

Topological AE(0)-groups

by

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Abstract. We investigate topological AE(0)-groups, a class which contains the class of Polish groups as well as the class of all locally compact groups. We establish the existence of a universal AE(0)-group of a given weight as well as the existence of a universal action of an AE(0)-group of a given weight on an AE(0)-space of the same weight. A complete characterization of closed subgroups of powers of the symmetric group S_∞ is obtained. It is also shown that every AE(0)-group is Baire isomorphic to a product of Polish groups. These results are obtained by using the spectral descriptions of AE(0)-groups which are presented in Section 3.

1. Introduction. One of the main structure theorems for compact groups (see [10, Chapters 6 & 9]) can be formulated as follows.

THEOREM A ([10], Theorem 9.24(ii)). *Let G be a connected compact group. Then there exists a continuous homomorphism*

$$p : Z_0(G) \times \prod \{L_t : t \in T\} \rightarrow G,$$

where $Z_0(G)$ stands for the identity component of the center of G , and L_t is a simple, connected and simply connected compact Lie group, $t \in T$, such that $\ker(p)$ is a zero-dimensional central subgroup of $Z_0(G) \times \prod \{L_t : t \in T\}$.

The above statement clearly shows that the classes of zero-dimensional groups, abelian groups and simple, simply connected Lie groups play a central role in the general theory of compact groups. Recently these classes of topological groups have been studied from the point of view of absolute extensors in dimension n (see [4] for a comprehensive introduction into the theory of absolute extensors in dimension n). The following three statements show that such an approach is quite effective.

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THEOREM B ([2]). *The following conditions are equivalent for a zero-dimensional topological group G :*

- (a) G is topologically equivalent to the product $(\mathbb{Z}_2)^\tau \times \mathbb{Z}^k$.
- (b) G is an $\text{AE}(0)$ -space.

THEOREM C ([6], Theorem E). *The following conditions are equivalent for a compact abelian group G :*

- (a) G is a torus group (both in topological and algebraic senses).
- (b) G is an $\text{AE}(1)$ -compactum.

THEOREM D ([5], Corollary 1). *The following conditions are equivalent for a non-trivial compact group G :*

- (a) G is a simple, connected and simply connected Lie group.
- (b) G is an $\text{AE}(2)$ -group with $\pi_3(G) = \mathbb{Z}$.

The full classification problem for non-abelian (see Theorem C) and for non-simply connected compact $\text{AE}(1)$ -groups remains open. On the other hand, the following two statements provide a complete classification of simply connected compact $\text{AE}(1)$ -groups.

THEOREM E ([5], Theorem C). *The following conditions are equivalent for a compact group G :*

- (a) G is a simply connected $\text{AE}(1)$ -compactum.
- (b) G is an $\text{AE}(2)$ -compactum.
- (c) G is an $\text{AE}(3)$ -compactum.
- (d) G is a product of simple, connected and simply connected compact Lie groups.

THEOREM F ([5], Corollary 2). *There is no non-trivial compact $\text{AE}(4)$ -group.*

We complete this brief survey by pointing out that, as shown by the following statement, for locally compact groups the restriction of being an $\text{AE}(0)$ -group is purely formal.

THEOREM G (Pontryagin–Haydon, [14], [8]; [4]). *Every locally compact group is an $\text{AE}(0)$ -space.*

Below we study $\text{AE}(0)$ -groups. The class of $\text{AE}(0)$ -groups contains the class of all (generally speaking, non-metrizable) locally compact groups (Theorem G) as well as the class of all Polish groups. Actually the class of Polish groups coincides with the class of metrizable $\text{AE}(0)$ -groups and forms a foundation of the entire theory of $\text{AE}(0)$ -groups. We hope that the results presented in Section 3 do indicate a potential for a non-trivial theory of $\text{AE}(0)$ -groups which unifies and generalizes the theories of locally compact

and Polish groups (thus offering a possible approach to the corresponding question posed in [13]).

In Section 3 we present a spectral characterization of AE(0)-groups in terms of well ordered continuous inverse spectra (Theorem 3.4). This characterization states that a non-metrizable topological group G of weight τ is a AE(0)-group if and only if it is the limit of a well ordered continuous inverse system $\mathcal{S}_G = \{G_\alpha, p_\alpha^{\alpha+1}, \alpha < \tau\}$ of length τ , consisting of AE(0)-groups G_α and 0-soft limit homomorphisms $p_\alpha^{\alpha+1} : G_{\alpha+1} \rightarrow G_\alpha, \alpha < \tau$, so that G_0 is a Polish group and each homomorphism $p_\alpha^{\alpha+1}, \alpha < \tau$, has a Polish kernel.

Obviously this result cannot be accepted as the one providing a satisfactory reduction of the non-metrizable case to the Polish case. Of course, everything is fine if the weight of G is ω_1 —in such a case all G_α 's, $\alpha < \omega_1$, (and not only the very first one, i.e. G_0) are indeed Polish. But if the weight of G is greater than ω_1 , then all G_α 's with $\alpha \geq \omega_1$ are non-metrizable.

In order to achieve our final goal and complete the reduction, we analyze 0-soft homomorphisms with Polish kernels between (generally speaking, non-metrizable) AE(0)-groups. A characterization of such homomorphisms, which is recorded in Proposition 3.6, states that a 0-soft homomorphism $f : G \rightarrow L$ of AE(0)-groups has a Polish kernel if and only if there exists a pullback diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & L \\ p \downarrow & & \downarrow q \\ G_0 & \xrightarrow{f_0} & L_0 \end{array}$$

where G_0 and L_0 are Polish groups and the homomorphisms $p : G \rightarrow G_0$ and $q : L \rightarrow L_0$ are 0-soft. Theorem 3.4 and Proposition 3.6 together complete the required reduction.

Polish groups and their actions have been extensively studied in a variety of directions (ergodic theory, group representations, operator algebras; see [1] for further discussion and references). Some of the central themes of the theory of Polish groups, counterparts of which (for arbitrary AE(0)-groups) are considered below, are the existence of universal groups, the existence of universal actions and characterization of closed subgroups of the symmetric group S_∞ .

In Section 4 we use the above mentioned spectral characterizations and present extensions of some of these results to AE(0)-groups. We prove the existence of universal AE(0)-groups of a given weight (Proposition 4.1) and the existence of universal actions of AE(0)-groups of a given weight on compact AE(0)-spaces of the same weight (Theorem 4.2).

Theorem 4.4 characterizes AE(0)-groups which are isomorphic to closed subgroups of powers S_∞^τ of the symmetric group S_∞ , the group of all bijec-

tions of \mathbb{N} under the relative topology inherited from $\mathbb{N}^{\mathbb{N}}$. This result extends the corresponding observation [1, Theorem 1.5.1] for the group S_{∞} itself. As a corollary (Corollary 4.5) we note that if a Polish group G can be embedded (as a closed subgroup) into S_{∞}^{τ} for some τ , then G can be embedded into S_{∞} as well. In light of [7] this shows that there exist zero-dimensional Polish groups which cannot be embedded into S_{∞}^{τ} as closed subgroups for any cardinal number τ .

