Model-theoretic consequences of a theorem of Campana and Fujiki

by

Anand Pillay (Urbana, IL, and Berlin)

Abstract. We give a model-theoretic interpretation of a result by Campana and Fujiki on the algebraicity of certain spaces of cycles on compact complex spaces. The model-theoretic interpretation is in the language of canonical bases, and says that if $b, c$ are tuples in an elementary extension $\mathcal{A}$ of the structure $\mathcal{A}$ of compact complex manifolds, and $b$ is the canonical base of $\text{tp}(c/b)$, then $\text{tp}(b/c)$ is internal to the sort $(\mathbb{P}^1)^*$. The Zilber dichotomy in $\mathcal{A}$ follows immediately (a type of $U$-rank 1 is locally modular or nonorthogonal to the field $\mathbb{C}^*$), as well as the “algebraicity” of any subvariety $X$ of a group $G$ definable in $\mathcal{A}$ such that $\text{Stab}(X)$ is trivial.

1. Introduction. This paper concerns the interaction between complex-geometric notions and model-theoretic notions in the structure theory of compact complex spaces. It has been known for some time that model-theoretic ideas yield a rather striking dichotomy for simple compact complex manifolds $M$: either $M$ is algebraic, or else there is no “2-parameter” family of finite-to-finite analytic correspondences between $M$ and itself. But, up to now, the only proof of this of which I was aware went through the results on Zariski geometries and their validity for compact complex manifolds, together with some other ingredients (see [6], [7] and [11]). It turns out that the dichotomy above and more are almost immediate consequences of a theorem proved independently by Campana [1] and Fujiki [3]. They prove, roughly speaking, that if $S$ is a compact space of cycles $(Z_s : s \in S)$ on a compact complex space $X$ then the natural morphism from the graph $\{(x, s) : x \in Z_s, s \in S\}$ of $S$ to $X$ is a Moishezon map. Via a translation established by Moosa ([9], [10]), this yields the following striking statement in the language of canonical bases (to be read in a saturated elementary extension $\mathcal{A}$ of the many-sorted structure $\mathcal{A}$ of compact complex spaces): ($\ast$) for any $b, c$, $\text{tp}(C_b(\text{tp}(c/b))/c)$ is “algebraic”, that is, internal to $\mathbb{C}^*$.

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The statement (⋆) (or the original Campana–Fujiki statement) yields the dichotomy for $U$-rank 1 types in $A^*$: they are modular or “algebraic”. Also the algebraicity of suitable subvarieties of meromorphic groups and homogeneous spaces follows directly, subsuming results of Ueno [15] as well as results from [8]. I guess that the benefit of the model-theoretic translation (⋆) lies in being able to work directly with bimeromorphic objects without worrying about specific compactifications. As the reader might surmise, the statement (⋆), when suitably re-interpreted, is also directly provable (using suitable jet spaces) in other algebraic/model-theoretic structures of interest, such as differential and difference fields. (See [14].)

The details of the observations above are given in the next section.

2. Results. For the theory of compact complex spaces see [4]. For the model-theoretic treatment of this subject see [11].

Let us work towards stating the Campana–Fujiki theorem. Let $X$ be a reduced, irreducible, compact complex space. There are two notions of cycle spaces on $X$. The Douady space $D(X)$ parametrizes pure-dimensional analytic subsets of $X$. The Barlet space $C(X)$ parametrizes integral linear combinations of irreducible analytic subsets (of the same dimension) of $X$. Campana works with Barlet spaces and Fujiki with Douady spaces. A morphism $f : Y \to X$ of compact complex spaces is said to be projective if there is a coherent analytic sheaf $\mathcal{F}$ over $X$ and an embedding (over $X$) $h$ of $Y$ into the projective linear space $\mathbb{P}(\mathcal{F})$ over $X$ associated with $\mathcal{F}$ such that $\pi \circ h = f$, where $\pi : \mathbb{P}(\mathcal{F}) \to X$ is the map realizing $\mathbb{P}(\mathcal{F})$ as a fibre space over $X$. A morphism $f : Y \to X$ is said to be Moishezon if it is bimeromorphic (over $X$) to a projective morphism $f' : Y' \to X$. Campana proves:

