\phi = \rho_\phi m_\phi$ is an f-invariant Borel probability measure equivalent to m_ϕ.*

Given $H \geq 0$ and $0 \leq t < \kappa_f$ let $\mathcal{P}_H^t(f)$ denote the class of all Hölder continuous potentials $\phi : J \to \mathbb{R}$ such that $\|\phi\|_\alpha \leq H$ and $\sup(\phi) - \inf(\phi) \leq t$. Call all such potentials (H,t)-*admissible*. By the Arzelà–Ascoli theorem, $\mathcal{P}_H^t(f)$ is a compact subset of $C(J)$. The following is a refinement of Proposition 5.3.

Lemma 5.6. *For every $H \geq 0$, $0 \leq t < \kappa_f$, and every relatively compact set $K \subset C(\mathbb{P}^k)$, the set $\{\hat{\mathcal{L}}_{\hat{\phi}_q}^n g : \phi \in \mathcal{P}_H^t(f),\, g \in K,\, n \geq 0\}$ is relatively compact.*

As a consequence of this lemma we get the following.

Lemma 5.7. *For every $H \geq 0$, $0 \leq t < \kappa_f$, and every relatively compact set $K \subset C(J)$, the set $\{\hat{\mathcal{L}}_\phi^n g : \phi \in \mathcal{P}_H^t(f),\, g \in K,\, n \geq 0\}$ is relatively compact.*

5.4. Uniqueness of the equilibrium measures; preparatory step. Let $H : C(J) \to C(J)$ be the bounded linear isomorphism defined by

$$Hg = \rho_\phi g \tag{5.5}$$

and let $\mathcal{L}_0 : C(J) \to C(J)$ be given by $\mathcal{L}_0 = H^{-1} \circ \hat{\mathcal{L}}_\phi \circ H$, or pointwise

$$\mathcal{L}_0 g(x) = \frac{1}{\rho_\phi(x)} \hat{\mathcal{L}}_\phi(\rho_\phi g)(x). \tag{5.6}$$

In particular $\mathcal{L}_0$ and $\hat{\mathcal{L}}_\phi$ are conjugate and

$$\mathcal{L}_0 \mathbb{1} = \frac{1}{\rho_\phi} \hat{\mathcal{L}}_\phi \rho_\phi = \frac{\rho_\phi}{\rho_\phi} = \mathbb{1}. \tag{5.7}$$

The proof of the proposition below is rather standard. In a somewhat different context it can be found in [Wa1].

Proposition 5.8. *For every $g \in C(J)$, the sequence $(\mathcal{L}_0^n g)_{n=0}^\infty$ converges uniformly to $\int g \, d\mu_\phi$.*

For a bounded operator $A : B \to B$ on a Banach space B, let B_u be the closure of the linear span of the unit norm eigenvectors of A, and let $B_0 = \{g \in B : \lim_{n\to\infty} A^n g = 0\}$. The results below follow easily from Proposition 5.8.

Theorem 5.9. *For the operator $\mathcal{L}_0 : C(J) \to C(J)$ we have*

$$C(J)_u = \mathbb{C}\mathbb{1}, \qquad C(J)_0 = \Big\{ g \in C(J) : \int g \, d\mu_\phi = 0 \Big\},$$

and

$$C(J) = C(J)_u \oplus C(J)_0,$$

i.e. $C(J)$ splits into the direct sum of its two closed vector subspaces $C(J)_u$ and $C(J)_0$. In addition, if $g = g_u + g_0$ with $g_u \in C(J)_u$ and $g_0 \in C(J)_0$, *then*

$$g_u = \left(\int g\, d\mu_\phi\right)\mathbb{1}, \qquad g_0 = g - \left(\int g\, d\mu_\phi\right)\mathbb{1}$$

and the sequence $(\mathcal{L}_0^n g)_{n=0}^\infty$ *converges to* $\int g\, d\mu_\phi$ *uniformly on* J. *In particular,* μ_ϕ *is the only Borel probability measure on* J *satisfying* $\mathcal{L}_0^* \mu_\phi = \mu_\phi$ *and* $\mathbb{1}$ *is the only nonnegative fixed point* g *of the operator* $\mathcal{L}_0$ *such that* $\int g\, d\mu_\phi = 1$.

REMARK 5.10. Since the operator $\hat{\mathcal{L}}_\phi$ is conjugate to $\mathcal{L}_0$ via the isomorphism $H : C(J) \to C(J)$ defined by (5.5), an immediate consequence of Theorem 5.9 is that it also holds for the operator $\hat{\mathcal{L}}_\phi : C(J) \to C(J)$ with $C(J)_u = \mathbb{C}\rho_\phi$ and $C(J)_0 = \{g \in C(J) : \int g\, dm_\phi = 0\}$ and the sequence $(\hat{\mathcal{L}}_\phi^n g)_{n=0}^\infty$ converges to $(\int g\, dm_\phi)\rho_\phi$ uniformly on J.

We shall record two mixing properties resulting from Theorem 5.9 and Remark 5.10. The proof of the first one is the same as the proof of Corollary 37 in [DU]; the second property is also its straightforward consequence.

THEOREM 5.11. *The dynamical system* (J, f, μ_ϕ) *is metrically exact. This means that* $\bigcap_{n=0}^\infty f^{-n}(\mathcal{B})$ *consists only of sets of measure* 0 *and* 1, *where* $\mathcal{B}$ *denotes the* σ*-algebra of Borel subsets of* J. *In particular, Rokhlin's natural extension of* (J, f, μ_ϕ) *is a* K*-system and the dynamical system* (J, f, μ_ϕ) *is mixing of any order. In particular, it is ergodic.*

THEOREM 5.12 (Mixing). *If* $g \in C(J)$ *and* $h \in L^1(m_\phi) = \mathcal{L}^1(\mu_\phi)$, *then*

$$\lim_{n\to\infty} \int h \circ f^n \cdot g\, d\mu_\phi = \int h\, d\mu_\phi \int g\, d\mu_\phi.$$

6. Pressure versus eigenvalue. In this section, developing the idea of Gromov [Gr], we prove the equality of the topological pressure $\mathrm{P}(\phi)$ and $\log\lambda$. A major part of our argument is contained in the following.

THEOREM 6.1. *Let* $\psi : \mathbb{P}^k \to \mathbb{R}$ *be a continuous function. Assume that there exist* $\lambda > 0$ *and* $Q > 0$ *such that* $\sup(\psi) + (k-1)\log d \le \log\lambda$ *and*

$$\mathcal{L}_\psi^n \mathbb{1}(x) \le Q\lambda^n$$

for all integers $n \ge 1$. *Then* $\mathrm{P}(\psi) \le \log\lambda$.

Proof. We shall follow the idea of the proof of the inequality

$$\mathrm{h}_{\mathrm{top}}(f) \le \log(\deg_{\mathrm{top}} f) = k\log d$$

which is due to M. Gromov [Gr]. Thus, Gromov's inequality corresponds to the case $\phi = 0$. Let us consider the integral

$$\int_{\mathbb{P}^k} \exp(S_n\psi)(\omega + f^*\omega + \cdots + (f^{n-1})^*\omega)^k.$$

As in [Gr] we consider the embedding, a generalized graph,

$$f_n : \mathbb{P}^k \to X_n = (\mathbb{P}^k)^n$$

given by the formula

$$f_n(x) = (x, f(x), \ldots, f^{n-1}(x)).$$

Let $\pi_i : X_n \to X^{(i)} = \mathbb{P}^k$ be the projection onto the ith coordinate. We endow X_n with a Kähler form η by putting $\omega_i = \pi_i^* \omega$ and $\eta = \omega_1 + \cdots + \omega_n$. Now, let E be an $(n, 2\varepsilon)$-separated set in $\mathbb{P}^k$, i.e. $d_n(x,y) > 2\varepsilon$ for $x, y \in E$, $x \neq y$, where d_n is the metric in $\mathbb{P}^k$ given by $d_n(x,y) = \max_{0 \le i < n} d(f^i x, f^i y)$. Then

$$d_\eta(f_n(x), f_n(y)) = \Big(\sum_{i=0}^{n-1} d(f^i x, f^i y)^2\Big)^{1/2} \ge \max_{0 \le i < n} d(f^i x, f^i y) > 2\varepsilon. \tag{6.1}$$

