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The number of $L_{\infty\kappa}$-equivalent nonisomorphic models for κ weakly compact

by

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Abstract. For a cardinal κ and a model M of cardinality κ let $\mathrm{No}(M)$ denote the number of nonisomorphic models of cardinality κ which are $L_{\infty,\kappa}$-equivalent to M. We prove that for κ a weakly compact cardinal, the question of the possible values of $\mathrm{No}(M)$ for models M of cardinality κ is equivalent to the question of the possible numbers of equivalence classes of equivalence relations which are Σ^1_1-definable over V_κ. By [SV] it is possible to have a generic extension where the possible numbers of equivalence classes of Σ^1_1-equivalence relations are in a prearranged set. Together these results settle the problem of the possible values of $\mathrm{No}(M)$ for models of weakly compact cardinality.

1. Introduction. Suppose κ is a cardinal and $\mathcal{M}$ is a model of cardinality κ. Let $\mathrm{No}(\mathcal{M})$ denote the number of nonisomorphic models of cardinality κ which are elementary equivalent to $\mathcal{M}$ over the infinitary language $L_{\infty\kappa}$. We study the possible values of $\mathrm{No}(\mathcal{M})$ for different models $\mathcal{M}$.

When $\mathcal{M}$ is countable, $\mathrm{No}(\mathcal{M}) = 1$ by [Sco65]. This result extends to all structures of singular cardinality λ provided that λ is of countable cofinality [Cha68]. The case where $\mathcal{M}$ is of singular cardinality λ with uncountable cofinality κ was first treated in [She85] and later on in [She86]. In these papers Shelah showed that if $\kappa > \aleph_0$, $\theta^\kappa < \lambda$ for every $\theta < \lambda$, and $0 < \mu < \lambda$ or $\mu = \lambda^\kappa$, then $\mathrm{No}(\mathcal{M}) = \mu$ for some model $\mathcal{M}$ of cardinality λ. In [SV00] of the present authors the singular case is revisited, and in particular, it is established, under the same assumptions as above, that the values μ with $\lambda \le \mu < \lambda^\kappa$ are possible for $\mathrm{No}(\mathcal{M})$ with $\mathcal{M}$ of cardinality λ.

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If $V = L$, κ is an uncountable regular cardinal which is not weakly compact, and $\mathcal{M}$ is a model of cardinality κ, then $\mathrm{No}(\mathcal{M}) \in \{1, 2^\kappa\}$ [She81]. For $\kappa = \aleph_1$ this result was first proved in [Pal77a]. The values $\mathrm{No}(\mathcal{M}) \in \{\aleph_0, \aleph_1\}$ for a model of cardinality $\aleph_1$ are consistent with ZFC + GCH as noted in [She81]. All the nonzero finite values of $\mathrm{No}(\mathcal{M})$ for models of cardinality $\aleph_1$ are proved to be consistent with ZFC + GCH in [SV01].

When κ is a weakly compact cardinal and μ is a nonzero cardinal $\le \kappa$ there is a model $\mathcal{M}$ of cardinality κ with $\mathrm{No}(\mathcal{M}) = \mu$ [She82]. In the present paper we show that for κ weakly compact, the possible values of $\mathrm{No}(\mathcal{M})$ for models of cardinality κ depend only on the possible numbers of equivalence classes of equivalence relations $\sim_{\phi,P}$ having the following form (Σ^1_1-definable over V_κ): for some first order sentence ϕ in the vocabulary $\{\in, R_0, R_1, R_2, R_3\}$ and a subset P of V_κ, it is the case that for all $s, t \in {}^\kappa 2$,

$$s \sim t \quad \text{iff} \quad \text{for some } r \in {}^\kappa 2,\ \langle V_\kappa, \in, P, s, t, r\rangle \models \phi,$$

where P, r, s, and t are the interpretations of the symbols R_0, R_1, R_2, and R_3 respectively.

THEOREM 1. *When κ is a weakly compact cardinal, the following two conditions are equivalent for all cardinals μ*:

(A) *there is an equivalence relation on ${}^\kappa 2$ which is Σ^1_1-definable over V_κ and has μ equivalence classes*;

(B) $\mathrm{No}(\mathcal{M}) = \mu$ *for some model $\mathcal{M}$ of cardinality κ.*

In [SV] we proved that for every nonzero cardinal $\mu \in \kappa \cup \{\kappa, \kappa^+, 2^\kappa\}$ there is a Σ^1_1-equivalence relation (as defined above) with μ equivalence classes. Moreover, it is possible to have a generic extension where the possible numbers of equivalence classes of any second order definable equivalence relations are completely controlled [SV, Theorem 1]. It follows that the question of possible value of $\mathrm{No}(\mathcal{M})$ is completely solved when $\mathcal{M}$ is of weakly compact cardinality. More formally, the conclusion is the following.

THEOREM 2. *Suppose that the following conditions are satisfied*:

- *κ is a weakly compact cardinal and $2^\kappa = \kappa^+$*;
- *κ remains a weakly compact cardinal in the standard Cohen forcing adding a new subset of κ*;
- *$\lambda > \kappa^+$ is a cardinal with $\lambda^\kappa = \lambda$*;
- *Ω is a set of cardinals below λ*;
- *Ω contains all nonzero cardinals $\le \kappa^+$*;
- *for every $\chi \in \Omega$ with $\chi > \kappa^+$ and $\theta < \kappa$, the inequality $\chi^\theta \le \chi^+$ holds*;
- *Ω is closed under unions and products of $\le \kappa$ cardinals.*

Then there is a forcing extension where there are no new sets of cardinality $< \kappa$, all cardinals and cofinalities are preserved, κ remains weakly compact,

$2^\kappa = \lambda$, *and for all cardinals* μ, *there exists a model* $\mathcal{M}$ *of cardinality* κ *with* $\mathrm{No}(\mathcal{M}) = \mu$ *if and only if* μ *is in* Ω.

When κ is a weakly compact cardinal, it is possible to have, using the upward Easton forcing, a generic extension where κ is still a weakly compact cardinal, and κ remains weakly compact in the Cohen forcing adding a subset of κ (by an unpublished result of Silver). The forcing needed in the conclusion is the standard way to add, for every $\mu \in \Omega$, a Kurepa tree of height κ having μ branches of length κ through it. As noted in [SV, Fact 5.1], this forcing is locally κ Cohen, and therefore, κ remains a weakly compact cardinal in the composite forcing of the upward Easton forcing and the addition of new Kurepa trees.

Note also that the closure properties mentioned in the conclusion are necessary by the fact that the possible numbers of equivalence classes of Σ^1_1-equivalence relations are always closed under unions of length $\leq \kappa$ and products of length $< \kappa$ [SV, Lemma 3.4].

There are several parts in the paper. In Section 2 we start with a definition of an Ehrenfeucht–Fraïssé game $\mathrm{EF}_{\kappa;\lambda}(\mathcal{M}, \mathcal{N})$ generalizing the elementary equivalence between two models over an infinitary language $L_{\infty\kappa}$. We note that if $\mathcal{M}$ is a model of cardinality κ, there is a Σ^1_1-equivalence relation having $\mathrm{No}(\mathcal{M})$ equivalence classes.

The remaining sections are dedicated to the other half of the proof of the theorem, namely to the proof that the existence of a Σ^1_1-equivalence relation with μ equivalence classes implies the existence of a model $\mathcal{M}$ with $\mathrm{card}(\mathcal{M}) = \kappa$ and $\mathrm{No}(\mathcal{M}) = \mu$ (Lemma 7.8 at the end of Section 7).

First in Section 3 we introduce a coding tree which is a skeleton for the basic functions defined in Section 4. The basic functions form partial isomorphisms between models. We want those partial isomorphisms to have a strong extension property ((1.2) below). The constructions of the partial isomorphisms and the models are similar to [She82]. The role of the coding tree is to provide control over which basic functions are extended. The basic functions are defined by induction along the branches of the coding tree. For example, a κ-branch through the coding tree yields an isomorphism between two models ((1.3) below).

In Section 5 we define a special family of functions, which is used to build models in the last section. Roughly speaking the family is the closure of the basic functions under composition. However, in order to manage the functions in the family, we cannot allow arbitrary compositions. For an explanation why we are forced to consider compositions, see the beginning of Section 5. This part might feel quite technical. The reader may skip all the lemmas of this section on the first reading, and return to them when they are referred to in the last section. The following intuition might help the reader: The basic functions are built from simple ordinal addition preserving "shifts" of ordinals by putting "a shift on top of another". A basic function consisting of finitely many shifts maps an interval $[0,\alpha)$ into finitely many intervals $[\beta_0,\beta_1), \dots, [\beta_{2n},\beta_{2n+1})$ only. Moreover, if the basic function consists of infinitely many shifts, then it is a permutation of an interval of the form $[0,\alpha)$. Similar type of properties are preserved under compositions of basic functions as proved in the last three lemmas of Section 5. This type of properties are mainly applied in the proof of (1.4) below.

In Section 6 we briefly sketch a proof that the extension properties of the basic functions are preserved under compositions.

The content of Section 7 is as follows. Assuming that κ is a strongly inaccessible cardinal and ϕ defines a Σ^1_1-equivalence relation $\sim_{\phi,P}$ on ${}^{\kappa}2$ with a parameter $P \subseteq V_\kappa$ we construct models $\mathcal{M}_t$ for $t \in {}^{\kappa}2$ satisfying:

(1.1) the models are of cardinality κ and they have a common vocabulary ϱ consisting of κ relation symbols each of arity $< \kappa$;

(1.2) all the models are pairwise $L_{\infty\kappa}$-equivalent, and even more, they are pairwise $M_{\infty\kappa;\lambda}$-equivalent for any previously fixed regular cardinal $\lambda < \kappa$ (Definition 2.1);

(1.3) for all $s,t \in {}^{\kappa}2$, the models $\mathcal{M}_s$ and $\mathcal{M}_t$ are isomorphic if, and only if, s and t are equivalent with respect to $\sim_{\phi,P}$.

Furthermore, when κ is a weakly compact cardinal the models have the additional property that

(1.4) if a model $\mathcal{N}$ has vocabulary ϱ, $\mathcal{N}$ is of cardinality κ, and $\mathcal{N}$ is $L_{\infty\kappa}$-equivalent to one (all) of the models $\mathcal{M}_t$, $t \in {}^{\kappa}2$, then $\mathcal{N}$ is isomorphic to $\mathcal{M}_s$ for some $s \in {}^{\kappa}2$.

This is the main difference between the strongly inaccessible non-weakly compact case and the weakly compact case: the Π^1_1-indescribability property of a weakly compact cardinal κ ensures that the "isomorphism type" of any model $\mathcal{N}$ with domain κ is determined by the isomorphism types of the bounded parts $\mathcal{N}\restriction\alpha$, $\alpha < \kappa$, alone (Lemma 7.7).

2. Preliminaries

Definition 2.1. Suppose μ is a cardinal and λ is an infinite regular cardinal. Let $\mathcal{M}$ and $\mathcal{N}$ be models of a common relational vocabulary. The *Ehrenfeucht–Fraïssé game* $\mathrm{EF}_{\mu;\lambda}(\mathcal{M},\mathcal{N})$ is defined as follows. The game has two players, player I and player II. A play of the game continues for at most λ rounds. In round $i < \lambda$ player I first chooses $X_i \in \{\mathcal{M},\mathcal{N}\}$ and $A_i \subseteq X_i$ of cardinality $< \mu$. Then player II replies with a partial isomorphism p_i such that

- $\mathrm{dom}(p_i) \subseteq \mathcal{M}$, $\mathrm{ran}(p_i) \subseteq \mathcal{N}$, $\bigcup_{j<i} p_j \subseteq p_i$, and
- $A_i \subseteq \mathrm{dom}(p_i)$ if $X_i = \mathcal{M}$, and $A_i \subseteq \mathrm{ran}(p_i)$ otherwise.

Player II wins if the play lasts λ rounds. Otherwise player I wins. We write $\mathcal{M} \equiv_{\infty\kappa;\lambda} \mathcal{N}$ when player II has a winning strategy in $\mathrm{EF}_{\kappa;\lambda}(\mathcal{M}, \mathcal{N})$.

Let $\mathcal{M}$ and $\mathcal{N}$ be models of a common relational vocabulary and κ be a cardinal. The game $\mathrm{EF}_{\kappa;\omega}(\mathcal{M}, \mathcal{N})$ is the usual Ehrenfeucht–Fraïssé game of length ω which characterizes the existence of a nonempty family of partial isomorphisms with the "fewer than κ at a time back-and-forth property". If $\mathcal{M}$ and $\mathcal{N}$ satisfy the same sentences of the infinitary language $L_{\infty\kappa}$, we write $\mathcal{M} \equiv_{\infty\kappa} \mathcal{N}$. By Karp's theorem [Kar65], player II has a winning strategy in $\mathrm{EF}_{\kappa;\omega}(\mathcal{M}, \mathcal{N})$ if, and only if, $\mathcal{M} \equiv_{\infty\kappa} \mathcal{N}$. So the game $\mathrm{EF}_{\kappa;\lambda}(\mathcal{M}, \mathcal{N})$, for an infinite regular cardinal $\lambda < \kappa$, is a generalized version of the "fewer than κ at a time back-and-forth property". There are so-called infinitely deep languages $M_{\infty\kappa;\lambda}$ with the property that $\mathcal{M} \equiv_{\infty\kappa;\lambda} \mathcal{N}$ if, and only if, $\mathcal{M}$ and $\mathcal{N}$ satisfy the same sentences of $M_{\infty\kappa;\lambda}$ [Hyt90, Kar84, Oik97].

For a model $\mathcal{N}$ we let $\mathrm{card}(\mathcal{N})$ denote the cardinality of the universe of $\mathcal{N}$. For any model $\mathcal{M}$ of cardinality κ and a regular cardinal $\aleph_0 \leq \lambda < \kappa$, we define $\mathrm{No}_\lambda(\mathcal{M})$ to be the cardinality of the set

$$\{\mathcal{N}/{\cong} \mid \mathrm{card}(\mathcal{N}) = \kappa \text{ and } \mathcal{N} \equiv_{\infty\kappa;\lambda} \mathcal{M}\},$$

where $\mathcal{N}/{\cong}$ is the equivalence class of $\mathcal{N}$ under the isomorphism relation.

Next we recall from [SV, Definition 3.1] the definition of an equivalence relation on ${}^{\kappa}2$ which is second order definable over the set $H(\kappa)$ (all sets hereditarily of cardinality $< \kappa$). In this paper we need only equivalence relations which are definable using one second order existential quantifier. Hence the definition is presented in a restricted form.

Definition 2.2. Suppose κ is a regular cardinal. We say that ϕ defines a Σ^1_1-equivalence relation $\sim_{\phi,P}$ on ${}^{\kappa}2$ with a parameter $P \subseteq H(\kappa)$ provided that

- ϕ is a first order sentence in a vocabulary consisting of $\in$, one unary relation symbol R_0, and binary relation symbols R_1, R_2, and R_3;
- the following definition yields an equivalence relation on ${}^{\kappa}2$: for all $s, t \in {}^{\kappa}2$,

$$s \sim_{\phi,P} t \quad \text{iff} \quad \text{for some } r \in {}^{\kappa}2,\ \langle V_\kappa, \in, P, s, t, r\rangle \models \phi,$$

where P, s, t, and r are the interpretations of the symbols R_0, R_1, R_2, and R_3 respectively.

By [SV, Lemma 3.4] the possible numbers of equivalence classes of a Σ^1_1-relation over $H(\kappa)$ include all nonzero cardinals in $\{\mu \mid \mu \leq \kappa^+ \text{ or } \mu = 2^\kappa\}$. In the next lemma we point out that the possible numbers of classes contain the cardinal $\mathrm{No}(\mathcal{M})$ for every model $\mathcal{M}$ of cardinality κ.

LEMMA 2.3. *Suppose κ is an uncountable regular cardinal, $\mathcal{M}$ is a model of cardinality κ, and μ is the cardinal* $\mathrm{No}(\mathcal{M})$. *Then there is an equivalence relation $\sim_{\phi,P}$ on ${}^{\kappa}2$ which is Σ^1_1-definable over $H(\kappa)$ and has μ equivalence classes.*

Proof. Without loss of generality $\mathcal{M}$ has domain κ and μ is infinite. We may also assume that the vocabulary of $\mathcal{M}$ consists of one n-place relation P with $n < \omega$ (Lemma 7.8).

Since there exists a definable bijection from κ into κ^3, we may define, for every $r \in {}^{\kappa}2$, that r codes a triple $\langle r_1, r_2, r_3\rangle$ so that for every $i \in \{1,2,3\}$, $\{r_i \mid r \in {}^{\kappa}2\} = {}^{\kappa}2$. Similarly, let R_s denote the n-place relation on κ coded by $s \in {}^{\kappa}2$, and let $h_{r'}$ denote the binary relation on κ coded by $r' \in {}^{\kappa}2$.

Since for all models $\mathcal{N}$, the game $\mathrm{EF}_{\kappa;\omega}(\mathcal{M},\mathcal{N})$ is determined, $\mathcal{M} \not\equiv_{\infty\kappa} \mathcal{N}$ iff player I has a winning strategy in $\mathrm{EF}_{\kappa;\omega}(\mathcal{M},\mathcal{N})$. Moreover, there is a definable bijection from κ into $[\kappa]^{<\aleph_0}$ (all finite sequences of ordinals below κ), and thus, for every $r' \in {}^{\kappa}2$, we may define $w_{r'}$ to be the function from $[\kappa]^{<\aleph_0}$ into κ coded by r'.