Finally we use Theorem 3.4 to prove (Theorem 4.8) that every $\text{AE}(0)$ -group is Baire isomorphic to a product of Polish groups. In light of Theorem G this result appears to be new even for compact groups.

2. Preliminaries. All topological spaces below are assumed to be Tikhonov (i.e. completely regular and Hausdorff) and all maps (except in Subsection 4.3) are continuous. Modified Lebesgue dimension \dim is defined in terms of finite functionally open covers so that $\dim X = \dim \beta X$ for any Tikhonov space X (βX denotes the Stone–Čech compactification of X). Definitions of concepts related to inverse spectra can be found in [4]. Finally, \mathbb{R} denotes the real line and \mathbb{Q} stands for the Hilbert cube.

2.1. Definitions of $\text{AE}(n)$ -spaces and n -soft maps. A comprehensive introduction to the general theory of $\text{AE}(n)$ -spaces and n -soft maps can be found in [4]. $C(X)$ denotes the set of all continuous real-valued functions defined on X .

DEFINITION 2.1. A space X is called an *absolute extensor in dimension n* (briefly, $\text{AE}(n)$ -space), $n = 0, 1, \dots$, if for each at most n -dimensional space Z and each subspace Z_0 of Z any map $f : Z_0 \rightarrow X$ such that $C(f)(C(X)) \subseteq \{\varphi|_{Z_0} : \varphi \in C(Z)\}$ can be extended to Z .

For compact spaces this definition is equivalent to the standard one.

PROPOSITION 2.2. *A compact space X is an $\text{AE}(n)$ -space if and only if for each at most n -dimensional compactum Z and for each closed subspace Z_0 of Z , any map $f : Z_0 \rightarrow X$ has an extension to Z .*

It is known [4, Chapter 6] that the class of metrizable $\text{AE}(0)$ -spaces coincides with the class of Polish spaces. Every $\text{AE}(0)$ -space is realcompact and has a countable Suslin number.

DEFINITION 2.3. A map $f : X \rightarrow Y$ between $\text{AE}(n)$ -spaces is *n -soft* if for each at most n -dimensional realcompact space Z , for its closed subspace Z_0 , and for any two maps $g : Z_0 \rightarrow X$ and $h : Z \rightarrow Y$ such that $f \circ g = h|_{Z_0}$ and $C(g)(C(X)) \subseteq \{\varphi|_{Z_0} : \varphi \in C(Z)\}$, there exists a map $k : Z \rightarrow X$ such that $k|_{Z_0} = g$ and $f \circ k = h$.

It is easy to check that X is an $\text{AE}(n)$ -space if and only if the constant map $X \rightarrow \{\text{pt}\}$ is n -soft. It is important to note that every 0-soft map

between AE(0)-spaces is surjective and open ([4, Lemma 6.1.13 & Proposition 6.1.26]) and that for surjections between Polish spaces the converse of this fact is also true. Note also that every surjective homomorphism between Polish groups is open and 0-soft.

We say (see [4, Section 6.3]) that a map $f : X \rightarrow Y$ has a *Polish kernel* if there exists a Polish space P such that X is C -embedded in the product $Y \times P$ so that f coincides with the restriction $\pi_Y|_X$ of the projection $\pi_Y : Y \times P \rightarrow Y$. Obviously any map between Polish spaces has a Polish kernel.

2.2. Set-theoretical preliminaries. Let A be a partially ordered directed set (i.e. for any two elements $\alpha, \beta \in A$ there exists an element $\gamma \in A$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$). We say that a subset $A_1 \subseteq A$ of A *majorizes* another subset $A_2 \subseteq A$ of A if for each element $\alpha_2 \in A_2$ there exists an element $\alpha_1 \in A_1$ such that $\alpha_1 \geq \alpha_2$. A subset which majorizes A is called *cofinal* in A . A subset of A is said to be a *chain* if any two of its elements are comparable. The symbol $\sup B$, where $B \subseteq A$, denotes the least upper bound of B (if such an element exists in A). Let now τ be an infinite cardinal. A subset B of A is said to be τ -*closed* in A if for each chain $C \subseteq B$ with $|C| \leq \tau$, we have $\sup C \in B$ whenever the element $\sup C$ exists in A . Finally, a directed set A is said to be τ -*complete* if for each chain B in A with $|C| \leq \tau$, the element $\sup C$ exists in A .

The standard example of a τ -complete set can be obtained as follows. For an arbitrary set A let $\exp A$ denote, as usual, the collection of all subsets of A . There is a natural partial order on $\exp A$: $A_1 \geq A_2$ if and only if $A_1 \supseteq A_2$. With this partial order $\exp A$ becomes a directed set. If we consider only those subsets of A which have cardinality $\leq \tau$, then the corresponding subcollection of $\exp A$, denoted by $\exp_\tau A$, serves as a basic example of a τ -complete set. The proofs of the following statements can be found in [4].

PROPOSITION 2.4. *Let $\{A_t : t \in T\}$ be a collection of τ -closed cofinal subsets of a τ -complete set A . If $|T| \leq \tau$, then $\bigcap \{A_t : t \in T\}$ is also cofinal (in particular, non-empty) and τ -closed in A .*

COROLLARY 2.5. *For each subset B with $|B| \leq \tau$ of a τ -complete set A there exists an element $\gamma \in A$ such that $\gamma \geq \beta$ for each $\beta \in B$.*

PROPOSITION 2.6. *Let A be a τ -complete set, $L \subseteq A^2$, and suppose the following three conditions are satisfied:*

- Existence: *For each $\alpha \in A$ there exists $\beta \in A$ such that $(\alpha, \beta) \in L$.*
- Majorization: *If $(\alpha, \beta) \in L$ and $\gamma \geq \beta$, then $(\alpha, \gamma) \in L$.*
- τ -closedness: *Let $\{\alpha_t : t \in T\}$ be a chain in A with $|T| \leq \tau$. If $(\alpha_t, \beta) \in L$ for some $\beta \in A$ and each $t \in T$, then $(\alpha, \beta) \in L$ where $\alpha = \sup\{\alpha_t : t \in T\}$.*

Then the set of all L -reflexive elements of A (an element $\alpha \in A$ is L -reflexive if $(\alpha, \alpha) \in L$) is cofinal and τ -closed in A .

2.3. Baire sets and Baire isomorphisms. Recall that elements of the σ -algebra generated by functionally open subsets of a space X are called *Baire sets* of X . A map $f : X \rightarrow Y$ is a *Baire map* if the inverse images of Baire sets are Baire sets. A bijection $f : X \rightarrow Y$ is a *Baire isomorphism* if both f and f^{-1} are Baire maps.

The following statement ([3, Proposition 2.5]) is used in the proof of Theorem 4.8.

PROPOSITION 2.7. *Let $f : X \rightarrow Y$ be a 0-soft map of $\text{AE}(0)$ -spaces. Then there exists a Baire map $g : Y \rightarrow X$ such that $f \circ g = \text{id}_Y$.*

If f is a continuous homomorphism of Polish groups, then the existence of such a g has been observed by Dixmier [1, Theorem 1.2.4], [11, Theorem 12.17]).

