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Embeddings of $C(K)$ spaces into $C(S,X)$ spaces with distortion strictly less than 3

by

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Abstract. In the spirit of the classical Banach–Stone theorem, we prove that if K and S are intervals of ordinals and X is a Banach space having non-trivial cotype, then the existence of an isomorphism T from $C(K,X)$ onto $C(S,X)$ with distortion $\|T\|\,\|T^{-1}\|$ strictly less than 3 implies that some finite topological sum of K is homeomorphic to some finite topological sum of S. Moreover, if X^n contains no subspace isomorphic to X^{n+1} for every $n \in \mathbb{N}$, then K is homeomorphic to S. In other words, we obtain a vector-valued Banach–Stone theorem which is an extension of a Gordon theorem and at the same time an improvement of a Behrends and Cambern theorem. In order to prove this, we show that if there exists an embedding T of a $C(K)$ space into a $C(S,X)$ space, with distortion strictly less than 3, then the cardinality of the αth derivative of S is finite or greater than or equal to the cardinality of the αth derivative of K, for every ordinal α.

1. Introduction. We follow the standard notation and terminology of Banach space theory that can be found in [17]. Let S be a compact Hausdorff space and X a Banach space. We denote by $C(S,X)$ the Banach space of all X-valued continuous functions on S endowed with the supremum norm. If X is the scalar field, this space will be denoted by $C(S)$. Moreover, when S is an ordinal interval $[1,\alpha]$ this space will be denoted by $C(\alpha,X)$. Given Banach spaces X and Y, the Banach–Mazur distance between X and Y is given by $\inf_T\{\|T\|\,\|T^{-1}\|\}$, where T runs through all isomorphisms of X onto Y. If there is an isomorphism T of X onto Y with $\|T^{-1}\|\,\|T\| < \lambda$ for some $1 < \lambda < \infty$, we will write $X \overset{<\lambda}{\sim} Y$.

The source of our research is the classical Banach–Stone theorem which states that if $C(K)$ and $C(S)$ are isometrically isomorphic (for short, $C(K) = C(S)$), then K and S are homeomorphic (written $K \approx S$). This result was obtained by Banach [4] for compact metric spaces and extended by Stone

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[20] to compact Hausdorff spaces. Amir [3] and independently Cambern [6] generalized this theorem as follows.

THEOREM 1.1. *Let K and S be compact Hausdorff spaces. Then*

$$C(K) \overset{<2}{\sim} C(S) \;\Rightarrow\; K \approx S.$$

Moreover, Amir conjectured that the number 2 may be replaced by 3 in this theorem. Cohen [9] disproved this conjecture for the class of uncountable compact metric spaces. However, this conjecture is true in the class of countable compact metric spaces. Indeed, Gordon [15] proved:

THEOREM 1.2. *Let K and S be countable compact metric spaces. Then*

$$C(K) \overset{<3}{\sim} C(S) \;\Rightarrow\; K \approx S.$$

The Banach–Stone theorem has also been generalized to real vector-valued continuous functions. In this setting, the farthest-reaching result is the following one, due to Behrends and Cambern [5] (see also [16]).

THEOREM 1.3. *Suppose that X is a uniformly non-square Banach space. Then there exists $1 < \lambda < 2$ such that for any compact Hausdorff spaces K and S we have*

$$C(K,X) \overset{<\lambda}{\sim} C(S,X) \;\Rightarrow\; K \approx S.$$

Recall that a Banach space X is said to be *uniformly non-square* if and only if there is a $\delta > 0$ such that there do not exist x and y in the the unit ball of X for which

$$\left\|\tfrac{1}{2}(x+y)\right\| > 1-\delta \quad \text{and} \quad \left\|\tfrac{1}{2}(x-y)\right\| > 1-\delta.$$

In the present paper we are mainly interested in getting vector-valued Banach–Stone type theorems for $C(S,X)$ spaces with Banach–Mazur distances strictly less than 3. The motivation for this work is to look for some improvements of Theorems 1.2 and 1.3. The difficulty in obtaining some results in this direction can be summarized in the following fact:

REMARK 1.4. Let S be an arbitrary countable compact metric space. By the classical Mazurkiewicz and Sierpiński theorem [19, Theorem 8.6.10], S is homeomorphic to an ordinal interval $[1, \omega^\alpha m]$, where ω is the first infinite ordinal, α is countable and m is finite and different from zero. Consequently, if X is a Banach space satisfying $X \overset{<d}{\sim} X \oplus_\infty X$, then

$$C(S,X) = C(\omega^\alpha m, X) \overset{<d}{\sim} C(\omega^\alpha m, X \oplus_\infty X) = C(\omega^\alpha m2, X).$$

In particular, if $X = l_p$, then $d = 2^{1/p}$ is arbitrarily close to 1 when p is very large. However, according to [19, Proposition 8.6.9], S is not homeomorphic to $[1, \omega^\alpha 2m]$.

