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ω-Limit sets for triangular mappings

by

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Abstract. In 1992 Agronsky and Ceder proved that any finite collection of non-degenerate Peano continua in the unit square is an ω-limit set for a continuous map. We improve this result by showing that it is valid, with natural restrictions, for the triangular maps $(x, y) \mapsto (f(x), g(x, y))$ of the square. For example, we show that a non-trivial Peano continuum $C \subset I^2$ is an orbit-enclosing ω-limit set of a triangular map if and only if it has a projection property. If C is a finite union of Peano continua then, in addition, a coherence property is needed. We also provide examples of two slightly different non-Peano continua C and D in the square such that C is and D is not an ω-limit set of a triangular map. In view of these examples a characterization of the continua which are ω-limit sets for triangular mappings seems to be difficult.

1. Introduction and main results. Let $\mathcal{C}(X)$ be the family of continuous mappings of a compact metric space X to itself. By a *trajectory* of a point x in X we mean the sequence $\{f^n(x)\}_{n=0}^{\infty}$, where f^0 is the identity map, and f^n is the nth iterate of f. A set $W \subset X$ is an *ω-limit set* of f provided, for some $x \in X$, W is the set of limit points of the trajectory of x; it is denoted by $\omega_f(x)$. A set W is an *orbit-enclosing* ω-limit set if $W = \omega_f(y)$ for some $y \in W$. Since any ω-limit set W is compact and invariant [i.e., $f(W) = W$], it is easy to see that any ω-limit set with non-empty interior must be orbit-enclosing.

To understand the structure of ω-limit sets is an interesting problem, but far from being solved. Among such sets, those having the orbit-enclosing property play a prominent role as they are apt to enclose non-trivial dynamics. If X is the unit real interval $I = [0, 1]$, the following characterization is

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well known ([1], cf. [7] for a shorter proof): A set $C \subset I$ is an ω-limit set of an $f \in \mathcal{C}(I)$ if and only if C is either a nowhere dense compact set, or the union of a finite number of (non-degenerate) compact intervals. Other properties of ω-limit sets for continuous maps of I can be found in [6]. However, if X is the compact k-dimensional unit interval I^k, with $k > 1$, only partial results are known. In fact, the following is true.

Theorem 1 (Agronsky and Ceder [2] and [3]). *Let $k > 1$. Then a compact set $C \subset I^k$ is an ω-limit set of a continuous map f of I^k provided one of the following conditions is satisfied*:

(i) *C is totally disconnected*;
(ii) *C is a continuum with empty interior*;
(iii) *C is a finite collection of non-degenerate Peano continua.*

Recall that a *continuum* is any compact and connected set, and a *Peano continuum* is any locally connected continuum, or equivalently, any continuous image of the unit interval I (cf., e.g., [13], p. 256). If $C \subset I^k$ is a finite collection of non-Peano continua, or a non-Peano continuum with non-empty interior, then no non-trivial sufficient condition is known for C to be an (orbit-enclosing) ω-limit set. We provide here a couple of examples of such continua.

Example 1a (Agronsky and Ceder [2]). Let H be a Cantor (i.e., nowhere dense perfect) set in $(0, 2\pi)$, and $C = \{re^{iy} : 0 \le r \le 1,\ y \in H\}$. Then C is a non-Peano continuum with empty interior. By Theorem 1 it is an ω-limit set for a continuous map of the square $[-1, 1]^2$. Further, it can be made orbit-enclosing.

Example 1b (Sivak [16]). Let $L = \{(x, \sin(1/x)) : x \in (0, 1]\}$, and let $C = L \cup \{(0, y) : -1 \le y \le 1\}$ be the closure of L. Then C is a non-Peano continuum with empty interior, which is an orbit-enclosing ω-limit set for a continuous (even triangular [10], see the definition below) map of the square.

Example 2. Let $L = \{(x, \sin(1/\varrho(x))) : 0 < x < 1\}$, where $\varrho(x) = \min\{x, 1 - x\}$, and let C_0 be the closure of L. Let $C_n = C_0 + (n, 0)$, i.e., to get C_n shift C_0 along the x-axis by n. Then $C = C_0 \cup C_1 \cup C_2$ is a non-Peano continuum with empty interior. Hence, by the above theorem, it is an ω-limit set for a continuous map F of the rectangle $[0, 3] \times [-1, 1]$. However it cannot be an orbit-enclosing ω-limit set. [To see this assume that $W = \omega_F(z)$ for some $z \in C$. Then for any i there is a j such that $F(C_i) \subset C_j$. This follows easily since the F-image of an arcwise connected set is arcwise connected. Since $F(C) = C$ it is easy to deduce $F(C_1) \subset C_1$. Since $F^n(z) \in C_1$ for some n, we have $F^{n+k}(z) \in C_1$ for any $k \ge 0$ and hence, $\omega_F(z) \subset C_1$—a contradiction.]

EXAMPLE 3 (Babilonová [4]; cf. also [10]). Let C be as in Example 1b and let $D = C \cup [1,2] \times [-1,1]$. Then D is a non-Peano continuum with non-empty interior which is a (necessarily orbit-enclosing) ω-limit set of a continuous map of the rectangle $[0,2] \times [-1,1]$. Moreover, this map can be taken triangular.

EXAMPLE 4 (Agronsky and Ceder [2]). Let $C = D \cup A \subset I^2$, where D is a closed disc and A a hereditarily indecomposable continuum (i.e., A is not the union of two proper subcontinua) such that $A \setminus D \neq \emptyset$ and $A \cap D = \{p\}$. Then C is a non-Peano continuum with non-empty interior which is an ω-limit set for no continuous map of the square I^2.