**Fact 2.1.** Let $S$ be an irreducible, compact, analytic subset of $C(X)$. Let $Z_s$ denote the cycle parameterized by $s \in S$. Assume that for general $s \in S$, $Z_s$ is irreducible. Let $Y = \{(x,s) \in X \times S : x \in Z_s, \ s \in S\}$, and let $f : Y \to X$ be the projection to the first coordinate. Then $f$ is a Moishezon map.

In fact the above statement comes from [2] (Theorem 3.6). The original theorem in [1] states that $f$ factors through an embedding in a suitable Grassmannian of $\mathcal{F}$ over $X$, where $\mathcal{F}$ is the coherent analytic sheaf of germs of differential operators of suitably bounded order. Such an $f$ is Moishezon. In Fujiki’s statement, $C(X)$ is replaced by $D(X)$.

We now work towards the model-theoretic interpretation. $\mathcal{A}$ is the many-sorted structure of compact complex spaces $X_i$ with predicates for analytic subvarieties of Cartesian products $X_{i_1} \times \ldots \times X_{i_n}$. Note that with this “language”, all elements of all sorts in $\mathcal{A}$ are named by constants. $\text{Th}(\mathcal{A})$ has
quantifier elimination, elimination of imaginaries and is stable with finite Morley rank (sort-by-sort). $\mathcal{A}^*$ will be a very saturated elementary extension of $\mathcal{A}$. Among the sorts in $\mathcal{A}$ is the projective line $\mathbb{P}_1$ over $\mathbb{C}$. There is no harm (via Chow’s theorem) in identifying this sort with the complex field $\mathbb{C}$ equipped with addition, multiplication, and constants for all elements (they are bi-interpretable). $\mathbb{C}^*$ denotes the “extension” of this sort in $\mathcal{A}^*$. Let $b, c$ be tuples from $\mathcal{A}^*$. Following [9], [10], we will say that $\text{tp}(c/b)$ is Moishezon if $\text{tp}(c/\text{acl}(b))$ is internal to $\mathbb{C}^*$, or equivalently if there is some finite tuple $b_0$ including $b$ and independent of $c$ over $b$ such that $c \in \text{dcl}(d,b_0)$ for some tuple $d$ from $\mathbb{C}^*$. (Note that $\text{tp}(c/b)$ is Moishezon just if every stationarization of it is Moishezon.) Moosa [10] observes the following:

**Fact 2.2.** Suppose that $X, Y$ are irreducible compact complex spaces and that $f : Y \to X$ is a Moishezon map. Let $c \in Y^*$ be a generic point of $Y$ (over $\mathcal{A}$). Then $\text{tp}(c/f(c))$ is Moishezon.

We can now obtain:

**Theorem 2.3.** Let $b, c$ be finite tuples from $\mathcal{A}^*$. Assume that $\text{tp}(c/b)$ is stationary and that $b = \text{Cb}(\text{tp}(c/b))$. Then $\text{tp}(b/c)$ is Moishezon.

**Proof.** Let $X, S, Z$ be irreducible compact complex spaces of which $c, b$ and $(c, b)$ respectively are generic points (over $\mathcal{A}$). Then $Z_b$ is irreducible with generic point $c$, and moreover $b$ is a canonical parameter for $Z_b$ (by quantifier elimination). Replacing $S$ by a suitable modification, we may assume that the projection $\pi : Z \to S$ is flat. The universal properties of the Douady space $D(X)$ of $X$ yield a morphism $p : S \to D(X)$ such that $p(s)$ corresponds to $Z_s$. Then $p(S)$ is a compact irreducible analytic subset of $D(X)$. Also, by compactness, there is a Zariski open subset $U$ of $S$ such that for $s_1, s_2 \in U$, $Z_{s_1} = Z_{s_2}$ iff $s_1 = s_2$. Thus $p : S \to p(S)$ is a modification. The end result is that, after replacing $b$ by something interdefinable with it (which we can do), we may assume that $S$ is a compact analytic subspace of $D(X)$ and that $Z$ is the associated subspace of $X \times S$. By Fact 2.1 (working with Douady spaces), the canonical morphism $f : Z \to X$ is Moishezon. By 2.2, $\text{tp}(b, c/c)$ is Moishezon, and thus $\text{tp}(b/c)$ is Moishezon.