In order to state our theorems, recall that a Banach space $X \neq \{0\}$ is said to have *cotype* $2 \le q < \infty$ [10] if there is a constant $\kappa > 0$ such that no matter how we select finitely many vectors $v_1, \ldots, v_n$ from X,

$$(1.1)\qquad \Big(\sum_{i=1}^{n} \|v_i\|^q\Big)^{1/q} \le \kappa\Big(\int_0^1 \Big\|\sum_{i=1}^{n} r_i(t)v_i\Big\|^2\,dt\Big)^{1/2},$$

where $r_i : [0,1] \to \mathbb{R}$ denote the *Rademacher functions*, defined by setting

$$r_i(t) = \operatorname{sign}(\sin 2^i \pi t).$$

Moreover, a Banach space X is said to have *non-trivial cotype* if it has cotype q for some $2 \le q < \infty$.

Our vector-valued Banach–Stone type theorems are as follows. For a Banach space X and $n \in \mathbb{N}$ we denote by X^n the finite sum of n copies of X.

Theorem 1.5. *Suppose that X is a Banach space having non-trivial cotype such that X^n contains no subspace isomorphic to X^{n+1} for every $n \in \mathbb{N}$. Then for any ordinal intervals K and S we have*

$$C(K,X) \overset{<3}{\sim} C(S,X) \ \Rightarrow\ K \approx S.$$

Remark 1.6. Notice that each finite-dimensional space X satisfies the assumptions of Theorem 1.5. Thus, if X is the scalar field and K and S are countable compact metric spaces, then by the above mentioned Mazurkiewicz and Sierpiński theorem, we see that Theorem 1.5 is exactly Theorem 1.2. On the other hand, it is well known that every uniformly non-square space is a Banach space having non-trivial cotype [10, Theorem 14.1]. Thus, Theorem 1.5 shows that, in the case of certain Banach spaces X and compact spaces K and S, we can replace the number $1 < \lambda < 2$ in Theorem 1.3 by 3.

Moreover, in Remark 4.1 we will show that there exist $2^{\aleph_0}$ infinite-dimensional separable Banach spaces satisfying the hypotheses of Theorem 1.5. In contrast with Remark 1.4, each of these spaces contains a complemented copy of some l_p space.

Remark 1.7. Without the hypothesis on finite sums of the Banach space X in the statement of Theorem 1.5, we can deduce that some finite topological sum of K is homeomorphic to some finite topological sum of S. This follows immediately from the next theorem and the Cantor normal form of an ordinal (see [19, Proposition 8.6.5]).

Theorem 1.8. *Suppose that X is a Banach space having non-trivial cotype, α and β are ordinals and $1 \le m, n < \omega$. Then*

$$C(\omega^\alpha m, X) \overset{<3}{\sim} C(\omega^\beta n, X) \ \Rightarrow\ \alpha = \beta.$$

REMARK 1.9. Notice that Remark 1.4 implies that Theorem 1.8 cannot be improved to state that also $m = n$. However, we do not know whether the number 3 in the above theorem may be replaced by 4 at least for the case where X is the scalar field.

The techniques developed to prove Theorems 1.5 and 1.8 allow us to get a general result on stability of the cardinality of the αth derivative of the compact space S via $C(S, X)$ spaces with Banach–Mazur distances strictly less than 3. This result is a vector-valued extension of another theorem of Gordon's [15, The Main Theorem]. To state it we recall that the *derivative* of a topological space S is the space $S^{(1)}$ obtained by deleting from S its isolated points. The αth derivative $S^{(\alpha)}$ is defined recursively by setting $S^{(0)} = S$, $S^{(\alpha+1)} = (S^{(\alpha)})^{(1)}$ and $S^{(\beta)} = \bigcap_{\gamma<\beta} S^{(\gamma)}$ for a limit ordinal β. The space S is said to be *scattered* if $S^{(\alpha)} = \emptyset$ for some ordinal α. In this case, the minimal α such that $S^{(\alpha)} = \emptyset$ is called the *height* of S (denoted $\mathrm{ht}(S)$). As usual, by $-1 + \alpha$ we denote the difference between the ordinal α and 1. The cardinality of a set Γ will be denoted by $|\Gamma|$.

