FUNDAMENTA
MATHEMATICAE
219 (2012)

A simultaneous selection theorem

by

Alexander D. Arvanitakis (Athens)

Abstract. We prove a theorem that generalizes in a way both Michael's Selection Theorem and Dugundji's Simultaneous Extension Theorem. We use it to prove that if K is an uncountable compact metric space and X a Banach space, then $C(K,X)$ is isomorphic to $C(\mathcal{C},X)$ where $\mathcal{C}$ denotes the Cantor set. For $X=\mathbb{R}$, this gives the well known Milyutin Theorem.

1. Introduction. We recall here two well-known and classical theorems. The first one is due to E. Michael [10], and it is known as *Michael's Selection Theorem*:

MICHAEL'S SELECTION THEOREM. *Let X be a paracompact space, Y a Banach space and $\Phi : X \to 2^Y$ a set-valued lower semicontinuous map with non-empty values and such that for every $x \in X$, $\Phi(x)$ is a closed convex subset of Y. Then there exists a continuous selection $F : X \to Y$ for Φ, i.e. F is continuous and $F(x) \in \Phi(x)$ for all $x \in X$.*

Recall that *lower semicontinuous* means that for any open subset U of Y, the set $\{x \in X : \Phi(x) \cap U \neq \emptyset\}$ is open in X.

The second theorem is due to J. Dugundji [5], and it is known as *Simultaneous Extension Theorem*:

DUGUNDJI'S SIMULTANEOUS EXTENSION THEOREM. *Let X be a metric space, $A \subset X$ a closed subset and E a locally convex linear topological space. Then there exists a linear operator $S : C(A,E) \to C(X,E)$ such that for any $f \in C(A,E)$, $S(f)$ is an extension of f. Furthermore S is continuous with respect to the topologies of uniform convergence and of uniform convergence on compact subsets.*

We will show that these two theorems are roughly speaking special cases of a more general one. Let us first recall the following definition (see [6]):

2010 *Mathematics Subject Classification*: Primary 54C65, 54C20; Secondary 46B03.
Key words and phrases: selection theorem, extension theorem.

DOI: 10.4064/fm219-1-1

We say that a topological space X is a *k-space* if it is a Hausdorff continuous image of a locally compact space. It is known ([6]) that if X is a k-space, then for every topological space Y and $\varphi : X \to Y$, φ is continuous if and only if it is continuous on every compact subspace of X.

For example, every metric space, every compact space, or more generally every paracompact and locally compact space is a k-space.

Using this definition, we prove the following:

THEOREM 1.1. *Let X be a paracompact k-space, Y a complete metric space, E a locally convex complete linear space and $\Phi : X \to 2^Y \setminus \{\emptyset\}$ a lower semicontinuous set-valued map. Then there exists a continuous linear operator $S : C(Y,E) \to C(X,E)$ such that*

$$S(f)(x) \in \overline{\operatorname{conv}}\, f(\Phi(x)).$$

Furthermore, S is continuous with respect to the topologies of uniform convergence on compact subsets.

Here $\overline{\operatorname{conv}}$ denotes the closed convex hull.

It is easy to see that any topological vector space E is a uniform space ([6]), the neighborhoods of the diagonal of $E \times E$ being the sets $\mathcal{D}_V = \{(e_1, e_2) \in E \times E : e_1 - e_2 \in V\}$, where V ranges over the open symmetric neighborhoods of 0 in E. We call E *complete* if it is complete as a uniform space.

Let us now indicate how we can obtain Michael's Selection Theorem and Dugundji's Simultaneous Extension Theorem as special cases of Theorem 1.1.

For the first one, assume that X is a paracompact k-space, Y is a Banach space and $\Phi : X \to 2^Y \setminus \{\emptyset\}$ is a lower semicontinuous set-valued map such that $\Phi(x)$ is closed and convex for any $x \in X$. Set $E = Y$. Then Theorem 1.1 ensures the existence of an operator $S : C(Y,Y) \to C(X,Y)$ such that $S(f)(x) \in \overline{\operatorname{conv}}\, f(\Phi(x))$ for any $f \in C(Y,Y)$ and $x \in X$. Take now f to be the identity, id, from Y to Y. Thus $S(\mathrm{id})(x) \in \overline{\operatorname{conv}}\, \Phi(x) = \Phi(x)$ since $\Phi(x)$ is convex and closed. Therefore $S(\mathrm{id})$ is a continuous selection for Φ.