3. $\text{AE}(0)$ -groups and actions of $\text{AE}(0)$ -groups—spectral representations. In this section we present spectral characterizations of $\text{AE}(0)$ -groups. We also present a spectral description of actions of $\text{AE}(0)$ -groups on $\text{AE}(0)$ -spaces. An inverse spectrum $\mathcal{S}_G = \{G_\alpha, p_\alpha^\beta, A\}$ is *factorizing* if for any continuous function $f : \lim \mathcal{S}_G \rightarrow \mathbb{R}$ there exist an index $\alpha \in A$ and a continuous function $f_\alpha : G_\alpha \rightarrow \mathbb{R}$ such that $f = f_\alpha \circ p_\alpha$.

PROPOSITION 3.1. *Each topological $\text{AE}(0)$ -group G is topologically and algebraically isomorphic to the limit of a factorizing ω -spectrum $\mathcal{S}_G = \{G_\alpha, p_\alpha^\beta, A\}$ consisting of Polish groups G_α , $\alpha \in A$, and 0-soft limit homomorphisms $p_\alpha : G \rightarrow G_\alpha$, $\alpha \in A$.*

Proof. By [4, Theorem 6.3.2 or Proposition 6.3.5], the space G can be represented as the limit space of a factorizing ω -spectrum $\mathcal{S} = \{G_\alpha, p_\alpha^\beta, \tilde{A}\}$ consisting of Polish spaces (i.e. $\text{AE}(0)$ -spaces of countable weight) and 0-soft limit projections. Let us show that this spectrum contains an ω -closed cofinal subspectrum consisting of topological groups and limit projections that are (continuous) homomorphisms.

Let $\mu : G \times G \rightarrow G$ and $\nu : G \rightarrow G$ be the continuous operations of multiplication and inversion coming from the topological group structure of G . We apply [4, Theorem 1.3.6] to both μ and ν . First consider the multiplication. Clearly, $G \times G$ is the limit space of the spectrum

$$\mathcal{S} \times \mathcal{S} = \{G_\alpha \times G_\alpha, p_\alpha^\beta \times p_\alpha^\beta, \tilde{A}\}.$$

All projections of the spectrum $\mathcal{S} \times \mathcal{S}$ are 0-soft, and hence, by [4, Proposition 6.1.26], open. The Suslin number of the product $G \times G$ is obviously countable (see [4, Proposition 6.1.8]). Consequently, by [4, Proposition 1.3.3],

the spectrum $\mathcal{S} \times \mathcal{S}$ is factorizing. Next we apply [4, Theorem 1.3.6] to the spectra $\mathcal{S} \times \mathcal{S}$, \mathcal{S} and to the map $\underline{\mu}$ between their limit spaces. Then we get an ω -closed cofinal subset A_μ of \tilde{A} such that for each $\alpha \in A_\mu$ there exists a continuous map $\mu_\alpha : G_\alpha \times G_\alpha \rightarrow G_\alpha$ such that the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ p_\alpha \times p_\alpha \downarrow & & \downarrow p_\alpha \\ G_\alpha \times G_\alpha & \xrightarrow{\mu_\alpha} & G_\alpha \end{array}$$

commutes. In other words, $p_\alpha \circ \mu = \mu_\alpha \circ (p_\alpha \times p_\alpha)$ for each $\alpha \in A_\mu$.

Next consider the continuous inversion $\nu : G \rightarrow G$. By [4, Theorem 1.3.6], applied to ν and the spectrum \mathcal{S} , there exists an ω -closed cofinal subset A_ν of \tilde{A} such that for each $\alpha \in A_\nu$ there exists a continuous map $\nu_\alpha : G_\alpha \rightarrow G_\alpha$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\nu} & G \\ p_\alpha \downarrow & & \downarrow p_\alpha \\ G_\alpha & \xrightarrow{\nu_\alpha} & G_\alpha \end{array}$$

commutes. In other words, $p_\alpha \circ \nu = \nu_\alpha \circ p_\alpha$ for each $\alpha \in A_\nu$.

By Proposition 2.4, $A = A_\mu \cap A_\nu$ is still ω -closed and cofinal in \tilde{A} . This guarantees that G is topologically and algebraically isomorphic to the limit of the factorizing ω -spectrum $\mathcal{S}_G = \{G_\alpha, p_\alpha^\beta, A\}$. Also for each $\alpha \in A$ we have two maps

$$\mu_\alpha : G_\alpha \times G_\alpha \rightarrow G_\alpha \quad \text{and} \quad \nu_\alpha : G_\alpha \rightarrow G_\alpha$$

which allow us to define:

- (a) a continuous multiplication operation on G_α by letting

$$x_\alpha \cdot y_\alpha = \mu_\alpha(x_\alpha, y_\alpha) \quad \text{for each } (x_\alpha, y_\alpha) \in G_\alpha \times G_\alpha;$$

- (b) a continuous inversion on G_α by letting

$$x_\alpha^{-1} = \nu_\alpha(x_\alpha) \quad \text{for each } x_\alpha \in G_\alpha.$$

It is easy to see that G_α , $\alpha \in A$, becomes a topological group with respect to these operations. Moreover, for each $\alpha \in A$ the limit projection $p_\alpha : G \rightarrow G_\alpha$ becomes a homomorphism with respect to the above defined operations. ■

COROLLARY 3.2. *Let G be an AE(0)-group such that $\dim G \leq n$. Then G is topologically and algebraically isomorphic to the limit of a factorizing ω -spectrum $\mathcal{S}_G = \{G_\alpha, p_\alpha^\beta, A\}$ consisting of at most n -dimensional Polish groups G_α , $\alpha \in A$, and 0-soft limit homomorphisms $p_\alpha : G \rightarrow G_\alpha$, $\alpha \in A$.*

Proof. By Proposition 3.1, $G = \lim \mathcal{S}_1$, where $\mathcal{S}_1 = \{G_\alpha, p_\alpha^\beta, A_1\}$ is a factorizing ω -spectrum consisting of Polish groups and 0-soft limit homomorphisms. By [4, Theorem 1.3.10], $G = \lim \mathcal{S}_2$, where $\mathcal{S}_2 = \{G_\alpha, p_\alpha^\beta, A_2\}$ is a factorizing ω -spectrum consisting of at most n -dimensional Polish spaces. It follows directly from the proofs of Proposition 3.1 and [4, Theorem 1.3.10] that both indexing sets A_1 and A_2 can be assumed to be closed ω -complete subsets of an ω -complete set B . By Proposition 2.4, $A = A_1 \cap A_2$ is still a cofinal ω -closed subset of B . Consequently, the factorizing ω -spectrum $\mathcal{S}_G = \{G_\alpha, p_\alpha^\beta, A\}$ consists of at most n -dimensional Polish groups and 0-soft limit homomorphisms. It only remains to note that $G = \lim \mathcal{S}_G$. ■

COROLLARY 3.3. *Let $\tau \geq \omega$. Every $\text{AE}(0)$ -group of weight $\tau \geq \omega$ is topologically and algebraically isomorphic to a closed C -embedded subgroup of the product $\prod\{G_t : t \in T\}$, where G_t , $t \in T$, is a Polish group and $|T| = \tau$.*

Proof. Let G be an $\text{AE}(0)$ -group of weight τ . If $\tau = \omega$, then G is Polish (see [4, Chapter 6]) and consequently there is nothing to prove.