In this paper we study the properties of ω-limit sets for triangular mappings of I^2. Recall that a *triangular map* is any $F \in \mathcal{C}(I^2)$ such that $F(x,y) = (f(x), g(x,y)) = (f(x), g_x(y))$ for any (x,y) in I^2; the map f is the *base map* of F. The dynamics of triangular maps is simpler than that of general continuous maps of the square. It is known [11] that, e.g., Sharkovsky's theorem is valid for such maps. However, information on the structure of ω-limit sets of triangular maps is scarce as well. We recall here only the paper by Kolyada and Snoha [12; cf. also our Remark 3].

Our main aim is to show that the result of Agronsky and Ceder [cf. (iii) of Theorem 1 above], with natural restrictions, is valid for triangular maps of the square. Actually, we characterize the finite unions of Peano continua which can be orbit-enclosing ω-limit sets for these maps (cf. Theorems 2 and 3 below). The restrictions are given by the well known fact that the *projection* $\Pi(W)$ of any ω-limit set W of a triangular map F to the x-axis is an ω-limit set of the base map f, i.e., $\Pi(W)$ is either a nowhere dense compact set or a finite union of non-degenerate compact subintervals of I. The "only if" parts of Theorems 2 and 3 are true for arbitrary non-trivial continua (while the argument is almost the same). Therefore we reformulate them in Proposition 4. The last Theorem 5 indicates that a similar characterization applicable to all continua (i.e., including non-Peano continua) is rather difficult.

To state the results we need other notions. A set $C \subset I^2$ is *non-trivial* if its projection $\Pi(C)$ onto the x-axis is not a point (and hence a closed interval if C is a continuum). It has the *projection property* if, for any $z \in C$ and any neighbourhood U of z in C, the projection $\Pi(U)$ contains more than one point. Finally, C has the *coherence property* if it has a finite number r of connected components and there are pairwise disjoint intervals $[a_j, b_j]$, $j = 0, 1, \ldots, s-1$, such that either (a) s divides r and for any j there are exactly r/s components of C each of which is projected by Π onto the interval $[a_j, b_j]$, or (b) $2s$ divides r and for any j there exists p_j in (a_j, b_j) such that there are exactly $r/(2s)$ components of C each of which is projected by Π onto $[a_j, p_j]$ and exactly $r/(2s)$ components of C each of which is projected onto $[p_j, b_j]$.

THEOREM 2. *A non-trivial Peano continuum $C \subset I^2$ is an orbit-enclosing ω-limit set of a triangular map $F \in \mathcal{C}(I^2)$ if and only if C has the projection property.*

THEOREM 3. *A finite union of non-trivial Peano continua $C \subset I^2$ is an orbit-enclosing ω-limit set of a triangular map $F \in \mathcal{C}(I^2)$ if and only if C has both the projection and coherence properties.*

PROPOSITION 4. *The projection and coherence properties are necessary for a finite union of non-trivial continua $C \subset I^2$ to be an orbit-enclosing ω-limit set of a triangular map $F \in \mathcal{C}(I^2)$. In particular, all ω-limit sets of triangular maps with non-empty interior must have both properties.*

THEOREM 5. *There are continua $C, D \subset I^2$ with the projection property (and with non-empty interior) such that C is and D is not an ω-limit set of a triangular map $F \in \mathcal{C}(I^2)$.*

REMARK 1. Putting emphasis on non-trivial sets implies of course no loss of generality. Indeed, let $C \subset I^2$ be a finite union of continua which is an orbit-enclosing ω-limit set for a triangular map $F \in \mathcal{C}(I^2)$ and assume that one of its components is trivial. Using the same ideas as in the "only if" part of the proof of Theorem 3 it is very easy to show that: (i) either all components of C are vertical segments or all components of C are singletons; and (ii) C has the coherence property (more exactly, there are points $c_j \in I$, $j = 0, 1, \dots, s-1$, with s dividing the number r of components of C, such that for any j there are exactly r/s components of C which are projected onto c_j). Conversely, if C is a set satisfying (i) and (ii) then it is an orbit-enclosing ω-limit set for an appropriate triangular map.

REMARK 2. When considering arbitrary continuous maps of I^2 the conditions from Proposition 4 are not necessary. As an example take $C = \{0\} \times [0,1] \cup [0,1] \times \{0\}$. Let f be any continuous transitive map $C \to C$, and let F be a continuous extension of f to I^2. Then C is an orbit-enclosing ω-limit set of F.

REMARK 3. As a corollary to Theorem 2 we get the following result by Kolyada and Snoha [12]: Let $c \in I$ and let B be a compact subset of the fibre $A = \{c\} \times I$. Then there is a triangular map $F \in \mathcal{C}(I^2)$ with an ω-limit set C such that $C \cap A = B$. To see this assume without loss of generality that $(c, 0), (c, 1) \in B$, and let $\{A_n\}_n$ be the countable family of pairwise disjoint, open subsegments of A complementary to B. For any n let D_n be the open disc having the segment A_n as a diameter, and put $R_n = D_n \cap I^2$. It is easy to check that $C = I^2 \setminus \bigcup_n R_n$ is a Peano continuum with the projection property, and $C \cap A = B$.

REMARK 4. As a corollary to Theorem 3 we get (iii) of Theorem 1. Indeed, for any $\alpha \in [0,\pi)$, let θ_α denote the rotation of the plane with centre $(0,0)$ and angle α. Let A be the set of α for which there are $z_\alpha \in C$ and an open neighbourhood U_α of z_α such that $\Pi(\theta_\alpha(U_\alpha \cap C))$ contains exactly one point. Since $z_\alpha \notin U_\beta$ for any $\beta \in A$ with $\beta \neq \alpha$, A is countable. Hence, there is a γ such that $D = \theta_\gamma(C)$ has the projection property. Now it is easy to find a homeomorphism φ from D into I^2 such that $E = \varphi(D)$ has both the projection and coherence properties. Let G be a triangular map of I^2 such that E is its orbit-enclosing ω-limit set. Put $F = \theta_\gamma^{-1} \circ \varphi^{-1} \circ G|_E \circ \varphi \circ \theta_\gamma$, and extend F continuously onto the whole of I^2.