Thus, the balls $B(f_n(x), \varepsilon)$, $x \in E$, (with respect to the metric d_η) are mutually disjoint. Now, we use Lelong's Theorem ([La], [McM]) for the form η and the embedded complex analytic variety $f_n(\mathbb{P}^k) \subset X_n$ to conclude that the η-volume of $f_n(\mathbb{P}^k) \cap B(p, \varepsilon)$, i.e.

$$\int\limits_{f_n(\mathbb{P}^k) \cap B(p,\varepsilon)} \eta^k$$

is bounded below by a constant c_ε, depending on ε only. Now, fix $\delta > 0$. Since the function ψ is uniformly continuous, there exists $\varepsilon > 0$ such that if $d(x,y) < \varepsilon$ then $|\psi(x) - \psi(y)| < \delta$, and consequently

$$e^{-\delta} < e^{\psi(x)}/e^{\psi(y)} < e^{\delta}.$$

Let E be an $(n, 2\varepsilon)$-separated set. Then we can write

$$\begin{aligned}
\sum_{x \in E} \exp(S_n \psi(x)) &\le e^{\delta n} \sum_{x \in E} \inf_{B_{d_n}(x,\varepsilon)} \{\exp(S_n \psi)\} \\
&\le e^{\delta n} \frac{1}{c_\varepsilon} \sum_{x \in E} \inf_{B_{d_n}(x,\varepsilon)} \{\exp(S_n \psi))\} \eta^k(B(f_n(x), \varepsilon)) \\
&= e^{\delta n} \frac{1}{c_\varepsilon} \sum_{x \in E} \inf_{B_{d_n}(x,\varepsilon)} \{\exp(S_n \psi)\} (f_n^* \eta^k)(f_n^{-1} B_{d_\eta}(f_n(x), \varepsilon)) \\
&\le e^{\delta n} \frac{1}{c_\varepsilon} \sum_{x \in E} \inf_{B_{d_n}(x,\varepsilon)} \{\exp(S_n \psi)\} (f_n^* \eta^k)(B_{d_n}(x, \varepsilon)) \\
&\le e^{\delta n} \frac{1}{c_\varepsilon} \int\limits_{\mathbb{P}^k} \exp(S_n \psi(f_n^* \eta^k))
\end{aligned} \tag{6.2}$$

where we have used (6.1) in the second inequality. Since $f_n^*(\omega_i) = (f^i)^* \omega$, the last integral takes on the form

$$\int\limits_{\mathbb{P}^k} \exp(S_n \psi)(\omega + f^* \omega + \cdots + (f^{(n-1)})^* \omega)^k. \tag{6.3}$$

We shall estimate this integral from above. First, notice that if, instead of the above integral, we had $\int_{\mathbb{P}^k} (f^{(n)})^*\omega^k$, then the integral would transform immediately to $\int_{\mathbb{P}^k} \mathcal{L}_0^n(\mathbb{1})\omega^k$ since the operator f^* acts on measures on $\mathbb{P}^k$ as a conjugate to the operator

$$f_*g(x) = \sum_{y\in f^{-1}(x)} g(y). \tag{6.4}$$

In our case, we have to write the integral (6.3) as a sum of integrals, and then to use the above observation:

$$\begin{aligned}\int_{\mathbb{P}^k} &\exp(S_n\psi)(\omega + f^*\omega + \cdots + (f^{(n-1)})^*\omega)^k \\ &= \int_{\mathbb{P}^k} \exp(S_n\psi)\Big(\sum_{0\le i_1,\dots,i_n\le n-1} (f^{i_i})^*\omega\wedge\cdots\wedge(f^{i_k})^*\omega\Big).\end{aligned}\tag{6.5}$$

Since all forms $(f^{i_i})^*\omega\wedge\cdots\wedge(f^{i_k})^*\omega$ are positive, we can treat them as measures and estimate

$$\begin{aligned}&\int_{\mathbb{P}^k} \exp(S_n\psi)(\omega + f^*\omega + \cdots + (f^{(n-1)})^*\omega)^k \\ &\le k!\int_{\mathbb{P}^k} \exp(S_n\psi) \sum_{0\le i_1\le\cdots\le i_k\le n-1} (f^{i_1})^*\omega\wedge\cdots\wedge(f^{i_k})^*\omega \\ &= k!\sum_{i=0}^{n-1}\int_{\mathbb{P}^k} \exp(S_n\psi) \sum_{0\le j_2\le\cdots\le j_k\le n-i-1} (f^{i})^*(\omega\wedge(f^{j_2})^*\omega\wedge\cdots\wedge(f^{j_k})^*\omega).\end{aligned}\tag{6.6}$$

Using the observation (6.4) again, one can rewrite the above sum as

$$k!\sum_{i=0}^{n-k}\int_{\mathbb{P}^k} \mathcal{L}_\psi^i\mathbb{1}(x)\exp(S_{n-i}\psi(x))\Big[\sum_{j_2\le\cdots\le j_k\le n-i} \omega\wedge(f^{j_2})^*\omega\wedge\cdots\wedge(f^{j_k})^*\omega\Big].\tag{6.7}$$

Recall that, according to our assumptions, we have

$$\mathcal{L}_\psi^i\mathbb{1}(x) \le Q\lambda^i$$

for all $x\in\mathbb{P}^k$. By our assumption on ψ we can estimate the above sum by

$$k!Q\sum_{i=0}^{n-k}\lambda^i\exp((n-i)\sup(\psi))\int_{\mathbb{P}^k} \sum_{j_2\le\cdots\le j_k\le n-i} \omega\wedge(f^{j_2})^*\omega\wedge\cdots\wedge(f^{j_k})^*\omega.\tag{6.8}$$

It remains to calculate the total mass of each measure $\omega\wedge(f^{j_2})^*\omega\wedge\cdots\wedge(f^{j_k})^*\omega$. Recall that $(f^j)^*\omega = d^j\omega$ in the de Rham cohomology group $H^2(\mathbb{P}^k)$. It is then straightforward to check that

$$\int_{\mathbb{P}^k} \omega\wedge(f^{j_2})^*\omega\wedge\cdots\wedge(f^{j_k})^*\omega = \int_{\mathbb{P}^k} d^{j_2+\cdots+j_k}\omega^k = d^{j_2+\cdots+j_k} \le d^{(k-1)(n-i)}$$

since $j_m \le n-i$ for all $2 \le m \le k$. Now, the number of all possible choices of $j_2, \dots, j_k$ can be estimated above by n^k. Finally, we can estimate (6.8) by

$$
\begin{aligned}
(6.9)\qquad c_k n^k \sum_{i=0}^{n-k} \lambda^i \exp((n-i)\sup(\psi)) d^{(k-1)(n-i)} \\
= c_k n^k \lambda^n \sum_{i=0}^{n-k} [\exp(-\log\lambda + \sup(\psi) + (k-1)\log d)]^{n-i}.
\end{aligned}
$$

The last sum is bounded by a constant depending on k but independent of n since $-\log\lambda + \sup(\psi) + (k-1)\log d < 0$. Therefore, we obtain the following. For every $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta)$ such that for every $\varepsilon < \varepsilon_0$ and every $(n, 2\varepsilon)$-separated set E we have

$$\sum_{x\in E} \exp(S_n\psi(x)) \le e^{\delta n} C(\varepsilon, k) n^k \lambda^n,$$

where $C(\varepsilon, k)$ is a constant depending on ε and k. This gives immediately $\mathrm{P}(\psi) \le \log\lambda + \delta$ and, as δ was arbitrarily small, $\mathrm{P}(\psi) \le \log\lambda$. ■

7. Existence and uniqueness of equilibrium states. Let J_{μ_ϕ} be the Jacobian of the map f with respect to the measure μ_ϕ.

Proposition 7.1. *The invariant measure $\mu_\phi = \rho_\phi m_\phi$ is an equilibrium state for the potential $\phi : J \to \mathbb{R}$. In addition,* $\mathrm{P}(\phi) = \log\lambda$.