Using the relation P as a parameter, we may form a first order sentence saying for fixed s, t, and r in ${}^{\kappa}2$ that

> either w_{r_1} is a winning strategy for player I in $\mathrm{EF}_{\kappa;\omega}(\langle\kappa,P\rangle,\langle\kappa,R_s\rangle)$ and w_{r_2} is a winning strategy for player I in $\mathrm{EF}_{\kappa;\omega}(\langle\kappa,P\rangle,\langle\kappa,R_t\rangle)$, or otherwise, h_{r_3} is an isomorphism between $\langle\kappa,R_s\rangle$ and $\langle\kappa,R_t\rangle$.

Such a sentence defines over $H(\kappa)$ a Σ^1_1-equivalence relation on ${}^{\kappa}2$ as required. ∎

3. The coding tree. Throughout the paper κ is a strongly inaccessible cardinal, i.e., a regular limit cardinal satisfying $2^{\mu} < \kappa$ for all $\mu < \kappa$. Additionally, λ is a fixed regular cardinal below κ.

During the next three sections we define a family of functions which is used to build the models $\mathcal{M}_t$, $t \in {}^{\kappa}2$ (Definition 7.1). There is a similar idea in [She82]. However, this time we want the functions to have more properties. Hence the definition of the family is more complicated.

To make our models strongly equivalent we shall guarantee that for every pair $\mathcal{M}_s$ and $\mathcal{M}_t$, $s,t \in {}^{\kappa}2$, a certain subfamily of all the functions forms a winning strategy for player II in the game $\mathrm{EF}_{\kappa;\lambda}(\mathcal{M}_s,\mathcal{M}_t)$ (Definition 3.5). So we shall also need a "stronger extension property" for the functions than was needed in [She82].

We want the models $\mathcal{M}_s$ and $\mathcal{M}_t$, $s,t \in {}^{\kappa}2$, to be isomorphic if, and only if, the indices s and t are equivalent with respect to some previously fixed Σ^1_1-equivalence relation (Lemma 7.4). Hence we "code" the equivalence relation into a special tree (Definitions 3.2 and 3.3). This tree is a steering apparatus in the construction of the family of functions (Definitions 4.3 and 5.2).

Throughout the paper ϕ denotes a sentence which defines a Σ_1^1-equivalence relation $\sim_{\phi,P}$ on ${}^{\kappa}2$ with a parameter $P \subseteq V_\kappa$ (Definition 2.2).

Without loss of generality we may assume that for every $s, t, r \in {}^{\kappa}2$, the same element r witnessing that $s \sim_{\phi,P} t$ also witnesses that $t \sim_{\phi,P} s$, i.e.,

$$\langle V_\kappa, \in, P, s, t, r\rangle \models \phi \quad \text{iff} \quad \langle V_\kappa, \in, P, t, s, r\rangle \models \phi. \tag{3.1}$$

Because $\sim_{\phi,P}$ is a relation on ${}^{\kappa}2$ and it has a Σ_1^1-definition, the most natural object to consider is the tree of triples such that the first two elements are initial segments of potentially equivalent functions in ${}^{\kappa}2$ and the third element is a potential witnessing function for the equivalence. For technical reasons we also have a fourth member in the nodes. It is convenient to use only a restricted part of ${}^{\kappa}2$ to get more "room for coding different initial segments".

Remark. The reader may skip the next definition, and safely think during the first reading of the paper that for every $\mu \leq \kappa$, the set $\mathrm{Fun}(\mu, 2)$ defined below equals the collection ${}^{\mu}2$ of all functions from μ into 2.

Definition 3.1. Suppose $\alpha \leq \kappa$ (remember $\kappa = \beth_\kappa$). We denote by $\mathrm{Fun}(\beth_\alpha, 2)$ the family of functions η from $\beth_\alpha$ into 2 such that

- $\eta(\xi) = 0$ if $\xi < \beth_0$;
- for every $\beta < \alpha$, for all $\xi \in \beth_{\beta+1} \smallsetminus \beth_\beta$, $\eta(\xi) = \eta(\beth_\beta)$.

Remark. This restriction is harmless: We want to show that it is possible to have a model $\mathcal{M}$ of cardinality κ such that $\mathrm{No}_\lambda(\mathcal{M})$ equals the number of equivalence classes with respect to $\sim_{\phi,P}$. Because κ is assumed to be a strongly inaccessible cardinal, we may assume that $\sim_{\phi,P}$ has the following property: for all $s, t \in {}^{\kappa}2$,

$$\text{if } s(\beth_\alpha) = t(\beth_\alpha) \text{ for every } \alpha < \kappa \text{ then } s \sim_{\phi,P} t. \tag{3.2}$$

The restriction turns out to be useful in Definition 4.3, and most importantly, in Lemma 5.7(e) (we want that for a successor $u \in T_\lambda^{1,2}$ with c^u increasing, the information $\mathrm{end}(p_u)$ is determined by a single point $\zeta \in \mathrm{ran}(c^u)$ alone; for the unexplained notation see Definitions 3.6, 4.3 and 5.2).

Definition 3.2. Let $T[0]$ be $\{\langle \emptyset, \emptyset, \emptyset, \emptyset\rangle\}$ and for every nonzero $\alpha < \kappa$ define $T[\alpha]$ to consist of all tuples $\langle \eta, \nu, \tau, C\rangle$ such that

- $\eta, \nu, \tau \in \mathrm{Fun}(\beth_\alpha, 2)$;
- $\eta \neq \nu$;
- C is a closed subset of α.

Then define

$$T = \bigcup_{\alpha<\kappa} T[\alpha].$$

Let $T[<\alpha]$ denote the set $\bigcup_{\beta<\alpha} T[\beta]$.

For every $u \in T$, we let $\mathrm{ord}(u)$, $\mathrm{fun}_1(u)$, $\mathrm{fun}_2(u)$, $\mathrm{fun}_3(u)$, and $\mathrm{cst}(u)$ be elements such that $u \in T[\mathrm{ord}(u)]$ and $u = \langle \mathrm{fun}_1(u), \mathrm{fun}_2(u), \mathrm{fun}_3(u), \mathrm{cst}(u)\rangle$. Furthermore, for every $u \in T$, define $u{\restriction}\beta$ to be $\langle \emptyset, \emptyset, \emptyset, \emptyset\rangle$ if $\beta = 0$, and when $\beta > 0$, set

$$u{\restriction}\beta = \langle \mathrm{fun}_1(u){\restriction}\beth_\beta, \mathrm{fun}_2(u){\restriction}\beth_\beta, \mathrm{fun}_3(u){\restriction}\beth_\beta, \mathrm{cst}(u) \cap \beta\rangle.$$

The elements $u, v \in T$ form a tree when they are ordered by

$$u \lhd v \quad \text{iff} \quad u = v{\restriction}\mathrm{ord}(u) \text{ and } \mathrm{ord}(u) \in \mathrm{cst}(v).$$

The notation $u \unlhd v$ stands for $u \lhd v$ or $u = v$. For a $\lhd$-increasing chain $\langle u_i \mid i < \theta\rangle$ of elements in T with $\theta < \kappa$, we write $\bigcup_{i<\theta} u_i$ for the following element in T:

$$\Big\langle \bigcup_{i<\theta} \mathrm{fun}_1(u_i), \bigcup_{i<\theta} \mathrm{fun}_2(u_i), \bigcup_{i<\theta} \mathrm{fun}_3(u_i), C\Big\rangle,$$

where C is the closure of $\bigcup_{i<\theta} \mathrm{cst}(u_i)$.

Remember that ϕ defines an equivalence relation. So in the next definition we fix the part T^1 of T whose "nodes are initial segments of equivalent functions".

Definition 3.3. Suppose s, t, r are in $\mathrm{Fun}(\kappa, 2)$. If $\langle V_\kappa, \in, P, s, t, r\rangle \models \phi$ does not hold, define $C_{s,t,r}$ to be $\{0\}$, and otherwise, define

$$C_{s,t,r} = \{0\} \cup \{\delta < \kappa \mid \langle V_\delta, \in, P \cap V_\delta, s{\restriction}\delta, t{\restriction}\delta, r{\restriction}\delta\rangle \prec \langle V_\kappa, \in, P, s, t, r\rangle \models \phi\}.$$

Let T^1 be the set of all $u \in T$ such that for some $s, t, r \in \mathrm{Fun}(\kappa, 2)$ the following conditions are satisfied:

- $\langle V_\kappa, \in, P, s, t, r\rangle \models \phi$;
- $\mathrm{fun}_1(u) \subseteq s$, $\mathrm{fun}_2(u) \subseteq t$, and $\mathrm{fun}_3(u) \subseteq r$;
- $\mathrm{ord}(u) \in C_{s,t,r}$ and $\mathrm{cst}(u) = C_{s,t,r} \cap \mathrm{ord}(u)$.

Note that for all nonzero $\delta \in C_{s,t,r}$, $\beth_\delta = \delta$. Notice also that if r witnesses that $s \sim_{\phi,P} t$, then the following elements form a κ-branch in the tree T^1:

$$\{\langle s{\restriction}\delta, t{\restriction}\delta, r{\restriction}\delta, C_{s,t,r} \cap \delta\rangle \mid \delta \in C_{s,t,r}\}.$$

In the definition below, we fix a "direction" for the first two functions in the nodes of T. This is needed to keep track of how the partial isomorphisms are extended between models.

Definition 3.4. For each $\alpha < \kappa$ define a lexicographic order $\lessdot_\alpha$ as follows: for all elements $\eta, \nu \in \mathrm{Fun}(\beth_\alpha, 2)$,

$$\eta \lessdot_\alpha \nu \quad \text{iff} \quad \eta \neq \nu \text{ and } \eta(\xi) < \nu(\xi) \text{ for } \xi = \min\{\zeta < \beth_\alpha \mid \eta(\zeta) \neq \nu(\zeta)\}.$$

Define

$$T[\lessdot] = \{u \in T \mid \mathrm{fun}_1(u) \lessdot_{\mathrm{ord}(u)} \mathrm{fun}_2(u)\}.$$

For every $u \in T$, denote the tuple $\langle \mathrm{fun}_2(u), \mathrm{fun}_1(u), \mathrm{fun}_3(u), \mathrm{cst}(u)\rangle$ by u^{-1} (the order of the first two elements is reversed).

Remark. For every $u \in T^1$, $u^{-1} \in T^1$ by the assumption (3.1).

In the next definition we choose another part of T. This time we use a fixed "bookkeeping function" π (which ensures that the models are strongly equivalent) and we do not restrict ourselves to the initial segments of equivalent functions. Remember that λ is a fixed regular cardinal below κ, and λ is the length of the Ehrenfeucht–Fraïssé game played between the forthcoming models.

Definition 3.5. Define Suc^+ to be the set

$$\{\beta + 1 \mid \beta \text{ is a successor ordinal}\}.$$

Choose a surjective function π from $\{0\} \cup \mathrm{Suc}^+$ onto $\{\langle \emptyset, \emptyset, \emptyset, \emptyset\rangle\} \cup T[\lessdot]$ such that

- if $\pi(\alpha) = u$ then either $\alpha = 0$ and $u = \langle \emptyset, \emptyset, \emptyset, \emptyset\rangle$, or else $\mathrm{ord}(u) < \alpha$;
- for every $u \in \{\langle \emptyset, \emptyset, \emptyset, \emptyset\rangle\} \cup T[\lessdot]$, the set $\{\alpha \in \mathrm{Suc}^+ \mid \pi(\alpha) = u\}$ is unbounded in κ.

We define T^2_λ, for fixed regular $\lambda < \kappa$, to be the smallest subset of T satisfying the following conditions:

(1) $\langle \emptyset, \emptyset, \emptyset, \emptyset\rangle$ is in T^2_λ.
(2) If $u \in T^2_\lambda$ then $u^{-1} \in T^2_\lambda$.
(3) T^2_λ contains every $u \in T[\lessdot]$ having the properties:
  (i) If $\sup \mathrm{cst}(u) < \mathrm{ord}(u)$ then $\mathrm{ord}(u) \in \mathrm{Suc}^+$ and for the maximal element $\gamma = \sup \mathrm{cst}(u)$ (which is in $\mathrm{cst}(u)$), we have $\pi(\mathrm{ord}(u)) = u{\restriction}\gamma$ (this element is in $T^1 \cup T^2_\lambda$);
  (ii) if $\sup \mathrm{cst}(u) = \mathrm{ord}(u)$, then $\mathrm{cst}(u) \cap \mathrm{Suc}^+$ is nonempty, $\mathrm{cf}(\mathrm{ord}(u)) < \lambda$, and for every $\beta \in \mathrm{cst}(u)$, $u{\restriction}\beta \in T^1 \cup T^2_\lambda$.

Remark. By Definition 3.5(3), the only common node of the parts T^1 and T^2_λ is $\langle \emptyset, \emptyset, \emptyset, \emptyset\rangle$. Moreover, in contrast to the property of T^1, there are no κ-branches through the tree T^2_λ by Definition 3.5(3)(ii), only branches of length λ (Fact 3.7).

Definition 3.6. We denote the set $T^1 \cup T^2_\lambda$ by $T^{1,2}_\lambda$. For all $\alpha < \kappa$, $T^{1,2}_\lambda[\alpha]$ denotes $T[\alpha] \cap T^{1,2}_\lambda$ and $T^{1,2}_\lambda[<\alpha]$ stands for $T[<\alpha] \cap T^{1,2}_\lambda$.

We say that $u \in T^{1,2}_\lambda$ is a *successor* of v when $v \in T^{1,2}_\lambda$ and there is no $w \in T^{1,2}_\lambda$ with $v \lhd w \lhd u$. An element $u \in T^{1,2}_\lambda$ is called a *successor node* if

$u = \langle \emptyset, \emptyset, \emptyset, \emptyset \rangle$ or there is $v \in T^{1,2}_\lambda$ such that u is a successor of v. If u is not a *successor node*, it is called a *limit node*.

When $u \in T^{1,2}_\lambda$, we write "for all $v \lhd u$" when we mean "for all $v \in T^{1,2}_\lambda$ with $v \lhd u$".

Note also that an element $u \in T^1$ is a successor of $v \in T^{1,2}_\lambda$ only if $v \in T^1$. However, for every $v \in T^1$ there is $u \in T^2_\lambda$ which is a successor of v. If $u \in T^1$ is a limit point then it is a limit of elements in T^1. Moreover, $u \in T^2_\lambda$ is a limit point only if it is a limit of elements in T^2_λ. See the proof of the fact below.

FACT 3.7. (a) *For all $\lhd$-increasing chains $\langle u_i \mid i < \theta \rangle$ of elements in T^2_λ, $\mathrm{cf}(\theta) < \lambda$ and the tuples $\bigcup_{i<\theta} u_i$ are in T^2_λ.*

(b) *For every $\lhd$-increasing chain $\bar{u} = \langle u_i \mid i < \theta \rangle$ of elements in T^1, if $\bar{u}$ has some upper bound in $T^{1,2}_\lambda$ (with respect to the order $\lhd$) then the tuple $\bigcup_{i<\theta} u_i$ is in T^1.*

Proof. (a) This is an obvious consequence of Definition 3.5(3)(ii).

(b) Suppose that $w \in T^{1,2}_\lambda$ is an upper bound for $\bar{u}$. Let v be the smallest element in $T^{1,2}_\lambda$ with $v \unlhd w$ and $u_i \lhd v$ for all $i < \theta$. Then v is a limit of the elements $u_i \in T^1$, $i < \theta$, and $\mathrm{ord}(v)$ must be a limit ordinal, say γ.

Suppose v is in T^2_λ. By Definition 3.5(3)(i), $\mathrm{cst}(v)$ must be unbounded in γ, and of course $\mathrm{cst}(v) = \bigcup_{i<\theta} \mathrm{cst}(u_i)$, since $\bar{u}$ is $\lhd$-increasing. Because Suc^+ contains only successor ordinals and each u_i is in T^1, $\mathrm{cst}(u_i) \cap \mathrm{Suc}^+ = \emptyset$ for every $i < \theta$. Hence $\mathrm{cst}(v)$ is disjoint from Suc^+ contrary to Definition 3.5(3)(ii).

It follows that v must be in T^1 and there are $s, t, r \in \mathrm{Fun}(\kappa, 2)$ such that

$$\begin{gathered}\mathrm{fun}_1(v) \subseteq s, \quad \mathrm{fun}_2(v) \subseteq t, \quad \mathrm{fun}_3(v) \subseteq r,\\ \mathrm{ord}(v) \in C_{s,t,r}, \quad \mathrm{cst}(v) = C_{s,t,r} \cap \mathrm{ord}(v).\end{gathered}$$

For every $i < \theta$ and $\alpha_i = \mathrm{ord}(u_i)$, $u_i \lhd v$ implies that $\alpha_i \in \mathrm{cst}(v) \subseteq C_{s,t,r}$. Hence, for each $i < \theta$,

$$\langle V_{\alpha_i}, \in, P \cap V_{\alpha_i}, s{\restriction}\alpha_i, t{\restriction}\alpha_i, r{\restriction}\alpha_i \rangle \prec \langle V_\kappa, \in, P, s, t, r \rangle \models \phi,$$

and for $\delta = \bigcup_{i<\theta} \alpha_i$,

$$\langle V_\delta, \in, P \cap V_\delta, s{\restriction}\delta, t{\restriction}\delta, r{\restriction}\delta \rangle \prec \langle V_\kappa, \in, P, s, t, r \rangle \models \phi.$$

Consequently, the tuple $\langle s{\restriction}\delta, t{\restriction}\delta, r{\restriction}\delta, C_{s,t,r} \cap \delta \rangle = \bigcup_{i<\theta} u_i$ is in T^1 (note that $\beth_{\alpha_i} = \alpha_i$, $\beth_\delta = \delta$, and by the choice of v, $v = \bigcup_{i<\theta} u_i$). ■

4. The basic functions. In this section we define those basic functions whose compositions form the family of functions used in the definition of the final models.