REMARK 5. A general result from [10] implies that the set D constructed in the proof of Theorem 5 is an ω-limit set of a continuous map of the square.

REMARK 6. Theorem 5 shows (see also Examples 3 and 4) that even a characterization of ω-limit sets with non-empty interior is difficult in the two-dimensional case. In contrast, for continuous mappings of the interval the ω-limit sets with non-empty interior are simply the finite collections of non-degenerate compact intervals; this result is due to Sharkovsky [14].

2. Proofs. In what follows, if φ is a map whose image lies in I^2 then φ_x, φ_y will denote its components. The diameter of a set $X \subset I$ or $X \subset I^2$ will be denoted by $|X|$. Recall that if C is a Peano continuum then there exists a continuous surjective map $\varphi : I \to C$. Any such map (or in general, any continuous map from a compact interval onto C) will be called a *parametrization* of C. In what follows, K will always denote a compact interval.

Let us say a few words about the proof of Theorem 2. The "only if" part is almost immediate while the "if" part essentially follows from Lemmas 1–3 below. The main ideas behind this last part of the proof can be well illustrated when applied to the typical Peano continuum—the square I^2.

We intend I^2 to be the ω-limit set for an appropriate triangular map $F : I^2 \to I^2$. Since I^2 can be parametrized via a certain map $\varphi : I \to I^2$, one could wonder whether the very map $F(u,v) = (\varphi_x(u), \varphi_y(u))$ may do the job. If φ_x is a transitive map we are done: Take a point x_0 having a dense orbit for φ_x, put $z_0 = (x_0, x_0)$ and note that the sequence $F^n(z_0) = \varphi(\varphi_x^{n-1}(x_0))$ is dense in I^2 since φ maps I onto I^2 and the sequence $\{\varphi_x^n(x_0)\}_n$ is dense in I.

However, φ_x need not be transitive. Of course, it is surjective but, e.g., it could have constant pieces. Thus we need from the beginning a parametrization φ of I^2 whose first coordinate has no intervals of constancy. The classical

Peano construction, for example, has this property, essentially because I^2 has the projection property (cf. Lemma 1).

Still in that case the transitivity of φ_x remains to be checked. A possible way to circumvent the problem would be to find a surjective map $\sigma \in \mathcal{C}(I)$ so that $\varphi_x \circ \sigma$ is transitive: then we could use the parametrization $\varphi \circ \sigma$ instead of φ. Somewhat unexpectedly, this map σ can be found just using the surjectivity of φ_x and its "no intervals of constancy" property (Lemma 3; cf. also Lemma 2). This concludes the proof.

Let us finally remark that the lemmas below are presented in a slightly stronger formulation than presently needed, in order to make them useful for the proof of Theorem 3. In what follows, we say that a map $\psi \in \mathcal{C}([a,b])$ is *proper* if $\psi(a) = a$, $\psi(b) = b$ and ψ has no intervals of constancy (that is, the ψ-image of any non-degenerate interval is non-degenerate as well).

LEMMA 1. *Let $C \subset I^2$ be a Peano continuum with the projection property. Write $K = \Pi(C)$. Then there is a parametrization $\varphi : K \to C$ such that φ_x is proper.*

Proof. Let $K = [a,b]$ and let $\phi : K \to C$ be a parametrization of C. We may assume that $\phi_x(a) = a$, $\phi_x(b) = b$, and that ϕ_x is not proper. Let $J \subset K$ be an interval of constancy of ϕ_x and let $\varepsilon > 0$ be such that $\varepsilon < \frac{1}{4}|J|$. Put $z_0 = \phi(t_0) = (x_0, y_0)$ where t_0 is the midpoint of J. Since C is a Peano continuum, for any $\nu > 0$ there is a $\delta > 0$ (depending only on C and ν) with the property that for z, z' in C such that $\|z - z'\| < \delta$ there is an arc in C connecting z and z' whose diameter is less than ν (cf. [13], p. 257). Use the projection property to find a point $z_1 = (x_1, y_1) \in C$, $x_1 \neq x_0$, and an arc $A \subset C$ with endpoints z_0 and z_1 such that $|A| < \varepsilon$.

Let $L \subset J$ be a compact interval containing t_0 such that $|L| < \varepsilon$ and $|\phi(L)| < \varepsilon$. It is easy to find a parametrization ϕ^* of C such that $\phi^*(L) = \phi(L) \cup A$, and $\phi^*(t) = \phi(t)$ for $t \notin L$. Then $\|\phi - \phi^*\| < 2\varepsilon$ since $|A| < \varepsilon$ and $|\phi(L)| < \varepsilon$. Moreover, since $|L| < \varepsilon < \frac{1}{4}|J|$, any interval of constancy of ϕ_x contained in J has diameter less than $\frac{3}{4}|J|$. Denote the above map ϕ^* by $\Phi(\phi, J, \varepsilon)$, and define inductively a sequence $\{\varphi_n\}_{n=0}^{\infty}$ of parametrizations of C such that $\varphi_0 = \phi$ and $\varphi_{n+1} = \Phi(\varphi_n, J_n, 2^{-n})$, where J_n is an interval of constancy of $(\varphi_n)_x$ of maximal length. Then $\lim_{n\to\infty} |J_n| = 0$ and $\lim_{n\to\infty} \varphi_n = \varphi$ uniformly. Therefore φ is the desired parametrization. ∎

