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Two cases when d_κ and d^*_κ are equal

by

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Abstract. We deal with two cardinal invariants and give conditions on their equality using Shelah's pcf theory.

In [C] Ciesielski asked whether d_c and d^*_c (see definition below) are equal. His proof that this is the case if $c^{<c} = c$ appeared in [J]. Taking the line of [Sh675] we investigate the problem for any cardinal κ. Using pcf notions and results we give sufficient conditions for the equality for regular cardinals in Theorem 1. For example, when $\kappa = \lambda^+$ we can relax the condition $2^\lambda = \lambda^+$ to $d_\lambda = \lambda^+$. In Theorem 2 we bound the value of d_κ for singular κ by d-numbers of smaller cardinals and by covering numbers. Also here we get a partial positive answer but actually we are doing much more than that: d_κ and d^*_κ are computed and shown to be equal to $\mathrm{pp}\,\kappa$. On cov and pp see [Sh-g, Ch II]. Trivial properties of cov which we use freely throughout the paper are listed (usually without a proof) in observation 5.3 there. $\exists^*\theta < \lambda$ means "for unboundedly many θ below λ".

Definition. For infinite cardinals κ:

$$d_\kappa = \min\{|A| : A \subseteq {}^\kappa\kappa,\ \forall f \in {}^\kappa\kappa \exists g \in A(|\{i < \kappa : f(i) = g(i)\}| = \kappa)\},$$

and

$$d^*_\kappa = \min\{|A| : A \subseteq {}^\kappa\kappa,\ \forall G \in [{}^\kappa\kappa]^\kappa \exists g \in A\, \forall f \in G(|\{i < \kappa : f(i) = g(i)\}| = \kappa)\}.$$

d^s_κ is defined similarly to d_κ but f is allowed to be also just a partial function with domain in $[\kappa]^\kappa$.

Remark. It is easy to see that $d^s_\kappa = \mathrm{cov}([\kappa]^\kappa, \supseteq)$.

Theorem 1. *For a* (*regular*) *infinite cardinal κ and for a sequence $\langle \alpha_i : i < \mathrm{cf}\,\kappa\rangle$ of ordinals increasing to κ, if every $\kappa_i = |\alpha_i|$ satisfies $d^*_{\kappa_i}, \mathrm{cov}(\kappa, \kappa_i^+, \kappa_i^+, 2) \le \kappa$, then $d_\kappa = d^*_\kappa$.*

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Proof. Represent κ as a disjoint union of intervals $\langle I_i : i < \kappa\rangle$ such that $|I_i| = \kappa_j$ if $i \in [\alpha_j, \alpha_{j+1})$. For every $i < \operatorname{cf}\kappa$ fix a cofinal set for $([\kappa]^{\kappa_i}, \subset)$ of cardinality less than or equal to κ, call it H_i, and for every $i < \kappa$ and $A \in H_j$ where $i \in [\alpha_j, \alpha_{j+1})$ let $R_{i,A} \subset {}^{I_i}A$ witness that $d^*_{\kappa_j} \le \kappa$. Let $R_i = \bigcup_{A\in H_j} R_{i,A}$ and fix a 1-1 function $F_i : R_i \to \kappa$. Let $G \subseteq {}^\kappa\kappa$ of cardinality d_κ be a witness for the definition of that cardinal invariant. For any $g \in G$ define $\widehat{g} \in {}^\kappa\kappa$ by $\widehat{g} = \bigcup\{F_i^{-1}(g(i)) : g(i) \in \operatorname{rang} F_i\} \cup \bigcup\{0\restriction I_i : g(i) \notin \operatorname{rang} F_i\}$. We prove that $\{\widehat{g} : g \in G\}$ witnesses that $d^*_\kappa \le |G|$. For any sequence $\langle f_i : i < \kappa\rangle \subseteq {}^\kappa\kappa$ and for every $i < \kappa$, cover $\bigcup_{\varepsilon<\alpha_j} \operatorname{rang} f_\varepsilon\restriction I_i$ where $i \in [\alpha_j, \alpha_{j+1})$ by some $A_i \in H_j$ and guess the sequence $\langle f_\varepsilon\restriction I_i : \varepsilon < \alpha_j\rangle$ by some $h_i \in R_i, A_i$. Define $h \in {}^\kappa\kappa$ by $h(i) = F_i(h_i)$ and guess it by $g \in G$. Now $\widehat{g}$ does the job, i.e. for every $i < \kappa$, $|\{\varepsilon < \kappa = \widehat{g}(\varepsilon) = f_i(\varepsilon)\}| = \kappa$.