Proof. Let $\hat{\phi}_q : \mathbb{P}^k \to \mathbb{R}$ be the extension of $\phi_q : J \to \mathbb{R}$ produced in Lemma 5.1. It follows from this lemma that the function $\hat{\phi}_q$ satisfies the assumptions of Theorem 6.1 for $f^q : \mathbb{P}^k \to \mathbb{P}^k$. Applying this theorem we get $\mathrm{P}(\phi) = (1/q)\mathrm{P}(S_q\phi) \le \mathrm{P}(\hat{\phi}_q) \le \log\lambda$. Therefore,

$$
\begin{aligned}
\mathrm{h}_{\mu_\phi} + \int \phi\, d\mu_\phi &\ge \int \log J_{\mu_\phi} + \int \phi\, d\mu_\phi \\
&= \int \log\rho_\phi\, d\mu_\phi - \int \log\rho_\phi \circ f\, d\mu_\phi + \log\lambda + \int \phi\, d\mu_\phi - \int \phi\, d\mu_\phi \\
&= \log\lambda \ge \mathrm{P}(\phi).
\end{aligned}
$$

Invoking the Variational Principle, i.e. formula (1.1), finishes the proof. ■

In order to demonstrate the uniqueness of equilibrium states (Theorem 7.6) we use an appropriate version of differentiability of topological pressure. The proof is based on Lemma 5.7 and two facts formulated below. It goes along the general scheme presented in [PU]. Therefore, only the steps of the proof are indicated below; the detailed proofs are omitted.

Proposition 7.2. *For every $H \ge 0$ and every $0 \le t < \kappa_f$ the function*

$$C(J) \supset \mathcal{P}_H^t(f) \ni \phi \mapsto \rho_\phi \in C(J)$$

is continuous with respect to the topology of uniform convergence on $C(J)$.

PROPOSITION 7.3. *For every $H \geq 0$ and every $0 \leq t < \kappa_f$ the function*

$$C(J) \supset \mathcal{P}_H^t(f) \ni \phi \mapsto m_\phi$$

is continuous with respect to the weak topology on the space of Borel probability measures on J.

LEMMA 7.4. *Fix $H \geq 0$ and $0 \leq t < \kappa_f$. If $\phi \in \mathcal{P}_H^t(f)$ and $g \in C(J)$, then*

$$\lim_{n\to\infty} \left\| \frac{\frac{1}{n}\hat{\mathcal{L}}_{\phi_q}^n(S_n g)}{\hat{\mathcal{L}}_{\phi_q}^n \mathbb{1}} - \left(\int g\, d\mu_\phi\right)\mathbb{1} \right\|_\infty = 0.$$

PROPOSITION 7.5. *Suppose that $\phi : J \to \mathbb{R}$ is a Hölder continuous potential with $\sup(\phi) - \inf(\phi) < \kappa_f$ and that $g : J \to \mathbb{R}$ is a Hölder continuous function. Then the function $\mathbb{R} \ni t \mapsto \mathrm{P}(\phi + tg) \in \mathbb{R}$ is differentiable on a sufficiently small open neighborhood of zero and*

$$\frac{d}{dt}\mathrm{P}(\phi + tg) = \int g\, d\mu_{\phi+tg}.$$

We are now ready for the main result of paper:

THEOREM 7.6. *If $f : \mathbb{P}^k \to \mathbb{P}^k$ is a regular holomorphic endomorphism of $\mathbb{P}^k$, and $\phi : J \to \mathbb{R}$ is a Hölder continuous potential satisfying $\sup(\phi) - \inf(\phi) < \kappa_f$, then there exists exactly one equilibrium state for ϕ. This equilibrium state is equal to $\mu_\phi = \rho_\phi m_\phi$ and it is metrically exact.*

Proof. The fact that $\mu_\phi = \rho_\phi m_\phi$ is an equilibrium state for ϕ was established in Proposition 7.1. Its uniqueness follows directly from Proposition 7.5 and Corollary 2.6.7 from [PU]. Metrical exactness of the dynamical system (f, μ_ϕ) coincides with Theorem 5.11. ∎

8. Local degree. Our main result in this section, Proposition 2.4, may be understood as a complement (a uniform version) of Favre's result, Theorem 2.5 from [F1] in the contex of local degrees (comp. [D2]). Denote by $e(f^n, x)$ the local degree of f^n at x. The statement in [F1] implies, in our setting, that the limit $e_\infty(x) = \lim n^{-1} \log e(f^n, x)$ exists everywhere. If the limit is nonzero then the trajectory of x falls eventually into a periodic irreducible variety V contained in the critical set. This result is not sufficient for our purposes.

We provide a more detailed (but elementary) analysis to obtain a uniform bound of the frequency of visits of a trajectory to the critical set. It follows immediately from our Proposition 2.4 that $e_\infty(x) = 0$ unless x falls into a periodic variety $V \subset \mathrm{Crit}(f)$ (and in this sense it can be understood as a strengthening of Favre's Theorem 2.5 in this particular context). However, we do not prove anything about the existence and the value of the limit $e_\infty(x)$ along these exceptional trajectories.

Let $C := \mathrm{Crit}(f)$. Recall two definitions from Section 2.

DEFINITION 8.1. Given an integer $n \geq 1$ the *periodic critical set* A_n is the union of the orbits of all irreducible varieties that are contained in the critical set and are periodic under an iterate f^l with some $l \leq n$. In particular, the orbit of a critical periodic point of period $l \leq n$ is in the critical periodic set A_n.

DEFINITION 8.2. E_n^p is the set of all points $x \in \mathbb{P}^k$ for which there exists a nonnegative integer $i \leq n-1$ such that $f^i(x) \in A_p$.

Our main result in this section is the following, stated above as Proposition 2.4.

PROPOSITION 8.3. *For every $\beta > 0$ there exist $p = p(\beta)$ and $N = N(\beta)$ such that for every $n > N$ and for every $x \notin E_n^p$ we have*

$$\#\{j \leq n : f^j(x) \in C\} \leq n\beta.$$

For its proof we need the following three simple lemmas.

LEMMA 8.4. *Let $h : X \to X$ be an arbitrary map and let $F \subset X$ be an arbitrary subset of X. Fix $\alpha > 0$ and consider the union*

$$F_\alpha = \bigcup_{i \leq [1/\alpha]} F \cap h^{-i}(F).$$

Next, consider a trajectory $x, h(x), \ldots, h^{M-1}(x)$ of length $M > [1/\alpha]$. If

$$\#\{s < M : h^s(x) \in F_\alpha\} \leq \alpha M,$$

then

$$\#\{s < M : h^s(x) \in F\} \leq 3\alpha M.$$

Proof. Divide the trajectory $x, h(x), \ldots, h^{M-1}(x)$ into blocks, ending at consecutive points in the trajectory, which are in F, i.e.

$$B_1 = [x, h(x), \ldots, h^{s_1}(x)]$$

where s_1 is the smallest iterate of x which falls into F, and, inductively,

$$B_{m+1} = [h^{s_m+1}(x), \ldots, h^{s_{m+1}}(x)]$$

while the last block is of the form $[h^{s_r+1}(x), h^{M-1}(x)]$. Let us choose all blocks B_m, $m < r$, of length $\leq [1/\alpha] - 1$. Notice that then $h^{s_m}(x) \in F_\alpha$ since the distance $s_{m+1} - s_m$ is not larger than $[1/\alpha]$. Consequently, by our assumption, the number of such blocks is not larger than αM. Moreover, in the remaining blocks B_m, $m < r$, every appearance of an element of F is followed by at least $[1/\alpha]$ elements which are not in F. Consequently, the total number of elements of F in the trajectory can be bounded from above by $\alpha M + \alpha M + 1 = 2\alpha M + 1 < 3\alpha M$ (since $M > [1/\alpha]$). ∎

LEMMA 8.5. *Let* $h : \mathbb{P}^k \to \mathbb{P}^k$ *be a holomorphic map and let* $\mathcal{D} = \{D_1, \ldots, D_t\}$ *be a collection of irreducible varieties in* $\mathbb{P}^k$ *of the same codimension* p. *If, for some* i, $h^{-i}(D_s) \cap D_r$ *has an irreducible component of codimension* p, *then* $h^i(D_r) = D_s$.

Proof. Let V be a component of $h^{-i}(D_s) \cap D_r$ of the same codimension. Since $V \subset D_r$ and D_r is irreducible, we have $V = D_r$, and consequently $D_r \subset h^{-i}(D_s)$. Thus, $h^i(D_r) \subset D_s$. Next, since D_s is also irreducible and $\dim h(D_s) = \dim D_r$, we get $h^i(D_r) = D_s$. ∎

LEMMA 8.6. *Under the assumptions of Lemma* 8.5, *there exists an integer* $l \geq 1$ *such that, for* $H = h^l$, *if* $H^i(D_r) = D_s$ *for some* $i \in \mathbb{N}$ *and some* $D_r, D_s \in \mathcal{D}$, *then* $H(D_s) = D_s$.