If $\tau > \omega$, then by Proposition 3.1, G is topologically and algebraically isomorphic to the limit of a factorizing ω -spectrum $\mathcal{S}_G = \{G_t, p_t', T\}$ with $|T| = \tau$. Clearly $\lim \mathcal{S}_G$ is isomorphic to a closed subgroup of the product $\prod\{G_t : t \in A\}$. Since the spectrum \mathcal{S}_G is factorizing, it follows that $\lim \mathcal{S}_G$ is C -embedded in $\prod\{G_t : t \in T\}$. ■

In light of Corollary 3.3 it is interesting to know whether every closed C -embedded subgroup of the product of Polish groups is an $\text{AE}(0)$ -group.

Another question related to Corollary 3.3 is whether $\text{AE}(0)$ -groups can be characterized as closed C -embedded subgroups G of products of Polish groups such that the projection of G onto every countable face of the product is closed.

THEOREM 3.4. *Let G be a topological group of weight $\tau > \omega$. Then the following conditions are equivalent:*

- (a) G is an $\text{AE}(0)$ -group.
- (b) *There exists a well ordered inverse spectrum $\mathcal{S}_G = \{G_\alpha, p_\alpha^{\alpha+1}, \tau\}$ with the following properties:*

- (1) G_α is an $\text{AE}(0)$ -group and $p_\alpha^{\alpha+1} : G_{\alpha+1} \rightarrow G_\alpha$ is a 0-soft homomorphism with Polish kernel, $\alpha < \tau$.
- (2) *If $\beta < \tau$ is a limit ordinal, then the diagonal product*

$$\Delta\{p_\alpha^\beta : \alpha < \beta\} : G_\beta \rightarrow \lim\{G_\alpha, p_\alpha^{\alpha+1}, \alpha < \beta\}$$

is a topological and algebraic isomorphism.

- (3) G is topologically and algebraically isomorphic to $\lim \mathcal{S}_G$.
- (4) G_0 is a Polish group.

Proof. (a) \Rightarrow (b). By Corollary 3.3, we may assume that G is a closed C -embedded subgroup of a product $\prod\{X_a : a \in A\}$, $|A| = \tau$, of Polish groups X_a , $a \in A$. There exists a proper, functionally closed 0-invertible map $f : Y \rightarrow \prod\{X_a : a \in A\}$, where Y is a spectrally complete (see [4, p. 247]) realcompact space of weight τ and dimension $\dim Y = 0$ (see [4, Proposition 6.2.13] for details). Consider the inverse image $f^{-1}(G) \subseteq Y$ of G and the map $f|f^{-1} : f^{-1}(G) \rightarrow G$. Since G is C -embedded in the product $\prod\{X_a : a \in A\}$, since $\dim Y = 0$ and since G , according to (a), is an AE(0)-space, there exists a map $g : Y \rightarrow G$ such that $g|f^{-1}(G) = f|f^{-1}(G)$.

Denote by

$$\pi_B : \prod\{X_a : a \in A\} \rightarrow \prod\{X_a : a \in B\}$$

and

$$\pi_C^B : \prod\{X_a : a \in B\} \rightarrow \prod\{X_a : a \in C\}$$

the natural projections onto the corresponding subproducts ($C \subseteq B \subseteq A$). We call a subset $B \subseteq A$ *admissible* (cf. the proof of [4, Theorem 6.3.1]) if

$$\pi_B(g(f^{-1}(x))) = \pi_B(x)$$

for each $x \in \pi_B^{-1}(\pi_B(G))$. We need the following properties of admissible sets.

CLAIM 1. *The union of every collection of admissible sets is admissible.*

Indeed, let $\{B_t : t \in T\}$ be a collection of admissible sets and $B = \bigcup\{B_t : t \in T\}$. Let $x \in \pi_B^{-1}(\pi_B(G))$. Clearly $x \in \pi_{B_t}^{-1}(\pi_{B_t}(G))$ for each $t \in T$ and consequently

$$\pi_{B_t}(g(f^{-1}(x))) = \pi_{B_t}(x) \quad \text{for each } t \in T.$$

Obviously, $\pi_B(x) \in \pi_B(g(f^{-1}(x)))$ and it therefore suffices to show that the set $\pi_B(g(f^{-1}(x)))$ contains only one point. Assuming that there is a point $y \in \pi_B(g(f^{-1}(x)))$ such that $y \neq \pi_B(x)$ we conclude (remembering that $B = \bigcup\{B_t : t \in T\}$) that there must be an index $t \in T$ such that $\pi_{B_t}^B(y) \neq \pi_{B_t}^B(\pi_B(x))$. But this is impossible:

$$\pi_{B_t}^B(y) \in \pi_{B_t}^B(\pi_B(g(f^{-1}(x)))) = \pi_{B_t}(g(f^{-1}(x))) = \pi_{B_t}(x) = \pi_{B_t}^B(\pi_B(x)).$$

CLAIM 2. *If $B \subseteq A$ is admissible, then the restriction $\pi_B|G : G \rightarrow \pi_B(G)$ is 0-soft.*

Let $\varphi : Z \rightarrow \pi_B(G)$ and $\varphi_0 : Z_0 \rightarrow G$ be defined on a realcompact space Z with $\dim Z = 0$ and its closed subset Z_0 respectively. Assume that $\pi_B\varphi_0 = \varphi|Z_0$ and $C(\varphi_0)(C(G)) \subseteq C(Z)|Z_0$. We wish to construct a map

$\phi : Z \rightarrow G$ such that $\phi|_{Z_0} = \varphi_0$ and $\pi_B \phi = \varphi$, i.e. ϕ makes the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\varphi_0} & G \\ \text{incl} \downarrow & \nearrow \phi & \downarrow \pi_B|_G \\ Z_0 & \xrightarrow{\varphi} & \pi_B(G) \end{array}$$

commutative. Since, according to our choice, all X_a 's are $\text{AE}(0)$ -spaces (recall that each X_a is a Polish space), so is the product $\prod\{X_a : a \in A - B\}$. This implies the 0-softness of the projection π_B and hence of its restriction

$$\pi_B|_{\pi_B^{-1}(\pi_B(G))} : \pi_B^{-1}(\pi_B(G)) \rightarrow \pi_B(G).$$

Then there exists a map $\phi'' : Z \rightarrow \pi_B^{-1}(\pi_B(G))$ such that $\phi''|_{Z_0} = \varphi_0$ and $\pi_B \phi'' = \varphi$. Since f is 0-invertible (and $\dim Z = 0$), there exists a map $\phi' : Z \rightarrow Y$ such that $f\phi' = \phi''$. Now let $\phi = g\phi'$. Since $g|_{f^{-1}(G)} = f|_{f^{-1}(G)}$, we have $\varphi_0 = \phi|_{Z_0}$. Finally, the admissibility of B implies $\varphi = \pi_B \phi$ as required.