CONCLUSION. (1) *If an infinite cardinal κ satisfies $d^*_\kappa = \kappa^+$ then $d_{\kappa^+} = d^*_{\kappa^+}$.*

(2) *If κ is inaccessible and for any singular $\lambda < \kappa$ we have $\operatorname{pp}_{\sigma\text{-com}}(\lambda) \le \kappa$ and $\operatorname{cf}\lambda = \aleph_0 \to \operatorname{pp}(\lambda) < \lambda^{+\omega}$ and $\operatorname{cf}\lambda = \aleph_1 \to \operatorname{pp}(\lambda) = \lambda^+$ then $\exists^*\theta < \kappa(d^*_\theta \le \kappa)$ implies that $d_\kappa = d^*_\kappa$.*

(3) *If κ is inaccessible and $0^\#$ does not exist then $\exists^*\theta < \kappa(d^*_\theta \le \kappa)$ implies that $d_\kappa = d^*_\kappa$.*

(4) *If $2^{\aleph_0} < \kappa$ is inaccessible, $\exists^*\theta < \kappa(d^*_\theta + \sup_{\lambda<\kappa} \operatorname{pp}(\lambda \le \kappa)$ and $\exists\theta < \kappa\forall\lambda(|\{\mu < \kappa : \operatorname{pp}_\theta(\mu) > \lambda\}| \le \theta)$ then $d_\kappa = d^*_\kappa$.*

Proof. (1) We only need $\operatorname{cov}(\kappa^+, \kappa^+, \kappa^+, 2) = \kappa^+$, which is trivial.

(2) Trivially, $\sup_{\theta<\kappa} \operatorname{cov}(\kappa, \theta^+, \theta^+, 2) = \sup_{\theta<\lambda<\kappa} \operatorname{cov}(\lambda, \theta^+, \theta^+, 2)$. Now for $\theta < \lambda$,

$$\operatorname{cov}(\lambda, \theta^+, \theta^+, 2) \le \operatorname{cov}(\operatorname{cov}(\lambda, \theta^+, \theta^+, \aleph_1), \theta^+, \aleph_1, 2).$$

By [Sh-g, Ch. II, S. 4],

$$\operatorname{cov}(\lambda, \theta^+, \theta^+, \aleph_1) \le \sup_{\theta<\chi\le\lambda,\, \operatorname{cf}\chi>\aleph_0} \operatorname{pp}_{\sigma\text{-com}}(\chi),$$

which is $\le \kappa$ by the assumption.

We continue:

$$\sup_{\theta<\lambda<\kappa} \operatorname{cov}(\lambda, \theta^+, \theta^+, 2) \le \sup_{\theta<\kappa} \operatorname{cov}(\kappa, \theta^+, \aleph_1, 2) \le \sup_{\lambda<\kappa,\, \operatorname{cf}\lambda>\aleph_0} \operatorname{cov}(\lambda, \lambda, \aleph_1, 2).$$

By [Sh-g, Ch. IX, 1.8] all these terms are equal to the respective $\operatorname{pp}(\lambda)$'s which are $\le \kappa$. Now apply Theorem 1.

(3) If $0^\#$ does not exist then $\forall\lambda(\operatorname{pp}(\lambda) = \lambda^+)$ (see [Sh-g]). In fact, it is enough that there is no inner model with a measurable χ such that $\circ(\chi) = \chi^{++}$. Now use (2).