Proof. We build a natural graph with vertices $D_1, \ldots, D_t$. We put an arrow from D_s to D_r if D_s is mapped onto D_r under some iterate h^i of h and, for all $1 \leq j < i$, the image $h^j(D_s)$ is not a variety in our family $\mathcal{D}$. Assign weight i to this arrow. To every maximal path in this graph, which is not eventually a loop, we assign a weight defined to be the product of the weights of all arrows forming this path. Let l be a multiple of the lengths of all simple loops, which, in addition, is larger than the weights of all maximal paths in the graph that do not contain loops. Then every variety D_s is mapped by $H = h^l$ either onto a variety which is not in $\mathcal{D}$, or onto a variety D_r which belongs to a loop, thus fixed by H. ∎

We now pass to the proof of Proposition 8.3. The proof is in two steps. First, we recursively construct appropriate families of irreducible varieties contained in the critical set. They are then used in the inductive proof of Proposition 8.3.

STEP I: *Construction of families of irreducible varieties* $\mathcal{D}_m^{(1,2,j_3,\ldots,j_m)}$ *and nonnegative integers* $l_1, \ldots, l_m$. Given $\beta > 0$, let $\beta_0 = \beta$, $\beta_1 = \beta/3$, and $\beta_m = \beta_{m-1}/(3k)$ for all $2 \leq m \leq k$. This choice of the sequence β_m will become clear in the second step of the proof. First, let us use Lemma 8.6 for the map f and for the collection $\mathcal{D}_1^{(1)}$ of irreducible components $C_1, \ldots, C_t$ of the critical set C. The superscript (1) stands here for the codimension 1 of all varieties in the family. We then replace the original map f by its iterate $g_1 = f^{l_1}$ such that the statement of Lemma 8.6 is satisfied for the family $\mathcal{D}_1^{(1)}$. Next we define $\mathcal{D}_2^{(1,2)}$ to consist of all irreducible components of the intersections $D_r \cap g_1^{-i}(D_s)$ of codimension 2 ($D_r, D_s \in \mathcal{D}_1^{(1)}$) where $i \leq [3/\beta]$. Notice that for $k = 2$ the varieties in $\mathcal{D}_2^{(1,2)}$ are just points, so our construction ends at this step.

For $k > 2$ we proceed as follows. Using Lemma 8.6 again, we find an iterate $g_2 = g_1^{l_2}$ such that the statement of this lemma is satisfied, i.e. for each

$D_r, D_s \in \mathcal{D}_2^{(1,2)}$, if $g_2^i(D_r) = D_s$ for some $i \geq 1$, then $g_2(D_s) = D_s$. Now, for every $3 \leq j_3 \leq k$ let $\mathcal{D}_3^{(1,2,j_3)}$ be the family of all irreducible components of all intersections $D_r \cap g_2^{-i}(D_s)$, $D_r, D_s \in \mathcal{D}_2$, that have codimension $3 \leq j_3 \leq k$ and $1 \leq i \leq [3/\beta_1]$. Let

$$\mathcal{D}_3 = \bigcup_{j_3=3}^{k} \mathcal{D}_3^{(1,2,j_3)}.$$

Again, notice that if $k = 3$ then only $j_3 = 3$ is possible and the procedure ends at this point.

For a general k, proceeding by induction, fix $m \geq 3$ and assume that for every $1 \leq j_m \leq k$ the family $\mathcal{D}_m^{(1,2,j_3,\ldots,j_m)}$ of irreducible varieties of codimension j_m has been defined. Also, assume that the map g_m has been defined (as an appropriate iterate of f). We assume by induction that g_m has the following property. If, for some $i \geq 1$, and $D_r, D_s \in \mathcal{D}_m^{(1,2,j_3,\ldots,j_m)}$, $g_m^i(D_r) = D_s$, then $g_m(D_s) = D_s$. Fix $1 \leq j_m \leq k$. If $j_m = k$ then the procedure ends at this point (note that the elements of $\mathcal{D}_m^{(1,2,j_3,\ldots,j_m)}$ are then just points). If $j_m < k$, we define the families $\mathcal{D}_{m+1}^{(1,2,j_3,\ldots,j_m,j_{m+1})}$ where $j_{m+1} > j_m$ as follows. The family $\mathcal{D}_{m+1}^{(1,2,j_3,\ldots,j_m,j_{m+1})}$ consists of all irreducible components of all intersections $D_r \cap g_m^{-i}(D_s)$, $i \leq [3/\beta_{m-1}]$, that have codimension j_{m+1}. Note that the range of admissible j_{m+1}'s is $\{j_m + 1, \ldots, k\}$ but some families $\mathcal{D}_{m+1}^{(1,2,j_3,\ldots,j_m,j_{m+1})}$ may be empty. Finally, for all families $\mathcal{D}_{m+1}^{(1,2,j_3,\ldots,j_{m+1})}$ defined in this way, we find, using Lemma 8.6, a common value l_{m+1} and an iterate $g_{m+1} = g_m^{l_{m+1}}$ of g_m such that if $D_s = g_{m+1}^i(D_r)$ for some $D_r, D_s \in \mathcal{D}_{m+1}^{(1,2,j_3,\ldots,j_{m+1})}$, then $g_{m+1}(D_s) = D_s$.

Step II. Fix $N = N_0 = l_1 \ldots l_k N_k$, where $N_k > N_k^0$ and $N_k^0 > [1/\beta_k]$ so that the statement of Lemma 8.4 is satisfied for all $M \geq N_k^0$ and all $\alpha \geq \beta_k$. Let $N_1 = l_2 \ldots l_k N_k, \ldots, N_m = l_{m+1} \ldots l_k N_k$ (so that $N_{m-1} = l_m N_m$). Take now an arbitrary point $x \in \mathbb{P}^k$ and assume that the trajectory of x up to $f^N(x)$ visits the critical set with a frequency larger than β:

$$\#\{n < N : f^n x \in C\} > \beta N.$$

Since $N = l_1 N_1$, the whole trajectory $x, f(x), \ldots, f^{N-1}(x)$ can be split in a natural way into l_1 trajectories of the points $f^i(x)$, $i \leq l_1$, under g_1:

$$f^i(x), g_1(f^i(x)), \ldots, g_1^{N_1-1}(f^i(x)),$$

and it is evident that there exists a point $\tilde{x} = f^j(x)$, $j < l_1$, such that

$$\#\{n < N_1 : g_1^n(\tilde{x}) \in C\} > \beta N_1.$$

Let us consider two cases. Either

(1) there exist $n < N_1$ and $i \le [3/\beta]$ with $n+i < N_1$ such that $g_1^n(\tilde{x}) \in C_r$, $g_1^{n+i}(\tilde{x}) \in C_s$ and the component of $C_r \cap g_1^{-i}(C_s)$ containing $\tilde{x}$ has codimension 1, or
(2) if $g_1^n(\tilde{x}) \in C_r$ and $g^{n+i}(\tilde{x}) \in C_s$ for some $n \le N_1$ and $i \le [3/\beta]$ with $n+i < N_1$ then the component of $C_r \cap g_1^{-i}(C_s)$ containing $g_1^n(\tilde{x})$ has codimension 2.

If case (1), we conclude that the point $g_1^{n+i}(\tilde{x})$, where $n+i < N_1, i \le [3/\beta]$, lands in a critical variety V which is fixed by g_1 thus periodic for f with period l_1. This implies that $f^m(x) \in V$, where $m \le N$, $f^{l_1}(V) = V$, and $x \in E_N^p$. The proof is then finished.

In case (2) we proceed as follows. Using Lemma 8.4 for $\alpha = \beta/3$, we conclude that

$$\#\Big\{0 \le n \le N_1 : g_1^n(\tilde{x}) \in \bigcup_{i \le [3/\beta]} (C \cap g^{-i}C)\Big\} > (\beta/3)N_1.$$

This means that the trajectory $g_1(\tilde{x}), g_1^2(\tilde{x}), \ldots, g_1^{N_1}(\tilde{x})$ of $\tilde{x}$ under g_1 visits the varieties D_r from the family $\mathcal{D}_2^{(1,2)}$ with frequency larger than $\beta/3$. This property is referred to as $M_1(\tilde{x})$ and reads as follows:

$$\#\Big\{0 \le n \le N_1 : g_1^n(\tilde{x}) \in \bigcup_{D_r \in \mathcal{D}_2^{(1,2)}} D_r\Big\} > (\beta/3)N_1 = \beta_1 N_1. \tag{8.1}$$

It is now easy to conclude the proof if $k = 2$. Indeed, the varieties in $\mathcal{D}^{(1,2)}$ are then just points, and we can proceed as follows. Let $R = R(\beta) = \#\mathcal{D}^{(1,2)}$. It is evident that if N_1 is large enough then there exists a point $z \in \mathcal{D}^{(1,2)}$ which is visited at least $\beta_1 N_1/R$ times. Thus, there are n, j with $n+j < N_1$ and $j \le R/\beta_1$ such that $g_1^n(x) = g^{n+j}(x) = z$, so $g^j(z) = z$ and z is a critical periodic point for f, with period $p = l_1 \cdot [1/\beta_1]$.