THEOREM 1.10. *Suppose that X is a Banach space having non-trivial cotype, and K and S are compact Hausdorff spaces. Then*

$$C(K, X) \overset{<3}{\sim} C(S, X) \ \Rightarrow\ |K^{(\alpha)}| = |S^{(\alpha)}|$$

for every ordinal α, different from $-1+\mathrm{ht}(S)$ in the case where S is scattered.

REMARK 1.11. Remark 1.4 also shows that we cannot remove the hypothesis on $\mathrm{ht}(S)$ in Theorem 1.10 even when $\mathrm{ht}(S) > 1$.

All the above results are direct consequences of our study of embeddings T of $C(K)$ spaces into $C(S, X)$ spaces with distortion $\|T\|\,\|T^{-1}\|$ less than 3, where X is a Banach space having non-trivial cotype. This study will be presented in the next section. In Section 3 we will prove Theorems 1.5, 1.8 and 1.10.

Finally observe that in this paper we deal only with $C(K, X)$ spaces for compact Hausdorff spaces K, but a somewhat related research has been done in the case where K are locally compact Hausdorff spaces, starting with a paper by Cambern [7] showing that if c denotes the space of complex-valued convergent sequences and c_0 the space of complex-valued sequences convergent to zero, then the Banach–Mazur distance between c and c_0 is 3. For further results in this direction see the recent paper [8].

2. On embbedings of $C(K)$ spaces into $C(S, X)$ spaces. The main aim of this section is to prove the following theorem.

THEOREM 2.1. *Let X be a Banach space having non-trivial cotype, and*

K and S compact Hausdorff spaces. Suppose that there exists an isomorphism T from $C(K)$ into $C(S,X)$ with $\|T\|\,\|T^{-1}\| < 3$. Then:

(a) *If S is scattered, then so is K and* $\mathrm{ht}(K) \le \mathrm{ht}(S)$.
(b) *For all ordinal α, $S^{(\alpha)}$ is finite or $|K^{(\alpha)}| \le |S^{(\alpha)}|$.*

Proof. Without loss of generality we may assume that $\|T^{-1}\| = 1$ and hence $\|T\| < 3$ and $\|f\| \le \|Tf\|$ for each $f \in C(S)$. Set $\epsilon = (1 - \|T\|/3)/2$ and put for $k \in K$,

$$F_k = \{f \in C(K) : 0 \le f \le 1 \text{ and } f(k) > \|T\|/3\},$$
$$\Lambda_k = \{s \in S : \|Tf(s)\| \ge \epsilon \text{ for every } f \in F_k\}.$$

In order to prove items (a) and (b) of the theorem we first prove three claims concerning the sets Λ_k.

Claim 1. *Λ_k is a non-empty closed set for every $k \in K$.*

Indeed, Λ_k is the intersection of closed sets, so it is closed. To see that is non-empty we check that it has the finite intersection property. Thus fix $f_1, \ldots, f_n \in F_k$ and take $g = \min_i f_i$. Then also $g \in F_k$ and $\|1+2g-2f_j\| \le 1$ for every j. Since

$$\|T(1+2g)\| \ge \|1+2g\| > 1 + 2\|T\|/3,$$

it follows that there is a point $s \in S$ such that

$$\|T(1+2g)(s)\| > 1 + 2\|T\|/3,$$

and we show that $\|Tf_j(s)\| \ge \epsilon$ for every j. Indeed, if this were false for some i, then

$$\begin{aligned}\|T\| &\ge \|T(1+2g-2f_i)\| \ge \|T(1+2g-2f_i)(s)\| \\ &\ge \|T(1+2g)(s)\| - 2\|Tf_i(s)\| > 1 + 2\|T\|/3 - 2\epsilon,\end{aligned}$$

contradicting the choice of ϵ.

Claim 2. *Let Λ be the set-valued map given by $\Lambda(k) = \Lambda_k$ for every $k \in K$. Then there exists an $m \in \mathbb{N}$ such that $|\Lambda^{-1}(s)| \le m$ for every $s \in S$.*

First of all observe that $\Lambda^{-1}(s) = \{k : s \in \Lambda_k\}$. Now pick $s \in S$ and suppose $\Lambda^{-1}(s)$ contains n distinct points $\{k_1, \ldots, k_n\}$. We shall derive an upper bound for n that is independent of s.

