FUNDAMENTA
MATHEMATICAE
222 (2013)

Extension properties of Stone–Čech coronas and proper absolute extensors

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Abstract. We characterize, in terms of X, the extensional dimension of the Stone–Čech corona $\beta X \setminus X$ of a locally compact and Lindelöf space X. The non-Lindelöf case is also settled in terms of extending proper maps with values in $I^\tau \setminus L$, where L is a finite complex. Further, for a finite complex L, an uncountable cardinal τ and a Z_τ-set X in the Tikhonov cube I^τ we find a necessary and sufficient condition, in terms of $I^\tau \setminus X$, for X to be in the class $\mathrm{AE}([L])$. We also introduce a concept of a proper absolute extensor and characterize the product $[0,1) \times I^\tau$ as the only locally compact and Lindelöf proper absolute extensor of weight $\tau > \omega$ which has the same pseudocharacter at each point.

1. Introduction. We study extension properties of Stone–Čech coronas of locally compact spaces, focusing on the following two problems:

(A) When, in terms of X, are maps, defined on closed subsets of $\beta X \setminus X$, into a finite complex L extendible to the whole $\beta X \setminus X$?
(B) When, in terms of Y, are maps, defined on closed subsets of nice spaces, into $\beta Y \setminus Y$ extendible to the whole domain?

When every map $f\colon A \to Y$, defined on a closed subset A of X, has an extension $\bar{f}\colon X \to Y$ we say that Y is an *absolute extensor* of X and write $Y \in \mathrm{AE}(X)$. Assuming that both f and $\bar{f}$ in this definition are proper we obtain the notion of a *proper absolute extensor* (for details see Definition 4.1). We then write $Y \in \mathrm{AE}_p(X)$. It turns out (Corollaries 4.2, 4.3) that for a locally compact and Lindelöf (e.g. separable and metrizable) space X and a finite complex L, $L \in \mathrm{AE}(\beta X \setminus X)$ precisely when $\mathrm{Cone}(L) \setminus L \in \mathrm{AE}_p(X)$ (here L is identified with the base $L \times \{0\}$ of $\mathrm{Cone}(L)$). For $L = S^n$, we obtain the following observation: $\dim(\beta X \setminus X) = \dim_p X - 1$, where $\dim_p X \le n$ is just a notation for $\mathbb{R}^n \in \mathrm{AE}_p(X)$. We point out that the problem of describing dimensions (covering, inductive) of the Stone–Čech (or

2010 *Mathematics Subject Classification*: Primary 54C20, 57N20; Secondary 54D35.
Key words and phrases: Stone–Čech corona, Z_τ-set, finite complex, absolute extensor.

DOI: 10.4064/fm222-2-3

Hewitt) coronas, using completely different approaches, has been considered by several authors (see, for example, [1], [2], [3], [15], [16], [5]).

However, non-Lindelöf spaces do not admit proper maps into $\mathbb{R}^n$ or into any space of the form $\mathrm{Cone}(L) \setminus L$, where L is a finite complex, and above observations need to be adjusted in order to remain valid in the general case. We start by noting that since L (i.e. $L \times \{0\} \subset \mathrm{Cone}(L)$) is a Z-set in $\mathrm{Cone}(L)$ it follows from Chapman's Complement Theorem that no matter how L is Z-embedded into the Hilbert cube, the complement $I^\omega \setminus L$ is homeomorphic to $I^\omega \times \mathrm{Cone}(L) \setminus I^\omega \times L = I^\omega \times (\mathrm{Cone}(L) \setminus L)$. Since $\mathrm{Cone}(L) \setminus L \in \mathrm{AE}_p(X)$ if and only if $I^\omega \times (\mathrm{Cone}(L) \setminus L) \in \mathrm{AE}_p(X)$, the observation made above can be reformulated as follows: $L \in \mathrm{AE}(\beta X \setminus X)$ if and only if $I^\omega \setminus L \in \mathrm{AE}_p(X)$. While the testing space $I^\omega \setminus L$ is still Lindelöf and hence is not suitable for the general situation, it does allow us to find its non-metrizable counterpart, which turns out to be the complement $I^\tau \setminus L$. The choice of the embedding $L \hookrightarrow I^\tau$, when $\tau > \omega$, is irrelevant since any metric compactum is a Z_τ-set in I^τ as long as $\tau > \omega$ [6, Corollary 8.5.7]. With this in mind we settle problem (A) by proving the following statement.

Theorem 4.1. *Let X be a locally compact space which can be covered by at most τ compact subsets and each regular closed subset of which is C^*-embedded. Let also L be a compact ANR-space embedded into the cube I^τ as a Z_τ-set. Then the following conditions are equivalent:*

(a) $L \in \mathrm{AE}(\beta X \setminus X)$;
(b) $I^\tau \setminus L \in \mathrm{AE}_p(X)$.

Problem (B), in some cases, can also be settled in a similar manner. Specifically, we consider spaces of the form $Y = I^\tau \setminus X$, where X is a Z_τ-set in I^τ. For $\tau > \omega$, I^τ is indeed the Stone–Čech compactification of Y (Lemma 2.1). In this situation problem (B) becomes a part of a general problem of recovering properties of X in terms of its complement $I^\tau \setminus X$. This leads us to considerations very similar to the study carried out in [10] for $\tau = \omega$.

However, there is a major difference between the metrizable ($\tau = \omega$) and non-metrizable ($\tau > \omega$) cases. Roots of this difference, one could argue, lie in the fact that the topological type of the complement $I^\omega \setminus X$ of a Z-set in the Hilbert cube, while determining X's shape, does not uniquely determine the topological type of X. But if $\tau > \omega$, the topological type of any Z_τ-set X in I^τ is completely determined by its complement. This is apparently why we need to exploit metric-uniform invariants in the metrizable case (see [10]) and why we could remain in the topological category if $\tau > \omega$.

Going back to problem (B), it turns out that—as in problem (A)—the complements $I^\tau \setminus L$ of finite complexes still play a critical role. In order to formulate our second result let us recall that the *extension class* $[L]$ of a complex is the collection of all extensionally equivalent complexes (K is equivalent to L if $K \in \mathrm{AE}(X)$ if and only if $L \in \mathrm{AE}(X)$ for any X). We say that $X \in \mathrm{AE}([L])$ if $X \in \mathrm{AE}(Y)$ whenever $L \in \mathrm{AE}(Y)$. Similarly, we can define a proper extensional class $\mathrm{AE}_p^\tau([I^\tau \setminus L])$ by agreeing that $Y \in \mathrm{AE}_p^\tau([I^\tau \setminus L])$, where Y is a locally compact space of weight $\leq \tau$, if $Y \in \mathrm{AE}_p(M)$ for any locally compact space M of weight $\leq \tau$ with $I^\tau \setminus L \in \mathrm{AE}_p(M)$. We prove the following statement.

THEOREM 4.4. *Let $\tau > \omega$, L be a compact ANR-space embedded into I^τ as a Z_τ-set and X be a Z_τ-set in I^τ. Then the following conditions are equivalent:*

(i) $X \in \mathrm{AE}([L])$;
(ii) $I^\tau \setminus X \in \mathrm{AE}_p^\tau([I^\tau \setminus L])$.

These considerations lead to the concept of a proper absolute extensor which we study in Section 6 (see [13], [12] for related results). Note that $\mathbb{R}^n$ is not a proper absolute extensor for any n (while it is, of course, an absolute extensor). To see this in case $n = 1$ note that the proper map $f\colon \mathbb{N} \to \mathbb{R}$, defined by

$$f(n) = \begin{cases} n, & n \text{ is odd,} \\ -n, & n \text{ is even,} \end{cases}$$

does not have a proper extension $\bar{f}\colon \mathbb{R} \to \mathbb{R}$. On the other hand, $\mathbb{R}^n_+ = \{(x_i)_{i=1}^n \in \mathbb{R}^n : x_n \geq 0\}$ is a proper absolute extensor for each n. Explanation of this fact (see Lemma 6.2) is that $\mathbb{R}^n_+$ has a compactification (namely, I^n) which is an absolute extensor and that the corresponding corona (I^{n-1}) is also an absolute extensor, sitting in I^n as a Z-set. We show that the only proper absolute extensor of countable weight satisfying DD^nP for each n is the product $[0,1) \times I^\omega$ (Proposition 6.5). In the non-metrizable case we have the following statement.

THEOREM 6.7. *A proper absolute extensor of weight $\tau > \omega$ is homeomorphic to the product $[0,1) \times I^\tau$ if and only if it has the same pseudocharacter at each point.*

The paper is organized as follows. In Section 3, based on modified versions of Shchepin's Spectral Theorem, we obtain a characterization of Z_τ-sets in the Tikhonov cube I^τ and prove the mapping replacement results (Propositions 3.9 and 3.12). These are then used in Section 4 to prove Theorems 4.1 and 4.4. In Section 5 we extend some results, obtained in [10] for the Hilbert cube, to the Tikhonov cube. Namely, we describe the topological and homotopy categories of Z_τ-sets in I^τ in terms of certain naturally defined categories of their complements. The considerations here involve certain concepts of coarse geometry which are still relevant in the topological setting. In the final Section 6 we investigate the concept of proper absolute extensor and prove Theorem 6.7.