LEMMA 2. *Let $h : [0,a] \to [0,|K|]$ be non-decreasing, continuous at 0 and such that $h(0) = 0$. Then there is a proper map $\sigma \in \mathcal{C}(K)$ such that $|\sigma(J)| \ge h(|J|)$ for any subinterval J of K with $|J| \le a$.*

Proof. We may assume $K = I$. By induction there is a sequence $\{r(n)\}_{n=1}^{\infty}$ of positive integers such that, for any n, $r(n+1) = 2p(n)r(n)$

where $p(n)$ is an odd integer, and $1/2^n \ge h(2/r(n))$. If we find a proper map σ such that

$$|\sigma(J)| \ge 1/2^{n-1} \quad \text{whenever} \quad |J| \ge 2/r(n) \tag{1}$$

we are done since $|J| \ge 2/r(1)$ implies $|\sigma(J)| = 1 \ge h(|J|)$ and, for any n, $2/r(n) > |J| \ge 2/r(n+1)$ gives $|\sigma(J)| \ge 1/2^n \ge h(2/r(n)) \ge h(|J|)$.

Define $J_n^i = [i/r(n), (i+1)/r(n)]$ for any n and $0 \le i < r(n)$. Since each interval J_n^i consists of $2p(n)$ intervals J_{n+1}^j and $p(n)$ is odd it is very easy to construct a proper piecewise affine map $\psi_n \in \mathcal{C}(I)$ with the following properties: For any j, ψ_n is monotone on J_{n+1}^j, $|\psi_n(J_{n+1}^j)| = 1/(2r(n))$, and $\psi_n(J_n^j) = J_n^j$. Define inductively a sequence $\{\sigma_n\}_{n=1}^\infty$ of maps in $\mathcal{C}(I)$ such that σ_1 is piecewise affine and maps monotonically each interval J_1^i onto I, and $\sigma_{n+1} = \sigma_n \circ \psi_n$ for $n \ge 1$.

It turns out that $\{\sigma_n\}_n$ converges uniformly to a map σ satisfying (1). In fact the construction above clearly implies that σ_n is affine on each interval J_n^i and $|\sigma_n(J_n^i)| = 1/2^{n-1}$. Since $\sigma_m = \sigma_n \circ \psi_n \circ \psi_{n+1} \circ \ldots \circ \psi_{m-1}$ and $\psi_m(J_n^i) = J_n^i$ for any $m \ge n$ [because $r(n)$ divides $r(m)$], we get $\sigma_m(J_n^i) = \sigma_n(J_n^i)$. In particular, $\|\sigma_m - \sigma_n\| \le 1/2^{n-1}$ for any $m \ge n$ and consequently, $\{\sigma_n\}_n$ converges uniformly to a map σ. Moreover, $|\sigma(J_n^i)| = 1/2^{n-1}$ and since $|J| \ge 2/r(n)$ implies that J must include an interval J_n^i, (1) is satisfied. Obviously, σ is a proper map. ∎

LEMMA 3. *Let $\psi \in \mathcal{C}(K)$ be a proper map. Then there is a proper map $\sigma \in \mathcal{C}(K)$ such that $\psi \circ \sigma$ is transitive in K.*

Proof. We can again assume $K = I$. Define $m : I \to I$ by putting $m(0) = 0$ and $m(t) = \min\{|\psi(J)| : J \text{ is an interval with } |J| = t\}$ for any $0 < t \le 1$. Obviously m is continuous and non-decreasing and, since ψ is surjective, it is surjective as well. Then there is a non-decreasing map $h : [0, 1/2] \to [0, 1]$ satisfying $m(h(t)) = 2t$ for any t. Since ψ has no intervals of constancy, $h(0) = 0$ and h is continuous at 0. Apply Lemma 2 to h and get the corresponding map σ. Then

$$|(\psi \circ \sigma)(J)| \ge \min\{2|J|, |K|\}, \tag{2}$$

for any subinterval J of K. Indeed, if $|J| \le 1/2$ then $|(\psi\circ\sigma)(J)| \ge m(|\sigma(J)|) \ge m(h(|J|)) = 2|J|$. If $|J| > 1/2$ then J contains an interval J' with $|J'| = 1/2$, hence $1 = |(\psi \circ \sigma)(J')| = |(\psi \circ \sigma)(J)|$. This proves (2).

It follows that for any subinterval J of K there is a positive integer n such that $(\psi \circ \sigma)^n(J) = K$. As is well known this condition implies the transitivity of $\psi \circ \sigma$. ∎

Proof of Theorem 2. Let $C \subset I^2$ be a non-trivial Peano continuum. Assume that there is a triangular map $F \in \mathcal{C}(I^2)$ and a point $z_0 = (x_0, y_0) \in C$ such that $\omega_F(z_0) = C$. If $K = \Pi(C)$ and f is the base map of F then K

is a non-degenerate interval and the orbit $\{f^n(x_0)\}_n$ is dense in K. Assume there is a point $z = (x, y)$ in C and a neighbourhood U of z in C such that $\Pi(U) = \{x\}$. Since $z \in \omega_F(z_0)$ and the orbit of z_0 lies in C, there are numbers $i < j$ such that $F^i(z_0), F^j(z_0) \in U$ and hence, $f^i(x_0) = x = f^j(x_0)$. Consequently, the orbit $\{f^n(x_0)\}_n$ is finite and cannot be dense in K. This contradiction proves the "only if" part of Theorem 2.