This concludes the proof if $k = 2$.

For an arbitrary k we use the following inductive procedure. Put $p = l_1 \ldots l_k$. Recall that $\beta_1 = \beta/3$. Let $m \le k$ and assume that for the point x the following property $M_{m-1}(x)$ holds:

$$\#\Big\{0 \le n < N_{m-1} : g_{m-1}^n(x) \in \bigcup_{D \in \mathcal{D}_m^{(1,2,j_3,\ldots,j_m)}} D\Big\} > \beta_{m-1} N_{m-1}$$

for some sequence $(1, 2, j_3, \ldots, j_m)$ which depends on x. Note that for $m = 2$ this is precisely formula (8.1). Let $\mathcal{D}_m$ be the union of all families of the form $\mathcal{D}_m^{(1,2,j_3,\ldots,j_m)}$. Recall that $g_m = g_{m-1}^{l_m}$ is the iterate of f such that (see Step 1) if for some $D_r, D_s \in \mathcal{D}_m$, $g_m^i(D_r) = D_s$ then $g_m(D_s) = D_s$. Since the trajectory

$$\{x, g_{m-1}(x), g_{m-1}^2(x), \ldots, g_{m-1}^{N_{m-1}}(x)\}$$

of the point x under g_{m-1} can be split into l_m trajectories of the points $g_{m-1}^j(x)$, $0 \le j \le l_m - 1$, under g_m, it is evident that there exists a point $\tilde{x} = g_{m-1}^j(x)$ with $j < l_m$ such that

$$\#\Big\{n \le N_m : g_m^n(\tilde{x}) \in \bigcup_{D \in \mathcal{D}_m^{(1,2,j_3,\ldots,j_m)}} D\Big\} \ge \beta_{m-1} N_m.$$

Recall that $N_{m-1} = l_m N_m$. Now, as in the case of $m = 1$, we consider two possibilities.

(Ind.1) There exist $n < N_m$, $i \le [3/\beta_{m-1}]$ and $D_r, D_s \in \mathcal{D}_m$ of the same codimension, say j_m, such that $g_m^n(\tilde{x}) \in D_r$, $g_m^{n+i}(\tilde{x}) \in D_s$, and the component of $D_r \cap g_m^{-i}(D_s)$ containing $\tilde{x}$ has codimension j_m.

(Ind.2) If $g_m^n(\tilde{x}) \in D_r$ and $g^{n+i}(\tilde{x}) \in D_s$ for some $n < N_m$ and $i \le [3/\beta_{m-1}]$, then the component of $D_r \cap g_m^{-i}(D_s)$ containing $g_m^n(\tilde{x})$ has codimension larger than j_m.

If (Ind.1) occurs then, by Lemmas 8.5 and 8.6, we get $g_m(D_s) = D_s$ and $g_m^{n+i}(\tilde{x})$ lands in an irreducible variety D_s which is fixed by g_m. In particular, this case always occurs when $j_m = k$, i.e. the varieties D_r, D_s are just points. Since $g_m^{n+i}(\tilde{x})$ lands in the variety D_s, fixed by g_m, we see that $g_{m-1}^{j+l_m(n+i)}(x) = g_m^{n+i}(\tilde{x})$ lands in the component D_s. Note that $j+l_m(n+i) < N_{m-1}$ and $g_{m-1}^{l_m}(D_s) = g_m(D_s) = D_s$. Thus, we conclude that for some $r < N_{m-1}$ the point $g_{m-1}^r(x)$ lands in the variety D_s which is contained in the critical set and which is periodic under g_{m-1} with period l_m. We are then done.

In the case (Ind.2) we proceed as follows. Using Lemma 8.4 for $\alpha = \beta_{m-1}/3$, we can write

$$\#\Big\{n \le N_m : g_m^n(\tilde{x}) \in \bigcup_{j \le [3/\beta_{m-1}]} \bigcup_{D_r, D_s \in \mathcal{D}_m^{(1,2,j_3,\ldots,j_m)}} (D_r \cap g_m^{-j} D_s)\Big\} \ge \frac{\beta_{m-1}}{3} N_m.$$

Recall that the family $\mathcal{D}_{m+1}^{(1,2,j_3,\ldots,j_m,j_{m+1})}$ consists of all intersections $D_r \cap g_m^{-j}(D_s)$, $1 \le j \le [3/\beta_{m-1}]$, $D_r, D_s \in \mathcal{D}_m^{(1,2,j_3,\ldots,j_m)}$, which have the same codimension j_{m+1},

$$\{D_r \cap g_m^{-i} D_s : D_r, D_s \in \mathcal{D}_m^{(1,2,j_3,\ldots,j_m)}\} = \bigcup_{j_{m+1}} \mathcal{D}_{m+1}^{(1,2,j_3,\ldots,j_m,j_{m+1})},$$

and since there are less than k possible j_{m+1}'s, we can choose one value of

j_{m+1} such that the following property $M_m(\tilde{x})$ holds:

$$\#\Big\{n \le N_m : g_m^n(\tilde{x}) \in \bigcup_{D_r \in \mathcal{D}_{m+1}^{(1,2,j_3,\ldots,j_{m+1})}} D_r\Big\} \ge \frac{\beta_{m-1}}{3k} N_m = \beta_m N_m.$$

Therefore, we have checked the following. If the condition $M_{m-1}(x)$ is satisfied for x then either there exists $r < N_{m-1}$ so that $g_{m-1}^r(x)$ falls into a variety which is contained in C and is periodic under g_{m-1} with period l_m, or else, the condition $M_m(\tilde{x})$ is satisfied for some $\tilde{x} = f^j(x)$, $j \le l_m$, and some family $\mathcal{D}_m^{(1,2,j_3,\ldots,j_{m+1})}$. This ends the inductive step.

Now, take an arbitrary point $x \in \mathbb{P}^k$ and consider the initial block of its trajectory $x, f(x), \ldots, f^N(x)$, where $N = pN_k$, $N_k \ge N_k^0$. Let us turn again to the beginning of Step II. Since case (1) ends the proof, we consider case (2). This leads to the condition $M_1(\tilde{x})$ (see (8.1)). We then apply the above inductive procedure, starting from the point $\tilde{x}$ (renamed x again) and $m = 1$. If, at some step, (Ind.1) occurs then the induction stops. It is evident that the number of inductive steps is at most k. Assume that the procedure ends for some $m = m_0 \le k$. At each step of the induction the starting point has been modified (with x replaced by $\tilde{x}$). Let us denote by $\hat{x}$ the resulting starting point, at the final step of the induction. It then follows that for some $r < N_{m_0-1}$ the point $g_{m_0-1}^r(\hat{x})$ lands in a variety D_s which is contained in the critical set and which is periodic under g_{m_0-1} with period l_{m_0}, thus periodic under f with period $l_{m_0} l_{m_0-1} \ldots l_1 \le p$. Observe that $\hat{x}$ is in the trajectory of x and in fact it is easy to see that $\hat{x} = f^t(x)$ for some $t < l_1 + l_1 l_2 + \cdots + l_1 \ldots l_{m_0-1}$. Since $g_{m_0-1} = f^{l_1 \ldots l_{m_0-1}}$, we get $f^t(x) \in D_s$ for some $t < N$. Finally, take an arbitrary $n > l_1 \ldots l_k N_k^0 = pN_k^0$. Then $n = pM + r$ for some $M \ge N_k^0$ and $0 \le r < p$. It is evident that if $\#\{i \le n : f^i(x) \in C\} > 2n\beta$ then $\#\{i \le pM : f^i(x) \in C\} > n\beta$ if M is large enough. Thus, the statement follows from the proven part for $N = pM$. ∎

9. Proof of almost periodicity

Proof of Proposition 5.3. Fix $\varepsilon > 0$. Recall that N was introduced in Lemma 2.10. Recall also that until now, we kept N fixed and we only required that $N > N_\beta$ (see the beginning of Section 4). However, all the estimates obtained until now work for an arbitrary $N > N_\beta$, with the same constants. The only value which does depend on N is the radius $R(\eta)$. So, now we assume in addition that $N \ge 1$ is so large that

$$(9.1) \qquad 2(1 - e^{-q\theta})^{-2}(1 - e^{-q\theta} + (\lambda^{-1} e^{\sup(\phi)} d^k)^q) e^{-\theta N q} < \varepsilon.$$