2. Preliminaries. Unless noted otherwise, below we consider only locally compact Tikhonov spaces and continuous maps. A map $f\colon X \to Y$ is *proper* if $f^{-1}(C)$ is compact for any compact $C \subset Y$. Note that the class of proper maps between locally compact spaces coincides with the class of perfect maps (a map is *perfect* if it is closed and has compact point inverses). A set $F \subset X$ is z-*embedded* in X if for every functionally closed (in F) set $Z \subset F$ there exists a functionally closed set $\widetilde{Z}$ in X such that $Z = F \cap \widetilde{Z}$. A set $F \subset X$ is C^*-*embedded* if every bounded real-valued continuous function, defined on F, has a bounded and continuous extension, defined on X.

LEMMA 2.1. *Let $\tau > \omega$ and X be an open and G_δ-dense subset of the Tikhonov cube I^τ. Then*

(i) *X is pseudocompact and $\beta X = I^\tau$;*
(ii) *if F is a functionally closed subset of I^τ, then $F \cap X$ is C^*-embedded in X;*
(iii) *if G is an open subset of X, then $\mathrm{cl}_X G$ is C^*-embedded in X.*

Proof. (i) Since X is dense in I^τ it follows (see [6, Corollary 6.4.7]) that X is z-embedded in I^τ. Since, by assumption, $I^\tau \setminus X$ does not contain functionally closed subsets of I^τ, we conclude ([6, Proposition 1.1.22]) that I^τ is the Hewitt realcompactification of X. Compactness of I^τ implies that I^τ is actually the Stone–Čech compactification of X and X is pseudocompact.

(ii) Since X is G_δ-dense, it follows that $F \cap X \neq \emptyset$. By (i) and [11, 8D.1], $F = \mathrm{cl}_{I^\tau}(F \cap X)$. Since I^τ is an AE(0)-space, it follows from [6, Propositions 6.1.8, 6.4.9] that F itself is an AE(0)-space. Consequently, by [6, Proposition 1.1.21(ii)], $F \cap X$ is z-embedded in F. Since $F \cap X$ is G_δ and dense in F, it follows from [6, Proposition 1.1.22] that F is the Stone–Čech compactification of $F \cap X$. Then $F \cap X$ is C^*-embedded in X.

(iii) Clearly, $\mathrm{cl}_X G = X \cap \mathrm{cl}_{I^\tau} G$. Since the latter set is functionally closed in I^τ, the conclusion follows from (ii). ∎

Extension theory—a generalization of the classical dimension theory—as developed by A. Dranishnikov, as well as certain facts from infinite-dimensional topology (see [8] for a unified treatment of both) are used without specific references.

3. Z_τ-sets in the Tikhonov cube. In this section we study certain properties of Z_τ-sets in I^τ introduced in [6].

3.1. Spectral Theorem—revisited. We begin by establishing some versions of Shchepin's Spectral Theorem [6, Theorem 1.3.4].

PROPOSITION 3.1. *Let $\tau \geq \omega$, $|T| > \tau$, $T_0 \subseteq T$, $|T_0| < |T|$ and $g\colon \prod_{t\in T} X_t \to \prod_{t\in T} X_t$ be a map of the product of compact metrizable spaces such that*

$\pi_{T_0} \circ g = \pi_{T_0}$. *Then the set*

$$\mathcal{M}_{(g,T_0)} = \Big\{R \subseteq \exp_\tau(T \setminus T_0)\colon \textit{there exists}$$
$$g_{T_0 \cup R}\colon \prod_{t \in T_0 \cup R} X_t \to \prod_{t \in T_0 \cup R} X_t \textit{ with } \pi_{T_0 \cup R} \circ g = g_{T_0 \cup R} \circ \pi_{T_0 \cup R}\Big\}$$

is cofinal and τ*-closed in* $\exp_\tau(T \setminus T_0)$.

Proof. By [6, Theorem 1.3.4], the set

$$\mathcal{M}_g = \Big\{R \in \exp_\tau T\colon \text{there exists}$$
$$g_R\colon \prod_{t \in R} X_t \to \prod_{t \in R} X_t \text{ with } \pi_R \circ g = g_R \circ \pi_R\Big\}$$

is cofinal and τ-closed in $\exp_\tau T$.

Let $S \in \exp_\tau(T \setminus T_0)$ and choose $\tilde{R} \in \mathcal{M}_g$ such that $S \subseteq \tilde{R}$. The corresponding $g_{\tilde{R}}$ does not change the X_t-coordinate for $t \in \tilde{R} \cap T_0$ (since $\pi_{T_0} \circ g = \pi_{T_0}$). Consequently, the diagonal product

$$g_{T_0 \cup \tilde{R}} = \pi^{T_0 \cup \tilde{R}}_{T_0} \bigtriangleup \pi^{\tilde{R}}_{\tilde{R} \setminus T_0} g_{\tilde{R}} \pi^{T_0 \cup \tilde{R}}_{\tilde{R}}\colon \prod_{t \in T_0 \cup \tilde{R}} X_t \to \prod_{t \in T_0} X_t \times \prod_{t \in \tilde{R} \setminus T_0} X_t$$

is well defined. Set $R = \tilde{R} \setminus T_0$. Obviously, $S \subseteq R$ and $R \in \mathcal{M}_{(g,T_0)}$, which proves that $\mathcal{M}_{(g,T_0)}$ is cofinal in $\exp_\tau(T \setminus T_0)$. The τ-completeness of $\mathcal{M}_{(g,T_0)}$ in $\exp_\tau(T \setminus T_0)$ is obvious. ∎

Corollary 3.2. *Let* $\tau \geq \omega$, $|T| > \tau$, $T_0 \subseteq T$, $|T_0| < |T|$ *and* $f\colon X \to Y$ *be a map between closed subspaces of the Tikhonov cube* I^T. *If* $\pi_{T_0} \circ f = \pi_{T_0}|X$, *then the set*

$$\mathcal{M}_{(f,T_0)} = \{R \subseteq \exp_\tau(T \setminus T_0)\colon \textit{there exists } f_{T_0 \cup R}\colon \pi_{T_0 \cup R}(X) \to \pi_{T_0 \cup R}(Y)$$
$$\textit{with } \pi_{T_0 \cup R} \circ f = f_{T_0 \cup R} \circ \pi_{T_0 \cup R}|X\}$$

is cofinal and τ*-closed in* $\exp_\tau(T \setminus T_0)$.

Proof. Let $g\colon I^T \to I^T$ be a map such that $g|X = f$ and $\pi_{T_0} \circ g = \pi_{T_0}$. By Proposition 3.1, the set $\mathcal{M}_{(g,T_0)}$ is cofinal and τ-closed in $\exp_\tau(T \setminus T_0)$.

For each $R \in \mathcal{M}_{(g,T_0)}$ let $f_{T_0 \cup R} = g_{T_0 \cup R}|g_{T_0 \cup R}(X)$. ∎

Proposition 3.3. *If, in Proposition* 3.1, *the map* g *is a homeomorphism, then the set*

$$\mathcal{H}_{(g,T_0)} = \{R \in \mathcal{M}_{(g,T_0)}\colon g_{T_0 \cup R} \textit{ is a homeomorphism}\}$$

is cofinal and τ*-closed in* $\exp_\tau(T \setminus T_0)$.

Proof. By Proposition 3.1 applied to both g and g^{-1}, the sets $\mathcal{M}_{(g,T_0)}$ and $\mathcal{M}_{(g^{-1},T_0)}$ are cofinal and τ-closed in $\exp_\tau(T \setminus T_0)$. By [6, Proposition

1.1.27], $\mathcal{M}_{(g,T_0)} \cap \mathcal{M}_{(g^{-1},T_0)}$ is still cofinal and τ-closed. It remains to note that for each R from this intersection the map $g_{T_0 \cup R}$ is a homeomorphism. ∎

Corollary 3.4. *If, in Corollary* 3.2, *the map f is a homeomorphism, then the set*

$$\mathcal{H}_{(f,T_0)} = \{R \in \mathcal{M}_{(f,T_0)} \colon f_{T_0 \cup R} \text{ is a homeomorphism}\}$$

is cofinal and τ-closed in $\exp_\tau(T \setminus T_0)$.