CLAIM 3. *For each countable subset $C \subseteq A$ there exists a countable admissible subset $B \subseteq A$ such that $C \subseteq B$.*

Since $w(Y) = \tau$ and $\dim Y = 0$, it follows (consult [4, Theorem 1.3.10]) that Y can be represented as the limit space of a factorizing ω -spectrum $\mathcal{S}_Y = \{Y_B, q_C^B, \exp_\omega A\}$ consisting of zero-dimensional Polish spaces Y_B , $B \in \exp_\omega A$, and continuous surjections $q_C^B : Y_B \rightarrow Y_C$, $C \subseteq B$, $C, B \in \exp_\omega A$. Consider also the standard factorizing ω -spectrum $\mathcal{S}_X = \{\prod\{X_a : a \in B\}, \pi_C^B, \exp_\omega A\}$ consisting of the countable subproducts of the product $\prod\{X_a : a \in A\}$ and the corresponding natural projections. Obviously the full product coincides with the limit of \mathcal{S}_X . One more factorizing ω -spectrum arises naturally. This is the spectrum $\mathcal{S}_G = \{\pi_B(G), \pi_C^B|_{\pi_B(G)}, \exp_\omega A\}$ whose limit coincides with G .

Consider the map $f : \lim \mathcal{S}_Y \rightarrow \lim \mathcal{S}_X$. By [4, Theorem 1.3.4], there is a cofinal ω -closed subset \mathcal{T}_f of $\exp_\omega A$ such that for each $B \in \mathcal{T}_f$ there is a map $f_B : Y_B \rightarrow \prod\{X_a : a \in B\}$ such that $f_B \circ q_B = \pi_B \circ f$. Moreover, these maps form a morphism

$$\{f_B; B \in \mathcal{T}_f\} : \mathcal{S}_Y \rightarrow \mathcal{S}_X$$

whose limit coincides with f . Since f is proper and functionally closed, we may assume (see [4, Proposition 6.2.9]), considering a smaller cofinal ω -closed subset of \mathcal{T}_f if necessary, that the above morphism is bicommutative. This simply means that $q_B f^{-1}(K) = f_B^{-1}(\pi_B(K))$ for any $B \in \mathcal{T}_f$ and any closed subset K of the product $\prod\{X_a : a \in A\}$.

Similarly, applying [4, Theorem 1.3.4] to the map $g : \lim \mathcal{S}_Y \rightarrow \lim \mathcal{S}_G$, we obtain a cofinal ω -closed subset \mathcal{T}_g of $\exp_\omega A$ and the associated morphism

$$\{g_B : Y_B \rightarrow \pi_B(G); B \in \mathcal{T}_g\} : \mathcal{S}_Y \rightarrow \mathcal{S}_G$$

whose limit coincides with g .

By Proposition 2.4, the intersection $\mathcal{T} = \mathcal{T}_f \cap \mathcal{T}_g$ is still a cofinal ω -closed subset of $\exp_\omega A$. Hence it suffices to show that each $B \in \mathcal{T}$ is an admissible subset of A . Let $x \in \pi_B^{-1}(\pi_B(G))$. First observe that the bicommutativity of the morphism associated with \mathcal{T}_f implies that $q_B(f^{-1}(x)) = f_B^{-1}(\pi_B(x))$. Since f_B and g_B coincide on $f_B^{-1}(\pi_B(G))$ we have

$$\begin{aligned} \pi_B(g(f^{-1}(x))) &= g_B(q_B(f^{-1}(x))) = g_B(f_B^{-1}(\pi_B(x))) \\ &= f_B(f_B^{-1}(\pi_B(x))) = \pi_B(x) \end{aligned}$$

as required.

CLAIM 4. *If C and B are admissible subsets of A and $C \subseteq B$, then the map $\pi_C^B | \pi_B(G) : \pi_B(G) \rightarrow \pi_C(G)$ is 0-soft.*

This property follows from Claim 2 and [4, Lemma 6.1.15].

Having all the needed properties of admissible subsets established, we proceed as follows. Since $|A| = \tau$ we can write $A = \{a_\alpha : \alpha < \tau\}$. By Claim 3, each $a_\alpha \in A$ is contained in a countable admissible subset $B_\alpha \subseteq A$. Let $A_\alpha = \bigcup \{B_\beta : \beta \leq \alpha\}$. We use these sets to define a transfinite inverse spectrum $\mathcal{S} = \{G_\alpha, p_\alpha^{\alpha+1}, \tau\}$ as follows. Let $G_\alpha = \pi_{A_\alpha}(G)$ and $p_\alpha^{\alpha+1} = \pi_{A_\alpha}^{A_{\alpha+1}} | G_{\alpha+1}$ for each $\alpha < \tau$. All the required properties of the spectrum \mathcal{S}_G are satisfied by construction.

Finally, (b) \Rightarrow (a) immediately follows from [4, Proposition 6.3.4]. ■

REMARK 3.5. Actually the 0-soft homomorphism $p_\alpha^{\alpha+1} : G_{\alpha+1} \rightarrow G_\alpha$, $\alpha < \tau$, in Theorem 3.4(b)(1) has Polish kernel in a somewhat stronger sense than in the original definition of Subsection 2.1. Namely the Polish space P (from the definition), such that $G_{\alpha+1}$ admits a C -embedding into the product $G_\alpha \times P$ in such a way that $p_\alpha^{\alpha+1}$ coincides with the restriction of the projection $\pi_{G_\alpha} : G_\alpha \times P \rightarrow G_\alpha$, can be chosen to be a Polish group and the embedding of $G_{\alpha+1} \rightarrow G_\alpha \times P$ can be assumed to be a homomorphism. Of course this implies that $\ker p_\alpha^{\alpha+1}$, as a closed subgroup of P , is itself a Polish group. It would be interesting to see whether the converse of this observation is also true, i.e. if it is true that if the kernel $\ker p$ of a 0-soft homomorphism $p : G \rightarrow L$ of AE(0)-groups is Polish, then p has a Polish kernel in the sense of Subsection 2.1.

Next we characterize 0-soft homomorphisms of AE(0)-groups with Polish kernels.

PROPOSITION 3.6. *A 0-soft homomorphism $f : G \rightarrow L$ between $\text{AE}(0)$ -groups has a Polish kernel if and only if there exist factorizing ω -spectra $\mathcal{S}_G = \{G_\alpha, p_\alpha^\beta, A\}$, $\mathcal{S}_L = \{L_\alpha, q_\alpha^\beta, A\}$ consisting of Polish groups and 0-soft limit homomorphisms, and a morphism $\{f_\alpha\} : \mathcal{S}_G \rightarrow \mathcal{S}_L$ consisting of 0-soft homomorphisms, such that the following conditions are satisfied:*

(1) $G = \lim \mathcal{S}_G$, $L = \lim \mathcal{S}_L$ and $f = \lim\{f_\alpha\}$.