Definition 4.1. For every $\beta < \kappa$ and $\gamma < \beth_{\beta+1}$ we define a function c^β_γ with domain $\beth_\beta$ as follows: for all $\xi < \beth_\beta$,

$$c^\beta_\gamma(\xi) = (\beth_\beta \cdot (\gamma + 1)) + \xi,$$

where $\cdot$ and $+$ are the ordinal multiplication and addition respectively. Write $\mathcal{E}$ for the family of functions $\{f \restriction A \mid f \in E \text{ and } A \subseteq \mathrm{dom}(f)\}$, where E is the set $\bigcup\{\{c^\beta_\gamma, (c^\beta_\gamma)^{-1}\} \mid \beta < \kappa \text{ and } \gamma < \beth_{\beta+1}\}$. The *reflection point* of $d \in \mathcal{E} \smallsetminus \{\emptyset\}$, denoted by $\mathrm{ref}(d)$, is the unique ordinal β for which there is $\gamma < \beth_{\beta+1}$ such that either $d \subseteq c^\beta_\gamma$ or $d \subseteq (c^\beta_\gamma)^{-1}$.

Fact 4.2. (a) *For all increasing $d, e \in \mathcal{E}$ and $Y = \mathrm{ran}(d) \cap \mathrm{ran}(e)$, $d^{-1}\restriction Y = e^{-1}\restriction Y$.*

(b) *For every $e \in \mathcal{E}$, either e is increasing and all the elements in* $\mathrm{ran}(e)$ *have the same cardinality, or otherwise, e is decreasing and all the elements in* $\mathrm{dom}(e)$ *have the same cardinality.*

(c) *For all $e \in \mathcal{E}$ and $\xi < \zeta \in \mathrm{dom}(e)$, $\zeta - \xi = e(\zeta) - e(\xi)$.*

Definition 4.3. First we need some auxiliary means used in this definition only. For all functions p and e, $p \uplus e$ is the function $p \cup (e\restriction(\mathrm{dom}(e) \smallsetminus \mathrm{dom}(p)))$. Let Ref^1 be the set of all limit ordinals below κ and Ref^2 be the set of all successor ordinals below κ. For every $\alpha < \kappa$ let $\ll_\alpha$ be a fixed well-ordering of $T^{1,2}_\lambda[\alpha]$ and define a well-ordering of $T^{1,2}_\lambda$ by

$$u \ll v \quad \text{iff} \quad \mathrm{ord}(u) < \mathrm{ord}(v) \text{ or } (\mathrm{ord}(u) = \mathrm{ord}(v) = \alpha \text{ and } u \ll_\alpha v).$$

For each $w \in T^{1,2}_\lambda$, denote the set $\{w' \mid w' \ll w\}$ by $T^{1,2}_\lambda[\ll w]$ (used in (1) only). Furthermore, let $\mathrm{id}(u)$, for $u \in T^{1,2}_\lambda$, denote the following identity function:

$$\{\langle \xi, \xi \rangle \mid \xi < \mathrm{ord}(u) \text{ and } \mathrm{fun}_1(u)\restriction\xi + 1 = \mathrm{fun}_2(u)\restriction\xi + 1\}.$$

Now define for each $u \in T^{1,2}_\lambda$ a function p_u as follows.

(a) First of all $p_u = \emptyset$ for $u = \langle \emptyset, \emptyset, \emptyset, \emptyset \rangle$.

(b) Suppose $u \in T[<]$, u is a successor of v, and for v the function p_v is already defined. Fix an ordinal $\beta < \mathrm{ord}(u)$ as follows:

(1) Suppose $u \in T^1$. For all $w \in T^{1,2}_\lambda$, define inductively

$$\beta'_w = \min(\mathrm{Ref}^1 \smallsetminus (\mathrm{dom}(\mathrm{id}(u)) \cup (\mathrm{ord}(w) + 1) \cup \{\beta'_{w'} \mid w' \in T^{1,2}_\lambda[\ll w]\})).$$

Fix β to be β'_v.

(2) Suppose $u \in T^2_\lambda$. By Definition 3.5, $\mathrm{ord}(u) \in \mathrm{Suc}^+$ and there is a unique $\beta' \in \mathrm{Ref}^2$ with $\mathrm{ord}(u) = \beta' + 1$. Fix β to be that β'. Note that necessarily $v = \pi(\mathrm{ord}(u))$.

Assume $\langle w^\beta_{\gamma'} \mid \gamma' < \theta\rangle$ with $\theta \le \beth_{\beta+1}$ is a fixed enumeration of $T[\beta]$ without repetition. Let γ be the ordinal for which $u{\restriction}\beta = w^\beta_\gamma$. We define

$$p_u = \begin{cases} \mathrm{id}(u) \uplus (p_v \uplus c^\beta_\gamma) & \text{if } \mathrm{ran}(p_v) \text{ is ordinal,}\\ \mathrm{id}(u) \uplus (p_v \uplus (c^\beta_\gamma)^{-1}) & \text{otherwise.}\end{cases}$$

(e) Suppose $u \in T[\lessdot]$, u is a limit, and for all $v \lhd u$, the functions p_v are defined. Then define p_u to be $\bigcup_{v\lhd u} p_v$. Note that by the definition of $\uplus$, $p_w \subseteq p_v$ for all $w \lhd v \lhd u$.

(f) For all $u \in T^{1,2}_\lambda \smallsetminus T[\lessdot]$ define p_u to be $(p_{u^{-1}})^{-1}$.

For every successor $u \in T^{1,2}_\lambda \smallsetminus \{\langle\emptyset,\emptyset,\emptyset,\emptyset\rangle\}$, say a successor of $v \in T^{1,2}_\lambda$, there is a unique $e \in \mathcal{E}$ such that either $e = \emptyset$, or otherwise $\mathrm{dom}(e) = \mathrm{dom}(p_u) \smallsetminus \mathrm{dom}(p_v)$ and $p_u = p_v \cup e$. We denote this e by c^u.

REMARK. The part $\mathrm{id}(u)$ in the definition above is needed first time in Lemma 6.3(d) to ensure that all the functions have arbitrary large extensions. Note also that p_u might be $\mathrm{id}(u) = \{\langle\xi,\xi\rangle \mid \xi < \beth_\beta\}$ when $\mathrm{ord}(u) = \beta+1$, $u \in T^2_\lambda$ is a successor of $\langle\emptyset,\emptyset,\emptyset,\emptyset\rangle$, and $\mathrm{fun}_1(u){\restriction}\beth_\beta = \mathrm{fun}_2(u){\restriction}\beth_\beta$.

FACT 4.4. (a) *If $u \in T^{1,2}_\lambda$, $\xi \in \mathrm{dom}(p_u)$, and $p_u(\xi) = \xi$, then $p_u(\zeta) = \zeta$ for all $\zeta \le \xi$.*

(b) *For every $u \in T^{1,2}_\lambda$, p_u is a partial function from $\beth_{\mathrm{ord}(u)}$ into $\beth_{\mathrm{ord}(u)}$.*

(c) *For all $u, v \in T^{1,2}_\lambda$, $u \lhd v$ implies $p_u \subsetneq p_v$.*

(d) *For every successor $u \in T^{1,2}_\lambda \smallsetminus \{\langle\emptyset,\emptyset,\emptyset,\emptyset\rangle\}$, $\mathrm{dom}(p_u)$ is the cardinal $\beth_{\mathrm{ref}(c^u)}$ iff c^u is $\emptyset$ or c^u is increasing, and $\mathrm{ran}(p_u)$ is the cardinal $\beth_{\mathrm{ref}(c^u)}$ iff c^u is $\emptyset$ or c^u is decreasing.*

(e) *For all limit points $u \in T^{1,2}_\lambda$, $\mathrm{dom}(p_u) = \bigcup_{v\lhd u}\mathrm{dom}(p_v) = \mathrm{ran}(p_u) = \bigcup_{v\lhd u}\mathrm{ran}(p_v) = \beth_{\mathrm{ord}(u)}$.*

(f) *For all limit points $u \in T^2_\lambda$, $\mathrm{dom}(p_u)$ is a cardinal of cofinality less than λ.*

(g) *Suppose that both u and v are successor elements in $T^{1,2}_\lambda$. If $c^u \cap c^v \neq \emptyset$ then u and v are successors of the same element, $c^u = c^v$, and for $\beta = \mathrm{ref}(c^u) = \mathrm{ref}(c^v)$, $u{\restriction}\beta = v{\restriction}\beta$.*

Proof. The proofs of (b)–(e) are straightforward inductions on the tree order $\lhd$. Note that for every limit $u \in T^{1,2}_\lambda$, $u = \bigcup_{v\lhd u} v$ by Fact 3.7. Note also that in Definition 4.3(1), when $u \in T^1$ is a successor of v, the following holds:

$$\beta_v < (\mathrm{card}(T^{1,2}_\lambda[\ll v]))^+ \le {\beth_{\mathrm{ord}(v)+1}}^+ < \mathrm{ord}(u),$$

since $\mathrm{card}(T^{1,2}_\lambda[\ll v]) \le \beth_{\mathrm{ord}(v)+1}$, $\mathrm{ord}(u) = \beth_{\mathrm{ord}(u)}$, and Ref^1 is the set of all limit ordinals.

(f) By (e), $\mathrm{dom}(p_u)$ is the cardinal $\beth_{\mathrm{ord}(u)}$. By Definition 3.5(3)(ii), $\mathrm{cf}(\mathrm{ord}(u)) < \lambda$.

(g) Let w^1 and w^2 be such that u is a successor of w^1 and v is a successor of w^2. By Definition 4.3(b), $p_u = p_{w^1} \cup c^u$ and $p_v = p_{w^2} \cup c^v$. If $w_1 = w_2$ and $c^u \cap c^v \neq \emptyset$ then there are $\beta < \min\{\mathrm{ord}(u), \mathrm{ord}(v)\}$ and $\gamma < \beth_{\beta+1}$ such that $c^u = c^v \subseteq c^\beta_\gamma$ and $u{\restriction}\beta = w^\beta_\gamma = v{\restriction}\beta$, where w^β_γ is given in Definition 4.3(b).

Suppose $u \in T^1$. If $v \in T^1$ and $w^1 \neq w^2$ then $\mathrm{ref}(c^u) \neq \mathrm{ref}(c^v)$ because the mapping $w \mapsto \beta'_w$, given in Definition 3.5, is injective. Hence $c^u \cap c^v = \emptyset$. Assume $v \in T^2_\lambda$. Then $\mathrm{ref}(c^v) \in \mathrm{Ref}^2 \setminus \mathrm{Ref}^1$, and because $\mathrm{ref}(c^u) \in \mathrm{Ref}^1$, we have $c^u \cap c^v = \emptyset$. Similarly, $c^u \cap c^v = \emptyset$ if $u \in T^2_\lambda$ and $v \in T^1$.

Suppose both $u \in T^2_\lambda$ and $v \in T^2_\lambda$. Then $\pi(\mathrm{ord}(u)) = w^1$ and $\pi(\mathrm{ord}(v)) = w^2$, and there are $\beta^1, \beta^2 \in \mathrm{Ref}^2$ with $\mathrm{ord}(u) = \beta^1 + 1$ and $\mathrm{ord}(v) = \beta^2 + 1$. If $\mathrm{ord}(u) \neq \mathrm{ord}(v)$ then $\mathrm{ref}(c^u) = \beta^1 \neq \beta^2 = \mathrm{ref}(c^v)$ and hence $c^u \cap c^v = \emptyset$. On the other hand, when $\mathrm{ord}(u)$ equals $\mathrm{ord}(v)$, we have $w^1 = \pi(\mathrm{ord}(u)) = \pi(\mathrm{ord}(v)) = w^2$. ∎

5. The family of functions. If the reader now quickly looks over Section 7, she or he realizes that the final models are such that the basic functions defined in the previous section become partial isomorphisms between the models. The basic functions have the extension property by the choice of the bookkeeping function π, the tree T^2_λ, and the definition of $\uplus$ in Definition 4.3 (Lemma 6.3). Furthermore, if s and t are equivalent functions with respect to $\sim_{\phi,P}$, then they determine a branch in T^1, which in turn yields a sequence of basic functions whose union forms an isomorphism between $\mathcal{M}_s$ and $\mathcal{M}_t$ (Lemma 7.4).

However, the basic functions alone do not suffice to ensure that the definition of the models (Definition 7.1) makes sense. To achieve that the basic functions really are partial isomorphisms, we have to "close" the interpretations of the relations under composition of basic functions. This leads to a technical problem: we must have a control over compositions of basic functions. In order to do that, we define a restricted collection of compositions. Unfortunately even the restricted set is a little bit confused. During the first reading of this section it might be helpful to the reader to think that $\lambda = \aleph_0$. Then T^2_λ does not contain limit nodes. However, remember that even if $\lambda = \aleph_0$, then T^1 does contain many limit nodes.

Remark. We want to point out again that the Π^1_1-indescribability plays no role in this section. The assumption that κ is a weakly compact cardinal is used in Lemma 7.7 only. Even there, the main point is to use Lemma 7.6(b), which, roughly speaking, says that the isomorphism types of small parts of the models defined are in a prearranged set of models (provided that the small part is strongly equivalent to a fixed model). The proof of Lemma 7.6

itself is straightforwardly based on the analysis of the composition of basic functions presented in the last three lemmas of this section (to be more precise, Lemmas 5.6(c), 5.7(e)–(g) and 5.8(d)).

Definition 5.1. For every set X of ordinals, the ordinal $\sup\{\alpha+1 \mid \alpha \in X\}$ is abbreviated by $\sup^+(X)$. For all sequences $\bar{\alpha} = \langle \alpha_i \mid i < \theta\rangle$ of ordinals, $\sup^+(\{\alpha_i \mid i < \theta\})$ is denoted by $\sup^+(\bar{\alpha})$, and the sequence $\langle f(\alpha_i) \mid i < \theta\rangle$ is abbreviated by $f(\bar{\alpha})$.

Definition 5.2. We define Seq to be the set of all pairs $\langle \bar{u}, W\rangle$ satisfying the following conditions:

(A) W is nonempty.

(B) For some $n < \omega$, $\bar{u}$ is a sequence $\langle u_i \mid i < n\rangle$ of elements in $T^{1,2}_\lambda \setminus \{\langle \emptyset, \emptyset, \emptyset, \emptyset\rangle\}$.

(C) Let W_0 be W. Inductively for every $i < n-1$, $W_i \subseteq \operatorname{dom}(p_{u_i})$ and $W_{i+1} = p_{u_i}[W_i]$.

(D) For every $i < n-1$, $\operatorname{fun}_2(u_i)\restriction\sup^+(W_{i+1}) = \operatorname{fun}_1(u_{i+1})\restriction\sup^+(W_{i+1})$.

For every $\langle \bar{u}, W\rangle$ in Seq there is a natural sequence $\bar{g}^{\bar{u},W}$ of functions defined as follows:

$$\bar{g}^{\bar{u},W} = \langle g^{\bar{u},W}_i \mid i < \operatorname{lh}(\bar{u})\rangle,$$

where each $g^{\bar{u},W}_i$ is a shorthand for $p_{u_i}\restriction W_i$. The composition $g^{\bar{u},W}_{\operatorname{lh}(\bar{u})-1} \circ \ldots \circ g^{\bar{u},W}_0$ is denoted by $g^{\bar{u},W}$. For all sequences $\bar{f} = \langle f_i \mid i < n\rangle$, $1 \le n < \omega$, which are of the form $\bar{g}^{\bar{u},W}$ for some fixed $\langle \bar{u}, W\rangle \in$ Seq with $\operatorname{lh}(\bar{u}) = n$, we shall use the following notation:

- for each $i < \operatorname{lh}(\bar{u})$,

$$\begin{aligned} \operatorname{ind}(f_i) &= u_i, \\ \operatorname{beg}(f_i) &= \operatorname{fun}_1(u_i)\restriction\sup^+(\operatorname{dom}(f_i)), \\ \operatorname{end}(f_i) &= \operatorname{fun}_2(u_i)\restriction\sup^+(\operatorname{ran}(f_i)); \end{aligned}$$

- $\operatorname{beg}(f) = \operatorname{beg}(f_0)$ and $\operatorname{end}(f) = \operatorname{end}(f_{\operatorname{lh}(\bar{u})-1})$;
- f is the composition $f_{\operatorname{lh}(\bar{u})-1} \circ \ldots \circ f_0$;
- for $i < \operatorname{lh}(\bar{u})$, $f_{\le i}$ is a shorthand for $f_i \circ \ldots \circ f_0$, and for all $\xi \in \operatorname{dom}(f)$,

$$f_{<i}(\xi) = \begin{cases} \xi & \text{if } i = 0, \\ f_{i-1} \circ \ldots \circ f_0(\xi) & \text{otherwise.} \end{cases}$$

Definition 5.3. A sequence $\bar{f} = \bar{g}^{\bar{u},W}$, $\langle \bar{u}, W\rangle \in$ Seq, is called *minimal* if the following two conditions are satisfied:

(A) for all $i < \operatorname{lh}(\bar{f})$ and $v \triangleleft \operatorname{ind}(f_i)$, $\operatorname{dom}(f_i) \not\subseteq \operatorname{dom}(p_v)$;

(B) there are no indices $i \le j < \operatorname{lh}(\bar{f})$ such that the composition $f_j \circ \ldots \circ f_i$ is identity and $\operatorname{beg}(f_i) = \operatorname{end}(f_j)$.