Conversely, let $C \subset I^2$ be a Peano continuum with the projection property and put $K = \Pi(C)$. According to Lemma 1, C admits a parametrization $\varphi : K \to C$ such that φ_x is proper. By Lemma 3 there is a map $\sigma \in \mathcal{C}(K)$ such that $\{(\varphi_x \circ \sigma)^n(x_0)\}_n$ is dense in K for some $x_0 \in K$. Find $z_0 \in C$ with $\Pi(z_0) = x_0$ and define $G : C \to C$ by $G(u, v) = (\varphi \circ \sigma)(u)$. Then $\{G^n(z_0)\}_n$ is dense in C. Since any triangular map defined on a compact subset of I^2 can be extended to a triangular map on I^2 (cf. [12]) there is a triangular map $F \in \mathcal{C}(I^2)$ whose restriction to C equals G. Since $\omega_F(z_0) = C$, the proof of the "if" part of Theorem 2 is finished. ∎

Let us next describe the key points of our proof of Theorem 3. The "only if" part is again simple enough; as we shall see it immediately follows from some standard properties of ω-limit sets and the following folklore lemma (whose easy proof is omitted):

Lemma 4. *Let $f \in \mathcal{C}(K)$ and let J be a compact subinterval of K. Assume that $f^r(J) = J$ for some positive integer r and that f^r* (*considered as a map from J into itself*) *is transitive. Then either* (a) *there is a number s dividing r such that $f^s(J) = J$ and the intervals $f^j(J)$ $(0 \leq j < s)$ are pairwise disjoint*; *or* (b) *there is a number s with $2s$ dividing r such that $f^{2s}(J) = J$, $f^j(J)$ and $f^{j+s}(J)$ have exactly one common point for any $0 \leq j < s$ and the intervals $f^j(J) \cup f^{j+s}(J)$ are pairwise disjoint.*

To fix ideas concerning the "if part" of Theorem 3, consider two disjoint Peano continua $C_0, C_1 \subset I^2$ with $\Pi(C_0) = \Pi(C_1) = I$ (i.e., with the coherence property), both having the projection property as well. We intend $C_0 \cup C_1$ to be an orbit-enclosing ω-limit set for a triangular map F.

We already know how to construct a parametrization $\varrho_i : I \to C_i$ whose first coordinate $\varrho_{i,x}$ is transitive. Assume for the moment $\varrho_{0,x} = \varrho_{1,x} =: f$. Then we can simply define $G : C_0 \cup C_1 \to C_0 \cup C_1$ by $G(u, v) = \varrho_1(u)$ for any $(u, v) \in C_0$ and $G(u, v) = \varrho_0(u)$ for any $(u, v) \in C_1$, and extend it as in the proof of Theorem 2 to a triangular map F defined on the whole square ($\varrho_{0,x} = \varrho_{1,x}$ implies that G is a triangular map). For $z_0 = (x_0, y_0) \in C_0$ we have $F^n(z_0) = \varrho_1(f^{n-1}(x_0))$ or $F^n(z_0) = \varrho_0(f^{n-1}(x_0))$ depending on whether n is odd or even. Hence, to guarantee $\omega_F(z_0) = C_0 \cup C_1$ we just need the sequence $\{f^{2m}(x_0)\}_m$ (and hence $\{f^{2m+1}(x_0)\}_m$) to be dense in I. Thus, we need f^2 to be transitive. Since our parametrizations are constructed by Lemma 3, because of (2) it turns out to be exactly the case.

It remains to find a way how to equalize the first coordinates of our parametrizations. A sensible global approach could be: (a) to use Lemma 1 to get parametrizations φ_i of C_i whose first coordinates are proper; (b) to construct maps $\tau_i \in \mathcal{C}(I)$ such that $\varphi_{0,x} \circ \tau_0 = \varphi_{1,x} \circ \tau_1 =: \kappa$, with κ being proper as well; (c) to apply Lemma 3 and find σ so that $(\kappa \circ \sigma)^2$ is transitive. Then $\varrho_i = \varphi_i \circ \tau_i \circ \sigma$ are the parametrizations we need. Only step (b) remains obscure but, surprisingly, the existence of the maps τ_i is just guaranteed by the following

LEMMA 5 (Homma [9]). *Let $\psi_0, \psi_1, \ldots, \psi_r \in \mathcal{C}(K)$ be proper maps. Then there are proper maps $\tau_0, \tau_1, \ldots, \tau_r \in \mathcal{C}(K)$ such that $\psi_i(\tau_i(x)) = \psi_1(\tau_1(x))$ for any $x \in K$ and any i.*

Proof of Theorem 3. Let C be an ω-limit set for a continuous map F. Then, for any $L \subset C$ which is both open and closed in C, $F(L) \subset L$ implies $L = \emptyset$ or $L = C$. (It is worthy to note that this property has been well known since Sharkovsky [15] proved it in 1966, cf. also [5]. However, it is not generally known that the same result had already been proved in 1954 by Dowker and Friedlander [8]; actually, they assume that F is a homeomorphism, but their argument is correct for any continuous map.) Since $F(C) = C$ it follows that if C has a finite number r of connected components then there is an ordering of these components $C_0, C_1, \ldots, C_{r-1}$ such that $F(C_i) = C_{i+1}$, for any i taken mod r. These facts are well known and easy to prove (cf., e.g., [5], p. 71).

If, in addition, F is a triangular map and C is orbit-enclosing and consists of non-trivial components then it has the projection property (just reason as in the first part of the proof of Theorem 2). Further, the projections $\Pi(C_i) =: K_i$ are then non-degenerate intervals and $f(K_i) = K_{i+1}$ for any i (where f denotes the base map of F). Since obviously F^r has a dense orbit in C_0, f^r has a dense orbit in K_0. By Lemma 4, C has the coherence property and the "only if" part of Theorem 3 is proved.

We now prove the "if" part of the theorem. Let C be the union of pairwise disjoint Peano continua $C_0, C_1, \ldots, C_{r-1}$ and suppose that C has both the projection and coherence properties. We may assume that C satisfies condition (b) of the definition of the coherence property; the case (a) is similar (but simpler). We write $K_j = [a_j, p_j]$ and $K_{j+s} = [p_j, b_j]$ for any $0 \le j < s$. After relabelling the components of C we may assume that $\Pi(C_i) = K_i$ for any $0 \le i < r$, where the indices "i" in K_i (and in f_i below) are taken mod $2s$.