Let $f_j \in F_{k_j}$ be disjointly supported functions with $\|Tf_j(s)\| \ge \epsilon$. Since $\epsilon \le \|Tf_j(s)\|$ for each $1 \le j \le n$ and X has cotype q for some $2 \le q < \infty$, there exists by (1.1) a constant $Q > 0$ such that for an appropriate choice of scalars $r_j = \pm 1$ we have

$$\epsilon Q \sqrt[q]{n} \le \Big\|\sum_{j=1}^{n} r_j \cdot Tf_j(s)\Big\|.$$

Disjointness of the functions f_j implies that $\| \sum_{j=1}^{n} r_j \cdot f_j \| \le 1$. Then

$$\epsilon Q \sqrt[q]{n} \le \Big\| T\Big(\sum_{j=1}^{n} r_j \cdot f_j\Big)(s)\Big\| \le \|T\| \le 3.$$

Consequently, $n \le \left(\frac{3}{\epsilon Q}\right)^q$. So we are done.

Claim 3. *Denote $F = \Lambda(K)$. Then F is a closed subset of S and $\Lambda^{-1}(F^{(\alpha)}) \supset K^{(\alpha)}$ for every ordinal α.*

First notice that $\Lambda^{-1}(F^{(\alpha)}) = \{k : \Lambda_k \cap F^{\alpha} \neq \emptyset\}$ for each ordinal α. Now, let $G = \{(k, s) \in K \times S : s \in \Lambda_k\}$ be the graph of Λ. Assume that $(k_i, s_i) \to (k, s)$ with $(k_i, s_i) \in G$ for each i in a directed set I. If $f \in F_k$ then $f(k) > \|T\|/3$ and there exists $i_0 \in I$ such that $f \in F_{k_i}$ for every $i \ge i_0$. Thus $\|Tf(s_i)\| \ge \epsilon$ for each $i \ge i_0$. Therefore $\|Tf(s)\| \ge \epsilon$ by continuity. So G is closed. In particular, the image $F = \Lambda(K)$, which is the projection of G on the S coordinate, is closed.

We show by induction that $\Lambda_k \cap F^{(\alpha)} \neq \emptyset$ for every $k \in K^{(\alpha)}$. If $\alpha = 0$ then $\Lambda_k \subset F$. So, by Claim 1 we infer that $\Lambda_k \neq \emptyset$ for every k.

Now, assume that α is a limit ordinal and fix $k \in K^{(\alpha)}$. Hence by the induction hypothesis $\Lambda_k \cap F^{(\beta)} \neq \emptyset$ for every $\beta < \alpha$. Since these sets are a decreasing family of closed non-empty sets, they have a non-empty intersection.

Next, suppose that $\alpha = \delta + 1$. Given $k \in K^{(\delta+1)}$ there is a net $(k_i)_I$ in $K^{(\delta)} \setminus \{k\}$ converging to k. By the induction hypothesis we can choose $s_i \in \Lambda_{k_i} \cap F^{(\delta)}$. Since $F^{(\delta)}$ is a compact set, there is a subnet, say $(s_j)_J$, converging to $s \in F^{(\delta)}$. Consider also the subnet $(k_j)_J$ which converges to k. For each $j_0 \in J$ there is $j \ge j_0$ such that $s_j \neq s$, because otherwise it would be possible to find infinitely many distinct elements $\{k_{j_1}, k_{j_2}, \ldots\}$ such that $s \in \bigcap_{n\ge 1} \Lambda_{k_{j_n}}$, contradicting Claim 2. Thus $s \in F^{(\delta+1)}$, and since G is closed, $s \in \Lambda_k$.

Now we pass to the proof of items (a) and (b) of the theorem.

(a) According to Claim 3, we deduce that if S is scattered then so is K and $\mathrm{ht}(K) \le \mathrm{ht}(F) \le \mathrm{ht}(S)$.

(b) By Claim 2, there is an $m \in \mathbb{N}$ such that $|\Lambda^{-1}(s)| \le m$ for each $s \in S$. Then, if $S^{(\alpha)}$ is infinite we can write

$$|K^{(\alpha)}| \le |\Lambda^{-1}(F^{(\alpha)})| = \Big| \bigcup_{s \in F^{(\alpha)}} \Lambda^{-1}(s)\Big| \le |F^{(\alpha)}| \cdot m \le |S^{(\alpha)}|. \ \blacksquare$$

3. The proofs of the vector-valued Banach–Stone type theorems. In this section, we first prove Theorem 1.10. Then, as a consequence, we deduce Theorem 1.8. Finally, we provide the proof of Theorem 1.10 by using Theorem 1.8 and Proposition 3.1.