Let $\bar{\mathcal{F}}$ be the set $\{\bar{g}^{\bar{u},W} \mid \langle \bar{u}, W\rangle \in \mathrm{Seq}$ and $\bar{g}^{\bar{u},W}$ is minimal$\}$. The collection $\{f \mid \bar{f} \in \bar{\mathcal{F}}$ and $\mathrm{lh}(\bar{f}) = 1\}$ is abbreviated by $\mathcal{F}_1$.

The following fact and lemma are basic properties needed in the proof of the last three lemmas.

FACT 5.4. (a) *For all* $\langle \bar{u}, W\rangle \in \mathrm{Seq}$, *either* $g^{\bar{u},W}$ *is the identity function and* $\mathrm{beg}(g^{\bar{u},W}) = \mathrm{end}(g^{\bar{u},W})$, *or otherwise*, *there is a sequence* $\bar{f} \in \bar{\mathcal{F}}$ *such that* $\mathrm{lh}(\bar{f}) \le \mathrm{lh}(\bar{u})$, $f = g^{\bar{u},W}$, $\mathrm{beg}(f) = \mathrm{beg}(g^{\bar{u},W})$, *and* $\mathrm{end}(f) = \mathrm{end}(g^{\bar{u},W})$.

(b) *For every* $q \in \mathcal{F}_1$ *there is* $\theta \in \mathrm{dom}(q)$ *such that for all* $\xi \in \mathrm{dom}(q) \smallsetminus \theta$, $q(\xi) \neq \xi$.

(c) *For every* $q \in \mathcal{F}_1$ *with* $\mathrm{ind}(q) = u$ *a successor*, c^u *is nonempty.*

Note that for all functions x and sets X, $x{\restriction}X$ means the restricted function $x{\restriction}(\mathrm{dom}(x) \cap X)$.

LEMMA 5.5. *For all nonempty* $q \in \mathcal{F}_1$,

$$\mathrm{ind}(q) \textit{ is a successor} \quad \textit{iff} \quad \sup{}^+(\mathrm{dom}(q)) \neq \sup{}^+(\mathrm{ran}(q)).$$

Moreover, *if* $\mathrm{ind}(q) = u$ *is a limit*, *then*

$$\sup{}^+(\mathrm{dom}(q)) = \sup{}^+(\mathrm{ran}(q)) = \mathrm{dom}(p_u) = \mathrm{ran}(p_u) = \beth_{\mathrm{ord}(u)}.$$

Proof. First of all recall that q is not identity (Fact 5.4(b)). Suppose first that $\mathrm{ind}(q) = u$ is a successor of $v \in T^{1,2}_\lambda$. Then $q \subseteq p_v \cup e$ for $e = c^u{\restriction}\mathrm{dom}(q)$. Abbreviate $\mathrm{ref}(e)$ by γ. We have $\gamma \ge \mathrm{ord}(v)$ and $\mathrm{dom}(p_v) \cup \mathrm{ran}(p_v) \subseteq \beth_{\mathrm{ord}(v)} \le \beth_\gamma$. Because e is nonempty, the claim follows from the facts that $\mathrm{dom}(e) \cap \beth_\gamma \neq \emptyset$ implies $\mathrm{dom}(e) \subseteq \beth_\gamma$ and $\mathrm{ran}(e) \cap \beth_\gamma = \emptyset$, and on the other hand, $\mathrm{dom}(e) \cap \beth_\gamma = \emptyset$ implies $\mathrm{ran}(e) \subseteq \beth_\gamma$.

Assume $\mathrm{ind}(q)$ is a limit. Denote $\mathrm{ind}(q)$ by u and $\beth_{\mathrm{ord}(u)}$ by μ. Because $\mathrm{dom}(p_u) = \mathrm{ran}(p_u) = \mu$ it suffices to prove that $\sup^+(\mathrm{dom}(q)) \ge \mu$ and $\sup^+(\mathrm{ran}(q)) \ge \mu$.

Let $\theta < \kappa$ be such that $\langle v_i \mid i < \theta\rangle$ is a $\lhd$-increasing enumeration of the elements $w \lhd u$. We know that for all ordinals i in $\{j + (2n+1) \mid j < \theta$ is a limit ordinal or $0,$ and $n < \omega\}$, $\mathrm{dom}(p_{v_i})$ is a cardinal. If $\sup^+(\mathrm{dom}(q)) < \mu$ then there would be $i < \theta$ with $\mathrm{dom}(q) \subseteq \mathrm{dom}(p_{v_i})$ contrary to Definition 5.3(A). So $\mathrm{dom}(q)$ must be unbounded in μ.

Moreover, we know that for all ordinals i in the set $I = \{j + 2n \mid j < \theta$ is a limit ordinal and $n < \omega\}$, $\mathrm{ran}(p_{v_i})$ is a cardinal. So if $\sup^+(\mathrm{ran}(q)) < \mu$ and $i \in I$ is such that $\mathrm{ran}(q) \subseteq \mathrm{ran}(p_{v_i})$, then $\mathrm{dom}(q) \subseteq \mathrm{dom}(p_{v_i})$ since q is injective and $\mathrm{ran}(p_{v_i})$ is an ordinal. Thus $\mathrm{ran}(q)$ is also unbounded in μ. ■

It remains to prove the last three main lemmas of the paper. The intuition behind the lemmas is simple. Just directly look at how a single ordinal is mapped under the relevant compositions. For example, because of the coding mechanism (Definition 4.3), every ordinal is mapped to a greater ordinal "in a unique way" as presented in Lemma 5.6(a). Recall also the explanations of the introduction.

The first lemma is mainly used in the proof of Lemma 7.4, where it is shown that the models $\mathcal{M}_s$ and $\mathcal{M}_t$ with $s, t \in {}^{\kappa}2$, are isomorphic if and only if s and t are equivalent with respect to $\sim_{\phi,P}$ (especially Lemma 5.6(e)). The third lemma, Lemma 5.8, is also applied in the proof of Lemma 7.4.

Lemma 5.6. (a) *Suppose* $p, q \in \mathcal{F}_1$ *and* $e \in \mathcal{E}$ *is a nonempty increasing function with* $e \subseteq p \cap q$ *and* $\mathrm{ref}(e) = \beta$. *Then for* $X = \mathrm{dom}(p) \cap \mathrm{dom}(q) \cap \beth_\beta$, $p \restriction X = q \restriction X$ *and* $\mathrm{ind}(p) \restriction \beta = \mathrm{ind}(q) \restriction \beta$. *In particular*, $\mathrm{beg}(p) \restriction \sup^+(X) = \mathrm{beg}(q) \restriction \sup^+(X)$.

(b) *For all* $q \in \mathcal{F}_1$, *if* $\sup^+(\mathrm{dom}(q))$ *is a cardinal*, *then* $\sup^+(\mathrm{dom}(q)) \leq \sup^+(\mathrm{ran}(q))$.

(c) *If* $\bar{f} \in \bar{\mathcal{F}}$ *and* $\sup^+(\mathrm{dom}(f_0)) < \sup^+(\mathrm{ran}(f_0))$ *then* $\sup^+(\mathrm{dom}(f_i)) < \sup^+(\mathrm{ran}(f_i))$ *for every* $i < \mathrm{lh}(\bar{f})$.

(d) *For all* $\bar{f} \in \bar{\mathcal{F}}$ *and* $i < \mathrm{lh}(\bar{f})$, $\sup^+(\mathrm{dom}(f_i) \cup \mathrm{ran}(f_i)) \leq \sup^+(\mathrm{dom}(f) \cup \mathrm{ran}(f))$.

(e) *For all* $\bar{f} \in \bar{\mathcal{F}}$, *if* $\sup^+(\mathrm{ran}(f))$ *is a cardinal* μ *and* $\mathrm{dom}(f) \subseteq \mu$ *then* $\sup^+(\mathrm{dom}(f)) = \mu$. *Furthermore*, *if* $\mathrm{dom}(f) = \mu$ *then* $\mathrm{dom}(f_i) = \mathrm{ran}(f_i) = \mu$ *for each* $i < \mathrm{lh}(\bar{f})$.

Proof. (a) By Definition 4.3, there are successors $u, v \in T^{1,2}_\lambda$ with $u \trianglelefteq \mathrm{ind}(p)$, $v \trianglelefteq \mathrm{ind}(q)$, $c^u \restriction \mathrm{dom}(e) = p_u \restriction \mathrm{dom}(e) = e = p_v \restriction \mathrm{dom}(e) = c^v \restriction \mathrm{dom}(e)$, and $\mathrm{dom}(p_u) = \mathrm{dom}(p_v) = \beth_\beta$. By Fact 4.4(g), u and v are successors of the same element, say $w \in T^{1,2}_\lambda$, $c^u = c^v$, and $u \restriction \beta = v \restriction \beta$. Therefore we have

$$p \restriction \beth_\beta \subseteq p_u = p_w \cup c^u = p_w \cup c^v = p_v \supseteq q \restriction \beth_\beta,$$

and

$$\mathrm{beg}(p) \restriction \beth_\beta \subseteq \mathrm{fun}_1(u) \restriction \beth_\beta = \mathrm{fun}_1(v) \restriction \beth_\beta \supseteq \mathrm{beg}(q) \restriction \beth_\beta.$$

(b) Suppose that $\sup^+(\mathrm{dom}(q)) > \sup^+(\mathrm{ran}(q))$. Then by Lemma 5.5, $\mathrm{ind}(q)$ must be a successor. Denote $c^{\mathrm{ind}(q)} \restriction \mathrm{dom}(q)$ by d. Necessarily d is decreasing and $\mathrm{dom}(d)$ is an end segment of $\mathrm{dom}(q)$. By Fact 4.2(b), we have $\mathrm{card}(\sup^+(\mathrm{dom}(d))) = \mathrm{card}(\min \mathrm{dom}(d))$. Thus the following ordinal is not a cardinal:

$$\sup^+(\mathrm{dom}(q)) = \sup^+(\mathrm{dom}(d)) > \min \mathrm{dom}(d).$$

(c) Suppose, contrary to the claim, that there is $j \in \{1, \ldots, \mathrm{lh}(\bar{f}) - 1\}$ with $\sup^+(\mathrm{dom}(f_j)) \geq \sup^+(\mathrm{ran}(f_j))$. We may assume that j is the smallest possible index with this property.

Suppose first that $\sup^+(\mathrm{dom}(f_j)) = \sup^+(\mathrm{ran}(f_j))$. Then for $u = \mathrm{ind}(f_j)$, $\sup^+(\mathrm{dom}(f_j))$ is the cardinal $\beth_{\mathrm{ord}(u)}$ by Lemma 5.5. It follows from the equalities $\mathrm{dom}(f_j) = \mathrm{ran}(f_{j-1}) = \mathrm{dom}(f^{-1}_{j-1})$, and by applying (b) to f^{-1}_{j-1}, that $\sup^+(\mathrm{dom}(f_{j-1}^{-1})) \le \sup^+(\mathrm{ran}(f_{j-1}^{-1}))$. However, then

$$\sup^+(\mathrm{dom}(f_{j-1})) \ge \sup^+(\mathrm{ran}(f_{j-1})),$$

contrary to the choice of j.

So suppose $\sup^+(\mathrm{dom}(f_j)) > \sup^+(\mathrm{ran}(f_j))$. Note that $\sup^+(\mathrm{dom}(f_{j-1})) < \sup^+(\mathrm{ran}(f_{j-1}))$. Abbreviate $\mathrm{ind}(f_{j-1})$ by u and $\mathrm{ind}(f_j)$ by v. By Lemma 5.5, there are $w^1, w^2 \in T_\lambda^{1,2}$ such that u is a successor of w^1 and v is a successor of w^2. Denote $\mathrm{ref}(c^u)$ by γ^1, $\mathrm{ref}(c^v)$ by γ^2, $c^u{\restriction}\mathrm{dom}(f_{j-1})$ by d^1, and $c^v{\restriction}\mathrm{dom}(f_j)$ by d^2. Then d^1, d^2 are nonempty, d^1 is increasing, d^2 is decreasing, and

$$\begin{gathered} f_{j-1} \subseteq p_u = p_{w^1} \cup d^1, \qquad \mathrm{ran}(p_{w^1}) \subseteq \beth_{\gamma^1}, \qquad \mathrm{ran}(d^1) \subseteq \beth_{\gamma^1+1} \setminus \beth_{\gamma^1}, \\ f_j \subseteq p_v = p_{w^2} \cup d^2, \qquad \mathrm{dom}(p_{w^2}) \subseteq \beth_{\gamma^2}, \qquad \mathrm{dom}(d^2) \subseteq \beth_{\gamma^2+1} \setminus \beth_{\gamma^2}. \end{gathered}$$

Because $\mathrm{ran}(f_{j-1}) = \mathrm{dom}(f_j)$, it follows that $\gamma_1 = \gamma_2$ and $\mathrm{ran}(d^1) = \mathrm{dom}(d^2)$. By Fact 4.2(a), $d^1 = (d^2)^{-1}$, and hence $c^u \cap c^{(v^{-1})} \ne \emptyset$ (the notation v^{-1} is explained in Definition 3.4). By Fact 4.4(g), $w^1 = (w^2)^{-1}$ and $u{\restriction}\gamma_1 = v^{-1}{\restriction}\gamma_1$. Consequently, $f_{j-1} = f_j^{-1}$ and for $\theta = \sup^+(\mathrm{dom}(f_{j-1})) = \sup^+(\mathrm{ran}(f_j)) \le \beth_{\gamma_1}$, $\mathrm{beg}(f_{j-1}) = \mathrm{fun}_1(u){\restriction}\theta = \mathrm{fun}_1(v^{-1}){\restriction}\theta = \mathrm{fun}_2(v){\restriction}\theta = \mathrm{end}(f_j)$ contrary to the minimality of $\bar f$.

(d) Denote $\sup^+(\mathrm{dom}(f) \cup \mathrm{ran}(f))$ by θ. To reach a contradiction let $j < \mathrm{lh}(\bar f)$ be the smallest index with $\sup^+(\mathrm{dom}(f_j) \cup \mathrm{ran}(f_j)) > \theta$. Then $\sup^+(\mathrm{dom}(f_j)) \le \theta < \sup^+(\mathrm{ran}(f_j))$ since $\mathrm{dom}(f_0) = \mathrm{dom}(f)$ and $\mathrm{ran}(f_i) = \mathrm{dom}(f_{i+1})$ for every $i < j$. However, from (c) it follows that $\sup^+(\mathrm{ran}(f_j)) \le \sup^+(\mathrm{dom}(f_k)) < \sup^+(\mathrm{ran}(f_k))$ for all $k \in \{j+1, \ldots, \mathrm{lh}(\bar f) - 1\}$, and so $\sup^+(\mathrm{ran}(f)) = \sup^+(\mathrm{ran}(f_{\mathrm{lh}(\bar f)-1})) > \theta$, a contradiction.

(e) By applying (b) to f_n^{-1}, for $n = \mathrm{lh}(\bar f) - 1$, we see that $\sup^+(\mathrm{dom}(f_n)) \ge \sup^+(\mathrm{ran}(f_n)) = \mu$. By (d), $\sup^+(\mathrm{dom}(f_n)) \le \mu$. Hence $\sup^+(\mathrm{dom}(f_n)) = \mu$. In the same way it can be shown that $\sup^+(\mathrm{dom}(f_i)) = \mu$ for all $i \le n$.

Suppose $\mathrm{dom}(f) = \mu$. By Lemma 5.5, $\mathrm{ind}(f_i)$ is a limit point, say $u_i \in T_\lambda^{1,2}$, and $\sup^+(\mathrm{ran}(f_i)) = \mu = \mathrm{dom}(p_{u_i}) = \mathrm{ran}(p_{u_i})$ for every $i \le n$. However, when $n > 1$, $f_0 \subseteq p_{u_0}$ together with $\mathrm{dom}(f_0) = \mathrm{dom}(f) = \mu = \mathrm{dom}(p_u)$ implies that $f_0 = p_{u_0}$ and $\mathrm{ran}(f_0) = \mu = \mathrm{dom}(f_1)$. A similar reasoning shows $\mathrm{dom}(f_i) = \mathrm{ran}(f_i) = \mu$ for every $i \le n$. ∎

The next lemma is the main tool in the proof of the most important lemma of the paper, namely Lemma 7.6(b) (in particular, the last three items).

Lemma 5.7. (a) *Suppose* $\bar f, \bar g \in \bar{\mathcal F}$ *are such that* $\mathrm{dom}(f) = \mathrm{dom}(g) = \{\xi\}$, $f(\xi) = g(\xi)$, *and* f_i *is increasing for every* $i < \mathrm{lh}(\bar f)$. *Then* $\mathrm{lh}(\bar f) \le \mathrm{lh}(\bar g)$ *and for* $k = \mathrm{lh}(\bar g) - \mathrm{lh}(\bar f)$, *both* $f_i = g_{k+i}$ *and* $\mathrm{beg}(f_i) = \mathrm{beg}(g_{k+i})$ *for every* $i < \mathrm{lh}(\bar f)$. *Moreover*, *if* $\mathrm{beg}(g) = \mathrm{beg}(f)$ *or for every* $j < \mathrm{lh}(\bar g)$, g_j *is increasing*, *then* $\mathrm{lh}(\bar f) = \mathrm{lh}(\bar g)$.