Apply Lemmas 1, 3 and 5 to get parametrizations $\varrho_i : K_i \to C_i$ such that all $\varrho_{i,x}$ are proper and transitive, and $\varrho_{i,x} = \varrho_{i',x}$ if $i \equiv i' \pmod{2s}$. Moreover, by (2), we may assume that $|\varrho_{i,x}(J)| \ge \min\{2|J|, |K_i|\}$ for any i and any subinterval J of K_i. Composing each ϱ_i with an affine bijection from K_{i-1} to K_i [decreasing if $i \equiv 0 \pmod{2s}$ or $i \equiv s \pmod{2s}$ and increasing otherwise] we obtain corresponding parametrizations $\varrho_i^* : K_{i-1} \to C_i$, still satisfying $f_i := \varrho_{i,x}^* = \varrho_{i',x}^*$ if $i \equiv i' \pmod{2s}$. Further, f_i carries the endpoints of K_{i-1} onto those of K_i decreasingly if $i \equiv 0$ or $i \equiv s$, and increasingly otherwise. So $\widetilde{f} : \bigcup_i K_i \to \bigcup_i K_i$ given by $\widetilde{f}(x) = f_i(x)$ for $x \in K_i$ is a well defined continuous map. [Note that this is the only place where we really need the property "$\psi(a) = a$ and $\psi(b) = b$" in the definition of a proper map; before, only the surjectivity and the absence of intervals of constancy were used.] Notice finally that

$$|\widetilde{f}(J)| \geq \frac{|K_{i+1}|}{|K_i|} \min\{2|J|, |K_i|\}$$

for any subinterval J of K_i, and then $|\widetilde{f}^{2s}(J)| \geq \min\{2^s|J|, |K_0|\}$ for any subinterval J of K_0. This implies there is a point $x_0 \in K_0$ whose $\widetilde{f}^{2s}$-orbit is dense in K_0. Consequently, the $\widetilde{f}$-orbit of x_0 is dense in $\bigcup_i K_i$.

We can conclude the proof. Define $G : C \to C$ by $G(x, y) = \varrho_{i+1}^*(x)$ if $(x, y) \in C_i$, for i taken mod r. Since G is triangular, it can be extended to a triangular map $F : I^2 \to I^2$ (cf. [12]). Let $z_0 \in C$ with $\Pi(z_0) = x_0$. Then $\omega_F(z_0) = C$ and we are done. ∎

Proof of Proposition 4. We omit it since the argument is just the same as that for the corresponding statements of Theorems 2 and 3; see the first parts of their proofs. Note that if C is an ω-limit set and one of its components C_0 has non-empty interior then C_0 is mapped into itself by some iterate of F and hence C has a finite number of components. ∎

Proof of Theorem 5. Let

$$I_n = \left[1 - \frac{1}{2n-1}, 1 - \frac{1}{2n}\right] \quad \text{and} \quad J_n = \left[1 - \frac{1}{2n}, 1 - \frac{1}{2n+1}\right],$$

for any positive integer n. Define

$$C = (\{1\} \times I) \cup \bigcup_{n=1}^{\infty} (I_n \times I) \cup (J_n \times \{0\}),$$

$$D = (\{1\} \times I) \cup \bigcup_{n=1}^{\infty} (I_n \times I) \cup (J_{2n-1} \times \{0\}) \cup (J_{2n} \times \{1\}).$$

We show that C is but D cannot be an (orbit-enclosing) ω-limit set for a triangular map (Lemmas 6 and 7 below). Clarifying the reasons why a continuum may or may not be an orbit-enclosing ω-limit set is presently beyond our ability, but it is sure that arcwise connectedness must play a prominent role.

In order to emphasize triangularity, we could begin by concentrating on continua $E \subset I^2$ with the property that the sets $E_x = E \cap I_x$, where I_x stands for $\{x\} \times I$, are connected for any $x \in I$ (as the sets C and D above). Assume that there is an arc in E connecting the sets E_x, or more precisely that there is a continuous map $h : I \to I$ whose graph $A = \{(x, h(x)) : x \in I\}$ is included in E. If E also has the projection property, there seems to be *a priori* no clear reason why such a set *should not* be an orbit-enclosing ω-limit set. We now try to explain how our desired map F could be devised. Let A_1 denote the set of points $x \in E$ such that E_x is a singleton, and let $A_2 = A \setminus A_1$. We should look for a transitive map $T : A \to A$ leaving A_2 "almost" invariant, and as a base map for F take $f = H^{-1} \circ T \circ H$ where $H : I \to A$ is given by $H(x) = (x, h(x))$. In this way we ensure that "almost" any non-degenerate segment E_x is mapped by F onto a non-degenerate segment. Next we can define $g_x : E_x \to E_{f(x)}$ by $g_x = \psi_{f(x)}^{-1} \circ \varrho \circ \psi_x$, where $\varrho \in \mathcal{C}(I)$ is a fixed transitive map, and $\psi_x : E_x \to [0, 1]$ is just the increasing affine function mapping E_x onto $[0, 1]$ (similarly for $\psi_{f(x)}$). When $F : (x, y) \mapsto (f(x), g_x(y))$ is properly defined at the points in A_1 which are mapped by T into A_2, we can reasonably expect that F will act transitively on $E \setminus A_1$. There is no choice for the remaining points in A_1 since F must coincide with T there, but the transitivity of T should hopefully guarantee that an ω-limit set containing $E \setminus A_1$ must contain A_1.