Fix $g \in C(\mathbb{P}^k)$. We use the same decomposition as in (4.22) for g instead of $\mathbb{1}$. We keep the notation of (4.23). For every $n \ge N$, every $z \in \mathbb{P}^k \setminus B(A_*, \varDelta_q)$, for a projective line Γ passing through z, and for $w \in \Gamma \cap B(z, R(\eta))$, set

$$\Sigma_1^{(n)}(g)(w) = \sum_{j=N}^{n} \sum_{x \in \Lambda_1^j(w)} \lambda^{-qj} \exp(S_j \hat{\phi}_q(x)) \hat{\mathcal{L}}_q^{n-j}(g)(x),$$

$$\Sigma_2^{(n)}(g)(w) = \sum_{j=N}^{n} \sum_{x_1 \in \Lambda^j(w)} \lambda^{-qj} \exp(S_j \hat{\phi}_q(x_1)) \sum_{i=0}^{n-j} \sum_{x_2 \in \Lambda_2^i(x_1)} \lambda^{-qi} \exp(S_i \hat{\phi}_q(x_2)) \cdot \sum_{x_3 \in \Lambda_3(x_2)} \lambda^{-q} e^{\hat{\phi}_q(x_3)} \hat{\mathcal{L}}_q^{n-(j+i+1)}(g)(x_3).$$

Note that, formally, the sum should start from $j = 1$, but as $B_j = \emptyset$ for $j < N$, the first N summands are equal to 0. Formulas (4.24) and (4.25) obviously imply

$$(9.2)\quad |\Sigma_1^{(n)}(g)(w)| \le \Sigma_1^{(n)}(w)\|g\|_\infty \le (1 - e^{-q\theta})^{-1} e^{-\theta N q} M_{n-1}^*(\tilde{\phi}_q)\|g\|_\infty,$$

$$(9.3)\quad |\Sigma_2^{(n)}(g)(w)| \le \Sigma_2^{(n)}(w)\|g\|_\infty \le (\lambda^{-1} e^{\sup(\phi)} d^k)^q (1 - e^{-q\theta})^{-2} e^{-\theta N q} M_{n-1}^*(\tilde{\phi}_q)\|g\|_\infty.$$

Now, if $w \in \Gamma \cap B(z, R(\eta))$ then using these two estimates and (9.1) we get, for every $n \ge 0$,

$$\begin{aligned}(9.4)\quad & |\hat{\mathcal{L}}_{\hat{\phi}_q}^n g(w) - \hat{\mathcal{L}}_{\hat{\phi}_q}^n g(z)| \\
&= |(\Sigma_1^{(n)}(g)(w) - \Sigma_1^{(n)}(g)(z)) + (\Sigma_2^{(n)}(g)(w) - \Sigma_2^{(n)}(g)(z)) \\
&\quad + (G_{\hat{\phi}_q,z,\Gamma}^{(n)}(g)(w) - G_{\hat{\phi}_q,z,\Gamma}^{(n)}(g)(z))| \\
&\le |\Sigma_1^{(n)}(g)(w)| + |\Sigma_1^{(n)}(g)(z)| + |\Sigma_2^{(n)}(g)(w)| + |\Sigma_2^{(n)}(g)(z)| \\
&\quad + |\hat{G}_{\hat{\phi}_q,z,\Gamma}^{(n)}(g)(w) - \hat{G}_{\hat{\phi}_q,z,\Gamma}^{(n)}(g)(z)| \\
&\le 2(1 - e^{-q\theta})^{-2}(1 - e^{-q\theta} + (\lambda^{-1} e^{\sup(\phi)} d^k)^q) \hat{Q}_q e^{-\theta N q}\|g\|_\infty \\
&\quad + |\hat{G}_{\hat{\phi}_q,z,\Gamma}^{(n)}(g)(w) - \hat{G}_{\hat{\phi}_q,z,\Gamma}^{(n)}(g)(z)| \\
&\le \hat{Q}_q \|g\|_\infty \varepsilon + |\hat{G}_{\hat{\phi}_q,z,\Gamma}^{(n)}(g)(w) - \hat{G}_{\hat{\phi}_q,z,\Gamma}^{(n)}(g)(z)|.\end{aligned}$$

Our goal is to estimate $|\hat{G}_{\hat{\phi}_q,z,\Gamma}^{(n)}(g)(w) - \hat{G}_{\hat{\phi}_q,z,\Gamma}^{(n)}(g)(z)|$. Denote $\hat{G}_{\hat{\phi}_q,z,\Gamma}^{(n)}(g)$ by $\hat{G}_n(g)$. As before, set

$$W_j = W_{qj}(\eta, z, \Gamma).$$

Let V be an arbitrary connected component in W_n. Since the map $f^{qn}|_V : V \to \Gamma \cap B(z, R(\eta))$ is proper, its degree is well defined and is constant throughout V. By Lemma 2.6(d_n) this degree is bounded above by γ^{-qN}. Let $f_\Gamma^{-qn}(w)$ be the collection of all points x from $f^{-qn}(w) \cap \bigcup W_n$, each repeated according to the local degrees defined above. Let σ be an arbitrary bijection from $f_\Gamma^{-qn}(z)$ to $f_\Gamma^{-qn}(w)$ respecting all components $V \in W_n$. Put $\delta_3 = \min\{R(\eta), \delta_1, \delta_2\}$, where δ_1 comes from Lemma 5.2 and δ_2 from its proof. Assume that $\varepsilon < \log 2$.

Using Lemmas 5.2 and 5.1, we thus deduce that for all $n \geq 0$ and $z \in \mathbb{P}^k \setminus B(A_*, \Delta_q)$, for a generic projective line Γ passing through z, and for $w \in \Gamma \cap B(z, \delta_3)$,

$$
\begin{aligned}
&|\hat{G}_n(g)(w) - \hat{G}_n(g)(z)| \\
&= \Big| \sum_{x \in f_\Gamma^{-qn}(z) \cap \bigcup W_n} \Big(\lambda^{-qn} \exp\big(S_n \hat{\phi}_q(\sigma(x))\big) g(\sigma(x)) - \lambda^{-qn} \exp(S_n \hat{\phi}_q(x)) g(x) \Big) \Big| \\
&\leq \sum_{x \in f_\Gamma^{-qn}(z) \cap \bigcup W_n} \lambda^{-qn} \big| \exp\big(S_n \hat{\phi}(\sigma(x))\big) - \exp(S_n \hat{\phi}(x)) \big| \, |g(\sigma(x))| \\
&\quad + \sum_{x \in f_\Gamma^{-qn}(z) \cap \bigcup W_n} \lambda^{-qn} \exp(S_n \hat{\phi}(x)) |g(\sigma(x)) - g(x)| \\
&\leq \|g\|_\infty \sum_{x \in \hat{f}_\Gamma^{-qn}(z) \cap \bigcup W_n} \lambda^{-qn} \max\big\{ \exp\big(S_n \hat{\phi}(\sigma(x))\big), \exp(S_n \hat{\phi}(x)) \big\} \\
&\qquad\qquad\qquad\qquad \cdot |S_n \tilde{\phi}(\sigma(x)) - S_n \hat{\phi}(x)| \\
&\quad + \varepsilon \|g\|_\infty \sum_{x \in f_\Gamma^{-qn}(z) \cap \bigcup W_n} \lambda^{-qn} \exp(S_n \hat{\phi}_q(x)) \\
&\leq \|g\|_\infty \sum_{x \in \hat{f}_\Gamma^{-qn}(z) \cap \bigcup W_n} \lambda^{-qn} e^\varepsilon \exp(S_n \hat{\phi}(x)) |S_n \tilde{\phi}(\sigma(x)) - S_n \hat{\phi}(x)| \\
&\quad + \varepsilon \|g\|_\infty \sum_{x \in f_\Gamma^{-qn}(z) \cap \bigcup W_n} \lambda^{-qn} \exp(S_n \hat{\phi}_q(x)) \\
&\leq \|g\|_\infty \Big(\varepsilon e^\varepsilon \sum_{x \in f_\Gamma^{-qn}(z) \cap \bigcup W_n} \lambda^{-qn} \exp(S_n \hat{\phi}(x)) + \varepsilon \hat{\mathcal{L}}_{\hat{\phi}_q} \mathbb{1}(z) \Big) \\
&\leq \varepsilon \|g\|_\infty (e^\varepsilon \hat{\mathcal{L}}_{\hat{\phi}_q} \mathbb{1}(z) + \hat{Q}_q) \leq 3 \hat{Q}_q \|g\|_\infty \varepsilon.
\end{aligned}
$$