(b) *Suppose* $q \in \mathcal{F}_1$, $\xi < \zeta \in \mathrm{dom}(q)$, *and that both* $q{\restriction}\{\xi\}$ *and* $q{\restriction}\{\zeta\}$ *are increasing. If there is no* $d \in \mathcal{E}$ *with* $\{\xi, \zeta\} \subseteq \mathrm{dom}(d)$ *then* $\mathrm{card}(\xi) < \beth_{\mathrm{ref}(q{\restriction}\{\xi\})} \le \mathrm{card}(\zeta)$.

(c) *Suppose* $\langle q_1, q_2\rangle \in \bar{\mathcal{F}}$ *and* $\sup^+(\mathrm{dom}(q_1)) < \sup^+(\mathrm{ran}(q_1))$. *Then* $\mathrm{ind}(q_1)$, $\mathrm{ind}(q_2)$ *are successors, the functions* $d^1 = c^{\mathrm{ind}(q_1)}{\restriction}\mathrm{dom}(q_1)$ *and* $d^2 = c^{\mathrm{ind}(q_2)}{\restriction}\mathrm{dom}(q_2)$ *are increasing, and* $\mathrm{dom}(q_2) \smallsetminus \min\mathrm{ran}(d^1) \subseteq \mathrm{dom}(d^2)$.

(d) *Suppose* $\bar{f} \in \bar{\mathcal{F}}$ *and* $\sup^+(\mathrm{dom}(f_0)) < \sup^+(\mathrm{ran}(f_0))$. *Then for every* $i < \mathrm{lh}(\bar{f})$, $\mathrm{ind}(f_i)$ *is a successor,* $c^{\mathrm{ind}(f_i)}$ *is increasing, and in particular, for* $d = c^{\mathrm{ind}(f_0)}{\restriction}\mathrm{dom}(f_0)$ *and for every* $i \in \{1, \ldots, \mathrm{lh}(\bar{f}) - 1\}$, $\mathrm{dom}(f_i) \smallsetminus f_{<i}(\min\mathrm{ran}(d)) \subseteq c^{\mathrm{ind}(f_i)}$.

(e) *Let* $\bar{f}$ *be as in* (d). *For* $\bar{u} = \langle \mathrm{ind}(f_i) \mid i < \mathrm{lh}(\bar{f})\rangle$ *and for every* θ *with* $\sup^+(\mathrm{dom}(f)) < \theta \le \beth_{\mathrm{ref}(d)}$, *the pair* $\langle \bar{u}, \theta\rangle$ *is in* Seq *and the sequence* $\bar{g}^{\bar{u},\theta}$ *is in* $\bar{\mathcal{F}}$ (see Definition 5.2). *Furthermore, if* $\mathrm{end}(f)$ *is a function with a constant value, then* $\mathrm{end}(g^{\bar{u},\theta}) \supseteq \mathrm{end}(f)$ *is also a constant function.*

(f) *For every* $\bar{f}$ *in* $\bar{\mathcal{F}}$ *there is* $\xi \in \mathrm{dom}(f)$ *satisfying* $\mathrm{ran}(f) \subseteq f(\xi) + \sup^+(\mathrm{dom}(f))$.

(g) *Suppose* $\bar{g} \in \bar{\mathcal{F}}$, $\mathrm{dom}(g)$ *is a cardinal* μ, $\mathrm{ind}(g_0) = u_0$ *is a successor, and* c^{u_0} *is increasing. Assume* $\bar{h} \in \bar{\mathcal{F}}$ *is such that* $\mathrm{dom}(h) = \{\xi, \xi'\} \subseteq \mu$, $\xi \in \mathrm{dom}(c^{u_0})$, $h(\xi) = g(\xi)$, *and* $\mathrm{beg}(h) \subseteq \mathrm{beg}(g)$. *Then either* $h(\xi') \in \mathrm{ran}(g)$, *or otherwise,* $h(\xi') \ge h(\xi) + \mu$.

Proof. (a) Denote $\mathrm{lh}(\bar{f})-1$ by n and $\mathrm{lh}(\bar{g})-1$ by m. If g_m were decreasing, then, by applying Lemma 5.6(c) to the sequence $\langle g_m^{-1}, \ldots, g_0^{-1}\rangle$ in $\bar{\mathcal{F}}$, g_i should be decreasing for every $i \le m$ and $\sup^+(\mathrm{dom}(g)) > \sup^+(\mathrm{ran}(g_m)) = \sup^+(\mathrm{ran}(f_n)) > \sup^+(\mathrm{dom}(f)) = \xi + 1$ contrary to the assumption $\mathrm{dom}(f) = \mathrm{dom}(g) = \{\xi\}$. Thus g_m is increasing. Since $\mathrm{ran}(f_n) = \mathrm{ran}(g_m)$, $f_n = g_m$ by Fact 4.2(a). By Lemma 5.6(a), $\mathrm{beg}(f_n) = \mathrm{beg}(g_m)$. When $n > 0$, $\mathrm{ran}(f_{n-1}) = \mathrm{dom}(f_n) = \mathrm{dom}(g_m) = \mathrm{ran}(g_{m-1})$. Hence we can repeat the same argument to deduce that $f_{n-i} = g_{m-i}$ and $\mathrm{beg}(f_{n-i}) = \mathrm{beg}(g_{m-i})$ for every $i \le \min\{m, n\}$. However $m \ge n$ since otherwise $\mathrm{dom}(g) = \{f_{<n-m}(\xi)\} \ne \{\xi\}$.

If $m > n$ and $\mathrm{beg}(g) = \mathrm{beg}(f)$ then $g_{\le m-n-1}(\xi) = \xi$ and $\mathrm{end}(g_{m-n-1}) = \mathrm{beg}(g_{m-n}) = \mathrm{beg}(f_0) = \mathrm{beg}(f) = \mathrm{beg}(g) = \mathrm{beg}(g_0)$ contrary to the minimality of $\bar{g}$.

If $m > n$ and g_j is increasing for every $j < \mathrm{lh}(\bar{g})$, then $\mathrm{ran}(g_{m-n-1}) = \mathrm{dom}(g_{m-n}) = \mathrm{dom}(f_0) = \{\xi\}$ and $\mathrm{dom}(g) = \{g_0^{-1} \circ \ldots \circ g_{m-n-1}^{-1}(\xi)\} \ne \{\xi\}$, a contradiction.

(b) Let $u \trianglelefteq \mathrm{ind}(q)$ be the smallest element with $\xi \in \mathrm{dom}(p_u)$, and $v \trianglelefteq \mathrm{ind}(q)$ be the smallest element with $\zeta \in \mathrm{dom}(p_v)$. Then u and v are successors, $u \trianglelefteq v$, $\xi \in \mathrm{dom}(c^u)$, $\zeta \in \mathrm{dom}(c^v)$, and $q{\restriction}\{\xi, \zeta\} \subseteq c^u \cup c^v \subseteq p_u \cup c^v$. Assume $c^u \ne c^v$. Then $u \triangleleft v$. Since c^u is increasing, $\mathrm{dom}(p_u) = \beth_\beta$ where

$\beta = \mathrm{ref}(c^u)$. Because $\zeta \in \mathrm{dom}(c^v) \setminus \mathrm{dom}(c^u)$, we have $\mathrm{dom}(c^v) \cap \beth_\beta = \emptyset$. So $\mathrm{card}(\xi) < \beth_\beta \le \mathrm{card}(\zeta)$.

(c) From Lemma 5.6(c) it follows that $\sup^+(\mathrm{dom}(q_2)) < \sup^+(\mathrm{ran}(q_2))$. The elements $\mathrm{ind}(q_1)$ and $\mathrm{ind}(q_2)$ are successors by Lemma 5.5. If $q_2(\xi) = \xi$ then q_2 should be identity contrary to Fact 5.4(b). So both d^1 and d^2 are increasing. If for some $\xi \in \mathrm{ran}(d^1)$, $q_2 \restriction \{\xi\}$ is decreasing, then $(d^1)^{-1} \restriction \{\xi\} = q_2 \restriction \{\xi\}$, and as in the proof of Lemma 5.6(c), $q_1^{-1} = q_2$ and $\mathrm{beg}(q_1) = \mathrm{end}(q_2)$ contrary to the minimality of the sequence $\langle q_1, q_2 \rangle$. Thus $q_2 \restriction \{\xi\}$ is increasing for all $\xi \in \mathrm{ran}(d^1)$. Now $\mathrm{card}(\xi) = \mathrm{card}(\min \mathrm{ran}(d^1))$ for each $\xi \in \mathrm{ran}(d^1)$. By (b), there is $e \in \mathcal{E}$ with $e \subseteq q_2$ and $\mathrm{dom}(e) = \mathrm{ran}(d^1)$. Because $\mathrm{ran}(d^1) = \mathrm{dom}(e)$ is an end segment of $\mathrm{ran}(q_1) = \mathrm{dom}(q_2)$, $\mathrm{dom}(q_2) \setminus \min \mathrm{ran}(d^1) = \mathrm{dom}(e)$. Since e is increasing, it follows from the definition of d^2 that $e \subseteq d^2$.

(d) The claim follows from (c) by induction on $i < \mathrm{lh}(\bar{f})$.

(e) It suffices to show that $\langle \bar{u}, \theta \rangle \in \mathrm{Seq}$ since then the minimality of $\bar{g}^{\bar{u},\theta}$ follows from the fact that $\bar{f}$ is in $\bar{\mathcal{F}}$. Let β_i denote $\mathrm{ref}(c^{u_i})$ for every $i < \mathrm{lh}(\bar{u}) = \mathrm{lh}(\bar{f})$. Abbreviate $\min \mathrm{dom}(d)$ by ξ. By (d), $\mathrm{ran}(p_{u_i}) \subseteq \beth_{(\beta_i)+1} \le \beth_{\beta_{(i+1)}} = \mathrm{dom}(p_{u_{i+1}})$ and $f_{\le i}(\xi) \in \beth_{\beta_i+1} \setminus \beth_{\beta_i}$ for every $i < \mathrm{lh}(\bar{u}) - 1$. So $\langle \bar{u}, \theta \rangle$ satisfies Definition 5.2(C). From $\langle \bar{u}, \mathrm{dom}(f) \rangle \in \mathrm{Seq}$ it follows that $\mathrm{fun}_2(u_i) \restriction (f_{\le i}(\xi) + 1) = \mathrm{fun}_1(u_{i+1}) \restriction (f_{\le i}(\xi) + 1)$ for all $i < \mathrm{lh}(\bar{u}) - 1$. These equations together with Definition 3.1 ensure that both of the functions $\mathrm{fun}_2(u_i)$ and $\mathrm{fun}_1(u_{i+1})$, for $i < \mathrm{lh}(\bar{u}) - 1$, have the same constant value on the interval $\beth_{\beta_i+1} \setminus \beth_{\beta_i}$. Hence the pair $\langle \bar{u}, \theta \rangle$ also satisfies Definition 5.2(D). Similarly, the latter claim, concerning $\mathrm{end}(f)$, is a consequence of the facts that for $n = \mathrm{lh}(\bar{u}) - 1$, $f(\xi) \in \mathrm{ran}(g^{\bar{u},\theta}) = \mathrm{ran}(g_n^{\bar{u},\theta}) \subseteq \beth_{\beta_n+1} \setminus \beth_{\beta_n}$ and $\mathrm{fun}_2(u_n)$ is a constant function on the interval $\beth_{\beta_n+1} \setminus \beth_{\beta_n}$.

(f) Abbreviate $\sup^+(\mathrm{dom}(f))$ by θ, $\mathrm{lh}(\bar{f}) - 1$ by n, and for every $i \le n$, $\mathrm{ind}(f_i)$ by u_i. If $\sup^+(\mathrm{ran}(f)) \le \theta$ there is nothing to prove. So assume $\sup^+(\mathrm{ran}(f)) > \theta$. There must be a smallest index $j \le n$ satisfying $\sup^+(\mathrm{dom}(f_j)) \le \theta < \sup^+(\mathrm{ran}(f_j))$, u_j is a successor, and $c^{u_j} \restriction \mathrm{dom}(f_j)$, abbreviated by d, is increasing. Let ξ be $\min \mathrm{dom}(d)$. Then for all $\zeta \in \mathrm{dom}(f_j) \setminus \mathrm{dom}(d)$, $f_j(\zeta) < f_j(\xi)$, by the definition of c^{u_j}. Moreover, $\mathrm{dom}(f_j) \subseteq \theta$ together with Fact 4.2(c) ensures that for all $\zeta \in \mathrm{dom}(d)$, $f_j(\zeta) - f_j(\xi) = d(\zeta) - d(\xi) = \zeta - \xi < \theta$. So the claim holds in case $j = n$.

Suppose $n > j$. From (d) it follows that for every $i \in \{j+1, \ldots, n\}$, u_i is a successor, c^{u_i} is increasing, and $\mathrm{dom}(f_i) \setminus f_{<i}(\xi) \subseteq \mathrm{dom}(c^{u_i})$. For all $\zeta \in \mathrm{dom}(f_j) \setminus \mathrm{dom}(d)$, $f(\zeta) < f(\xi)$ since $f_j(\zeta) < f_j(\xi)$ and the property "$f_i \restriction \{\xi\}$ is increasing for every $i \in \{j+1, \ldots, n\}$" implies $f_{\le i}(\zeta) < f_{\le i}(\xi)$ for every $i \in \{j+1, \ldots, n\}$. Suppose $\zeta \in \mathrm{dom}(d)$, $i \in \{j+1, \ldots, n\}$, and $f_{<i}(\xi) < f_{<i}(\zeta) < f_{<i}(\xi) + \theta$. Then $\{f_{<i}(\xi), f_{<i}(\zeta)\} \subseteq \mathrm{dom}(c^{u_i})$ and by Fact 4.2(c),

$$f_{\le i}(\zeta) - f_{\le i}(\xi) = c^{u_i}(f_{<i}(\zeta)) - c^{u_i}(f_{<i}(\xi)) = f_{<i}(\zeta) - f_{<i}(\xi) < \theta.$$

The claim follows from the fact that $\mathrm{ran}(f) \smallsetminus f(\xi) = \mathrm{ran}(f_n) \smallsetminus f_{\leq n}(\xi) \subseteq \mathrm{ran}(c^{u_n})$ (remember $\mathrm{dom}(f_n) \smallsetminus f_{<n}(\xi) \subseteq \mathrm{dom}(c^{u_n})$ and c^{u_n} is increasing).

(g) Denote $\mathrm{lh}(\bar{g}) - 1$ by n, $\mathrm{lh}(\bar{h}) - 1$ by m and for each $i \leq m$ abbreviate $h_{<i}(\xi)$ by ξ_i and $h_{<i}(\xi')$ by ξ'_i. Write ξ_{m+1} for $h(\xi)$ and ξ'_{m+1} for $h(\xi')$. Note that by (d), for every $i \leq n$ and for $u_i = \mathrm{ind}(g_i)$, u_i is a successor, c^{u_i} is increasing, and $g_i\restriction\{g_{<i}(\xi)\} = c^{u_i}\restriction\{g_{<i}(\xi)\}$.

There exists a smallest index $j \leq m$ with $h_j\restriction\{\xi_j\} = g_0\restriction\{\xi\}$, because otherwise for the minimal reduct $\bar{d}$ of the sequence $\langle h_i\restriction\{\xi_i\} \mid i \leq m\rangle$ (see Fact 5.4), $\bar{d}$ differs from the minimal sequence $\bar{e} = \langle g_i\restriction\{g_{<i}(\xi)\} \mid i \leq n\rangle$ and $e_n \circ \ldots \circ e_0 = d_{\mathrm{lh}(\bar{d})-1} \circ \ldots \circ d_0$ contrary to (a). We have two cases to consider:

(1) $\xi'_j \geq \mu$;
(2) $\xi'_j < \mu$.

(1) Suppose first that $\xi'_j \geq \mu$. Note that $\mu > \xi = \xi_j$. Note that $h_j(\xi'_j) \neq \xi'_j$ since otherwise also $h_j(\xi_j) = \xi_j$. The function $h_j\restriction\{\xi'_j\}$ must be increasing; otherwise, we reach a contradiction in the following manner. Assume $h_j\restriction\{\xi'_j\}$ is decreasing. There are two subcases:

(i) Assume that $\sup^+(\mathrm{dom}(h_j)) > \sup^+(\mathrm{ran}(h_j))$ or $\sup^+(\mathrm{dom}(h_{j-1})) > \sup^+(\mathrm{ran}(h_{j-1}))$.

(ii) By Lemma 5.5,

$$\sup^+(\mathrm{dom}(h_j)) \neq \sup^+(\mathrm{ran}(h_j)), \qquad \sup^+(\mathrm{dom}(h_{j-1})) \neq \sup^+(\mathrm{ran}(h_{j-1})).$$

So suppose that both $\sup^+(\mathrm{dom}(h_j)) < \sup^+(\mathrm{ran}(h_j))$ and $\sup^+(\mathrm{dom}(h_{j-1})) < \sup^+(\mathrm{ran}(h_{j-1}))$.

(i) It would follow from the assumption $\xi'_j \geq \mu$ and by applying (d) to the sequence $\langle h_j^{-1}, \ldots, h_0^{-1}\rangle$ or $\langle h_{j-1}^{-1}, \ldots, h_0^{-1}\rangle$ that $\sup^+(\mathrm{dom}(h)) = \sup^+(\mathrm{dom}(h_0)) > \mu$, a contradiction.