Up to some unavoidable technical details, the construction in Lemma 6 is exactly as described above, with $A = I \times \{0\}$. One could try and reason similarly in the case of the set D, then constructing a triangular map $F : [0, 1)^2 \to [0, 1)^2$ having $D \setminus (\{1\} \times I)$ as an ω-limit set, but the problem arises with the vertical line $\{1\} \times I$. It is shown in Lemma 7 that its presence forces a kind of non-expansiveness behaviour [best illustrated by sentence labelled "(9)" there] which turns out to be incompatible with transitivity. The set D is not arcwise connected, of course, but it must be emphasized that a connected set E of the type described above, even when it is not arcwise connected, *may* be an orbit-enclosing ω-limit set as well (see Example 3). Thus things are rather unclear even in this restrictive setting.

LEMMA 6. *The set C is an ω-limit set of a triangular map $F \in \mathcal{C}(I^2)$.*

Proof. Let $f \in \mathcal{C}(I)$ be a continuous map with the following properties:

(i) $f(1) = 1$,

$$f\left(\frac{k+1}{k+2}\right) = \frac{k-1}{k} \quad \text{for } k \geq 1, \qquad f\left(\frac{1}{2}\right) = f(0) = 0;$$

(ii) the restriction of f to any of the intervals I_n and J_n consists of a finite number of affine pieces, and each of them has slope greater than 2 (in absolute value);

(iii) if $M \subset I$ is a maximal interval with $f|_M$ affine then both its endpoints are mapped by f into $\{k/(k+1)\}_{k=0}^{\infty}$;

(iv) $f(I_{n+1}) = I_n$ for any $n \geq 1$;
(v) $f(I_1) \supset J_1$ and $f(J_n) \supset J_{n+1}$ for any $n \geq 1$.

Such a map f obviously exists. Further, f is transitive. In fact, if J is an arbitrary subinterval of I then there is an iterate $f^m(J)$ such that, due to (ii), $f|_{f^m(J)}$ cannot be affine. Hence, by (iii), $f^{m+1}(J)$ intersects $\{k/(k+1)\}_{k=0}^{\infty}$ and by (i), $0 \in f^r(J)$ for some r. Since 0 is fixed by f, (ii) and (iii) imply that $f^s(J)$ must cover I_1 for some s. Finally, by (v), $\bigcup_{n=0}^{\infty} f^n(J)$ covers $[0,1)$. This proves that f is transitive.

The above map f is the base map of the triangular map $F(x,y) = (f(x), g(x,y))$ we are looking for. Next we define g. For any $a \in I$, let $h_a \in \mathcal{C}(I)$ be given by $h_a(y) = a$ if $y \in [0, a/2]$ and $h_a(y) = 1 - |2y-1|$ otherwise; thus, h_0 is the full tent map. By induction there are sequences $\{a_m\}_{m=1}^{\infty}$ converging to 0 and $0 = k(1) < k(2) < \ldots$ such that, for any m, the set

$$\{h_0^i(a_m) : k(m) \leq i < k(m+1)\} \text{ is } 1/m\text{-dense in } I. \tag{3}$$

Thus, for any $y \in I$ there is an i such that $k(m) \leq i < k(m+1)$ and $|h_0^i(a_m) - y| < 1/m$ (this is clearly allowed by the transitivity of h_0). Let $\{K_n\}_{n=1}^{\infty}$ be an enumeration of the compact intervals with rational endpoints contained in the interior $(0, 1/2)$ of I_1 such that

$$K_i = K_j \quad \text{whenever} \quad k(m) < i, j \leq k(m+1). \tag{4}$$

By (i), f^{n-1} carries the endpoints of I_n onto the endpoints of I_1 and, by (i) and (v), $f(J_n) \supset I_n$. Hence, for any $n \geq 1$ there are compact intervals $L_n \subset M_n \subset J_n$ such that $f^n(L_n) = K_n$ and $f(M_n) = I_n$. Finally, let b be the left endpoint of M_1. Since any K_n is in the interior of I_1, L_n must be in the interior of M_n and hence, there is a continuous map $\varrho : [b,1] \to I$ vanishing outside $\bigcup_{i=1}^{\infty} M_i$ such that

$$\varrho(x) = a_m \quad \text{if } x \in L_n \text{ and } k(m) < n \leq k(m+1). \tag{5}$$

Now define g by

$$g(x,y) = \begin{cases} 0 & \text{if } x \in I_1, \\ \dfrac{2x-1}{2b-1} h_0(y) & \text{if } x \in [1/2, b], \\ h_{\varrho(x)}(y) & \text{otherwise.} \end{cases}$$

We may assume that $b > 1/2$ (changing slightly f on J_1 otherwise) so that g, and hence F, is continuous and well defined. Moreover, $F(C) \subset C$. Indeed, if $x \in I_{n+1}$ then by (iv), $F(x,y) \in I_n \times I$, if $x \in I_1$ then $F(x,y) = (f(x), 0)$, and if $x \in M_n$ then $F(x,0) \in I_n \times I$ since $f(M_n) = I_n$. Finally, if $x \in J_n \setminus M_n$ then $F(x,0) = (f(x), 0)$.

Since f is transitive, there is a $z_0 \in C$ such that $\Pi(z_0)$ has a dense f-orbit, and since $F(C) \subset C$, we have $\omega_F(z_0) \subset C$. To finish the proof it suffices to show that $C \subset \omega_F(z_0)$. To do this we first prove that $I_1 \times I \subset \omega_F(z_0)$ by showing that, for any m and any interval $U \subset I$ with $|U| > 2/m$, the trajectory $\{z_j\}_{j=0}^{\infty}$ of z_0 visits the set $W := K_{k(m+1)} \times U$ (see (4)). Define $z_j = (x_j, y_j)$ for any j. Since $\{x_j\}_{j=0}^{\infty}$ is dense in I, for any i with $k(m) < i \le k(m+1)$ there is a number $s(i)$ with $x_{s(i)} \in L_i$. Hence

$$x_{s(i)+i} \in K_i = K_{k(m+1)}$$

by the definition of L_i and (4). Further, (5) and the fact that $x_{s(i)+r} \in I_{i+1-r}$ for any $1 \le r \le i$ imply

$$y_{s(i)+i} = h_0^{i-1}(a_m).$$

By (3) the points $h_0^{i-1}(a_m)$, $k(m) < i \le k(m+1)$, are $1/m$-dense in I, and hence $z_{s(i)+i} \in W$ for some i with $k(m) < i \le k(m+1)$, as we wanted to show.