**Remark 2.4.** Let $b, c$ be as in the above theorem. Then for any tuple $a$ from $\mathcal{A}^*$, $\text{tp}(b/ca)$ is Moishezon.

**Proof.** The conclusion of Theorem 2.3 tells us that there is some set $C$ of parameters containing $c$ and such that $b$ is independent of $C$ over $c$ and $b \in \text{dcl}(C, d)$ for some tuple $d$ from $\mathbb{C}^*$. Without loss of generality (that is, by automorphism) $C$ is independent of $a$ over $bc$, so independent of $bca$ over $c$, and so independent of $b$ over $ca$.

Let us now give some applications.
Corollary 2.5. Let \( p(x) \) be a stationary type of \( U \)-rank 1 over some set in \( \mathcal{A}^* \). Then \( p \) is either modular or nonorthogonal to (the generic type of) \( C^* \).

Proof. As \( \mathcal{A}^* \) has finite Morley rank (sort-by-sort) we may assume that \( p(x) \) is over a finite tuple \( a \) of parameters. If \( p \) were not modular then by 2.2.6 of [12] there would be a tuple \( c \) of realizations of \( p \) and some tuple \( b \) such that \( b = \text{Cb}(tp(c/ba)) \) and \( b \notin \text{acl}(ca) \). By Remark 2.4, \( \text{tp}(b/ca) \) is Moishezon. In particular \( \text{tp}(b/ca) \) is nonorthogonal to the generic type of \( C^* \). But \( b \) lives in \( p^{eq} \), and thus \( p \) is nonorthogonal to \( C^* \).

The next application concerns definable subsets of groups and homogeneous spaces. Definable groups in \( \mathcal{A} \) have naturally the structure of “meromorphic groups” (see [13] and [8]). A definable group in \( \mathcal{A}^* \) can be considered as the generic fibre of a meromorphic family of meromorphic groups.

Let us start with a general lemma about stable groups.

Lemma 2.6. Let \( G \) be a connected group (type)-definable in a saturated stable structure \( \bar{M} \). Let \( c \in G \) be such that \( p(x) = \text{tp}(c) \) is stationary. Let \( H \) be the left stabilizer of \( p \). Let \( a \in G \) be generic over \( c \). Let \( c/H \) denote \( Hc \) as an element of the right coset space \( H \setminus G \). Then \( c/H \) is interdefinable over \( a \) with \( \text{Cb}(\text{tp}(a/ca)) \). (Similarly with left and right interchanged.)

Proof. So \( H \setminus G \) denotes the space \( \{Hg : g \in G\} \) of right cosets of \( H \) in \( G \), and as above we write \( Hg \) as \( g/H \) when we want to treat it as an element rather than a definable set. Some more notation: given a stationary type \( q \) over \( \emptyset \) of some element of \( G \), and given \( a \in G \), by \( qa \) we mean the restriction to \( a \) of the translate \( q'a \) of \( q' \) by \( a \) where \( q' \) is the global nonforking extension of \( q \). Note that \( qa \) is stationary and \( q'a \) is its global nonforking extension. Moreover \( qa = \text{tp}(da/a) \) for \( d \) realizing \( q|a \).

Now let us make some observations.

(i) \( a \) realizes \( p^{-1}ca \).