(ii) The function $h_{j-1}\restriction\{\xi'_{j-1}\}$ is increasing, since otherwise,

$$\sup^+(\mathrm{dom}(h_{j-1})) \geq \xi'_{j-1} + 1 \geq h_{j-1}(\xi'_{j-1}) + 1 = \xi'_j + 1 = \sup^+(\mathrm{ran}(h_{j-1})).$$

Let β be $\mathrm{ref}(h_{j-1}\restriction\{\xi'_{j-1}\})$. If $\xi_{j-1} \geq \beth_\beta$, then $\xi_{j-1} > \xi'_{j-1}$ and $h(\xi_{j-1}) \neq \xi_{j-1}$. Moreover, $\mathrm{ref}(h_{j-1}\restriction\{\xi_{j-1}\}) > \beta$ and $\sup^+(\mathrm{dom}(h_{j-1})) = \xi_{j-1} + 1 > \beth_{\beta+1} > \xi'_j + 1 = \sup^+(\mathrm{ran}(h_{j-1}))$, a contradiction. On the other hand, if $\xi_{j-1} < \beth_\beta$, then it follows from the assumption $\xi'_j > h_j(\xi'_j)$ that $h_{j-1}\restriction\{\xi'_{j-1}\} = (h_j\restriction\{\xi'_j\})^{-1}$. By Lemma 5.6(a), $h_j = h_{j-1}^{-1}$ and $\mathrm{beg}(h_{j-1}) = \mathrm{end}(h_j)$ contrary to the minimality of $\bar{h}$.

Hence $h_j\restriction\{\xi'_j\}$ is increasing, and by (d), $\mathrm{ind}(h_i)$, abbreviated by v_i, is a successor, c^{v_i} is increasing, and $\xi'_i \in \mathrm{dom}(c^{v_i})$ for every $i \in \{j, \ldots, m\}$. We show by induction on $i \in \{j, \ldots, m\}$ that $\xi_i + \mu \leq \xi'_i$ where $+$ is the ordinal addition.

Since $\xi_j = \xi < \mu$, $\xi'_j \geq \mu$, and μ is cardinal, we have $\xi_j + \mu \leq \xi'_j$. Suppose $i < m$ and $\xi_i + \mu \leq \xi'_i$. If $h_i(\xi_i) = \xi_i = \xi_{i+1}$ then $\xi_{i+1} + \mu = \xi_i + \mu \leq \xi'_i < h_i(\xi'_i) = \xi'_{i+1}$.

If $\xi_i \in \mathrm{dom}(c^{v_i})$, then $\xi'_{i+1} = c^{v_i}(\xi'_i)$, $\xi_{i+1} = c^{v_i}(\xi_i)$, and $\xi'_{i+1} - \xi_{i+1} = c^{v_i}(\xi'_i) - c^{v_i}(\xi_i) = \xi'_i - \xi_i \geq \mu$.

If $\xi_i \notin \mathrm{dom}(c^{v_i})$, then the reflection point of $h_i{\restriction}\{\xi_i\}$, say β, is smaller than $\mathrm{ref}(c^{v_i})$ by the definition of c^{v_i}. Since $\xi_{i+1} \leq \beth_{\beta+1} \leq \beth_{\mathrm{ref}(c^{v_i})}$ and $\mu < \beth_{\mathrm{ref}(c^{v_i})}$, it follows that $\xi_{i+1} + \mu < \beth_{\mathrm{ref}(c^{v_i})} < \xi'_{i+1}$.

(2) Suppose then that $\xi'_j < \mu$. Abbreviate $\mathrm{ref}(c^{u_i})$, for $i \leq n$, by γ_i. Since $h_j{\restriction}\{\xi_j\} = c^{u_0}{\restriction}\{\xi\} = g_0{\restriction}\{\xi\}$ is increasing and $\{\xi_j, \xi'_j\} \subseteq \mu = \mathrm{dom}(g_0) \leq \beth_{\gamma_0}$, we see by Lemma 5.6(a) that $\mathrm{beg}(h_j) \subseteq \mathrm{beg}(g_0)$ and $h_j{\restriction}\{\xi'_j\} = g_0{\restriction}\{\xi'_j\}$. By (d), $h_i{\restriction}\{\xi_i\}$ is increasing for all $i \in \{j, \ldots, m\}$. It follows from $\xi = \xi_j$ and $h(\xi_j) = h(\xi) = g(\xi)$ together with (a) that $\langle h_i{\restriction}\{\xi_i\} \mid j \leq i \leq m\rangle = \langle g_k{\restriction}\{g_{<k}(\xi)\} \mid k \leq n\rangle$.

To show that $h(\xi') = h_m \circ \ldots \circ h_j(\xi'_j) \in \mathrm{ran}(g)$ we prove by induction on $k \leq n$ that $h_{j+k}{\restriction}\{\xi'_{j+k}\} = g_k{\restriction}\{\xi'_{j+k}\}$. Note that $m = j + n$ and it is possible that $\xi'_j \neq \xi'$. We already proved the case $k = 0$. Suppose $k > 0$ and for every $i < k$ the subclaim holds. Then $\{\xi_{j+k}, \xi'_{j+k}\} = \mathrm{dom}(h_{j+k}) \subseteq \mathrm{ran}(g_{k-1}) = \mathrm{dom}(g_k)$. Since $h_{j+k}{\restriction}\{\xi_{j+k}\} = g_k{\restriction}\{g_{<k}(\xi)\}$ is increasing and $\xi'_{j+k} \in \mathrm{dom}(g_k) \subseteq \beth_{\gamma_k}$, we conclude by Lemma 5.6(a) that $h_{j+k}{\restriction}\{\xi'_{j+k}\} = g_k{\restriction}\{\xi'_{j+k}\}$. ∎

The last properties below, Lemma 5.8(c), (d), are needed in the proof of Lemmas 7.4(b) and 7.6(a).

LEMMA 5.8. (a) *Suppose* $p, q \in \mathcal{F}_1$ *are such that* $\mathrm{ind}(p)$ *is a limit*, $\mathrm{dom}(p) = \mathrm{dom}(q) = X$, *and the set* $Y = \{\zeta \in X \mid p(\zeta) = q(\zeta)\}$ *is unbounded in* X. *Then* $\mathrm{ind}(q) = \mathrm{ind}(p)$, *and in particular*, $p = q$, $\mathrm{beg}(p) = \mathrm{beg}(q)$, *and* $\mathrm{end}(p) = \mathrm{end}(q)$.

(b) *Suppose* $\bar{f}$ *in* $\bar{\mathcal{F}}$ *and the set* $I_0 = \{\xi \in \mathrm{dom}(f_0) \mid \xi < f_0(\xi)\}$ *is unbounded in* $\mathrm{dom}(f_0)$. *Then there is an end segment* J *of* I_0 *such that for every* $\xi \in J$ *and* $i < \mathrm{lh}(\bar{f})$, $f_i{\restriction}\{f_{<i}(\xi)\}$ *is increasing.*

(c) *Suppose* $\bar{f}, \bar{g} \in \bar{\mathcal{F}}$ *and* $n < \omega$ *are such that* $\mathrm{lh}(\bar{f}) = \mathrm{lh}(\bar{g}) = n$, $\mathrm{dom}(f) = \mathrm{ran}(f)$ *is a cardinal*, *and* $f \subseteq g$. *Then* $\mathrm{ind}(f_i) \trianglelefteq \mathrm{ind}(g_i)$ *and* $f_i \subseteq g_i$ *for every* $i < n$.

(d) *Suppose* $\bar{\alpha}$ *is an increasing sequence* $\langle \alpha_l \mid l < \omega\rangle$ *of ordinals below* κ *such that* $\sup^+(\bar{\alpha})$ *is a cardinal. Then for every* $\bar{f} \in \bar{\mathcal{F}}$ *such that* $\mathrm{dom}(f) = \{\alpha_l \mid l < \omega\}$, *there are infinitely many indices* $l < \omega$ *with* $f(\alpha_l) \neq \alpha_l$.

Proof. (a) Let u be $\mathrm{ind}(p)$ and v be $\mathrm{ind}(q)$. We may assume that $p(\zeta) = q(\zeta) \neq \zeta$ for all $\zeta \in Y$. Let Z be the set $\{\min\{\zeta, p(\zeta)\} \mid \zeta \in Y\}$. By Lemma 5.6(a), $u{\restriction}\xi = v{\restriction}\xi$ for every $\xi \in Z$. Since Y is unbounded in X and $\mathrm{ind}(p)$ is a limit, also Z is unbounded in X. By Lemma 5.5, $\sup^+(X) =$

$\sup^+(\mathrm{dom}(p)) = \mathrm{dom}(p_u) = \beth_{\mathrm{ord}(u)}$. Hence, as in the proof of Fact 3.7, $u \trianglelefteq v$. By Definition 5.3(A), $u = v$. Because p and q have the common domain X, it follows that $p = q$, $\mathrm{beg}(p) = \mathrm{beg}(q)$ and $\mathrm{end}(p) = \mathrm{end}(q)$.

(b) If $\mathrm{ind}(f_0)$ is a successor then $\sup^+(\mathrm{dom}(f_0)) < \sup^+(\mathrm{ran}(f_0))$, and the claim follows from Lemma 5.7(d). Suppose $\mathrm{ind}(f_0)$ is a limit. By Lemma 5.5, $\sup^+(\mathrm{dom}(f_0))$ is a cardinal. Suppose, contrary to the claim, that $j < n$ is the smallest index for which there is an unbounded $J \subseteq I_0$ such that for every ζ in the set $Y = \{f_{<j}(\xi) \mid \xi \in J\}$, $f_j \restriction \{\zeta\}$ is increasing and $f_{j+1} \restriction \{f_j(\zeta)\}$ is not increasing. Then $\mathrm{ind}(f_i)$ is a limit for every $i \leq j$, since otherwise the existence of the chosen j contradicts Lemma 5.7(d). Therefore $\sup^+(\mathrm{dom}(f_j))$ is the cardinal $\sup^+(\mathrm{dom}(f_0))$, and necessarily Y is unbounded in $\mathrm{dom}(f_j)$. So we may assume that $f_{j+1} \restriction \{f_j(\zeta)\}$ is decreasing for every $\zeta \in Y$. Since $f_j \restriction \{\zeta\}$ must equal $(f_{j+1} \restriction \{f_j(\zeta)\})^{-1}$ for every $\zeta \in Y$ it follows from (a) that $f_j = f_{j+1}^{-1}$ and $\mathrm{beg}(f_j) = \mathrm{end}(f_{j+1})$ contrary to the minimality of $\bar{f}$.

(c) In the case $n = 1$ the claim is proved in (a). Assume $n > 1$. Let θ be the cardinal $\mathrm{dom}(f) = \mathrm{ran}(f)$. For each $i < n$, $\mathrm{dom}(f_i) = \mathrm{ran}(f_i) = \theta$ by Lemma 5.6(e). By Lemma 5.5, $\mathrm{ind}(f_i)$ is a limit point and $f_i = p_{\mathrm{ind}(f_i)}$ for every $i < n$. Denote the set $\{\xi < \theta \mid f_0(\xi) > \xi\}$ by I_0. Then I_0 must be unbounded in θ by Definition 4.3. For each $i < n - 1$ define I_{i+1} to be $\{\xi \in f_i[I_i] \mid f_{i+1}(\xi) > \xi\}$.

By (b), there is an end segment K of I_0 such that for every $i < n - 1$, $f_{\leq i}[K]$ is an end segment of I_{i+1}. Now $\mathrm{lh}(\bar{f}) = \mathrm{lh}(\bar{g}) = n$ and $f(\xi) = g(\xi)$ together with Lemma 5.7(a) imply that $f_{\leq i}(\xi) = g_{\leq i}(\xi)$ for all $\xi \in K$ and $i < n$. Since K is unbounded in θ and for each $i < n$, $f_i \restriction f_{<i}[K]$ is increasing, also $f_{\leq i}[K]$ is unbounded in θ for every $i < n$. By (a), $\mathrm{ind}(f_i) \trianglelefteq \mathrm{ind}(g_i)$ and $f_i = f_i \restriction \theta = g_i \restriction \theta$ for every $i < n$.

(d) Let θ be the cardinal $\sup^+(\bar{\alpha})$ and let n denote the length of $\bar{f}$. For every $l < \omega$ and $i < n$ write d_i^l for the function $f_i \restriction \{f_{<i}(\alpha_l)\}$. For every $i < n$ define I_i to be the set $\{l < \omega \mid d_i^l \text{ is increasing}\}$.

Suppose, contrary to the claim, that there is $m < \omega$ such that $f(\alpha_l) = \alpha_l$ for all $l \in \omega \setminus m$. By (b), I_0 is finite. There must be a smallest $j \in \{1, \ldots, n-1\}$ such that I_j is infinite. By Lemma 5.6(b), $\sup^+(\mathrm{dom}(f_0)) = \sup^+(\mathrm{ran}(f_0))$ and so $\mathrm{ind}(f_0)$ is a limit. Since $\mathrm{dom}(f_{i+1}) = \mathrm{ran}(f_i)$ for all $i < j$, we infer, by applying Lemma 5.6(b) repeatedly, that $\mathrm{ind}(f_i)$ is a limit point for all $i < j$. By the choice of j, there is an end segment J of ω such that $\min J \geq m$ and d_i^l is decreasing for all $l \in J$ and $i < j$ (d_i^l cannot be identity for unboundedly many $l < \omega$). The set $Y = \{f_{<j}(\alpha_l) \mid l \in J\}$ is unbounded in $\mathrm{dom}(f_j)$.

If $\mathrm{ind}(f_j)$ is a successor, then $Y \cap \mathrm{dom}(c^{\mathrm{ind}(f_j)})$ is infinite, and by Lemma 5.7(d), $f(\alpha_l) > \theta > \alpha_l$ for infinitely many $l \in J$, a contradiction. Hence

$\mathrm{ind}(f_j)$ is a limit. By (b), there is an end segment K of J such that for every $l \in J$ and $k \in \{j, \ldots, n-1\}$, d^l_k is increasing. For every $l \in K$, we have $f(\alpha_l) = \alpha_l$, and thus the compositions $(d^l_0)^{-1} \circ \ldots \circ (d^l_{j-1})^{-1}$ and $d^l_{n-1} \circ \ldots \circ d^l_j$ are equal. Since $\mathrm{end}(f_{j-1}) = \mathrm{beg}(f_j)$ and the sequences $\langle (d^l_{j-1})^{-1}, \ldots, (d^l_0)^{-1} \rangle$ and $\langle d^l_j, \ldots, d^l_{n-1} \rangle$ are in $\bar{\mathcal{F}}$ (in both, all the functions are increasing), it follows from Lemma 5.7(a), that these sequences are equal. In particular, $d^l_j = (d^l_{j-1})^{-1}$ for every $l \in K$. From (a) it would follow that $f_{j-1} = f_j$ and $\mathrm{beg}(f_{j-1}) = \mathrm{end}(f_j)$, contrary to the minimality of $\bar{f}$. ■

6. The back-and-forth properties of the family. To make sure that certain submodels are already strongly equivalent, we need the following closed unbounded set.

Definition 6.1. We define D to be the following closed unbounded subset of κ:

$$\{\mu \in \kappa \smallsetminus (\lambda+1) \mid \langle V_\mu, \in, \pi \cap V_\mu, X \cap V_\mu, Y \cap V_\mu \rangle \prec \langle V_\kappa, \in, \pi, X, Y \rangle\},$$

where π is the function from Definition 3.5 and

$$X = \{\langle p_u, \mathrm{beg}(p_u), \mathrm{end}(p_u)\rangle \mid u \in T^{1,2}_\lambda\},$$
$$Y = \{\langle v, p_v, \mathrm{beg}(p_v), \mathrm{end}(p_v)\rangle \mid v \in T^{1,2}_\lambda\}.$$

Note that $\beth_\mu = \mu$ for all $\mu \in D$.

The following notation is used for "all compositions of basic functions beginning from η and ending with ν". This is needed to define the "closure" of relations in each model $\mathcal{M}_\eta$ as described at the beginning of Section 5 (and formally presented in Definition 7.1).

Definition 6.2. For all $\mu \in D \cup \{\kappa\}$ and $\eta, \nu \in \mathrm{Fun}(\mu, 2)$ define

$$\bar{\mathcal{F}}[\eta,\nu] = \{\bar{f} \in \bar{\mathcal{F}} \mid \mathrm{ind}(f_i) \in T^{1,2}_\lambda[<\mu] \text{ for all } i < \mathrm{lh}(\bar{f}),$$
$$\mathrm{beg}(f) \subseteq \eta, \text{ and } \mathrm{end}(f) \subseteq \nu\};$$
$$\mathcal{F}_1[\eta,\nu] = \{f \mid \bar{f} \in \bar{\mathcal{F}}[\eta,\nu] \text{ and } \mathrm{lh}(\bar{f}) = 1\}.$$

In the lemma below we present the promised back-and-forth properties of the basic functions. Naturally the properties extend to compositions of basic functions too. The lemma is applied in the proof of Lemma 7.5.

Lemma 6.3. *Suppose* $\mu \in D \cup \{\kappa\}$.

(a) *For every* $u \in T^{1,2}_\lambda$, *if* u *is a limit point or a successor in* T^2_λ, *then* $p_u \in V_\mu$ *implies* $u \in T^{1,2}_\lambda[<\mu]$. *For all successors* $u \in T^1$, *if* p_u *is in* V_μ *then there is* $v \in T^{1,2}_\lambda[<\mu]$ *such that* $p_v = p_u$, $\mathrm{beg}(p_v) = \mathrm{beg}(p_u)$, *and* $\mathrm{end}(p_v) = \mathrm{end}(p_u)$.