3.2. Properties of Z_τ-sets in I^τ. We denote by $\operatorname{cov}(X)$ the collection of all countable functionally open covers of the space X. We set

$$B(f, \{\mathcal{U}_t \colon t \in T\}) = \{g \in C(X,Y) \colon g \text{ is } \mathcal{U}_t\text{-close to } f \text{ for each } t \in T\},$$

Let τ be an infinite cardinal. If X and Y are Tikhonov spaces then $C_\tau(X,Y)$ denotes the space of all continuous maps $X \to Y$ with the topology defined as follows ([7], [6, p. 273]): a set $G \subseteq C_\tau(X,Y)$ is open if for each $h \in G$ there is a collection $\{\mathcal{U}_t \colon t \in T\} \subseteq \operatorname{cov}(Y)$, with $|T| < \tau$, such that

$$h \in B(f, \{\mathcal{U}_t \colon t \in T\}) \subseteq G.$$

Obviously if $\tau = \omega$, then the above topology coincides with the limitation topology (see [18]). For $\tau > \omega$, this topology is conveniently described in the following statement [9, Lemma 3.1].

Lemma 3.5. *Let $\tau > \omega$ and X be a z-embedded subspace of a product $\prod_{t \in T} X_t$ of separable metrizable spaces. If $|T| = \tau$, then basic neighborhoods of a map $f \colon Y \to X$ in $C_\tau(Y,X)$ are of the form $B(f,S) = \{g \in C_\tau(Y,X) \colon \pi_S \circ g = \pi_S \circ f\}$, $S \subset T$, $|S| < \tau$, where $\pi_S \colon \prod_{t \in T} X_t \to \prod_{t \in S} X_t$ denotes the projection.*

Now we are ready to define Z_τ-sets [6, Definition 8.5.1].

Definition 3.1. Let $\tau \geq \omega$. A closed subset $A \subset X$ is a *Z_τ-set* in X if the set $\{f \in C_\tau(X,X) \colon f(X) \cap A = \emptyset\}$ is dense in the space $C_\tau(X,X)$.

Clearly Z_ω-sets are the same as standard Z-sets. We also need the following concept.

Definition 3.2. Let $\pi \colon X \to Y$ be a map. A closed subset $A \subset X$ is a *fibered Z-set* in X if the set $\{f \in C^\pi_\tau(X,X) \colon f(X) \cap A = \emptyset\}$ is dense in the space $C^\pi_\tau(X,X) = \{f \in C_\tau(X,X) \colon \pi \circ f = \pi\}$.

Lemma 3.6. *Let $\tau > \omega$ and $|T| = \tau$. For a closed set $M \subset I^T$ the following conditions are equivalent:*

(1) *M is a Z_τ-set in I^T,*
(2) *for each $T_0 \subset T$ with $|T_0| < \tau$, the set*

$$\mathcal{Z}_{(M,T_0)} = \{S \subset \exp_\omega(T \setminus T_0) \colon \pi_{T_0 \cup S}(M) \text{ is a fibered } Z\text{-set in } I^{T_0 \cup S} \text{ with respect to } \pi^{T_0 \cup S}_{T_0}\}$$

is cofinal and ω-closed in $\exp_\omega(T \setminus T_0)$.

Proof. (1)⇒(2). Let $S_0 = R \in \exp_\omega(T \setminus T_0)$. Since M is a Z_τ-set in I^T and $|T_0 \cup S_0| < \tau$, there exists, by Lemma 3.5, a map $f_1\colon I^T \to I^T$ such that $\pi_{T_0\cup S_0} \circ f_1 = \pi_{T_0\cup S_0}$ and $f_1(I^T) \cap M = \emptyset$. By Proposition 3.1, there exist a countable subset $S_1 \subseteq T \setminus T_0$, with $S_0 \subseteq S_1$, and a map $g_1\colon I^{T_0\cup S_1} \to I^{T_0\cup S_1}$ such that $g_1(I^{T_0\cup S_1}) \cap \pi_{T_0\cup S_1}(M) = \emptyset$ and $\pi_{T_0\cup S_1} \circ f_1 = g_1 \circ \pi_{T_0\cup S_1}$. Note that $\pi^{T_0\cup S_1}_{T_0} \circ g_1 = \pi^{T_0\cup S_1}_{T_0}$.

Continuing this process we construct an increasing sequence $\{S_n\colon n\in\omega\}$ of countable subsets of $T \setminus T_0$ and maps $g_n\colon I^{T_0\cup S_n} \to I^{T_0\cup S_n}$ so that $\pi^{T_0\cup S_{n+1}}_{T_0\cup S_n} \circ g_{n+1} = \pi^{T_0\cup S_{n+1}}_{T_0\cup S_n}$ and $g_n(I^{T_0\cup S_n}) \cap \pi_{T_0\cup S_n}(M) = \emptyset$ for each $n \geq 1$. Let $S = \bigcup_{n\in\omega} S_n$. We leave to the reader the verification of the fact that $\pi_{T_0\cup S}(M)$ is a fibered Z-set in $I^{T_0\cup S}$ with respect to the projection $\pi^{T_0\cup S}_{T_0}$ (cf. [9, proof of Proposition 2.3]). This proves the cofinality of $\mathcal{Z}_{(M,T_0)}$. The ω-completeness of this set is obvious.

(2)⇒(1). According to Lemma 3.5 it suffices to find, for any $T_0 \subset T$ with $|T_0| < \tau$, a map $f\colon I^T \to I^T$ such that $\pi_{T_0} \circ f = \pi_{T_0}$ and $f(I^T) \cap M = \emptyset$. By (2), there exist a countable subset $S \subset T\setminus T_0$ and a map $g\colon I^{T_0\cup S} \to I^{T_0\cup S}$ such that $\pi^{T_0\cup S}_{T_0} \circ g = \pi^{T_0\cup S}_{T_0}$ and $g(I^{T_0\cup S}) \cap \pi_{T_0\cup S}(M)$. Let $j\colon I^{T_0\cup S} \to I^T$ be a section of the projection $\pi_{T_0\cup S}\colon I^T \to I^{T_0\cup S}$. It only remains to note that the map $f = j \circ g \circ \pi_{T_0\cup S}$ has the required properties. ∎

Proposition 3.7. *Let $\tau > \omega$ and $|T| = \tau$. For a closed set $M \subset I^T$ the following conditions are equivalent:*

1. *M is a Z_τ-set in I^T;*
2. *If $F \subset M$ is a closed subset, then $\psi(F, I^T) = \tau$;*
3. *T can be represented as the increasing union $T = \bigcup_{\alpha<\tau} T_\alpha$ of its subsets so that*
   - (a) *$|T_0| = \omega$;*
   - (b) *$|T_{\alpha+1} \setminus T_\alpha| = \omega$;*
   - (c) *$T_\alpha = \bigcup_{\beta<\alpha} T_\beta$ for each limit ordinal $\alpha < \tau$.*
   - (d) *$\pi_{T_{\alpha+1}}(Z)$ is a fibered Z-set in $I^{T_{\alpha+1}}$ with respect to the projection $\pi^{T_{\alpha+1}}_{T_\alpha}\colon I^{T_{\alpha+1}} \to I^{T_\alpha}$.*

Proof. (1)⇔(2) is proved in [6, Proposition 8.5.5], and (2)⇒(3) follows from Lemma 3.6.

To prove (3)⇒(1), let $A \subset T$ with $|A| < \tau$. Then, by (b), there exists $\alpha < \tau$ such that $A \subset T_\alpha$. By (d), there exists a map $f\colon I^{T_{\alpha+1}} \to I^{T_{\alpha+1}}$ such that $\pi^{T_{\alpha+1}}_{T_\alpha} \circ f = \pi^{T_{\alpha+1}}_{T_\alpha}$ and $\mathrm{Im}(f) \cap \pi_{T_{\alpha+1}}(Z) = \emptyset$. Define $g\colon I^T \to I^T$ by $g = i_{T_{\alpha+1}} \circ f \circ \pi_{T_{\alpha+1}}$, where $i_{T_{\alpha+1}}$ is a section of the projection $\pi_{T_{\alpha+1}}$. It is clear that $\pi_A \circ g = \pi_A$ and $\mathrm{Im}(g) \cap Z = \emptyset$. ∎

Corollary 3.8. *Let $\tau \geq \omega$ and X be a compact space of weight $\leq \tau$. Then there exists a Z_τ-embedding of X into the cube I^τ.*

Proof. For $\tau = \omega$ the statement is known. Let $|T| = \tau > \omega$ and represent I^τ as $\prod_{t\in T} Y_t$, where each Y_t is a copy of I^ω. Embed X into a product $\prod_{t\in T} X_t$ of compact metrizable spaces. Since each X_t admits a Z-embedding into Y_t, it follows from Proposition 3.7 that $\prod_{t\in T} X_t$, and hence X, admits a Z_τ-embedding into $\prod_{t\in T} Y_t$. ■

3.3. Mapping replacement. For a Z-set $Z \subset I^\omega$ and a closed subset Y of a metrizable compactum X, the standard mapping replacement allows us to approximate a map $f\colon X \to I^\omega$ by a map $g\colon X \to I^\omega$ in such a way that $g|Y = f|Y$ and $g(X \setminus Y) \cap Z = \emptyset$. The key here is that the space $C(X, I^\omega)$ is completely metrizable and hence possesses the Baire property. Below we prove versions of mapping replacement for Z_τ-sets in Tikhonov cubes by using the spectral technique.