(2) *All limit square diagrams generated by limit projections of the spectra \mathcal{S}_G and \mathcal{S}_L , by elements of the morphism $\{f_\alpha\}$ and by the map f are pullback squares.*

Proof. By Proposition 3.1, we may assume that both G and L are topologically and algebraically isomorphic to the limits of factorizing ω -spectra $\mathcal{S}_G = \{G_\alpha, p_\alpha^\beta, A\}$ and $\mathcal{S}_L = \{L_\alpha, q_\alpha^\beta, A\}$, consisting of Polish groups and 0-soft limit homomorphisms. By [4, Theorem 1.3.6], we may also assume that the map f is the limit of a morphism $\{f_\alpha : G_\alpha \rightarrow L_\alpha; A\}$, consisting of continuous maps $f_\alpha : G_\alpha \rightarrow L_\alpha$. Since f itself and all the limit projections $p_\alpha : G \rightarrow G_\alpha$ and $q_\alpha : L \rightarrow L_\alpha$ are group homomorphisms, it follows easily that so is $f_\alpha : G_\alpha \rightarrow L_\alpha$, $\alpha \in A$. Finally since f is 0-soft and has a Polish kernel, it follows, by [4, Theorem 6.3.1(vi)], that all limit square diagrams

$$\begin{array}{ccc} G & \xrightarrow{f} & L \\ p_\alpha \downarrow & & \downarrow q_\alpha \\ G_\alpha & \xrightarrow{f_\alpha} & L_\alpha \end{array}$$

generated by the limit projections of the spectra \mathcal{S}_G and \mathcal{S}_L , by elements of the morphism $\{f_\alpha\}$ and by the map f , are pullback squares. ■

In Subsection 4.1, we consider actions of $\text{AE}(0)$ -groups on $\text{AE}(0)$ -spaces. The main tool is the following statement (see [4, Theorem 8.7.1]).

PROPOSITION 3.7. *Let $\lambda : G \times X \rightarrow X$ be a continuous action of an $\text{AE}(0)$ -group G on an $\text{AE}(0)$ -space X . Suppose that X is homeomorphic to the limit space of a factorizing ω -spectrum $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ consisting of Polish spaces and 0-soft limit projections. Suppose also that G is topologically and algebraically isomorphic to the limit of a factorizing ω -spectrum $\mathcal{S}_G = \{G_\alpha, s_\alpha^\beta, A\}$ consisting of Polish groups and 0-soft limit homomorphisms. Then λ is the limit of “level actions”, i.e. $\lambda = \lim \lambda_\alpha$, where*

$$\{\lambda_\alpha : G_\alpha \times X_\alpha \rightarrow X_\alpha, B\} : \mathcal{S}_G|B \times \mathcal{S}_X|B \rightarrow \mathcal{S}_X|B$$

is a morphism between the spectra $\mathcal{S}_G|B \times \mathcal{S}_X|B$ and $\mathcal{S}_X|B$, and B is a cofinal ω -closed subset of the indexing set A .

4. Applications

4.1. Universal $\text{AE}(0)$ -groups and universal actions of $\text{AE}(0)$ -groups. In this subsection we prove the existence of universal $\text{AE}(0)$ -groups of a given weight as well as the existence of a universal action of an $\text{AE}(0)$ -group of a given weight on a compact $\text{AE}(0)$ -space of the same weight.

PROPOSITION 4.1. *Let $\tau \geq \omega$. The class of $\text{AE}(0)$ -groups of weight $\leq \tau$ contains a universal element. More formally, every $\text{AE}(0)$ -group is topologically and algebraically isomorphic to a closed C -embedded subgroup of the power $(\text{Aut}(\mathbb{Q}))^\tau$, where $\text{Aut}(\mathbb{Q})$ denotes the group of autohomeomorphisms of the Hilbert cube \mathbb{Q} .*

Proof. Let G be an $\text{AE}(0)$ -group of weight τ . By Corollary 3.3, G is topologically and algebraically isomorphic to a closed C -embedded subgroup of the product $\prod\{G_t : t \in T\}$, where G_t is a Polish group for each $t \in T$ and $|T| = \tau$. By Uspenskii's theorem [15], G_t can be identified with a closed subgroup of $\text{Aut}(\mathbb{Q})$. Obviously $\prod\{G_t : t \in T\}$, and consequently G , is a closed C -embedded subgroup of $(\text{Aut}(\mathbb{Q}))^\tau$. ■

Let $\tau > \omega$. Clearly the $\text{AE}(0)$ -group $(\text{Aut}(\mathbb{Q}))^\tau$ (i.e. the τ th power of the group $\text{Aut}(\mathbb{Q})$) acts continuously on the Tikhonov cube \mathbb{Q}^τ via the natural action ("coordinatewise evaluation")

$$\text{ev}_\tau : (\text{Aut}(\mathbb{Q}))^\tau \times \mathbb{Q}^\tau \rightarrow \mathbb{Q}^\tau,$$

defined by letting

$$\text{ev}_\tau(\{g_\alpha : \alpha < \tau\}, \{q_\alpha : \alpha < \tau\}) = \{g_\alpha(q_\alpha) : \alpha < \tau\}$$

for each $(\{g_\alpha : \alpha < \tau\}, \{q_\alpha : \alpha < \tau\}) \in (\text{Aut}(\mathbb{Q}))^\tau \times \mathbb{Q}^\tau$.

THEOREM 4.2. *Let $\tau > \omega$. The action $\text{ev}_\tau : (\text{Aut}(\mathbb{Q}))^\tau \times \mathbb{Q}^\tau \rightarrow \mathbb{Q}^\tau$ is universal in the category of actions of $\text{AE}(0)$ -groups of weight τ on compact $\text{AE}(0)$ -spaces of weight τ . More formally, let $\lambda : G \times X \rightarrow X$ be a continuous action of an $\text{AE}(0)$ -group G of weight τ on a compact $\text{AE}(0)$ -space X of weight τ . Then there exists a topological and algebraic embedding $i_G : G \rightarrow (\text{Aut}(\mathbb{Q}))^\tau$ with closed image and an embedding $i_X : X \rightarrow \mathbb{Q}^\tau$ such that the following diagram commutes:*

$$\begin{array}{ccc} (\text{Aut}(\mathbb{Q}))^\tau \times \mathbb{Q}^\tau & \xrightarrow{\text{ev}_\tau} & \mathbb{Q}^\tau \\ \uparrow i_G \times i_X & & \uparrow i_X \\ G \times X & \xrightarrow{\lambda} & X \end{array}$$

Proof. By Proposition 3.1, G can be represented as the limit of a factorizing ω -spectrum $\mathcal{S}_G = \{G_\alpha, s_\alpha^\beta, A\}$ consisting of Polish groups and 0-soft limit homomorphisms. Similarly, by [4, Proposition 6.3.5], X can be represented as the limit space of a factorizing ω -spectrum $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ consisting

of metrizable compacta and 0-soft limit projections. Without loss of generality we may assume that these spectra \mathcal{S}_G and \mathcal{S}_X have the same indexing set A and $|A| = \tau$. By Proposition 3.7, the given action $\lambda : G \times X \rightarrow X$ is the limit of level actions, i.e. $\lambda = \lim \lambda_\alpha$, where

$$\{\lambda_\alpha : G_\alpha \times X_\alpha \rightarrow X_\alpha, B\} : \mathcal{S}_G|B \times \mathcal{S}_X|B \rightarrow \mathcal{S}_X|B$$

is a morphism between the spectra $\mathcal{S}_G|B \times \mathcal{S}_X|B$ and $\mathcal{S}_X|B$, and B is a cofinal ω -closed subset of the indexing set A . We may also assume that $|B| = \tau$.