(b) *For every* $v \in T^{1,2}_\lambda[<\mu]$ *and* $\gamma < \mu$, *there is* $\alpha \in (\mathrm{Suc}^+ \cap \mu) \smallsetminus (\gamma + 1)$ *such that for every* $\eta', \nu' \in \mathrm{Fun}(\beth_\alpha, 2)$ *with* $\mathrm{fun}_1(v) \subseteq \eta'$ *and* $\mathrm{fun}_2(v) \subseteq \nu'$, *we can find* $u^{\eta',\nu'} \in T^2_\lambda$ *such that* $\mathrm{fun}_1(u^{\eta',\nu'}) = \eta'$, $\mathrm{fun}_2(u^{\eta',\nu'}) = \nu'$, *and* $u^{\eta',\nu'}$ *is a successor of* v.

(c) *If* $\eta, \nu \in \mathrm{Fun}(\mu, 2)$ *and* $q \in \mathcal{F}_1[\eta, \nu]$ *is such that for* $v = \mathrm{ind}(q)$ *both* $\mathrm{fun}_1(v) \subseteq \eta$ *and* $\mathrm{fun}_2(v) \subseteq \nu$, *then there is* $u \in T^{1,2}_\lambda[<\mu]$ *such that* $\mathrm{ind}(q) \trianglelefteq u$ (*implying* $q \subseteq p_u$), $p_u \in \mathcal{F}_1[\eta, \nu]$, *and* $\theta \subseteq \mathrm{dom}(p_u) \cap \mathrm{ran}(p_u)$.

(d) *Suppose* $\eta, \nu \in \mathrm{Fun}(\mu, 2)$, $q \in \mathcal{F}_1[\eta, \nu]$, *and* $\theta < \mu$. *There is* $\bar{f} \in \bar{\mathcal{F}}[\eta, \nu]$ *such that* $q \subseteq f$ *and* $\theta \subseteq \mathrm{dom}(f) \cap \mathrm{ran}(f)$.

(e) *Suppose* $\eta, \nu \in \mathrm{Fun}(\mu, 2)$, $\bar{f} \in \bar{\mathcal{F}}[\eta, \nu]$, *and* $\theta < \mu$. *There is* $\bar{g} \in \bar{\mathcal{F}}[\eta, \nu]$ *with* $g \supseteq f$ *and* $\theta \subseteq \mathrm{dom}(g) \cap \mathrm{ran}(g)$.

Proof. Properties (a)–(c) are straightforward consequences of the definition of the functions p_u. We sketch the proofs of the remaining properties.

(d) Here we need the small detail that we used $\mathrm{id}(u)$ in Definition 4.3. Denote $\mathrm{ind}(q)$ by v. If both $\mathrm{fun}_1(v) \subseteq \eta$ and $\mathrm{fun}_2(v) \subseteq \nu$, then the claim follows from (c).

Let $\eta', \nu' \in \mathrm{Fun}(\mu, 2)$ be such that $\mathrm{fun}_1(v) \subseteq \eta'$ and $\mathrm{fun}_2(v) \subseteq \nu'$. Fix elements u^0, u^1, u^2 from T^2_λ so that

- u^0 is $\triangleleft$-smallest with $\mathrm{fun}_1(u^0) \subseteq \eta$, $\mathrm{fun}_2(u^0) \subseteq \eta'$, and $\theta \subseteq \mathrm{dom}(p_{u^0})$;
- u^1 is $\triangleleft$-smallest with $\mathrm{fun}_1(u^1) \subseteq \eta'$, $\mathrm{fun}_2(u^1) \subseteq \nu'$, $v \triangleleft u^1$ and $\mathrm{ran}(p_{u^0} \restriction \theta) \subseteq \mathrm{dom}(p_{u^1})$;
- u^2 is $\triangleleft$-smallest with $\mathrm{fun}_1(u^2) \subseteq \nu'$, $\mathrm{fun}_2(u^2) \subseteq \nu$, $\mathrm{ran}(p_{u^1} \restriction \mathrm{ran}(p_{u^0} \restriction \theta)) \subseteq \mathrm{dom}(p_{u^2})$.

Define $\bar{f}$ to be $\bar{g}^{\bar{w},W}$, where $\bar{w} = \langle u^i \mid 0 \leq i \leq 2 \rangle$. Then $\bar{f}$ is in $\bar{\mathcal{F}}[\eta, \nu]$.

Define ξ_1 to be $\min\{\zeta + 1 \mid \zeta \in \mathrm{dom}(q)$ and $\eta(\zeta) \neq \mathrm{fun}_1(v)(\zeta)\}$, and ξ_2 to be $\min\{\zeta + 1 \mid \zeta \in \mathrm{dom}(q)$ and $\nu(\zeta) \neq \mathrm{fun}_2(v)(\zeta)\}$. Since $\mathrm{beg}(q) \subseteq \eta$ and $\mathrm{end}(q) \subset \nu$, we have $\xi_1 \geq \sup^+(\mathrm{dom}(q))$ and $\xi_2 \geq \sup^+(\mathrm{ran}(q))$. So $\eta \restriction \xi_1 = \eta' \restriction \xi_1$ and $\nu' \restriction \xi_2 = \nu \restriction \xi_2$ ensure that $f_0^{-1} \restriction \mathrm{dom}(q)$ is identity and $f_2 \restriction \mathrm{ran}(q)$ is identity. Therefore $q \subseteq f$.

(e) Since $\mathrm{dom}(f) \cup \mathrm{ran}(f)$ is bounded in μ it follows from Lemma 5.6(d) that $\mathrm{dom}(f_i) \cup \mathrm{ran}(f_i)$ is bounded in μ for all $i < \mathrm{lh}(\bar{f})$. Hence for every $i < \mathrm{lh}(\bar{f})$, $f_i \in V_\mu$, and by (a) we may assume $\mathrm{ind}(f_i) \in V_\mu$. The claim follows from (d) by induction on $i < \mathrm{lh}(\bar{f})$. ∎

7. The strongly equivalent nonisomorphic models. Recall that κ is a fixed strongly inaccessible cardinal and λ is a fixed regular cardinal below κ.

For ordinals $\theta < \mu$ and subsets A of μ, $[A]^\theta$ is the set of all θ-sequences of ordinals in A. For every $\theta < \mu < \kappa$ define ($\sup^+(\boldsymbol{a})$ is defined in Definition 5.1)

$$[\mu]^\theta_B = \{\boldsymbol{a} \in [\mu]^\theta \mid \sup^+(\boldsymbol{a}) < \mu \text{ and for all } i < j < \theta, \boldsymbol{a}_i \neq \boldsymbol{a}_j\}.$$

Denote the union $\bigcup_{\theta<\mu} [\mu]^\theta_B$ by $[\mu]^{<\mu}_B$. We write $\mathbf{0}$ for the constant function having domain κ and range $\{0\}$.

DEFINITION 7.1. For every $\mu \in D \cup \{\kappa\}$ (D is defined in Definition 6.1), $\theta < \mu$, and $\boldsymbol{a} \in [\mu]^\theta_B$ we define a family $\langle R^\eta_{\boldsymbol{a}} \mid \eta \in \mathrm{Fun}(\mu, 2)\rangle$ of relations (having arity θ) on μ as follows: The relations $R^\eta_{\boldsymbol{a}}$, $\eta \in \mathrm{Fun}(\mu, 2)$, are the smallest subsets of $[\mu]^\theta$ closed under the following operations:

- if $\eta = \mathbf{0}{\restriction}\mu$ then $\boldsymbol{a} \in R^\eta_{\boldsymbol{a}}$;
- if there is $\bar{f} \in \bar{\mathcal{F}}[\mathbf{0}{\restriction}\mu, \eta]$ with $\mathrm{dom}(f) = \boldsymbol{a}$, then $f(\boldsymbol{a}) \in R^\eta_{\boldsymbol{a}}$.

Suppose $\mu \in D \cup \{\kappa\}$. Define ϱ_μ to be the vocabulary $\{R_{\boldsymbol{a}} \mid \boldsymbol{a} \in [\mu]^{<\mu}_B\}$ where each $R_{\boldsymbol{a}}$ is a relation symbol of arity $\mathrm{lh}(\boldsymbol{a})$. For every $\eta \in \mathrm{Fun}(\mu, 2)$, let $\mathcal{M}_\eta$ be the ϱ_μ-structure with domain μ and interpretations $(R_{\boldsymbol{a}})^{\mathcal{M}_\eta} = R^\eta_{\boldsymbol{a}}$ for all $\boldsymbol{a} \in [\mu]^{<\mu}_B$. For every $\chi \in D \cap \mu$ and $A \subseteq \mu$, we write $\mathcal{M}^{\varrho_\chi}_\eta{\restriction}A$ for the model having vocabulary ϱ_χ, domain A, and the interpretations $(R_{\boldsymbol{a}})^{\mathcal{M}^{\varrho_\chi}_\eta{\restriction}A} = R^\eta_{\boldsymbol{a}} \cap [A]^{\mathrm{lh}(\boldsymbol{a})}$ for each $\boldsymbol{a} \in [\chi]^{<\chi}_B$.

FACT 7.2. *Assume $\mu \in D \cup \{\kappa\}$ and $\eta \in \mathrm{Fun}(\mu, 2)$.*

(a) *For every $\boldsymbol{a} \in [\mu]^{<\mu}_B$, $R^\eta_{\boldsymbol{a}}$ is a subset of $[\mu]^{<\mu}_B$.*
(b) *For all $\chi \in D \cap \mu$, $\mathcal{M}_{\eta\restriction\chi} = \mathcal{M}^{\varrho_\chi}_\eta{\restriction}\chi$.*

Proof. (a) Assume that for some $\boldsymbol{b} \in R^\eta_{\boldsymbol{a}}$, $\sup^+(\boldsymbol{b}) = \mu$. Then there should be $\bar{f} \in \bar{\mathcal{F}}[\mathbf{0}{\restriction}\mu, \eta]$ with $\mathrm{dom}(f) = \boldsymbol{a}$ and $f(\boldsymbol{a}) = \boldsymbol{b}$ contrary to Lemma 5.6(e) and the fact $\sup^+(\boldsymbol{a}) < \mu$.

(b) Abbreviate $\eta{\restriction}\chi$ by ν and let $\boldsymbol{a}$ be a sequence from $[\chi]^{<\chi}_B$. The interpretation $(R_{\boldsymbol{a}})^{\mathcal{M}_\nu} = R^\nu_{\boldsymbol{a}}$ is a subset of the interpretation $(R_{\boldsymbol{a}})^{\mathcal{M}^{\varrho_\chi}_\eta{\restriction}\chi}$ since $\bar{\mathcal{F}}[\mathbf{0}{\restriction}\chi, \nu] \subseteq \bar{\mathcal{F}}[\mathbf{0}{\restriction}\mu, \eta]$. Suppose $\boldsymbol{b} \in (R_{\boldsymbol{a}})^{\mathcal{M}^{\varrho_\chi}_\eta{\restriction}\chi}$ and let $\bar{f} \in \bar{\mathcal{F}}[\mathbf{0}{\restriction}\mu, \eta]$ be such that $\mathrm{dom}(f) = \boldsymbol{a}$ and $f(\boldsymbol{a}) = \boldsymbol{b}$. By Lemma 6.3(a), we may assume $\mathrm{ind}(f_i) \in T^{1,2}_\lambda[{<}\chi]$ for every $i < \mathrm{lh}(\bar{f})$. Consequently, $\bar{f} \in \bar{\mathcal{F}}[\mathbf{0}{\restriction}\mu', \nu]$ and $\boldsymbol{b} \in R^\nu_{\boldsymbol{a}}$. ∎

FACT 7.3. *Suppose $\mu \in D \cup \{\kappa\}$ and $\eta, \nu \in \mathrm{Fun}(\mu, 2)$.*

(a) *For all $v \in T^{1,2}_\lambda$ with $\mathrm{fun}_1(v) \subseteq \eta$ and $\mathrm{fun}_2(v) \subseteq \nu$, the function p_v is a partial isomorphism from $\mathcal{M}_\eta$ into $\mathcal{M}_\nu$.*

(b) *For every $\theta < \mu$ and $\boldsymbol{b} \neq \boldsymbol{c} \in [\mu]^\theta_B$, if there exists $\boldsymbol{a} \in [\mu]^\theta_B$ satisfying*

$$\mathcal{M}_\eta \models R_{\boldsymbol{a}}(\boldsymbol{b}) \quad \text{and} \quad \mathcal{M}_\nu \models R_{\boldsymbol{a}}(\boldsymbol{c}),$$

then there is $\bar{f} \in \bar{\mathcal{F}}[\eta, \nu]$ with $f(\boldsymbol{b}) = \boldsymbol{c}$.

Proof. Both of these properties are direct consequences of Definition 7.1 and Fact 5.4. ∎

Lemma 7.4. *For all* $s, t \in \mathrm{Fun}(\kappa, 2)$,

(a) $s \sim_{\phi,P} t$ *implies* $\mathcal{M}_s \cong \mathcal{M}_t$ ($\sim_{\phi,P}$ *is given in Definition* 2.2), *and*
(b) *if* $\mathcal{M}_s \cong \mathcal{M}_t$ *then* $s \sim_{\phi,P} t$.

Proof. (a) Suppose $s \sim_{\phi,P} t$, and let $r : \kappa \to 2$ be such that $\langle V_\kappa, \in, P, s, t, r\rangle \models \phi$. For every $\delta \in C' = C_{s,t,r} \cap D$ define u_δ to be the tuple $\langle s\restriction\delta, t\restriction\delta, r\restriction\delta, C_{s,t,r} \cap \delta\rangle$ ($C_{s,t,r}$ is given in Definition 3.3). Directly by Definition 3.3, for all $\delta < \varepsilon \in C'$, u_δ, u_ε are in T^1 and $u_\delta \lhd u_\varepsilon$. Hence $p_{u_\delta} \subseteq p_{u_\varepsilon}$ for $\delta < \varepsilon \in C'$, and moreover, for the function $h = \bigcup_{\delta \in C'} p_{u_\delta}$ both $\mathrm{dom}(h) = \kappa$ and $\mathrm{ran}(h) = \kappa$. Consequently, h is an isomorphism from $\mathcal{M}_s$ onto $\mathcal{M}_t$.

(b) Suppose $s \neq t$ and for fixed $\xi < \kappa$, $s(\xi) \neq t(\xi)$. Let h be an isomorphism from $\mathcal{M}_s$ onto $\mathcal{M}_t$, and let S' be the set $\{\delta \in \kappa \setminus (\xi+1) \mid h[\delta] = \delta$ is a cardinal of cofinality $\geq \lambda\}$. Since h is an isomorphism and $s\restriction\delta \neq t\restriction\delta$ for all $\delta \in S'$, it follows from Fact 7.3(b) that for every $\delta \in S'$ there is a sequence $\bar{f}^\delta \in \bar{\mathcal{F}}[s,t]$ such that $f^\delta = h\restriction\delta$. For all $\delta < \varepsilon \in S'$, $f^\delta = h\restriction\delta \subseteq h\restriction\varepsilon = f^\varepsilon$. Since S' is stationary in κ, there are $n < \omega$ and a stationary subset S of S' such that $\mathrm{lh}(\bar{f}^\delta) = n$ for every $\delta \in S$.

Consider some $\delta \in S$ and $i < n$. Abbreviate $\mathrm{ind}(f_i^\delta)$ by u_i^δ. By Lemma 5.6(e), $\mathrm{dom}(f_i^\delta) = \mathrm{ran}(f_i^\delta) = \delta$. Moreover, by Lemma 5.8(c), $u_i^\delta \trianglelefteq u_i^\varepsilon$ and $f_i^\delta \subseteq f_i^\varepsilon$ for all $\varepsilon \in S \setminus \delta$. By Fact 4.4(f), u_i^δ is in T^1. So $f_i^\delta = p_{u_i^\delta}$ and $\mathrm{dom}(f_i^\delta) = \delta = \mathrm{ord}(u_i^\delta) = \beth_\delta$. Define, for each $i < n$,

$$s_i = \bigcup_{\delta \in S} \mathrm{fun}_1(u_i^\delta) \qquad r_i = \bigcup_{\delta \in S} \mathrm{fun}_3(u_i^\delta),$$

and let s_n be $\bigcup_{\delta \in S} \mathrm{fun}_2(u_{n-1}^\delta)$. Then $s = s_0$ and $t = s_n$.

We claim that $s \sim_{\phi,P} t$. By the transitivity of $\sim_{\phi,P}$ it is enough to show that for every $i < n$, r_i witnesses $s_i \sim_{\phi,P} s_{i+1}$. Contrary to this subclaim assume that for some $i < n$,

$$\langle V_\kappa, \in, P, s_i, s_{i+1}, r_i\rangle \not\models \phi.$$

Then there is $\delta \in S$ for which

$$\langle V_\delta, \in, P \cap V_\delta, s_i\restriction\delta, s_{i+1}\restriction\delta, r_i\restriction\delta\rangle \prec \langle V_\kappa, \in, P, s_i, s_{i+1}, r_i\rangle.$$

However $s_i\restriction\delta = \mathrm{fun}_1(u_i^\delta)$, $s_{i+1}\restriction\delta = \mathrm{fun}_2(u_i^\delta)$, and $r_i\restriction\delta = \mathrm{fun}_3(u_i^\delta)$, and so

$$\langle V_\delta, \in, P \cap V_\delta, \mathrm{fun}_1(u_i^\delta), \mathrm{fun}_2(u_i^\delta), \mathrm{fun}_3(u_i^\delta)\rangle \not\models \phi,$$

contrary to the fact that u_i^δ is in T^1. ∎

In the following two lemmas we assume existence of a regular cardinal μ in D. Such a μ does not necessarily exist if κ is an arbitrary strongly inaccessible cardinal. However, these lemmas are only preliminaries for the main lemma, Lemma 7.7, where we assume κ to be a weakly compact cardinal. Note that when $\mu = \kappa$, in Lemma 7.5(a) below, it suffices that κ is a strongly inaccessible cardinal.