(4) By the proof of [Sh420, 6.4], $\forall\lambda > 2^{\aleph_0}\forall\theta \ge 2^{\aleph_0} + \operatorname{cf}\lambda(\operatorname{cov}(\lambda, \lambda, \theta^+, 2) = \operatorname{pp}_\theta(\lambda))$. If for some $\lambda, \theta \ge 2^{\aleph_0}$ we have $\operatorname{pp}_\theta(\lambda) > \kappa$ then for the minimal

such λ, $\mathrm{pp}(\lambda)=\mathrm{pp}_\theta(\lambda)$ ([Sh-g, Ch. VIII, 1.6]). Together we have

$$\sum_{\theta<\kappa}\mathrm{cov}(\kappa,\theta^+,\theta^+,2)=\sum_{\theta<\lambda<\kappa}\mathrm{cov}(\lambda,\lambda,\theta^+2)$$
$$=\sum_{\theta<\lambda<\kappa}\mathrm{pp}_\theta(\lambda)=\sum_{\lambda<\kappa}\mathrm{pp}(\lambda)\le\kappa.$$

Now use Theorem 1.

REMARK. In all the known models of ZFC, for every inaccessible κ, $\sup_{\lambda<\kappa}\mathrm{pp}(\lambda)=\kappa$. Notice that in (1) above both the assumptions $\forall\lambda<\kappa$ $(\mathrm{pp}_{\sigma\text{-com}}(\lambda)\le\kappa)$ and $\mathrm{cf}\,\lambda=\aleph_1\to\mathrm{pp}(\lambda)=\lambda^+$ can hold even just from some point on. Also the assumption $\forall\theta\forall\lambda(|\{\mu:\mathrm{pp}_\theta(\mu)>\lambda\}|\le\aleph_0)$ is not violated in any known model of ZFC.

THEOREM 2. *If κ is a singular cardinal and $\langle\kappa_i:i<\mathrm{cf}\,\kappa\rangle$ increases to κ then for $\mu=\sup_{i<\mathrm{cf}\,\kappa}[d_{\kappa_i}+\mathrm{cov}(\kappa,\kappa_i^+,\kappa_i^+,2)]$ and $\mu^s=\sup_{1<\mathrm{cf}\,\kappa}[d^s_{\kappa_i}+\mathrm{cov}(\kappa,\kappa_i^+,\kappa_i^+,\kappa_i)]$ we have*:

(1) $d_\kappa\le\mathrm{cov}(\mu,(\mathrm{cf}\,\kappa)^+,(\mathrm{cf}\,\kappa)^+,2)d_{\mathrm{cf}\,\kappa}$.

(2) $d_\kappa\le\mathrm{cov}(\mu,(\mathrm{cf}\,\kappa)^+,(\mathrm{cf}\,\kappa)^+,\mathrm{cf}\,\kappa)d^s_{\mathrm{cf}\,\kappa}$.

(3) *The claim of* (1) *and* (2) *holds for μ^s instead of μ if the κ_i's are regular.*

(4) $d^*_\kappa\le\mathrm{cov}(\mu^*,(\mathrm{cf}\,\kappa)^+,(\mathrm{cf}\,\kappa)^+,2)d_{\mathrm{cf}\,\kappa}$ *where*

$$\mu^*=\sup_{i<\mathrm{cf}\,\kappa}[d^*_{\kappa_i}+\mathrm{cov}(\kappa,\kappa_i^+,\kappa_i^+,2)].$$

(5) $d^*_\kappa\le\mathrm{cov}(\mu^*,(\mathrm{cf}\,\kappa)^+,(\mathrm{cf}\,\kappa)^+,\mathrm{cf}\,\kappa)d^s_{\mathrm{cf}\,\kappa}$.