Proof of Theorem 1.10. Let T be an isomorphism of $C(K,X)$ onto $C(S,X)$ with $\|T\|\,\|T^{-1}\|<3$. We distinguish two cases:

CASE 1: *S is scattered.* By restricting T to $C(K)$ and applying Theorem 2.1(a) we deduce that K is scattered and $\mathrm{ht}(K)\leq \mathrm{ht}(S)$. Moreover, by restricting T^{-1} to $C(S)$, again according to Theorem 2.1(a) we conclude that $\mathrm{ht}(S)\leq \mathrm{ht}(K)$. Hence $\mathrm{ht}(K)=\mathrm{ht}(S)$.

Now, let α be an ordinal different from $-1+\mathrm{ht}(K)$. If $\alpha\geq \mathrm{ht}(K)$, then by the definition of height we infer that $|K^{(\alpha)}|=|S^{(\alpha)}|=0$ and we are done. Otherwise, $\alpha<-1+\mathrm{ht}(K)$. Thus $|K^{(\alpha)}|$ and $|S^{(\alpha)}|$ are infinite. Therefore applying Theorem 2.1(b) twice it follows that $|K^{(\alpha)}|=|S^{(\alpha)}|$ and we are also done.

CASE 2: *S is not scattered.* In this case, proceeding as in Case 1 we prove that K is not scattered either. Since $|K^{(\alpha)}|$ and $|S^{(\alpha)}|$ are infinite for every α, it suffices to apply Theorem 2.1(b) twice to see that $|K^{(\alpha)}|=|S^{(\alpha)}|$. This completes the proof of Theorem 1.10. ∎

Proof of Theorem 1.8. Denote $K=[1,\omega^\alpha m]$ and $S=[1,\omega^\beta n]$. By [19, Theorem 8.6.6] we know that $\mathrm{ht}(K)=\alpha+1$ and $\mathrm{ht}(S)=\beta+1$. Then, by what we have just stated at the beginning of the proof of Case 1 of Theorem 1.10, we have $\alpha+1=\mathrm{ht}(K)=\mathrm{ht}(S)=\beta+1$. Thus the theorem is proved. ∎

Finally, we turn to the proof of Theorem 1.5. We recall that a closed subset $A\subset K$ admits a *regular simultaneous extension operator* if there is a bounded linear operator $L: C(A)\to C(K)$ satisfying $Lf|_A=f$ for all $f\in C(A)$, with $\|L\|=1$ and $L(1_A)=1_K$. It is well known that such extensions exist whenever K is metrizable [19, Theorem 21.1.4], and a direct construction shows this is also true when K is an interval of ordinals (see for instance [1, Proposition 1.1.c]).

PROPOSITION 3.1. *Suppose that X is Banach space having non-trivial cotype, K and S compact Hausdorff spaces, T an isomorphism of $C(K,X)$ onto $C(S,X)$ satisfying $\|T\|<3$ and $\|f\|\leq\|Tf\|$, $\epsilon=(1-\|T\|/3)/2$ and $A\subset K$ a closed set admitting a regular simultaneous extension operator. For an ordinal α, if $A^{(\alpha)}\neq\emptyset$, then for every $\varphi\in C(K,X)$ such that $\|\varphi\|=1$ and $\|\varphi(k)\|=1$ for every $k\in A$, there is an $s\in S^{(\alpha)}$ satisfying $\|T\varphi(s)\|\geq\epsilon$.*

Proof. Assume that $\varphi\in C(K,X)$, $\|\varphi\|=1$ and $\|\varphi(k)\|=1$ for every $k\in A$. Let $L: C(A)\to C(K)$ be a regular simultaneous extension operator. Define $T^A: C(A)\to C(S,X)$ by

$$T^Af=T(\varphi Lf).$$

Then $\|T^A\|<3$ and $\|f\|\leq\|T^Af\|$ for every $f\in C(A)$. Indeed, for every $k\in A$,

$$\|T^Af\|=\|T(\varphi Lf)\|\geq\|\varphi(k)f(k)\|=|f(k)|.$$

Set $\epsilon_A = (1 - \|T^A\|/3)/2$. We can thus define for every $k \in A$, as in the proof of Theorem 2.1, the sets F_k^A and Λ_k^A associated with T^A, and the image $F_A \subset S$ of Λ^A. So, according to Claim 3 of Theorem 2.1 we see that $A^{(\alpha)} \subset (\Lambda^A)^{-1}(F_A^{(\alpha)})$ for every ordinal α. In particular, if $A^{(\alpha)} \neq \emptyset$, there is an $s \in F_A^{(\alpha)} \subset S^{(\alpha)}$ such that $\|T\varphi(s)\| = \|T^A 1(s)\| \geq \epsilon_A \geq \epsilon$. ∎