This is because \( c^{-1} \) realizes \( p^{-1} \), and (as \( a \) is generic in \( G \) over \( c \)), \( ca \) is generic over \( c^{-1} \) so independent of \( c^{-1} \).

(ii) \( ca/H \) is interdefinable with \( \text{Cb}(p^{-1}ca) \).

Indeed, \( H \) is the left stabilizer of \( p \) so the right stabilizer of \( p^{-1} \), which clearly yields (ii): an automorphism \( f \) fixes \( ca/H \) iff \( p^{-1}ca \) is parallel to \( f(p^{-1}ca) = p^{-1}f(ca) \).

Note that \( c/H \) and \( ca/H \) are interdefinable over \( a \). So by (i) and (ii) we deduce that \( c/H \) and \( \text{Cb}(\text{tp}(a/ca)) \) are interdefinable over \( a \).

Remark 2.7. Note that Lemma 2.6 includes the old result [5] that in modular (or 1-based) groups any stationary type is a translate of the generic
type of a subgroup: if $\text{Cb}(\text{tp}(a/\text{ca})) \subseteq \text{acl}(a)$ then $c/H \in \text{acl}(\emptyset)$ and so $\text{tp}(c)$ is a translate of the generic type of $H$.

Lemma 2.6 applies to groups definable in $\mathcal{A}^*$. So together with Theorem 2.3 we obtain:

**Corollary 2.8.** Let $G$ be a group definable in $\mathcal{A}^*$. Work over some algebraically closed set of parameters $\mathcal{A}$ over which $G$ is definable. Let $c \in G$ and let $H$ be the left stabilizer of $\text{tp}(c/\mathcal{A})$. Then $\text{tp}((c/H)/\mathcal{A})$ is Moishezon.

For groups definable in $\mathcal{A}$, that is, *meromorphic groups*, Corollary 2.9 below gives a more geometric looking statement. As a matter of notation, if $X$ and $Y$ are complex (not necessarily compact) spaces definable in $\mathcal{A}$, we will say that $X$ and $Y$ are *meromorphically isomorphic* if there is a bihomolorphic map between $X$ and $Y$ which is also definable in $\mathcal{A}$.

**Corollary 2.9.** Let $G$ be an arbitrary meromorphic group (not necessarily commutative). Let $X$ be an irreducible meromorphic subvariety of $G$, and let $H = \{g \in G : g \cdot X = X\}$ be the set-theoretic left stabilizer of $X$ in $G$. Let $H \backslash X$ denote the Zariski closure of $H \backslash X$ in the meromorphic homogeneous space $H \backslash G$ of right cosets $Hg$ ($g \in G$). Then $H \backslash X$ is meromorphically isomorphic to an algebraic variety. (Likewise with left and right interchanged.)

**Proof.** Let $p = \text{tp}(c/\mathcal{A})$ be the generic type of $X$. (So $c \in X^* \subseteq G^*$. ) Then $H$ (or rather its extension $H^*$) as defined in the statement to be proved identifies with the left stabilizer of $p$. By Corollary 2.8, $\text{tp}((c/H^*)/\mathcal{A})$ is Moishezon. As $\mathcal{A}$ is a model, this is witnessed over $\mathcal{A}$, namely $c/H^* \in \text{dcl}(d)$ for some tuple from $\mathbb{C}^*$. It follows that $c/H^*$ is interdefinable with some tuple in $\mathbb{C}^*$. But $c/H^*$ is a generic point over $\mathcal{A}$ of $(\overline{H \backslash X})^* \subseteq (H \backslash G)^*$. As $H \backslash G$ is a homogeneous space it follows that $\overline{H \backslash X}$ is meromorphically isomorphic to an algebraic variety.

**References**


Department of Mathematics
University of Illinois at Urbana-Champaign
Altgeld Hall
1409 W. Green St.
Urbana, IL 61801, U.S.A.
E-mail: pillay@math.uiuc.edu

Institut für Mathematik
Humboldt Universität
D-10099 Berlin, Germany

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