Proof. (1) Represent κ as a disjoint union of intervals $\langle I_i:i<\mathrm{cf}\,\kappa\rangle$ such that $|I_i|=\kappa_i$. For every $i<\mathrm{cf}\,\kappa$ let H_i be cofinal in $([\kappa]^{\kappa_i},\subseteq)$ of cardinality $\mathrm{cov}(\kappa,\kappa_i^+,\kappa_i^+,2)$ and for every $A\in H_i$ let $R_{i,A}\subseteq{}^{I_i}A$ be of cardinality d_{κ_i} such that for every $f\in{}^{I_i}A$ there is $g\in R_{i,A}$ for which $|\{\alpha\in I_i:f(\alpha)=g(\alpha)\}|=\kappa_i$. Define $R=\bigcup_{i<\mathrm{cf}\,\kappa}\bigcup_{A\in H_i}R_{i,A}$, let H be cofinal in $([R]^{\mathrm{cf}\,\kappa,\subseteq})$ of cardinality $\mathrm{cov}(\mu,(\mathrm{cf}\,\kappa)^+,(\mathrm{cf}\,\kappa)^+,2)$ (notice that $|R|=\mu$) and for every $C\in H$ of cardinality $\mathrm{cf}\,\kappa$ fix an order $<_c$ on c of order type $\mathrm{cf}\,k$. Let P be of cardinality $d_{\mathrm{cf}\,\kappa}$ such that for every $f\in{}^{\mathrm{cf}\,\kappa}\mathrm{cf}\,\kappa$ there is $g\in P$ for which $|\{\alpha<\mathrm{cf}\,\kappa:f(a)=g(\alpha)\}|=\mathrm{cf}\,\kappa$.

It is enough to show that we can guess a function in ${}^\kappa\kappa$ by the members of $G=\{f\in{}^\kappa\kappa$: for some $C\in H$ and $g\in P$ for every $i<\mathrm{cf}\,\kappa$, $f\restriction I_i$ is the $g(i)$th element in $(C\cap{}^{I_i}\kappa,<_c)\}$. For any function $f\in{}^\kappa\kappa$ for any $i<\mathrm{cf}\,\kappa$, cover $f''[I_i]$ by a set from H_i, call it A_i, and guess $f\restriction I_i$ as a function in ${}^{I_i}A_i$ by some $g_i\in R_{i,A_i}$. Next cover $\{g_i:i<\mathrm{cf}\,\kappa\}$ by some $C\in H$ and guess f', which is defined as a function in ${}^{\mathrm{cf}\,\kappa}\mathrm{cf}\,\kappa$, by $f'(i)=\mathrm{otp}(\{j\in C\cap{}^{I_i}\kappa:j<_c g_i\},<_c)$ by some function $h\in P$. The function in G which is defined from C and h does the job.

(2) The only differences are that H is of cardinality $\operatorname{cov}(\mu, (\operatorname{cf}\kappa)^+, (\operatorname{cf}\kappa)^+, \operatorname{cf}\kappa)$ and only the sets of unions of less than $\operatorname{cf}\kappa$ elements from it are cofinal in $([R]^{\operatorname{cf}\kappa}, \subseteq)$, that P is now of cardinality $d^s_{\operatorname{cf}\kappa}$, that it guesses also partial functions in ${}^{\operatorname{cf}\kappa}\operatorname{cf}\kappa$ with domains of cardinality $\operatorname{cf}\kappa$ and that we define G by $G = \{f \in {}^\kappa\kappa :$ for some $C \in H$ and a partial function $g \in P$, for every $i \in \operatorname{dom} g$, $f\restriction I_i$ is the $g(i)$th element in $(C \cap {}^{I_i}\kappa, <_c)\}$. For any function $f \in {}^\kappa\kappa$ we get $\langle g_i : i < \operatorname{cf}\kappa\rangle$ as in (1). Next we cover this set by the union of less than $\operatorname{cf}\kappa$ elements from H and pick one of them, call it c, such that $|C \cap \langle g_i : i < \operatorname{cf}\kappa\rangle| = \operatorname{cf}\kappa$. Define the partial function of ${}^{\operatorname{cf}\kappa}\operatorname{cf}\kappa$ by $f'(i) = \operatorname{otp}(\{j \in C \cap {}^{I_i}\kappa : j <_c g_i\}, <_c)$ if $g_i \in C$, and guess it by some $\kappa \in P$. The function in G which is defined from C and h does the job.

(3) is proved by repeating the argument from (2) $\operatorname{cf}\kappa$ many times for any $R_{i,A}$, $A \in H$.

(4), (5) Easy.

Conclusion. *Let κ be a singular cardinal which is not a fixed point, i.e. $\kappa = \aleph_{\alpha+\beta}$, $\beta < \aleph_\alpha$, and $\langle \kappa_i : i < \operatorname{cf}\kappa\rangle$ an unbounded set of cardinals below it, $\aleph_\alpha < \kappa_0$. Then*:

(1) *If* $\sum_{i<\operatorname{cf}\kappa} d_{\kappa_i} \le \kappa^{|\beta|}$ *then* $d_\kappa \le \kappa^{|\beta|} + d_{\operatorname{cf}\kappa}$.