PROPOSITION 3.9. *Let $\tau \geq \omega$ and Z be a Z_τ-set in I^τ. Suppose also that $Y \subset X$ is a closed subset of a compactum X. Then for any map $f\colon X \to I^\tau$ and any collection $\{K_\alpha\colon 1 \leq \alpha < \tau\}$ of compact subsets of X with $\bigcup_{1\leq\alpha<\tau} K_\alpha \cap Y = \emptyset$ there exists a map $g\colon X \to I^\tau$ such that $g|Y = f|Y$ and $f(\bigcup_{1\leq\alpha<\tau} K_\alpha) \subset I^\tau \setminus Z$.*

Proof. Let $|T| = \tau$ and $\{T_\alpha\colon \alpha < \tau\}$ be a collection of subsets supplied by Proposition 3.7 corresponding to the Z_τ-set Z.

We construct maps $g_\alpha\colon X \to I^{T_\alpha}$ as follows. Let $g_0 = \pi_{T_0} \circ f\colon X \to I^{T_0}$. Suppose that for each $\beta < \alpha$ we have already constructed g_β satisfying the following conditions:

(i) $\pi^{T_\beta}_{T_\delta} \circ g_\beta = g_\delta$, whenever $\delta < \beta < \alpha$;
(ii) $g_\beta = \lim\{g_\delta\colon \delta < \beta\}$, whenever $\beta < \alpha$ is a limit ordinal;
(iii) $g_\beta(K_\beta) \cap \pi_{T_\beta}(Z) = \emptyset$, whenever $1 \leq \beta < \alpha$;
(iv) $g_\beta|Y = \pi_{T_\beta} \circ f|Y$, whenever $\beta < \alpha$.

First consider the case $\alpha = \beta + 1$. Since $\pi_{T_\alpha}(Z)$ is a fibered Z-set in I^{T_α} with respect to the projection $\pi^{T_\alpha}_{T_\beta}\colon I^{T_\alpha} \to I^{T_\beta}$ there exists a map $h_\alpha\colon I^{T_\alpha} \to I^{T_\alpha}$ such that $h_\alpha(K_\alpha) \cap \pi_{T_\alpha}(Z) = \emptyset$ and $\pi^{T_\alpha}_{T_\beta} \circ h_\alpha = \pi^{T_\alpha}_{T_\beta}$. Let $s\colon I^{T_\beta} \to I^{T_\alpha}$ be a section of $\pi^{T_\alpha}_{T_\beta}$. Consider the map $r_\alpha\colon Y \cup K_\alpha \to I^{T_\alpha}$ which coincides with $\pi_{T_\alpha} \circ f$ on Y and with $h_\alpha \circ s \circ g_\alpha$ on K_α. Straightforward verification shows that the following diagram of unbroken arrows commutes:

$$\begin{tikzcd}
Y \cup K_\alpha \arrow[r, "r_\alpha"] \arrow[d, hook, "\mathrm{incl}"'] & I^{T_\alpha} \arrow[d, "\pi^{T_\alpha}_{T_\beta}"] \\
X \arrow[r, "g_\beta"] \arrow[ur, dotted, "g_\alpha"] & I^{T_\beta}
\end{tikzcd}$$

Consequently, by softness of $\pi^{T_\alpha}_{T_\beta}$, there is a map $g_\alpha\colon X \to I^A$ (the dotted arrow in the diagram) such that $g_\alpha|(Y \cup K_\alpha) = r_\alpha$ and $\pi^{T_\alpha}_{T_\beta} \circ g_\alpha = g_\beta$. It is also clear that $g_\alpha(K_\alpha) \cap \pi_{T_\alpha}(Z) = \emptyset$ and $g_\alpha|Y = \pi_{T_\alpha} \circ f$.

If $\alpha = \lim\{\beta\colon \beta < \alpha\}$, then let $g_\alpha = \lim\{g_\beta\colon \beta < \alpha\}\colon X \to I^{T_\alpha}$.

This completes the inductive construction. Let $g = \lim\{g_\alpha\colon \alpha < \tau\} : X \to I^T$ be the limit map. It is clear that $g|Y = f|Y$ and $g(K_\alpha) \cap Z = \emptyset$ for each α. ∎

Corollary 3.10. *Let $\tau > \omega$. For any map $f\colon X \to Y$ between Z_τ-sets in I^τ there exists a proper map $g\colon I^\tau \setminus X \to I^\tau \setminus Y$ such that $f = \bar{g}|X$, where $\bar{g}\colon I^\tau \to I^\tau$ is the extension of g.*

Proof. Let $\bar{f}\colon I^\tau \to I^\tau$ be an extension of f. By Proposition 3.9, there exists a map $\bar{g}\colon I^\tau \to I^\tau$ such that $\bar{g}|X = \bar{f}|X = f$ and $\bar{g}(I^\tau \setminus X) \subset I^\tau \setminus Y$. Clearly, $g = \bar{g}|(I^\tau \setminus X)\colon I^\tau \setminus X \to I^\tau \setminus Y$ is a proper map with the required properties. ∎

Lemma 3.11. *Let $B \subset A$ and $|A \setminus B| = \omega$. Suppose that Z is a fibered Z-set in I^A with respect to the projection $\pi^A_B\colon I^A \to I^B$. Suppose also that X is closed in I^A and we are given an embedding $f\colon X \to Z$ and a map $g\colon I^B \to I^B$ such that $\pi^A_B \circ f = g \circ \pi^A_B|X$. Then there exists a fibered Z-embedding $h\colon I^A \to I^A$ such that $\pi^A_B \circ h = g \circ \pi^A_B$, $h|X = f|X$ and $h(I^A \setminus X) \subset I^A \setminus Z$.*

Proof. Using [6, Theorem 1.3.4] for ω-spectra, one can find a countable subset $C \subset B$, an embedding $f_0\colon \pi^A_{C\cup(A\setminus B)}(X) \to \pi^A_{C\cup(A\setminus B)}(Z)$ and a map $g_0\colon I^C \to I^C$, satisfying the following conditions:

(i) $\pi^A_{C\cup(A\setminus B)}(Z)$ is a fibered Z-set in $I^{C\cup(A\setminus B)}$ with respect to the projection $\pi^{C\cup(A\setminus B)}_C\colon I^{C\cup(A\setminus B)} \to I^C$;

(ii) $f_0 \circ \pi^A_{C\cup(A\setminus B)}|X = \pi^A_{C\cup(A\setminus B)} \circ f$;

(iii) $g_0 \circ \pi^A_{C\cup(A\setminus B)} = \pi^A_{C\cup(A\setminus B)} \circ g$;

(iv) $\pi^{C\cup(A\setminus B)}_C \circ f_0 = g_0 \circ \pi^{C\cup(A\setminus B)}_C|\pi^A_{C\cup(A\setminus B)}(X)$.

Let $\bar{h}\colon I^{C\cup(A\setminus B)}\colon I^{C\cup(A\setminus B)} \to I^A$ be such that $\pi^{C\cup(A\setminus B)}_C \circ \bar{h} = g_0 \circ \pi^{C\cup(A\setminus B)}_C$ and $\bar{h}|\pi^A_{C\cup(A\setminus B)}(X) = f_0$. Next consider the space (with the compact-open topology) $C^{\bar{h}}$ of all maps $h\colon I^{C\cup(A\setminus B)} \to I^{C\cup(A\setminus B)}$ such that $h \circ \pi^{C\cup(A\setminus B)}_C = \bar{h} \circ \pi^{C\cup(A\setminus B)}_C$ and $h|\pi^A_{C\cup(A\setminus B)}(X) = f_0$. It follows from [19] that the set S of fibered Z-embeddings is dense and G_δ in $C^{\bar{h}}$. Moreover, the set R of maps h with $h(I^{C\cup A\setminus B)} \setminus \pi^A_{C\cup(A\setminus B)}(X)) \cap \pi^A_{C\cup(A\setminus B)}(Z) = \emptyset$ is also dense and G_δ in $C^{\bar{h}}$. Consequently, since $C^{\bar{h}}$ is completely metrizable, $S \cap R \neq \emptyset$. Take any $h_0 \in S \cap R$. There is precisely one map $h\colon I^A \to I^A$ such that $\pi^A_{C\cup(A\setminus B)} \circ h = h_0 \circ \pi^A_{C\cup(A\setminus B)}$ and $\pi^A_B \circ h = \pi^A_B$. It follows from the construction that h has the required properties. ∎