Since $G = \lim \mathcal{S}_G|B$ it follows that the diagonal product

$$s = \Delta\{s_\alpha : G \rightarrow G_\alpha, \alpha \in B\} : G \rightarrow \prod\{G_\alpha : \alpha \in B\}$$

is a topological and algebraic isomorphism with a closed image. Similarly the diagonal product

$$p = \Delta\{p_\alpha : X \rightarrow X_\alpha, \alpha \in B\} : X \rightarrow \prod\{X_\alpha : \alpha \in B\}$$

is an embedding. Consider also the product action

$$\tilde{\lambda} : \prod\{G_\alpha : \alpha \in B\} \times \prod\{X_\alpha : \alpha \in B\} \rightarrow \prod\{X_\alpha : \alpha \in B\}$$

defined by letting

$$\tilde{\lambda}(\{g_\alpha : \alpha \in B\}, \{x_\alpha : \alpha \in B\}) = \{\lambda_\alpha(g_\alpha, x_\alpha) : \alpha \in B\}$$

for each $(\{g_\alpha : \alpha \in B\}, \{x_\alpha : \alpha \in B\}) \in \prod\{G_\alpha : \alpha \in B\} \times \prod\{X_\alpha : \alpha \in B\}$. Note that $\tilde{\lambda} \circ (s \times p) = p \circ \lambda$, i.e. the following diagram is commutative:

$$\begin{array}{ccc} \prod\{G_\alpha : \alpha \in B\} \times \prod\{X_\alpha : \alpha \in B\} & \xrightarrow{\tilde{\lambda}} & \prod\{X_\alpha : \alpha \in B\} \\ s \times p \uparrow & & \uparrow p \\ G \times X & \xrightarrow{\lambda} & X \end{array}$$

Since for each $\alpha \in B$ the group G_α is Polish, it follows by [12] (see also [1, Theorem 2.6.7]) that there exist a topological and algebraic embedding $j_\alpha : G_\alpha \rightarrow \text{Aut}(\mathbb{Q}_\alpha)$ with closed image and an embedding $i_\alpha : X_\alpha \rightarrow \mathbb{Q}_\alpha$ (here \mathbb{Q}_α denotes a copy of the Hilbert cube \mathbb{Q}) such that $\text{ev}_\alpha \circ (j_\alpha \times i_\alpha) = i_\alpha \circ \lambda_\alpha$, i.e. the diagram

$$\begin{array}{ccc} \text{Aut}(\mathbb{Q}_\alpha) \times \mathbb{Q}_\alpha & \xrightarrow{\text{ev}_\alpha} & \mathbb{Q}_\alpha \\ j_\alpha \times i_\alpha \uparrow & & \uparrow i_\alpha \\ G_\alpha \times X_\alpha & \xrightarrow{\lambda_\alpha} & X_\alpha \end{array}$$

commutes for each $\alpha \in B$. Here $\text{ev}_\alpha : \text{Aut}(\mathbb{Q}_\alpha) \times \mathbb{Q}_\alpha \rightarrow \mathbb{Q}_\alpha$ is the evaluation action, i.e. $\text{ev}_\alpha(g_\alpha, x_\alpha) = g_\alpha(x_\alpha)$ for each $(g_\alpha, x_\alpha) \in \text{Aut}(\mathbb{Q}_\alpha) \times \mathbb{Q}_\alpha$. Let

$$j = \times\{j_\alpha : \alpha \in B\} : \prod\{G_\alpha : \alpha \in B\} \rightarrow \prod\{\text{Aut}(\mathbb{Q}_\alpha) : \alpha \in B\}$$

and

$$i = \prod \{i_\alpha : \alpha \in B\} : \prod \{X_\alpha : \alpha \in B\} \rightarrow \prod \{Q_\alpha : \alpha \in B\}.$$

Finally consider the commutative diagram

$$\begin{array}{ccc}
 \prod \{\text{Aut}(\mathbb{Q}_\alpha) : \alpha \in B\} \times \prod \{\mathbb{Q}_\alpha : \alpha \in B\} & \xrightarrow{\times \{\text{ev}_\alpha : \alpha \in B\}} & \prod \{\mathbb{Q}_\alpha : \alpha \in B\} \\
 \uparrow j \times i & & \uparrow i \\
 \prod \{G_\alpha : \alpha \in B\} \times \prod \{X_\alpha : \alpha \in B\} & \xrightarrow{\tilde{\lambda}} & \prod \{X_\alpha : \alpha \in B\} \\
 \uparrow s \times p & & \uparrow p \\
 G \times X & \xrightarrow{\lambda} & X
 \end{array}$$

and note that since $|B| = \tau$ the upper horizontal arrow is actually the action $\text{ev}_\tau : (\text{Aut}(\mathbb{Q}))^\tau \times \mathbb{Q}^\tau \rightarrow \mathbb{Q}^\tau$. Clearly it suffices to let $i_G = (j \times i) \circ (s \times p)$ and $i_X = i \circ p$. This completes the proof. ■

It would be very interesting to prove that for an AE(0)-group G of weight $\tau > \omega$ the category of AE(0)-spaces (of weight τ) admitting actions of the group G and their G -maps contains a universal object. For $\tau = \omega$ this fact has recently been proved in [9].

4.2. Closed subgroups of powers of the symmetric group S_∞ . The following result gives an embeddability criterion into the symmetric group S_∞ , the group of all bijections of \mathbb{N} under the relative topology inherited from $\mathbb{N}^\mathbb{N}$. It is important to note [7] that there exist zero-dimensional Polish groups which cannot be embedded into S_∞ as closed subgroups.

THEOREM 4.3 ([1]). *Let G be a Polish group. Then the following conditions are equivalent:*

- (i) G is isomorphic to a closed subgroup of S_∞ ;
- (ii) G admits a (countable) neighborhood basis at the identity consisting of open subgroups;
- (iii) G admits a (countable) basis closed under left multiplication (or a countable basis closed under right multiplication);
- (iv) G admits a compatible left-invariant ultrametric.

If a topological group G has countable Suslin number and admits a neighborhood basis at the identity consisting of open subgroups, then G is isomorphic to a topological subgroup of S_∞^τ , where $\tau = w(G)$. To see this consider left actions of G on countable discrete spaces of the form G/H , where H is an open subgroup of G . If G is Raïkov-complete, then G is

closed in S_∞^τ . Since $\text{AE}(0)$ -groups are Raïkov-complete and have countable Suslin number, we obtain ⁽¹⁾ the following result.

THEOREM 4.4. *Let $\tau \geq 1$ be a cardinal number. The following conditions are equivalent for any topological $\text{AE}(0)$ -group G of weight $\tau \geq \omega$:*

- (i) G is isomorphic to a closed subgroup of S_∞^τ ;
- (ii) G admits a neighborhood basis at the identity consisting of open subgroups.