Similarly, we can prove that $I_n \times I \subset \omega_F(z_0)$ for any n, and consequently, $\{1\} \times I \subset \omega_F(z_0)$. Since $J_n \times \{0\} \subset \omega_F(z_0)$ is trivial, the lemma is proved. ∎

Lemma 7. *The set D is an ω-limit set of no triangular map $F \in \mathcal{C}(I^2)$.*

Proof. Suppose that D is an ω-limit set of a triangular map $F \in \mathcal{C}(I^2)$, $F(x,y) = (f(x), g(x,y))$. Define $A_n = I_n \times I$ for any n, and $B_n = J_n \times \{0\}$ or $B_n = J_n \times \{1\}$ depending on whether n is odd or even. Also, put $A = \{1\} \times I$. We show that

$$F(A) = A \quad \text{and} \quad F(D \setminus A) = D \setminus A. \tag{6}$$

Indeed, A and $D \setminus A$ are the arcwise connected components of D. By the continuity of F, $F(D \setminus A) \subset A$ or $F(D \setminus A) \subset D \setminus A$. Since $F(D) = D$, the first inclusion would imply $D \setminus A \subset F(A) \subset \{f(1)\} \times I$, which is impossible. Thus $F(D \setminus A) \subset D \setminus A$, and $F(A) \subset A$ together with $F(D) = D$ imply (6).

Since $F(A) = A$ and F is continuous, we have $\lim_{x\to 1} F(\{x\} \times I) = A$. Hence, for any $\varepsilon > 0$, $F(A_n) \supset f(I_n) \times [\varepsilon, 1-\varepsilon]$ whenever n is sufficiently large, and consequently, there is a k_0 such that

$$F(A_n) \subset A_m \quad \text{for some } m \text{ if } n \ge k_0. \tag{7}$$

Moreover, there is a k_1 such that, for any m,

$$F(B_n) \cap B_m = \emptyset \quad \text{or} \quad F(B_n) \cap B_{m+1} = \emptyset \quad \text{if } n \ge k_1. \tag{8}$$

This follows since $\operatorname{dist}(B_m, B_{m+1}) > 1$, while $\lim_{n\to\infty} |F(B_n)| = 0$, by the uniform continuity of F. Let $k = \max\{k_0, k_1\}$. We claim that

$$\text{if } n \ge k \text{ then } F(A_n \cup B_n \cup A_{n+1}) \subset A_m \cup B_m \cup A_{m+1} \text{ for some } m. \tag{9}$$

Indeed, (7) implies $F(A_n) \subset A_{m_1}$ and $F(A_{n+1}) \subset A_{m_2}$. Since $A_n \cup B_n \cup A_{n+1}$ is connected, (8) gives $|m_1 - m_2| \le 1$. Now there are three possibilities. If $m_2 = m_1 + 1$ then the connectedness argument implies $F(B_n) \cap B_{m_1} \ne \emptyset$. Hence, by (8), $F(B_n) \subset A_{m_1} \cup B_{m_1} \cup A_{m_1+1}$ and (9) holds for $m = m_1$. The case $m_1 = m_2 + 1$ is similar. Finally, if $m_1 = m_2$ then $F(B_n)$ intersects A_{m_1}. Hence, by (8), $F(A_n \cup B_n \cup A_{n+1})$ is contained in $A_{m_1-1} \cup B_{m_1-1} \cup A_{m_1}$ or in $A_{m_1} \cup B_{m_1} \cup A_{m_1+1}$, which proves (9).

For any n define $D_n = \bigcup_{i=1}^n A_i \cup B_i$. Since D_k is and $D \setminus A$ is not compact, (6) implies $F(D_k) \subset D_r$ for some $r > k$. Since f is transitive, for any n there is a minimal $l(n)$ such that $F^{l(n)}(A_n \cup B_n \cup A_{n+1}) \cap \operatorname{Int} D_{k-1} \neq \emptyset$. By (9) and the minimality of $l(n)$ there is an $m(n)$ such that $F^{l(n)}(A_n \cup B_n \cup A_{n+1}) \subset A_{m(n)} \cup B_{m(n)} \cup A_{m(n)+1}$. Thus, we get $F^{l(n)}(A_n \cup B_n \cup A_{n+1}) \subset D_k$. Put $s = \max\{l(n) : k < n \leq r\}$. Then

$$
\begin{aligned}
F^{s+1}(D_k) \subset F^s(D_r) &= F^s(D_k) \cup \bigcup_{n=k+1}^{r} F^s(A_n \cup B_n) \\
&\subset F^s(D_k) \cup \bigcup_{n=k+1}^{r} F^{s-l(n)}(D_k) \subset \bigcup_{n=0}^{s} F^n(D_k).
\end{aligned}
$$

Consequently, $F(\bigcup_{n=0}^s F^n(D_k)) = \bigcup_{n=1}^{s+1} F^n(D_k) \subset \bigcup_{n=0}^s F^n(D_k)$ so the set $\bigcup_{n=0}^s F^n(D_k)$ is invariant for F. Since it is compact and strictly included in D and has non-empty interior, D cannot be an ω-limit set for F, a contradiction. ∎

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