Proposition 3.12. *Let $|T| = \tau \geq \omega$ and Z be a Z_τ-set in I^T. Suppose that Y is closed in a compactum X of weight $\leq \tau$ and $f\colon X \to I^T$ is a map such that $f(Y) \subset Z$ and $f|Y\colon Y \to Z$ is an embedding. Then there exists a Z_τ-embedding $h\colon X \to I^T$ such that $h|Y = f|Y$ and $h(X \setminus Y) \subset I^T \setminus Z$.*

Proof. Without loss of generality we may assume that $X = I^T$. Using Corollaries 3.2, 3.4 and Lemma 3.6 it is easy to construct subsets $T_\alpha \subset T$ and maps $f_\alpha\colon I^{T_\alpha} \to I^{T_\alpha}$, $\alpha < \tau$, with the following properties:

(a) $|T_0| = \omega$;
(b) $T_\alpha \subset T_{\alpha+1}$ and $|T_{\alpha+1} \setminus T_\alpha| = \omega$;
(c) $T = \bigcup_{\alpha<\tau} T_\alpha$ and $T_\alpha = \bigcup_{\beta<\alpha} T_\beta$ for each limit ordinal $\alpha < \tau$;
(d) $\pi_{T_{\alpha+1}}(Z)$ is a fibered Z-set in $I^{T_{\alpha+1}}$ with respect to the projection $\pi^{T_{\alpha+1}}_{T_\alpha}\colon I^{T_{\alpha+1}} \to I^{T_\alpha}$;
(e) $\pi^{T_{\alpha+1}}_{T_\alpha} \circ f_{\alpha+1} = f_\alpha \circ \pi^{T_{\alpha+1}}_{T_\alpha}$;
(f) $f = \lim\{f_\alpha\colon \alpha < \tau\}$ and $f_\alpha = \lim\{f_\beta\colon \beta < \alpha\}$ for each limit ordinal $\alpha < \tau$;
(g) $f_\alpha|\pi_{T_\alpha}(Y)\colon \pi_{T_\alpha}(Y) \to \pi_{T_\alpha}(Z)$ is an embedding.

In order to construct the required Z_τ-embedding $h\colon I^T \to I^T$ we proceed by induction. Let $h_0 = f_0$. Supposing that f_β's have been constructed for all $\beta < \alpha$, the construction of f_α for non-limit α is straightforward by using Lemma 3.11. For a limit α, we set $h_\alpha = \lim\{h_\beta\colon \beta < \alpha\}$. Finally, the required embedding is defined by letting $h = \lim\{h_\alpha\colon \alpha < \tau\}$. Proposition 3.7 guarantees that h is a Z_τ-embedding. By construction, $h|X = f|X$ and $h(I^T \setminus X) \subset I^T \setminus Z$. ∎

4. Extension properties of the Stone–Čech corona. We begin by introducing the following concept (cf. [13], [12]).

Definition 4.1. A locally compact space Y is a *proper absolute extensor* for a locally compact space X (notation: $Y \in \mathrm{AE}_p(X)$) if any proper map $f\colon A \to Y$, defined on a closed C^*-embedded subset A of X, admits a proper extension $\bar{f}\colon X \to Y$.

Recall that *regular closed* subsets are closures of open subsets.

Theorem 4.1. *Let X be a locally compact space which can be covered by at most τ compact subsets and each regular closed subset of which is C^*-embedded. Let also L be a compact ANR-space embedded into I^τ as a Z_τ-set. Then the following conditions are equivalent:*

(a) *$L \in \mathrm{AE}(\beta X \setminus X)$;*
(b) *$I^\tau \setminus L \in \mathrm{AE}_p(X)$.*

Proof. (a)⇒(b). Let $f\colon A \to I^\tau \setminus L$ be a proper map defined on a closed C^*-embedded subspace $A \subset X$. Then $\beta A = \mathrm{cl}_{\beta X} A$ and there is an extension $\bar{f}\colon \mathrm{cl}_{\beta X} A \to I^\tau$ of f. Since f is proper, we have $\bar{f}(\mathrm{cl}_{\beta X} A \setminus A) \subset L$. By (a), $\bar{f}|(\mathrm{cl}_{\beta X} A \setminus A)\colon (\mathrm{cl}_{\beta X} A \setminus A) \to L$ can be extended to a map $g\colon \beta X \setminus X \to L$. Since $A \cup (\beta X \setminus X)$ is closed in βX there exists a map $G\colon \beta X \to I^\tau$ such that $G|(\beta X \setminus X) = g$ and $G|A = f$.

Using the spectral theorem for τ-spectra [6, Theorem 1.3.4], we can find a compact space Y of weight $\le \tau$, and maps $p\colon \beta X \to Y$, $q\colon Y \to I^\tau$ such that $G = q \circ p$ and $\beta X \setminus X = p^{-1}(p(\beta X \setminus X))$. By Proposition 3.9, there exists a map $H\colon Y \to I^\tau$ such that $H|p(A \cup (\beta X \setminus X)) = q|p(A \cup (\beta X \setminus X))$ and $H(Y \setminus p(A \cup (\beta X \setminus X))) \subset I^\tau \setminus L$.

It remains to note that the map $F = H \circ p\colon \beta X \to I^\tau$ has the following properties: $F|(A \cup (\beta X \setminus X)) = G|(A \cup (\beta X \setminus X))$ and $F(X \setminus A) \subset I^\tau \setminus L$. Consequently, $\widetilde{f} = F|X\colon X \to I^\tau \setminus L$ is a proper map extending f.

(b)⇒(a). Let $f\colon A \to L$ be defined on a closed subspace $A \subset \beta X \setminus X$. Since L is an ANR-space, we may assume that f is already defined on the closure $\mathrm{cl}_{\beta X} U$ of an open neighborhood U of A in βX. Note that $\mathrm{cl}_{\beta X} U = \mathrm{cl}_{\beta X}(U \cap X) = \mathrm{cl}_{\beta X}(\mathrm{cl}_X(U \cap X))$ and that according to our assumption $\mathrm{cl}_X(X \cap U)$ is C^*-embedded in X. Since $\kappa(\mathrm{cl}_X(U \cap X)) \le \tau$, we conclude, by Proposition 3.9, that there exists a map $g\colon \mathrm{cl}_{\beta X} U \to I^\tau$ such that $g|(\mathrm{cl}_{\beta X} U \setminus \mathrm{cl}_X(U \cap X)) = f|(\mathrm{cl}_{\beta X} U \setminus \mathrm{cl}_X(U \cap X))$ and $g(\mathrm{cl}_X(U \cap X)) \subset I^\tau \setminus L$. By (b), the proper map $g|\mathrm{cl}_X(U \cap X)\colon \mathrm{cl}_X(U \cap X) \to I^\tau \setminus L$ has a proper extension $G\colon X \to I^\tau \setminus L$. Since G is proper, its Stone–Čech extension $\widetilde{G}\colon \beta X \to I^\tau$ sends $\beta X \setminus X$ into L. Straightforward verification shows that $\widetilde{F}|A = f$. ∎

Corollary 4.2. *Let L be metrizable ANR-compact space embedded into I^ω as a Z-set. Then the following conditions are equivalent for any locally compact and Lindelöf space X:*

(a) $L \in \mathrm{AE}(\beta X \setminus X)$;
(b) $I^\omega \setminus L \in \mathrm{AE}_p(X)$;
(c) $\mathrm{Cone}(L) \setminus L \in \mathrm{AE}_p(X)$.

Proof. The equivalence of (a) and (b) follows from Theorem 4.4 since $\kappa(X) \le \omega$ for any locally compact and Lindelöf space X.

To prove the remaining equivalence, first note that by Edwards' theorem [6, Corollary 2.3.23], $I^\omega \times \mathrm{Cone}(L)$ is homeomorphic to I^ω. Further, by Chapman's Complement Theorem [4], the complements $I^\omega \setminus L$ and $I^\omega \times \mathrm{Cone}(L) \setminus I^\omega \times L = I^\omega \times (\mathrm{Cone}(L) \setminus L)$ are homeomorphic. Finally, note that $I^\omega \times (\mathrm{Cone}(L) \setminus L) \in \mathrm{AE}_p(X)$ precisely when $\mathrm{Cone}(L) \setminus L \in \mathrm{AE}_p(X)$. ∎

Corollary 4.3. *The following conditions are equivalent for any locally compact and Lindelöf space X:*

(a) $\dim(\beta X \setminus X) \leq n$;
(b) $\dim_p X \leq n+1$, *i.e. any proper map* $f\colon A \to \mathbb{R}^{n+1}$, *defined on a closed subspace* $A \subset X$, *can be extended to a proper map* $\bar{f}\colon X \to \mathbb{R}^{n+1}$.