Combining this with (9.4) we get

$$
|\hat{\mathcal{L}}_{\hat{\phi}_q}^n g(w) - \hat{\mathcal{L}}_{\hat{\phi}_q}^n g(z)| \leq \hat{Q}_q \|g\|_\infty \varepsilon + 3 \hat{Q}_q \|g\|_\infty \varepsilon = 4 \hat{Q}_q \|g\|_\infty \varepsilon \tag{9.5}
$$

for a generic line Γ and $w \in \Gamma \cap B(z, \delta_3)$. By continuity,

$$
|\hat{\mathcal{L}}_{\hat{\phi}_q}^n g(w) - \hat{\mathcal{L}}_{\hat{\phi}_q}^n g(z)| \leq 4 \hat{Q}_q \|g\|_\infty \varepsilon \tag{9.6}
$$

for all $n \geq 0$, $z \in \mathbb{P}^k \setminus B(A_*, \Delta_q)$, and $w \in B(z, \delta_3)$. Thus, it remains to check the equicontinuity condition only in the Δ_q-neighborhood of A_*. To do this, fix $u \geq 1$ so large that

$$\max\{1, (\lambda^{-1} d^k e^{\sup(\phi)})^q\}(1 - e^{-q\theta})^{-1} e^{-\theta q u} \leq \varepsilon. \tag{9.7}$$

Let $z \in B(A_*, \Delta_q)$. Recall that we denoted $\Lambda_j = \bigcap_{l=0}^{j} f^{-ql}(B(A_*, \Delta_q))$ for $j \geq 0$. According to (4.32), we have

$$\begin{aligned}
(9.8) \quad & \hat{\mathcal{L}}^n_{\hat{\phi}_q} g(z) \\
&= \sum_{j=0}^{u-1} \sum_{x \in f^{-qj}(z) \cap \Lambda_j} \lambda^{-qj} \exp(S_j \hat{\phi}_q(x)) \sum_{y \in f^{-q}(x) \cap B^c(A_*, \Delta_q)} \lambda^{-q} e^{\hat{\phi}_q(y)} \hat{\mathcal{L}}^{n-j-1}_q g(y) \\
&\quad + \sum_{j=u}^{n-1} \sum_{x \in f^{-qj}(z) \cap \Lambda_j} \lambda^{-qj} \exp(S_j \hat{\phi}_q(x)) \sum_{y \in f^{-q}(x) \cap B^c(A_*, \Delta_q)} \lambda^{-q} e^{\hat{\phi}_q(y)} \hat{\mathcal{L}}^{n-j-1}_q g(y) \\
&\quad + \sum_{x \in f^{-qn}(z) \cap \Lambda_n} \lambda^{-qn} \exp(S_n \hat{\phi}_q(x)) = S + R_1 + R_2.
\end{aligned}$$

Now for every $\xi \in \mathbb{P}^k$ and every integer $n \geq 0$, let $\hat{f}^{-n}(\xi)$ denote the collection of all points in $f^{-n}(\xi)$ repeated according to the respective local degrees of the map $f^n : \mathbb{P}^k \to \mathbb{P}^k$. From (9.6) and Lemma 5.1(c) along with the standard continuity argument, we conclude that there exists $\delta_4 \in (0, \delta_3]$ so small that if $z, w \in \mathbb{P}^k$, for all $0 \leq u - 1$ some set E_j is contained in $\hat{f}^{-q(j+1)}(y) \cap B^c(A_*, \Delta_q)$ and $\tau : E_j \to \hat{f}^{-q(j+1)}(z)$ is an arbitrary function such that $\rho(\tau(y), y) < \delta_4$, then

$$\begin{aligned}
(9.9) \quad \Big| \sum_{j=0}^{u-1} \sum_{y \in E_j} & \big(\lambda^{-q(j+1)} \exp(S_{j+1} \hat{\phi}_q(y)) \hat{\mathcal{L}}^{n-(j+1)}_q g(y) \\
& - \lambda^{-q(j+1)} \exp(S_{j+1} \hat{\phi}_q(\tau(y))) \hat{\mathcal{L}}^{n-(j+1)}_q g(\tau(y))\big) \Big| \leq \hat{Q}_q \|g\|_\infty \varepsilon.
\end{aligned}$$

Now, take $\delta_5 \in (0, \delta_4)$ so small that for all $j = 0, 1, \ldots, u-1$ and all $a, b \in \mathbb{P}^k$ with $\rho(a, b) < \delta$ there exists a bijection $\tau^j_{a,b} : \hat{f}^{-qj}(a) \to \hat{f}^{-qj}(b)$ such that $\tau^j_{a,b} \circ \tau^j_{b,a} = \mathrm{Id}$ and $\rho(\tau^j_{a,b}(x), x) < \delta_4$ for all $x \in \hat{f}^{-qj}(a)$. If now $z \in B(A_*, \Delta_q)$ and $w \in B(z, \delta_5)$, then looking at (9.8) we can write

$$\begin{aligned}
(9.10) \quad & \hat{\mathcal{L}}^n_{\hat{\phi}_q} g(w) - \hat{\mathcal{L}}^n_{\hat{\phi}_q} g(z) \\
&= \sum_{(1)} \big(\lambda^{-qj} \exp(S_j \hat{\phi}_q(\tau^j_{z,w}(x))) \lambda^{-q} \exp(\hat{\phi}_q(\tau^1_{x, \tau^j_{z,w}(x)}(y))) \hat{\mathcal{L}}^{n-j-1}_q g(\tau^1_{x, \tau^j_{z,w}(x)}(y)) \\
&\quad - \lambda^{-qj} \exp(S_j \hat{\phi}_q(x)) \lambda^{-q} \exp(\hat{\phi}_q(y)) \hat{\mathcal{L}}^{n-j-1}_q g(y)\big) \\
&\quad + \sum_{(2)} \big(\lambda^{-qj} \exp(S_j \hat{\phi}_q(\tau^j_{z,w}(x))) \lambda^{-q} \exp(\hat{\phi}_q(y)) \hat{\mathcal{L}}^{n-j-1}_q g(y)
\end{aligned}$$

$$- \lambda^{-qj}\exp(S_j\hat\phi_q(x))\lambda^{-q}\exp(\hat\phi_q(\tau^1_{\tau^j_{z,w}(x),x}(y))\hat{\mathcal L}_q^{n-j-1}g(\tau^1_{\tau^j_{z,w}(x),x}(y)))\big)$$
$$+\sum_{(3)}\big(\lambda^{-qj}\exp(S_j\hat\phi_q(x))\lambda^{-q}\exp(\hat\phi_q(y))\hat{\mathcal L}_q^{n-j-1}g(y)$$
$$- \lambda^{-qj}\exp(S_j\hat\phi_q(\tau^j_{w,z}(x)))\lambda^{-q}\exp(\hat\phi_q(\tau^1_{x,\tau^j_{w,z}(x)}(y))\hat{\mathcal L}_q^{n-j-1}g(\tau^1_{x,\tau^j_{z,w}(x)}(y)))\big)$$
$$+ \Sigma^*_{2,s}(g)(w) - \Sigma^*_{2,s}(g)(z),$$

where $\sum_{(1)}$ is the sum over j, x, y such that $0\le j\le u-1$, $x\in\hat f^{-qj}(z)\cap\Lambda_j$, $y\in\hat f^{-q}(x)\cap B^c(A_*,\Delta_q)$; $\sum_{(2)}$ is over $0\le j\le u-1$, $x\in f^{-qj}(z)\cap\Lambda_j$, $y\in\hat f^{-q}(\tau^j_{z,w}(x)))\setminus\tau^1_{x,\tau^j_{z,w}(x)}(f^{-q}(x)\cap B^c(A_*,\Delta_q))$; and $\sum_{(3)}$ is over $0\le j\le u-1$, $x\in(\hat f^{-qj}(w)\cap\Lambda_j)\setminus\tau^j_{z,w}(\hat f^{-qj}(z)\cap\Lambda_j)$, $y\in f^{-q}(x)\cap B^c(A_*,\Delta_q)$. Here $\Sigma^*_{2,u}(g)(z)$ is a subsum of