Proof of Theorem 1.5. Let T be an isomorphism of $C(K,X)$ onto $C(S,X)$. Without loss of generality we may assume that $\|T^{-1}\| = 1$. Therefore $\|T\| < 3$ and $\|f\| \leq \|Tf\|$ for each $f \in C(K,X)$. By [19, Proposition 8.6.5] we may assume that $K = [1, \omega^\alpha m]$ and $S = [1, \omega^\beta n]$ for some ordinals α, β, m and n different from zero, where m and n are finite numbers. It follows from Theorem 1.8 that $\alpha = \beta$. So, it is enough to show that $m = n$.

To see this, write $K^{(\alpha)} = \{k_1, \ldots, k_m\}$ and let $A_1, \ldots, A_m$ be mutually disjoint clopen sets such that $k_j \in A_j$ for each j and $K = \bigcup_{j=1}^m A_j$.

Now identify X^m and X^n with $C(K^{(\alpha)}, X)$ and $C(S^{(\alpha)}, X)$ respectively. Next assign to each $z = (z_1, \ldots, z_m) \in X^m$ the function $\varphi_z = \sum_{j=1}^m 1_{A_j} z_j \in C(K,X)$. Clearly, $\|\varphi_z\| = \|z\|$ and the linear operator $\Phi : X^m \to X^n$ given by

$$\Phi(z) = T\varphi_z|_{S^{(\alpha)}}$$

satisfies $\|\Phi\| \leq \|T\|$. Fix $z = (z_1, \ldots, z_m)$ in the unit sphere of X^m. Without loss of generality we may suppose that $\|z\| = \|z_1\|$.

Set $\epsilon = (1 - \|T\|/3)/2$. Then, by applying Proposition 3.1 with $A = A_1$ and $\varphi = \varphi_z$, we conclude that there is an $s \in S^{(\alpha)}$ such that

$$\|\Phi z\| \geq \|\Phi z(s)\| = \|T\varphi_z(s)\| \geq \epsilon.$$

Hence, for every $z \in X^m$,

$$\|\Phi z\| \geq \epsilon \|z\|.$$

Since by our hypothesis X^n contains no subspace isomorphic to X^{n+1} we must have $m \leq n$. Switching the roles of K and S in the above proof we also obtain $n \leq m$. ∎

4. Final remark. In this last section we show that there exist many Banach spaces satisfying the conditions of Theorem 1.5.

Remark 4.1. Denote by H the separable uniformly convex hereditarily indecomposable Banach space introduced by Ferenczi in [12]. Then each of the isomorphically different spaces $X = l_p \oplus H$ having non-trivial cotype, with $1 \leq p < \infty$ [10, Theorem 14.1], satisfies the conditions of Theorem 1.5. Indeed, first suppose that $l_p \oplus H$ is isomorphic to $l_q \oplus H$. Since l_p and l_q are essentially incomparable with H [2, Proposition 4.11], it follows by [14, Remark 3.3] that there exist m and n in $\mathbb{N}$ such that $l_p \oplus \mathbb{R}^m$ is isomorphic to $l_q \oplus \mathbb{R}^n$. Hence $p = q$.

Now assume that $(l_p \oplus H)^n$ contains a subspace isomorphic to $(l_p \oplus H)^{n+1}$ for some $n \in \mathbb{N}$. Then there exists $T : H^{n+1} \to l_p \oplus H^n$ which is an isomorphism onto its image. Let P be the natural projection of $l_p \oplus H^n$ onto l_p. Since PT is strictly singular, it follows by [18, Proposition 2.c.10] that $(I - P)T : H^{n+1} \to H^n$ has a closed range and its kernel is a finite-dimensional space V. Fix a Banach space W such that $H^{n+1} = W \oplus V$. Then the restriction of $(I - P)T$ to W is an isomorphism onto its image. This means that H^n contains a subspace isomorphic to W. Now, let Z be a subspace of H such that $H = V \oplus Z$. Then $H^{n+1} \oplus Z \sim W \oplus V \oplus Z$ is isomorphic to a subspace of $H^n \oplus V \oplus Z \sim H^{n+1}$, which is absurd by [13, Corollary 2].

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