THEOREM 4.4. *Let* $\tau > \omega$, L *be a compact ANR embedded into* I^τ *as a* Z_τ*-set and* X *be a* Z_τ*-set in* I^τ. *Then the following conditions are equivalent:*

(i) $X \in \mathrm{AE}([L])$;
(ii) $I^\tau \setminus X \in \mathrm{AE}_p^\tau([I^\tau \setminus L])$.

Proof. (ii)⇒(i). We need to show that a map $f\colon A \to X$ defined on a closed subset $A \subset Y$, where Y is a compact space of weight $\leq \tau$ such that $L \in \mathrm{AE}(Y)$, has a continuous extension $g\colon Y \to X$. By Corollary 3.8, we may assume that A is embedded into $K = I^\tau$ as a Z_τ-set. Note that K is then the Stone–Čech compactification of $K \setminus A$ (Lemma 2.1). Similarly, we may assume that Y is also embedded into (a different copy of) I^τ as a Z_τ-set. By Proposition 3.12, it is possible to embed K into I^τ in such a way that $K \cap Y = A$. Let $\bar{f}\colon A \to I^\tau$ denote an extension of f. Using Proposition 3.9 we can find a map $h\colon K \to I^\tau$ such that $h|A = f$ and $h(K \setminus A) \subset I^\tau \setminus X$. Note that $h|(K \setminus A)\colon K \setminus A \to I^\tau \setminus X$ is proper. Since K is the Stone–Čech compactification of $K \setminus A$, it follows that $K \setminus A$ is C^*-embedded in $I^\tau \setminus Y$. Next note that, by Lemma 2.1(iii), $I^\tau \setminus Y$ satisfies the assumptions of Theorem 4.1. Consequently, since $L \in \mathrm{AE}(Y)$, we conclude by Theorem 4.1 that $I^\tau \setminus L \in \mathrm{AE}(I^\tau \setminus Y)$. Then, by (ii), the proper map $h|(K \setminus A)\colon K \setminus A \to I^\tau \setminus X$ admits a proper extension $\bar{h}\colon I^\tau \setminus Y \to I^\tau \setminus X$. Let $\bar{g}\colon I^\tau \to I^\tau$ be the extension of $\bar{h}$. Properness of $\bar{h}$ implies that $\bar{g}(Y) \subset X$. Then $g = \bar{g}|Y\colon Y \to X$ is the required extension of f.

(i)⇒(ii). Let now $f\colon B \to I^\tau \setminus X$ be a proper map defined on a closed and C^*-embedded subset B of a locally compact space Y of weight $\leq \tau$ such that $I^\tau \setminus L \in \mathrm{AE}_p(Y)$. We need to construct a proper extension $\bar{f}\colon Y \to I^\tau \setminus X$ of f. Since B is C^*-embedded in Y it follows that $\mathrm{cl}_{\beta Y} B$ is the Stone–Čech compactification of B. Consequently, there is an extension $g\colon \mathrm{cl}_{\beta Y} B \to I^\tau$ of f. Properness of f implies that $g(\mathrm{cl}_{\beta Y} B \setminus B) \subset X$. Since $I^\tau \setminus L \in \mathrm{AE}_p(Y)$ we conclude, by Theorem 4.1, that $L \in \mathrm{AE}(\beta Y \setminus Y)$. Thus, by (i), there exists an extension $h\colon \beta Y \setminus Y \to X$ of $g|(\mathrm{cl}_{\beta Y} B \setminus B)\colon \mathrm{cl}_{\beta Y} B \setminus B \to X$. Now consider the closed subset $A = (\beta Y \setminus Y) \cup B$ of βY and the map $h'\colon A \to I^\tau$ defined be

$$h'(y) = \begin{cases} h(y) & \text{if } y \in \beta Y \setminus Y, \\ f(y) & \text{if } y \in B. \end{cases}$$

Next consider any extension $\bar{h}\colon \beta Y \to I^\tau$ of h'. By Proposition 3.9, we can find a map $\bar{g}\colon \beta Y \to I^\tau$ such that $\bar{g}|A = h'$ and $\bar{g}(\beta Y \setminus A) \subset I^\tau \setminus X$. Straightforward verification shows that $\bar{g}(Y) \subset I^\tau \setminus X$ and consequently $\bar{f} = \bar{g}|Y : Y \to I^\tau \setminus X$ is proper. It only remains to note that $\bar{f}|B = f$. ∎

5. Categorical isomorphisms. Recall (Corollary 3.10) that for any map $f\colon X \to Y$ between Z_τ-sets in I^τ we can find a proper map $g\colon I^\tau \setminus X \to I^\tau \setminus Y$ such that $f = \widetilde{g}|X$, where $\widetilde{g}\colon I^\tau \to I^\tau$ is the unique extension of g. If f is a homeomorphism, we may assume that so is g (this a Z_τ-set unknotting theorem, [6, Theorem 8.5.4]). In other words, any map between Z_τ-sets of the Tikhonov cube can be obtained as the restriction of the Stone–Čech extension of a proper map between their complements, i.e. the correspondence $\lambda\colon C_p(I^\tau \setminus X, I^\tau \setminus Y) \to C(X,Y)$ defined by $\lambda(g) = \widetilde{g}|X$ is surjective. Below we show that (up to a certain equivalence relation) λ is in fact a bijection. Here we extend the considerations of [10], carried out for the Hilbert cube, to the Tikhonov cube.

Let $\mathcal{Z}_\tau$ denote the category of Z_τ-sets in I^τ and their continuous maps. Let also $\mathcal{C}_p(\mathcal{Z}_\tau)$ denote the category whose objects are complements of Z_τ-sets in I^τ and whose morphisms are the equivalence classes of proper maps with respect to the following relation: two proper maps are equivalent if they are close in the continuously controlled (by the compactification I^τ) coarse structure ([14, Remark 2.29]). Recall that two proper maps $g_1, g_2\colon I^\tau \setminus X \to I^\tau \setminus Y$ are *close* if $\widetilde{g}_1(x) = \widetilde{g}_2(x)$ for any $x \in X$. The equivalence class with representative g will be denoted by $\{g\}$. With this in mind we have

Proposition 5.1. *Let $\tau > \omega$. Then the correspondence $\lambda\colon \mathcal{C}_p(\mathcal{Z}_\tau) \to \mathcal{Z}_\tau$ defined by letting:*

(i) *for $I^\tau \setminus X \in \mathcal{OB}(\mathcal{C}_p(\mathcal{Z}_\tau))$, $\lambda(I^\tau \setminus X) = X$,*

(ii) *for $\{g\}\colon I^\tau \setminus X \to I^\tau \setminus Y \in \mathcal{MOR}(\mathcal{C}_p(\mathcal{Z}_\tau))$, $\lambda(\{g\}) = \widetilde{g}|X$,*

is an isomorphism of categories.

Proof. Structurally the proof follows that of [10, Theorem 2], but is much simpler and is left to the reader. The fact that λ is well defined on morphisms is a direct consequence of the definition of the closeness relation. The fact that λ is surjective on morphisms, as noted above, follows from Corollary 3.10. ∎

Next we consider homotopy categories. Let $\mathcal{H}(\mathcal{Z}_\tau)$ denote the category whose objects are the same as in $\mathcal{Z}_\tau$ and morphisms are the homotopy classes of maps. Similarly $\mathcal{H}_p(\mathcal{Z}_\tau)$ denotes the category whose objects are the same as in $\mathcal{C}_p(\mathcal{Z}_\tau)$ and morphisms are the proper homotopy classes of proper maps. First, we need the following observation.

Lemma 5.2. *Let $\tau > \omega$ and $f_0, f_1\colon X \to Y$ be two maps between Z_τ-sets in I^τ. Suppose also that $g_0, g_1\colon I^\tau \setminus X \to I^\tau \setminus Y$ are proper maps such that $f_k = \widetilde{g}_k|X$, $k = 0,1$. Then $f_0 \simeq f_0$ iff $g_0 \simeq_p g_1$.*

Proof. Let $F\colon X \times [0,1] \to Y$ be a homotopy between f_0 and f_1. Consider the map $H\colon X \times [0,1] \cup I^\tau \times \{0,1\} \to I^\tau$ defined by letting

$$H(z,t) = \begin{cases} F(z,t) & \text{if } (z,t) \in X \times [0,1], \\ \widetilde{g}_k(z) & \text{if } z \in I^\tau \times \{0,1\},\ k = 0,1. \end{cases}$$

By Proposition 3.9, there exists a map $\widetilde{G}\colon I^\tau \times [0,1] \to I^\tau$ such that $\widetilde{G}|(X\times[0,1]\cup I^\tau\times\{0,1\}) = H$ and $\widetilde{G}(I^\tau\times[0,1]\setminus(X\times[0,1]\cup I^\tau\times\{0,1\}))\subset I^\tau\setminus Y$. Clearly, $G = \widetilde{G}|((I^\tau\setminus X)\times[0,1])\colon (I^\tau\setminus X)\times[0,1]\to I^\tau\setminus Y$ is a proper homotopy between g_0 and g_1.