$$\Sigma^*_{3,u}(g)(z)$$
$$:=\sum_{j=u}^{n-1}\sum_{x\in f^{-qj}(z)\cap\Lambda_j}\lambda^{-qj}\exp(S_j\hat\phi_q(x))\sum_{y\in f^{-q}(x)\cap B^c(A_*,\Delta_q)}\lambda^{-q}e^{\hat\phi_q(y)}\hat{\mathcal L}_q^{n-j-1}g(y)$$
$$+\sum_{x\in f^{-qn}(z)\cap\Lambda_n}\lambda^{-qn}\exp(S_n\hat\phi_q(x)),$$

and likewise, $\Sigma^*_{2,u}(g)(w)$ is a subsum of

$$\Sigma^*_{3,u}(g)(w)$$
$$:=\sum_{j=u}^{n-1}\sum_{x\in f^{-qj}(w)\cap\Lambda_j}\lambda^{-qj}\exp(S_j\hat\phi_q(x))\sum_{y\in f^{-q}(x)\cap B^c(A_*,\Delta_q)}\lambda^{-q}e^{\hat\phi_q(y)}\hat{\mathcal L}_q^{n-j-1}g(y)$$
$$+\sum_{x\in f^{-qn}(w)\cap\Lambda_n}\lambda^{-qn}\exp(S_n\hat\phi_q(x)).$$

The same calculation as in (9.2) and (9.3) gives that

$$|\Sigma^*_{2,u}(g)(w)|\le|\Sigma^*_{2,u}(|g|)(w)|\le|\Sigma^*_{3,u}(|g|)(w)|\le\hat Q_q\|g\|_\infty\varepsilon, \tag{9.11}$$

and similarly,

$$|\Sigma^*_{2,u}(g)(z)|\le|\Sigma^*_{2,u}(|g|)(z)|\le|\Sigma^*_{3,u}(|g|)(z)|\le\hat Q_q\|g\|_\infty\varepsilon. \tag{9.12}$$

In view of (9.9) each of the first three differences in (9.10) is bounded above by $\hat Q_q\|g\|_\infty\varepsilon$. Combining this with (9.11), (9.12), and applying (9.10), we thus get

$$|\hat{\mathcal L}^n_{\hat\phi_q}g(w)-\hat{\mathcal L}^n_{\hat\phi_q}g(z)|\le 5\hat Q_q\|g\|_\infty\varepsilon$$

for all $n\ge 0$, all $z\in B(A_*,\Delta_q)$, and all $w\in B(z,\delta_5)$. In turn, combining this with (9.6), we obtain

$$|\hat{\mathcal{L}}^n_{\hat{\phi}_q} g(w) - \hat{\mathcal{L}}^n_{\hat{\phi}_q} g(z)| \le 5\hat{Q}_q \|g\|_\infty \varepsilon \tag{9.13}$$

for all $n \ge 0$ and all $z, w \in \mathbb{P}^k$ with $\rho(z,w) < \delta_5$. Thus, the family $\big(\hat{\mathcal{L}}^n_{\hat{\phi}_q} g\big)_{n=0}^\infty$ is equicontinuous. Hence, invoking also Lemma 5.1(c), it follows from Arzelà–Ascoli's theorem that the family $(\hat{\mathcal{L}}^n_{\hat{\phi}_q} g)_{n=0}^\infty$ is relatively compact. ∎

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References

[BD] J.-Y. Briend et J. Duval, *Deux caractérisations de la mesure d'équilibre d'un endomorphisme de* $\mathrm{P}^k(\mathbf{C})$, Publ. Math. IHES 93 (2001), 145–159; erratum, ibid. 109 (2009), 295–296.

[DPU] M. Denker, F. Przytycki and M. Urbański, *On the transfer operator for rational functions on the Riemann sphere*, Ergodic Theory Dynam. Systems 16 (1996), 255–266.

[DU] M. Denker and M. Urbański, *Ergodic theory of equilibrium states for rational maps*, Nonlinearity 4 (1991), 103–134.

[D1] T.-C. Dinh, *Attracting current and equilibrium measure for attractors on* $\mathbb{P}^k$, J. Geom. Anal. 17 (2007), 227–244.

[D2] T.-C. Dinh, *Analytic multiplicative cocycles over holomorphic dynamical systems*, Complex Var. Elliptic Equations 54 (2009), 243–251.

[DNS] T.-C. Dinh, V.-A. Nguyen and N. Sibony, *Exponential estimates for plurisubharmonic functions*, J. Differential Geom. 84 (2010), 465–488.

[DS1] T.-C. Dinh and N. Sibony, *Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings*, in: Holomorphic Dynamical Systems, Lecture Notes in Math. 1998, Springer, 2010, 165–294.

[DS2] T.-C. Dinh and N. Sibony, *Equidistribution towards the Green current for holomorphic maps*, Ann. Sci. École Norm. Sup. 41 (2008), 307–336.

[DS3] T.-C. Dinh and N. Sibony, *Equidistribution speed for endomorphisms of projective spaces*, Math. Ann. 347 (2010), 613–626.

[Du2] C. Dupont, *Large entropy measures for endomorphisms of* $\mathbb{CP}^k$, Israel J. Math., to appear.

[F1] C. Favre, *Multiplicity of holomorphic functions*, Math. Ann. 316 (2000), 355–378.

[F2] C. Favre, *Dynamique des applications rationnelles*, PhD thesis, 2002.

[FS1] J. E. Fornæss and N. Sibony, *Complex dynamics in higher dimension I*, Astérisque 222 (1994), 201–231.

[FS2] J. E. Fornæss and N. Sibony, *Complex dynamics in higher dimension*, in: Complex Potential Theory (Montreal, PQ, 1993), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 439, Kluwer, 1994, 131–186.

[Gr] M. Gromov, *On the entropy of holomorphic maps*, preprint IHES.

[Gu] V. Guedj, *Propriétés ergodiques des applications rationnelles*, in: Panoramas et Synthèses 30, Soc. Math. France, 2010, 97–202.

[Ha] N. Haydn, *Convergence of the transfer operator for rational maps*, Ergodic Theory Dynam. Systems 19 (1999), 657-669.

[HP] J. H. Hubbard and P. Papadopol, *Superattractive fixed points in* $\mathbb{C}^n$, Indiana Univ. Math. J. 43 (1994), 321–365.

[J] M. Jonsson, *Dynamics of polynomial skew products on* $\mathbb{C}^2$, Math. Ann. 314 (1999), 403–447.

[La] S. Lang, *Introduction to Complex Hyperbolic Spaces*, Springer, 1987.

[McM] C. McMullen, *Algebra and Dynamics*, lecture notes, 2004.

[Ne] S. Newhouse, *Continuity properties of entropy*, Ann. of Math. 129 (1989), 215–235.

[Pr] F. Przytycki, *On the Perron–Frobenius–Ruelle operator for rational maps on the Riemann sphere and for Hölder continuous functions*, Bol. Soc. Brasil. Mat. 20 (1990), 95–125.

[PU] F. Przytycki and M. Urbański, *Conformal Fractals—Ergodic Theory Methods*, Cambridge Univ. Press, 2010.

[PUZ, I] F. Przytycki, M. Urbański and A. Zdunik, *Harmonic, Gibbs and Hausdorff measures on repellers for holomorphic maps I*, Ann. of Math. 130 (1989), 1–40.

[PUZ, II] F. Przytycki, M. Urbański and A. Zdunik, *Harmonic, Gibbs and Hausdorff measures on repellers for holomorphic maps II*, Studia Math. 97 (1991), 189–225.

[Ru1] D. Ruelle, *Statistical mechanics on a compact set with* $\mathbb{Z}^\nu$ *action satisfying expansiveness and specification*, Trans. Amer. Math. Soc. 185 (1973), 237–251.

[dT1] H. de Thélin, *Sur la construction de mesures selles*, Ann. Inst. Fourier (Grenoble) 56 (2006), 337–372.

[dT2] H. de Thélin, *Sur les exposants de Lyapounov des applications méromorphes*, Invent. Math. 172 (2008), 89–116.

[Wa1] P. Walters, *Invariant measures and equilibrium states for some mappings which expand distances*, Trans. Amer. Math. Soc. 236 (1978), 121–153.

[Wa2] P. Walters, *An Introduction to Ergodic Theory*, Springer, 1982.

[Yo] Y. Yomdin, *Volume growth and entropy*, Israel J. Math. 57 (1987), 285–300.

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