(2) *If* $\sum_{i<\operatorname{cf}\kappa} d^*_{\kappa_i} \le \kappa^{|\beta|}$ *then* $d^*_\kappa \le \kappa^{|\beta|} + d_{\operatorname{cf}\kappa}$.

(3) *If* $2^{\operatorname{cf}\kappa}, \sum_{i<\operatorname{cf}\kappa} d_{\kappa_i} \le \operatorname{pp}(\kappa)$ *and* $\forall \kappa' < \kappa(\operatorname{cf}\kappa' \le |\beta| \to \operatorname{pp}_{|\beta|}(\kappa) < \kappa)$ *then* $d_\kappa = \operatorname{pp}(\kappa)$.

(4) *If in* (3) *also* $\sum_{i<\operatorname{cf}\kappa} d^*_{\kappa_i} \le \operatorname{pp}(\kappa)$ *then* $d_\kappa = d^*_\kappa = \operatorname{pp}(\kappa)$.

(5) *If κ is below the first fixed point then in* (3) *we can replace* $2^{\operatorname{cf}\kappa}$ *by* $d_{\operatorname{cf} k}$.

Proof. (1) By [Sh-g, Ch. II, 3.6],

$$\mu = \sup_{1<\operatorname{cf}\kappa} [d_{\kappa_i} + \operatorname{cov}(\kappa, \kappa_i^+, \kappa_i^+, 2)] \le \kappa^{|\beta|} + \max\operatorname{pcf}\operatorname{Reg}\cap[\aleph_\alpha, \kappa) \le \kappa^{|\beta|}.$$

As $\operatorname{cf}\kappa \le |\beta|$, we have $\operatorname{cov}(\mu, (\operatorname{cf}\kappa)^+, (\operatorname{cf}\kappa)^+, 2) \le \mu^{\operatorname{cf}\kappa} = \kappa^{|\beta|}$. Now use (1) of Theorem 2.

(2) Use (4) of Theorem 2.

(3) If κ_0 is large enough below κ then $\mu \le \operatorname{pp}_{|\beta|}(\kappa) + \max\operatorname{pcf}\operatorname{Reg}\cap[\kappa_0, \kappa) = \operatorname{pp}_{|\beta|}(\kappa)$.

Now by [Sh-g, Ch. VIII, 1.6], $\operatorname{pp}_{|\beta|}(\kappa) = \operatorname{pp}(\kappa)$ and by [Sh-g, Ch. II, 5.4],

$$\begin{aligned}\operatorname{cov}(\mu, (\operatorname{cf}\kappa)^+, (\operatorname{cf}\kappa)^+, \operatorname{cf}\kappa) &= \sup\{\operatorname{pp}(\theta) : \theta \le \mu, \operatorname{cf}\theta = \operatorname{cf}\kappa\} \\ &= \sup\{\operatorname{pp}(\theta) : \theta \le \kappa, \operatorname{cf}\theta = \operatorname{cf}\kappa\} = \operatorname{pp}(\kappa)\end{aligned}$$

(the second equality follows from $\operatorname{cf}\operatorname{pp}(\kappa) > \operatorname{cf}\kappa$ and [Sh-g, Ch. II, 2.3(2)]). By (2) of Theorem 2, $d_\kappa \le \operatorname{pp}(\kappa) + d^s_{\operatorname{cf}\kappa} = \operatorname{pp}(\kappa)$. The inequality $d_\kappa \ge \operatorname{pp}(\kappa)$ holds by [Sh-g, Ch. VIII, 1.6] and [Sh675, 2.2(2)].

(4) Use (5) of Theorem 2, and the proof of (3) here to get $\mu^* \le \operatorname{pp}(\kappa)$.

(5) In computing $\operatorname{cov}(\mu, (\operatorname{cf}\kappa)^+, (\operatorname{cf}\kappa)^+, 2)$ we use [Sh-g, Ch. IX, 3.7) and then apply (1) of Theorem 2.

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