Conversely, suppose that $G\colon (I^\tau\setminus X)\times[0,1]\to I^\tau\setminus Y$ is a proper homotopy between $g_0, g_1\colon I^\tau\setminus Y\to I^\tau\setminus X$. Note that by Lemma 2.1, $I^\tau\times[0,1]$ is the Stone–Čech compactification of $(I^\tau\setminus X)\times[0,1]$. Consequently, G admits an extension $\widetilde{G}\colon I^\tau\times[0,1]$ such that $\widetilde{G}(X\times[0,1])\subset Y$. It is clear that $H = \widetilde{G}|X\times[0,1]\colon X\times[0,1]\to Y$ is a homotopy between f_0 and f_1. ∎

Corollary 5.3. *Let $\tau > \omega$ and X and Y be Z_τ-sets in I^τ. If proper maps $g_0, g_1\colon T^\tau\setminus X\to I^\tau\setminus Y$ are close with respect to the continuously controlled coarse structure induced by I^τ, then g_0 and g_1 are properly homotopic.*

Proof. [14, Theorem 2.27] implies that coarsely close proper maps coincide on the Stone–Čech corona. Consequently, by Lemma 5.2, they are properly homotopic. ∎

Now let us define a functor $\mu\colon \mathcal{H}_p(\mathcal{Z}_\tau)\to\mathcal{H}(\mathcal{Z}_\tau)$ between these homotopy categories. The following statement is parallel to [10, Proposition 10].

Proposition 5.4. *Let $\tau > \omega$. Then the correspondence $\mu\colon \mathcal{H}_p(\mathcal{Z}_\tau)\to\mathcal{H}(\mathcal{Z}_\tau)$, defined by letting:*

(i) *for $I^\tau\setminus X\in\mathcal{OB}(\mathcal{H}_p(\mathcal{Z}_\tau))$, $\mu(I^\tau\setminus X) = X$,*
(ii) *for $[g]\colon I^\tau\setminus X\to I^\tau\setminus Y\in\mathcal{MOR}(\mathcal{H}_p(\mathcal{Z}_\tau))$, $\mu([g]) = [\widetilde{g}|X]$,*

is an isomorphism of categories.

Proof. One part of Lemma 5.2 shows that μ is well defined. The other part guarantees that μ is surjective on morphisms. The rest is straightforward and left to the reader. ∎

In light of the above considerations and the role of the closeness relation associated to the continuously controlled coarse structure (induced by the Stone–Čech compactification), we would like to investigate this concept a little further. For locally compact and paracompact spaces [14, Theorem 2.27] characterizes close proper maps between such spaces as those whose extensions to the Stone–Čech compactifications coincide on the Stone–Čech corona. For Lindelöf spaces we have the following statement.

Proposition 5.5. *Let $f, g\colon X\to Y$ be proper maps between locally compact and Lindelöf spaces. Then the following conditions are equivalent:*

(i) *f and g are close in the continuously controlled (by the Stone–Čech compactification) coarse structure;*
(ii) *$\widetilde{f}(x) = \widetilde{g}(x)$ for any $x\in\beta X\setminus X$;*

(iii) *there is a compact subset* $C \subset X$ *such that* $f(x) = g(x)$ *for any* $x \in X \setminus C$.

Proof. As mentioned above, [14, Theorem 2.27] implies (i)⇔(ii).

Let us prove (ii)⇒(iii). Take $x \in \beta X \setminus X$. Since X is Lindelöf (actually realcompactness of X suffices here), we can find a sequence $\{U_n : n \in \omega\}$ of open neighborhoods of x in βX with the following properties:

(1) $\mathrm{cl}_{\beta X} U_{n+1} \subset U_n$;
(2) $\bigcap_{n\in\omega} U_n \subset \beta X \setminus X$.

We need the following

Claim. *There exists* $i \in \omega$ *such that* $\widetilde{f}|_{U_i} = \widetilde{g}|_{U_i}$.

To prove the claim, assume the contrary. By (ii), our assumption implies that $f|_{U_i\cap X} \neq g|_{U_i\cap X}$ for each $i \in \omega$. Assume that for each $k < n$ we have found $z_k \in U_k \cap X$ such that

$$(*)\qquad \{f(z_i) : i \le k\} \cap \{g(z_i) : i \le k\} = \emptyset.$$

Next, let us construct a desired z_n. First, for each $i \ge n$, fix $a_i \in U_i \cap X$ such that

(a) $f(a_i) \neq g(a_i)$.

Such a_i's exist because we assumed that $f|_{U_i\cap X} \neq g|_{U_i\cap X}$ for each $i \in \omega$. Due to (1) and (2), no infinite subset of $\{a_i : i \ge n\}$ is compact. Since f is proper, there exists n_1 such that

(b) $f(a_i) \notin \{g(z_k) : k \le n\}$ for each $i > n_1$.

Similarly, there exists n_2 such that

(c) $g(a_i) \notin \{f(z_k) : k \le n\}$ for each $i > n_2$.

Pick any $i > \max\{n_1, n_2\}$ and let $z_n = a_i$. Since $i \ge n$, $z_n \in U_n$. By (a)–(c), the formula $(*)$ holds for $k = n$. Our construction is complete.

Let $Z = \{z_n : n \in \omega\}$. Clearly Z is closed in X. Since f and g are closed maps, $f(Z)$ and $g(Z)$ are closed in Y. By $(*)$, they are disjoint. Since Y is normal, $\mathrm{cl}_{\beta Y} f(Z)$ and $\mathrm{cl}_{\beta Y} g(Z)$ are also disjoint. Therefore $\widetilde{f}(x) \neq \widetilde{g}(x)$ contradicting the hypothesis of the lemma. The claim is proved.

By the Claim, for each $x \in \beta X \setminus X$, we can select an open neighborhood U_x of x in βX such that $\widetilde{f}|_{U_x} = \widetilde{g}|_{U_x}$. Let $U = \bigcup_{x\in\beta X\setminus X} U_x$. Then $\widetilde{f}|_U = \widetilde{g}|_U$. The set $C = \beta X \setminus U$ is a compact subset of X and $f|_{X\setminus C} = g|_{X\setminus C}$.

The implication (iii)⇒(ii) is trivial and valid for any spaces. Indeed, let C be a compact subset of X such that $f|(X \setminus C) = g|(X \setminus C)$. Fix $x \in \beta X \setminus X$. Since C is closed in βX and $x \notin C$, we can find an open neighborhood U of x in βX such that $\mathrm{cl}_{\beta X} U \cap C = \emptyset$. The functions $\widetilde{f}|\mathrm{cl}_{\beta X} U$

and $\widetilde{g}|\mathrm{cl}_{\beta X} U$ coincide on $U \cap X$. Since $U \cap X$ is dense in $\mathrm{cl}_{\beta X} U$, we conclude that $\widetilde{f}|\mathrm{cl}_{\beta X} U = \widetilde{g}|\mathrm{cl}_{\beta X} U$. Consequently, $\widetilde{f}(x) = \widetilde{g}(x)$. ■

6. Proper absolute extensors. We begin by a local version of Definition 4.1.

Definition 6.1. A locally compact space X is a *proper absolute neighborhood extensor* for a locally compact space Y (notation: $X \in \mathrm{ANE}_p(Y)$) if any proper map $f: A \to X$ defined on a closed subset A of Y, admits a proper extension $\bar{f}: G \to X$, where G is a closed neighborhood of A in Y.

Below let $\mathcal{LCL}$ denote the class of locally compact and Lindelöf spaces.

Definition 6.2. A space $X \in \mathcal{LCL}$ is a *proper absolute* (*neighborhood*) *extensor* (notation: $X \in \mathrm{A(N)E}_p$) if $X \in \mathrm{A(N)E}_p(Y)$ for any $Y \in \mathcal{LCL}$.

Proposition 6.1. *Every proper absolute* (*neighborhood*) *extensor is an absolute* (*neighborhood*) *extensor.*

Proof. Let X be a proper absolute neighborhood extensor. Since X is locally compact and Lindelöf, there exists a proper map $p: X \to Y$, where Y is a locally compact space with countable base. We may assume that Y is a closed subspace of $[0,1) \times I^{\omega}$. Let also $i: X \to I^{\tau}$ denote an embedding of X into I^{τ}, where $\tau = w(X) \geq \omega$. Then the diagonal product $q = p \triangle i: X \to [0,1) \times I^{\omega} \times I^{\tau} \approx [0,1) \times I^{\tau}$ is an embedding with $q(X)$ closed in $[0,1) \times I^{\tau}$. We will identify X with $q(X) \subset [0,1) \times I^{\tau}$. Since X is a proper absolute neighborhood extensor, there exist a functionally open neighborhood G of X in $[0,1) \times I^{\tau}$ and a proper retraction $r: \mathrm{cl}_{[0,1)\times I^{\tau}} G \to X$. Since, by [6, Proposition 6.1.4, Lemma 7.1.3], G is an absolute neighborhood extensor, it follows that so is X. ■

As noted in the Introduction, $\mathbb{R}^n_+$ is a proper absolute extensor. The next statement makes this observation formal.