LEMMA 7.5. *Suppose $\mu \in D$ is a regular cardinal or $\mu = \kappa$, and that η, ν are functions from μ into 2.*

(a) $\mathcal{M}_\eta \equiv_{\infty\mu;\lambda} \mathcal{M}_\nu$.

(b) *For every $\theta < \mu$, the model $\mathcal{M}_\eta$ satisfies the $L_{\infty\mu}$-sentence*

$$\forall \langle x_i \mid i < \theta \rangle \Big(\bigvee_{\boldsymbol{a} \in [\mu]^\theta_B} R_{\boldsymbol{a}}(\langle x_i \mid i < \theta \rangle) \Big).$$

(c) *For all $\boldsymbol{a} \in [\mu]^{<\mu}_B$ and $\xi < \mu$, the following $L_{\infty\mu}$-sentence holds in $\mathcal{M}_\eta$:*

$$\forall \bar{x} \ (R_{\boldsymbol{a}}(\bar{x}) \to \exists y (R_{\langle \xi \rangle \frown \boldsymbol{a}}(\langle y \rangle \frown \bar{x}))).$$

(d) *For all $\boldsymbol{a} \in [\mu]^{<\mu}_B$, $\mathcal{M}_\eta$ satisfies the $L_{\infty\mu}$-sentence*

$$\forall \bar{x} \ \forall y \ \Big(R_{\boldsymbol{a}}(\bar{x}) \to \bigvee_{\xi < \mu} R_{\langle \xi \rangle \frown \boldsymbol{a}}(\langle y \rangle \frown \bar{x}) \Big).$$

Proof. (a) We show that there exists a winning strategy for player II in the game $\mathrm{EF}_{\mu;\lambda}(\mathcal{M}_\eta, \mathcal{M}_\nu)$ (Definition 2.1). Suppose $i < \lambda$ and for each $j \le i$, player I has chosen $X_j \in \{\mathcal{M}_\eta, \mathcal{M}_\nu\}$ and $A_j \subseteq \mu$ (where μ is the domain of both $\mathcal{M}_\eta$ and $\mathcal{M}_\nu$). Suppose that for every $j < i$, player II has replied with a partial isomorphism p_{u_j} such that $u^j \in T^2_\lambda$, $\mathrm{fun}_1(u^j) \subseteq \eta$, $\mathrm{fun}_2(u^j) \subseteq \nu$, $\bigcup_{k \le j} A_k \subseteq \mathrm{dom}(p_{u_j}) \cap \mathrm{ran}(p_{u_j})$, and $u^k \lhd u^j$ for all $k < j$. Since $i < \lambda$ and $u^j \in T^2_\lambda$ for each $j < i$, the tuple $v = \bigcup_{j<i} u^j$ is in T^2_λ by Fact 3.7(a). Let θ be the smallest ordinal which is strictly greater than any ordinal in $\bigcup_{j \le i} A_j$ ($\theta < \mu$ since μ is regular, $i < \mu$, and $\mathrm{card}(A_j) < \mu$ for every $j \le i$). By Lemma 6.3(d), there is u^i in $T^{1,2}_\lambda[{<}\mu]$ such that $v \lhd u^i$, $\mathrm{fun}_1(u^i) \subseteq \eta$, $\mathrm{fun}_2(u_i) \subseteq \nu$, and $\theta \subseteq \mathrm{dom}(p_{u^i}) \cap \mathrm{ran}(p_{u^i})$. Since $\bigcup_{j<i} p_{u^j} = p_v \subseteq p_{u^i}$, the partial isomorphism p_{u^i} is a valid reply for player II in round i.

(b) By Definition 7.1, for every $b \in [\mu]^\theta_B$, $R_{\boldsymbol{b}}(\boldsymbol{b})$ is satisfied in $\mathcal{M}_{\mathbf{0}\restriction\mu}$. The claim follows from (a).

(c) By (a) we may assume $\eta = \mathbf{0}\restriction\mu$. For $\bar{x} = \boldsymbol{a}$ the claim holds directly by Definition 7.1. For any $\bar{x} = \boldsymbol{b} \in R^{\mathbf{0}}_{\boldsymbol{a}} \smallsetminus \{\boldsymbol{a}\}$ there is some $\bar{f} \in \bar{\mathcal{F}}[\mathbf{0}\restriction\mu, \mathbf{0}\restriction\mu]$ such that $\mathrm{dom}(f) = \boldsymbol{a}$ and $f(\boldsymbol{a}) = \boldsymbol{b}$. Since $\mu \in D$ there is, by Lemma 6.3(e), $\bar{g} \in \bar{\mathcal{F}}[\mathbf{0}\restriction\mu, \mathbf{0}\restriction\mu]$ with $g \supseteq f$ and $\mathrm{dom}(g) = \langle \xi \rangle \frown \boldsymbol{a}$.

(d) Analogously to the proof of (c), if $\bar{x} = \boldsymbol{b}$ and $y = \zeta$ then there is some $\bar{f} \in \bar{\mathcal{F}}[\mathbf{0}\restriction\mu, \mathbf{0}\restriction\mu]$ such that $f(\boldsymbol{a}) = \boldsymbol{b}$. Moreover by Lemma 6.3(e), there is $\bar{g} \in \bar{\mathcal{F}}[\mathbf{0}\restriction\mu, \mathbf{0}\restriction\mu]$ with $g \supseteq f$ and $\mathrm{ran}(g) = \langle \zeta \rangle \frown \boldsymbol{b}$. ∎

LEMMA 7.6. *Suppose μ is a regular cardinal in D and A is a subset of κ having cardinality μ.*

(a) *Suppose $\eta \in \mathrm{Fun}(\mu, 2)$, $A \subseteq \mu$, and $\mathcal{M}^{\varrho_\mu}_\eta \restriction A \equiv_{\infty\mu} \mathcal{M}_\eta$. Then $A = \mu$.*

(b) *If $\mathcal{M}^{\varrho_\mu}_{\mathbf{0}} \restriction A \equiv_{\infty\mu} \mathcal{M}_{\mathbf{0}\restriction\mu}$, then there is $\eta \in \mathrm{Fun}(\mu, 2)$ for which $\mathcal{M}^{\varrho_\mu}_{\mathbf{0}} \restriction A \cong \mathcal{M}_\eta$.*

Proof. (a) Suppose, contrary to the claim, that $\xi < \mu$ is not in A. Let $\boldsymbol{b}$ in $[\mu]^{\omega}_{B}$ be such that $\boldsymbol{b} \subseteq A$, $\boldsymbol{b}_0 > \xi$ and for every $i < \omega$, $\boldsymbol{b}_i < \mathrm{card}(\boldsymbol{b}_{i+1})$. This is possible since $\mu = \mathrm{cf}(\mu)$, $\mu \in D$ implies μ is an uncountable limit cardinal, and $\mathrm{card}(A) = \mu$ implies that A is unbounded in μ. By Lemma 7.5(b), there is $\boldsymbol{a} \in [\mu]^{\omega}_{B}$ such that $\mathcal{M}_\eta \models R_{\boldsymbol{a}}(\boldsymbol{b})$. Since $\boldsymbol{b} \subseteq A$ also $\mathcal{M}^{\varrho_\mu}_{\eta}\restriction A$ satisfies $R_{\boldsymbol{a}}(\boldsymbol{b})$. By Lemma 7.5(d), there is $\xi' < \mu$ such that $\mathcal{M}_\eta \models R_{\langle\xi'\rangle^\frown \boldsymbol{a}}(\langle\xi\rangle \frown \boldsymbol{b})$. By Lemma 7.5(c), there should be $\zeta \in A$ with $\mathcal{M}^{\varrho_\mu}_{\eta}\restriction A \models R_{\langle\xi'\rangle^\frown \boldsymbol{a}}(\langle\zeta\rangle \frown \boldsymbol{b})$. However, then by Fact 7.3(b), there should be $\bar{f} \in \bar{\mathcal{F}}$ satisfying $\mathrm{dom}(f) = \{\xi\} \cup \boldsymbol{b}$, $f(\xi) = \zeta \neq \xi$, and $f(\boldsymbol{b}) = \boldsymbol{b}$ contrary to Lemma 5.8(d).

(b) Our proof has the following structure:

- When $A \subseteq \mu$ the claim follows from (a) and Fact 7.2(b).
- The case that A is not a subset of $\zeta + \mu$ for any $\zeta \in A$ is shown to be impossible.
- Lastly we prove that if $A \subseteq \zeta + \mu$ for some $\zeta \in A$, then there are $\eta \in \mathrm{Fun}(\mu, 2)$ and $\bar{g} \in \bar{\mathcal{F}}$ such that $\mathrm{dom}(g) = \mu$, $\mathrm{ran}(g) = A$, $\mathrm{beg}(g) = \eta$, and $\mathrm{end}(g) \subseteq \mathbf{0}$. So g is an isomorphism between $\mathcal{M}_\eta$ and $\mathcal{M}^{\varrho_\mu}_{\mathbf{0}}\restriction A$.

Suppose there is an ω-sequence $\boldsymbol{b}$ such that $\boldsymbol{b}_0 > \mu$ and for all $l < \omega$, $\boldsymbol{b}_l \in A$ and $\boldsymbol{b}_{l+1} \geq \boldsymbol{b}_l + \mu$. By the equivalence $\mathcal{M}^{\varrho_\mu}_{\mathbf{0}}\restriction A \equiv_{\infty\mu} \mathcal{M}_{\mathbf{0}\restriction\mu}$ and Lemma 7.5(b), there is $\boldsymbol{a} \in [\mu]^{\omega}_{B}$ such that $\mathcal{M}^{\varrho_\mu}_{\mathbf{0}}\restriction A \models R_{\boldsymbol{a}}(\boldsymbol{b})$. Hence there should be $\bar{f} \in \bar{\mathcal{F}}$ with $\mathrm{dom}(f) = \boldsymbol{a}$ and $f(\boldsymbol{a}) = \boldsymbol{b}$ contrary to Lemma 5.7(f).

Suppose $\zeta \in A$ and $A \subseteq \zeta + \mu$. As above, there are $\bar{f} \in \bar{\mathcal{F}}$ and $\gamma < \mu$ with $\mathrm{dom}(f) = \{\gamma\}$, $f(\gamma) = \zeta$, $\mathrm{beg}(f) = \mathbf{0}\restriction(\gamma + 1)$, and $\mathrm{end}(f) = \mathbf{0}\restriction(\zeta + 1)$. Since $\gamma < \mu \leq \zeta = f(\gamma)$, there is the smallest index $k < \mathrm{lh}(\bar{f})$ such that $f_{<k}(\gamma) < \mu$ and $f_{\leq k}(\gamma) \geq \mu$. By Lemma 5.6(c), f_j is increasing for all $j \in \{k, \ldots, \mathrm{lh}(\bar{f}) - 1\}$. Let $\bar{u}$ be the sequence $\langle \mathrm{ind}(f_j) \mid j \in \{k, \ldots, \mathrm{lh}(\bar{f}) - 1\}\rangle$. By Lemma 5.7(e), the sequence $\bar{g}^{\bar{u},\mu}$ is a well-defined member of $\bar{\mathcal{F}}$, and moreover, $\mathrm{end}(g^{\bar{u},\mu}) \subseteq \mathbf{0}$. Abbreviate this sequence by g and the ordinal $f_{<k}(\gamma)$ by ξ. We define the required η to be $\mathrm{beg}(g)$.

Finally we show that for this g we have $A = \mathrm{ran}(g)$. Suppose ζ' is in A but not in $\mathrm{ran}(g)$. By the equivalence and Lemma 7.5(b), there are $\varepsilon, \varepsilon' < \mu$ such that $R_{\langle\varepsilon\rangle}(\zeta)$ and $R_{\langle\varepsilon',\varepsilon\rangle}(\zeta', \zeta)$ hold in $\mathcal{M}^{\varrho_\mu}_{\mathbf{0}}\restriction A$. By Lemma 7.5(c), there is $\xi' < \mu$ for which $R_{\langle\varepsilon',\varepsilon\rangle}(\xi', \xi)$ holds in $\mathcal{M}_\eta$. However, by Fact 7.3(b), there should be $\bar{h} \in \bar{\mathcal{F}}$ with $h(\xi) = \zeta$ and $h(\xi') = \zeta'$, contrary to Lemma 5.7(g). On the other hand, if $A \subsetneq \mathrm{ran}(g)$, then η and the set $B = g^{-1}[A] \subsetneq \mu$ contradict (a), since by Lemma 7.5(a), $\mathcal{M}_\eta \equiv_{\infty\mu} \mathcal{M}_{\mathbf{0}\restriction\mu}$, by our assumption, $\mathcal{M}_{\mathbf{0}\restriction\mu} \equiv_{\infty\mu} \mathcal{M}^{\varrho_\mu}_{\mathbf{0}}\restriction A$, and $g^{-1}\restriction A : \mathcal{M}^{\varrho_\mu}_{\mathbf{0}}\restriction A \cong \mathcal{M}^{\varrho_\mu}_{\eta}\restriction B$. ∎

Lemma 7.7. *Suppose κ is a weakly compact cardinal and $\mathcal{M}$ is a model of cardinality κ with $\mathcal{M} \equiv_{\infty\kappa} \mathcal{M}_{\mathbf{0}}$. Then there is $s \in \mathrm{Fun}(\kappa, 2)$ for which $\mathcal{M} \cong \mathcal{M}_s$.*

Proof. Without loss of generality we may assume that the domain of $\mathcal{M}$ is κ. By the $\equiv_{\infty\kappa}$-equivalence of the models $\mathcal{M}$ and $\mathcal{M}_{\mathbf{0}}$, let, for every regular cardinal $\mu < \kappa$, A_μ be a subset of κ such that $\mathcal{M}^{\varrho_\mu}{\restriction}\mu \cong \mathcal{M}_{\mathbf{0}}^{\varrho_\mu}{\restriction}A_\mu$. Let Y be the set given in Definition 6.1. Note that for all $\mu \in D \cup \{\kappa\}$ and $\eta \in \mathrm{Fun}(\mu, 2)$, the model $\mathcal{M}_\eta$ is definable from η and $Y \cap V_\mu$. Let τ be a winning strategy for player II in the game $\mathrm{EF}_{\kappa;\omega}(\mathcal{M}, \mathcal{M}_{\mathbf{0}})$. Assume now, contrary to the claim, that $\mathcal{M} \ncong \mathcal{M}_s$ for all $s \in \mathrm{Fun}(\kappa, 2)$. Because κ is Π^1_1-indescribable, there is a regular cardinal $\mu < \kappa$ such that $\langle V_\mu, \in, \mathcal{M}^{\varrho_\mu}{\restriction}\mu, \tau \cap V_\mu, Y \cap V_\mu\rangle$ satisfies the following:

$$\text{for all } \eta \in \mathrm{Fun}(\mu, 2), \quad \mathcal{M}^{\varrho_\mu}{\restriction}\mu \ncong \mathcal{M}_\eta.$$

Then $\mathcal{M}_{\mathbf{0}{\restriction}\mu} = \mathcal{M}_{\mathbf{0}}^{\varrho_\mu}{\restriction}\mu \equiv_{\infty\mu} \mathcal{M}^{\varrho_\mu}{\restriction}\mu$, and by the isomorphism $\mathcal{M}^{\varrho_\mu}{\restriction}\mu \cong \mathcal{M}_{\mathbf{0}}^{\varrho_\mu}{\restriction}A_\mu$, we have $\mathcal{M}_{\mathbf{0}{\restriction}\mu} \equiv_{\infty\mu} \mathcal{M}_{\mathbf{0}}^{\varrho_\mu}{\restriction}A_\mu$ and for all $\eta \in \mathrm{Fun}(\mu, 2)$, $\mathcal{M}_{\mathbf{0}}^{\varrho_\mu}{\restriction}A_\mu \ncong \mathcal{M}_\eta$. This contradicts Lemma 7.6(b). ■

LEMMA 7.8. *Suppose κ is a weakly compact cardinal, $\lambda < \kappa$ is a regular cardinal, and there is a Σ^1_1-equivalence relation on ${}^\kappa 2$ having μ different equivalence classes. Then there exists a model $\mathcal{M}$ such that the vocabulary of $\mathcal{M}$ consists of one relation symbol of finite arity*, $\mathrm{card}(\mathcal{M}) = \kappa$, *and* $\mathrm{No}_\lambda(\mathcal{M}) = \mu$.

Proof. By the preceding lemmas the model $\mathcal{M}_{\mathbf{0}}$ defined as in Definition 7.1 satisfies the claim, except that the vocabulary of $\mathcal{M}$ is overly large. However, by [She85, Claim 1.3(1)], the inaccessibility of κ ensures that there is a model $\mathcal{N}$ of cardinality κ with λ relations of finite arity satisfying $\mathrm{No}(N) = \mathrm{No}(\mathcal{M}_{\mathbf{0}})$ (the proof is a simple coding). Furthermore, by [She85, Claim 1.4(2)], λ relations can be coded by one relation so that the other properties are preserved. Actually, the claims cited concern the case $\lambda = \aleph_0$, but there is also no problem to preserve $\mathrm{No}_\lambda(\mathcal{M}_{\mathbf{0}})$ in the cases $\aleph_0 < \lambda < \kappa$. ■

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