COROLLARY 4.5. *Let $\tau \geq 2$. The following conditions are equivalent for any Polish group G :*

- (i) G is topologically isomorphic to a closed subgroup of S_∞ ;
- (ii) G is topologically isomorphic to a closed subgroup of S_∞^τ .

COROLLARY 4.6. *There exists a zero-dimensional Polish group which is not topologically isomorphic to a closed subgroup of S_∞^τ for any cardinal number τ .*

Proof. It is known [7] that there exists a zero-dimensional Polish group G which is not topologically isomorphic to a closed subgroup of S_∞ . By Corollary 4.5, G cannot be topologically isomorphic to a closed subgroup of S_∞^τ for any cardinal $\tau \geq 2$. ■

4.3. Baire isomorphisms. The main result of this subsection (Theorem 4.8) allows us to reduce in many instances (descriptive) set-theoretical considerations of general $\text{AE}(0)$ -groups to those for Polish groups.

LEMMA 4.7. *Let $f : X \rightarrow Y$ be a 0-soft homomorphism between $\text{AE}(0)$ -groups. Then there exists a Baire isomorphism $h : Y \times \ker f \rightarrow X$ such that $f \circ h = \pi_Y$, where $\pi_Y : Y \times \ker f \rightarrow Y$ stands for the projection onto the first coordinate.*

Proof. By Proposition 2.7, there exists a Baire map $g : Y \rightarrow X$ such that $f \circ g = \text{id}_Y$. The required Baire isomorphism $h : Y \times \ker f \rightarrow X$ (not a homomorphism unless g is a homomorphism itself) can now be defined by letting

$$h(y, a) = g(y) \cdot a \quad \text{for each } (y, a) \in Y \times \ker f,$$

where \cdot denotes the multiplication operation in X . ■

THEOREM 4.8. *Every $\text{AE}(0)$ -group is Baire isomorphic to a product of Polish groups.*

Proof. Let X be an $\text{AE}(0)$ -group. If $w(X) = \omega$, then X itself is Polish and there is nothing to prove.

⁽¹⁾ The proof presented here has been suggested by the referee and is significantly shorter than the original one.

Let now $w(X) = \tau > \omega$. According to Theorem 3.4, X is topologically and algebraically isomorphic to the limit of a well ordered continuous spectrum $\mathcal{S}_X = \{X_\alpha, p_\alpha^{\alpha+1}, \tau\}$ such that X_0 is a Polish group and the 0-soft homomorphism $p_\alpha^{\alpha+1} : X_{\alpha+1} \rightarrow X_\alpha$ has a Polish kernel for each $\alpha < \tau$.

Our goal is to prove that X is Baire isomorphic to the product $X_0 \times \prod\{\ker p_\alpha^{\alpha+1} : \alpha < \tau\}$. We proceed by induction. By Lemma 4.7, there exists a Baire isomorphism $h_1 : X_0 \times \ker p_0^1 \rightarrow X_1$ such that $p_0^1 \circ h_1 = \pi_{X_0}$. Suppose that for each α , where $1 \leq \alpha < \beta < \tau$, we have already constructed a Baire isomorphism $h_\alpha : X_0 \times \prod\{\ker p_\delta^{\delta+1} : \delta < \alpha\} \rightarrow X_\alpha$ in such a way that

1. If $\alpha + 1 < \beta$, then the diagram

$$\begin{array}{ccc} X_0 \times \prod\{\ker p_\delta^{\delta+1} : \delta < \alpha + 1\} & \xrightarrow{h_{\alpha+1}} & X_{\alpha+1} \\ \text{id}_{X_0} \times \pi_\alpha^{\alpha+1} \downarrow & & \downarrow p_\alpha^{\alpha+1} \\ X_0 \times \prod\{\ker p_\delta^{\delta+1} : \delta < \alpha\} & \xrightarrow{h_\alpha} & X_\alpha \end{array}$$

commutes, where

$$\begin{aligned} \pi_\alpha^{\alpha+1} &: \prod\{\ker p_\delta^{\delta+1} : \delta < \alpha + 1\} \\ &= \prod\{\ker p_\delta^{\delta+1} : \delta < \alpha\} \times X_\alpha \rightarrow \prod\{\ker p_\delta^{\delta+1} : \delta < \alpha\} \end{aligned}$$

is the natural projection.

2. $h_\alpha = \lim\{h_\gamma : \gamma < \alpha\}$ whenever $\alpha < \beta$ is a limit ordinal.

We now construct a Baire isomorphism $h_\beta : X_0 \times \prod\{\ker p_\delta^{\delta+1} : \delta < \beta\} \rightarrow X_\beta$. If β is a limit ordinal, then we let $h_\beta = \lim\{h_\alpha : \alpha < \beta\}$. If $\beta = \alpha + 1$, then consider the commutative diagram

$$\begin{array}{ccccc} (X_0 \times \prod\{\ker p_\delta^{\delta+1} : \delta < \alpha\}) \times \ker p_\alpha^{\alpha+1} & \xrightarrow{h_\alpha \times \text{id}} & X_\alpha \times \ker p_\alpha^{\alpha+1} & \xrightarrow{h} & X_{\alpha+1} \\ \pi_1 \downarrow & & \pi_{X_\alpha} \downarrow & & \downarrow p_\alpha^{\alpha+1} \\ X_0 \times \prod\{\ker p_\delta^{\delta+1} : \delta < \alpha\} & \xrightarrow{h_\alpha} & X_\alpha & \xrightarrow{\text{id}_{X_\alpha}} & X_\alpha, \end{array}$$

where

- (a) $\text{id} : \ker p_\alpha^{\alpha+1} \rightarrow \ker p_\alpha^{\alpha+1}$ stands for the identity map;
- (b)

$$\pi_1 : \left(X_0 \times \prod\{\ker p_\delta^{\delta+1} : \delta < \alpha\} \right) \times \ker p_\alpha^{\alpha+1} \rightarrow X_0 \times \prod\{\ker p_\delta^{\delta+1} : \delta < \alpha\}$$

denotes the projection onto the first coordinate and

- (c) $h : X_\alpha \times \ker p_\alpha^{\alpha+1} \rightarrow X_{\alpha+1}$ is a Baire isomorphism whose existence is guaranteed by Lemma 4.7.

The required Baire isomorphism $h_{\alpha+1} : X_0 \times \prod\{\ker p_\delta^{\delta+1} : \delta < \alpha + 1\} \rightarrow X_{\alpha+1}$ can now be defined as the composition $h_{\alpha+1} = h \circ (h_\alpha \times \text{id})$. This

completes the induction and finishes the construction of Baire isomorphisms h_α , $\alpha < \tau$. It is now easy to see that

$$h = \lim\{h_\alpha : \alpha < \tau\} : X_0 \times \prod\{\ker p_\alpha^{\alpha+1} : \alpha < \tau\} \rightarrow X$$

is the required Baire isomorphism. Note here that X_0 as well as $\ker p_\alpha^{\alpha+1}$, $\alpha < \tau$, are Polish groups. The proof is complete. ■

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