Lemma 6.2. *Let M be a compact metrizable* A(N)E*-space and N be a Z-set in M. If N is also an* A(N)E*-compactum, then $M \setminus N \in \mathrm{A(N)E}_p$.*

Proof. We only prove the parenthetical part since the absolute case is simpler. Let $f: A \to M \setminus N$ be a proper map, defined on a closed subset of a locally compact space X. Without loss of generality we may assume that A is functionally closed in X. Consider the Stone–Čech extension $\widetilde{f}: \mathrm{cl}_{\beta X} A \to M$ of f. Since f is proper, it follows that $\widetilde{f}(\mathrm{cl}_{\beta X} A \setminus A) \subset N$. Since N is an ANE-compactum, the map $\widetilde{f}|(\mathrm{cl}_{\beta X} A \setminus A): \mathrm{cl}_{\beta X} A \setminus A \to N$ can be extended to a map $g: G \to N$, defined on an open neighborhood G of $\mathrm{cl}_{\beta X} A \setminus A$ in $\beta X \setminus X$. Since $(\beta X \setminus X) \setminus G$ and $\mathrm{cl}_{\beta X} A$ are disjoint closed subsets of βX, we can find an open neighborhood U of $\mathrm{cl}_{\beta X} A$ in βX such that $\mathrm{cl}_{\beta X} U \cap (\beta X \setminus X) \subset G$.

Next consider the map $h\colon (\mathrm{cl}_{\beta X} U \setminus X) \cup \mathrm{cl}_{\beta X} A \to M$ defined by

$$h(x) = \begin{cases} g(x) & \text{if } x \in \mathrm{cl}_{\beta X} U \setminus X, \\ f(x) & \text{if } x \in A. \end{cases}$$

Note that h is well defined since $\widetilde{f}$ and g coincide on $\mathrm{cl}_{\beta X} A \setminus A$. Since M is an ANE-compactum, we can extend h to a map $\widetilde{h}\colon \mathrm{cl}_{\beta X} V \to M$, where V is an open subset of βX such that $(\mathrm{cl}_{\beta X} U \setminus X) \cup \mathrm{cl}_{\beta X} A \subset V$ and $\mathrm{cl}_{\beta X} V \subset U$. Next choose $\alpha\colon \mathrm{cl}_{\beta X} V \to [0,1]$ such that $\alpha^{-1}(0) = (\mathrm{cl}_{\beta X} V \setminus V) \cup A$. This is possible since the Stone–Čech corona $\beta X \setminus X$ (and consequently $\mathrm{cl}_{\beta X} V \setminus \mathrm{cl}_X V$) is functionally closed in βX (respectively, in $\mathrm{cl}_{\beta X} V$). Also, since by our assumption, N is a Z-set in M, there is a homotopy $H\colon M \times [0,1] \to M$ such that $H(m,0) = m$ for any $m \in M$ and $H(m,t) \in M \setminus N$ for any $(m,t) \in M \times (0,1]$. Finally consider the map $f'\colon \mathrm{cl}_{\beta X} V \to M$ defined by $f'(x) = H(\widetilde{h}(x), \alpha(x))$ for $x \in \mathrm{cl}_{\beta X} V$. Note that $f'(\mathrm{cl}_X V) \subset M \setminus N$ and $f'(\mathrm{cl}_{\beta X} V \setminus \mathrm{cl}_X V) \subset N$. Consequently, $f'|\mathrm{cl}_X V\colon \mathrm{cl}_X V \to M \setminus N$ is proper. It remains to observe that, by construction, $f'|A = f$. ∎

LEMMA 6.3. *Let X be an AE_p-space with countable base. Then its one-point compactification $\alpha X = X \cup \{\infty\}$ is an* AE*-compactum and $\{\infty\}$ is a Z-set in αX.*

Proof. Embed αX into the Hilbert cube I^ω. Since X is a proper absolute neighborhood extensor, there exists a proper retraction $r\colon I^\omega \setminus \{\infty\} \to X$. Properness of r guarantees that r has an extension $\widetilde{r}\colon I^\omega \to \alpha X$ such that $\widetilde{r}|(I^\omega \setminus \{\infty\}) = r$ and $\widetilde{r}(\{\infty\}) = \{\infty\}$. Since $\widetilde{r}$ is also a retraction, it follows that αX is an absolute extensor. Since $\widetilde{r}^{-1}(\{\infty\}) = \{\infty\}$ and $\{\infty\}$ is a Z-set in I^ω, we conclude that $\{\infty\}$ is a Z-set in αX as well. ∎

COROLLARY 6.4. *Let X be a locally compact space with countable base. Then the following conditions are equivalent:*

(i) *X is a proper absolute extensor;*
(ii) *X is a proper retract of $[0,1) \times I^\omega$;*
(iii) *the one-point compactification $\alpha X = X \cup \{\infty\}$ of X is an absolute extensor in which $\{\infty\}$ is a Z-set;*
(iv) *there exists a metrizable compactification $\widetilde{X}$ of X such that $\widetilde{X}$ and $\widetilde{X} \setminus X$ are absolute extensors and the corona $\widetilde{X} \setminus X$ is a Z-set in $\widetilde{X}$.*

Proof. (i)⇒(ii). Any locally compact space with countable base, in particular, X, admits a closed embedding into $[0,1) \times I^\omega$. By (i), the identity map id_X has a proper extension $r\colon [0,1) \times I^\omega \to X$, which obviously is a retraction.

(ii)⇒(iii) follows from Lemma 6.3, since $[0,1) \times I^\omega$ (and hence X as its proper retract) is a proper absolute extensor.

(iii)⇒(iv) is trivial and (iv)⇒(i) follows from Lemma 6.2. ∎

Proposition 6.5. *Let X be a proper absolute extensor of countable weight. Then the following conditions are equivalent:*

(i) *X satisfies DD^nP for each n;*
(ii) *X is homeomorphic to $[0,1) \times I^\omega$.*

Proof. (i)⇒(ii). By Corollary 6.4(iii), the one-point compactification $\alpha X = X \cup \{\infty\}$ of X is an absolute extensor in which $\{\infty\}$ is a Z-set. Then, by (i), αX has the DD^nP for each n and by Toruńczyk's theorem [17], $\alpha X \approx I^\omega$. Therefore $X \approx I^\omega \setminus \{\infty\}$. Finally, by Chapman's Complement Theorem, $I^\omega \setminus \{\infty\} \approx [0,1) \times I^\omega$.

(ii)⇒(i). Trivial. ■

Corollary 6.6. *If X is a proper absolute extensor of countable weight, then $X \times I^\omega \approx [0,1) \times I^\omega$.*

Proof. Note that $X \times I^\omega$ is a proper absolute extensor satisfying DD^nP for each n and apply Proposition 6.5 ■

Theorem 6.7. *A proper absolute extensor of weight $\tau > \omega$ is homeomorphic to $[0,1) \times I^\tau$ if and only if it has the same pseudocharacter at each point.*

Proof. Obviously the pseudocharacter of each point of $[0,1)\times I^\tau$ equals τ. Let now X be a proper absolute retract of weight τ. As in the proof of Proposition 6.1 we may assume that X is closed in $[0,1) \times I^A$, where $|A| = \tau$. Since X is a proper absolute extensor, there exists a proper retraction $r\colon [0,1) \times I^A \to X$. Proceeding as in the proof of [6, Theorem 7.2.8], we can construct a continuous well ordered inverse spectrum $\mathcal{S} = \{X_\alpha, p_\alpha^{\alpha+1}, \tau\}$ of length τ, satisfying the following conditions:

(i) $X = \lim \mathcal{S}$;
(ii) all X_α are locally compact and Lindelöf proper absolute extensors;
(iii) all short projections $p_\alpha^{\alpha+1}\colon X_{\alpha+1} \to X_\alpha$ are trivial bundles with fiber I^ω;
(iv) X_0 is a locally compact space of countable weight.

Then X is homeomorphic to $X_0 \times I^\tau$. By (ii), (iv) and Corollary 6.6, $X_0 \times I^\omega \approx [0,1) \times I^\omega$, and consequently $X \approx [0,1) \times I^\tau$. ■

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Received 14 August 2012;
in revised form 11 April 2013