For the second theorem, let X be a metric space (therefore a paracompact k-space), A a closed subset of X, and E a locally convex complete linear space. Define a set-valued map $\Phi : X \to 2^E$ by

$$\Phi(x) = \begin{cases} \{x\} & \text{if } x \in A, \\ A & \text{if } x \in X \setminus A. \end{cases}$$

It is easy to check that Φ is lower semicontinuous. Therefore there exists a continuous linear map $S : C(A,E) \to C(X,E)$ such that for any $f \in C(A,E)$ and $x \in X$, $S(f)(x) \in \overline{\operatorname{conv}}\, f(\Phi(x))$. For $x \in A$, we get

$$S(f)(x) \in \overline{\operatorname{conv}}\, f(\{x\}) = \{f(x)\}.$$

Thus $S(f)$ is an extension of f.

The proof of Theorem 1.1 is different from the proofs of the two above mentioned theorems and it relies on ideas from the study of regular averaging and regular extension operators and their applications to the classification of spaces of continuous functions, as initiated by Bade's students, S. Ditor and A. Etcheberry in [2, 7], and further developed by S. Ditor and R. Haydon [3, 4, 8, 9] and more recently in [1]. Our proof is elementary and uses no results from that theory. For a further study of regular extension and regular averaging operators we refer to [13, 14, 15].

In Section 3, we give an application of Theorem 1.1 by proving the following generalization of Milyutin's Theorem [11, 12]:

THEOREM 1.2. *Let K be an uncountable compact metric space and X a Banach space. Then $C(K,X)$ is isomorphic to $C(\mathcal{C},X)$ where $\mathcal{C}$ denotes the Cantor set with its product topology.*

Recently, V. Valov [16] obtained an interesting generalization of Theorem 1.1 using a different proof, and T. Yamauchi [17], basing on this proof, dropped the assumption of X being a k-space and also gave some applications of the theorem.

2. The proof of Theorem 1.1. If $(T,\prec)$ is a tree, we denote by $[T]$ the infinite branches of T. Then $[T]$ is naturally topologized by the clopen sets $V_t=\{b\in[T]:t\in b\}$, where $t\in T$. Also we denote by $S(t)$ the immediate successors of t. We say that T is *finitely branching* if $S(t)$ is finite for all $t\in T$.

We begin with the following lemma:

LEMMA 2.1. *Assume that T is a finitely branching rooted tree and that to every $t\in T$ a number $\lambda_t\geq 0$ has been assigned such that $\lambda_{r(T)}=1$, where $r(T)$ is the root of T, and $\lambda_t=\sum\{\lambda_s:s\in S(t)\}$ for every $t\in T$. Assume moreover that E is a locally convex complete linear topological space. Then there is a unique linear function $u:C([T],E)\to E$ such that for every $t\in T$ and $e\in E$, if $\chi(V_t,e):[T]\to E$ is the function*

$$\chi(V_t,e)(b)=\begin{cases} e & \text{if } b\in V_t,\\ 0 & \text{otherwise,}\end{cases}$$

then $u(\chi(V_t,e))=\lambda_t e$. Moreover u is continuous with respect to the uniform topology of $C([T],E)$.

Proof. A function $f:[T]\to E$ is called *simple* if for every $b\in[T]$, there is a basic neighborhood V_t of b such that f restricted to V_t is constant. Let $C_{\mathrm{s}}([T],E)$ denote the set of simple functions from $[T]$ to E. Then $C_{\mathrm{s}}([T],E)$ is clearly a linear subspace of $C([T],E)$ and every $f\in C_{\mathrm{s}}([T],E)$ can be (non-uniquely) written in the form

$$f = \sum_{i=1}^{n} \mu_i \chi(V_{t_i}, e_i)$$

where $\mu_i \in \mathbb{R}$, $t_i \in T$, $e_i \in E$ and $V_{t_i}, i = 1, \dots, n$, is a partition of $[T]$. We call such a form a *normal* form for f.

We define $u' : C_{\mathrm{s}}([T], E) \to E$ by

$$u'(f) = u'\Big(\sum_{i=1}^{n} \mu_i \chi(V_{t_i}, e_i)\Big) = \sum_{i=1}^{n} \mu_i \lambda_{t_i} e_i.$$

We have to show that u' is independent of the normal form used. So let $f = \sum_{i=1}^{n} \mu_i \chi(V_{t_i}, e_i) = \sum_{j=1}^{m} \nu_j \chi(V_{s_j}, r_j)$ be two normal forms for f. Set $V_{t_{ij}} = V_{t_i} \cap V_{s_j}$. Then whenever $V_{t_{ij}} \neq \emptyset$ we have $\mu_i e_i = \nu_j r_j$. Moreover using our assumption that $\lambda_t = \sum\{\lambda_s : s \in S(t)\}$ and a simple induction, one can easily prove that $\sum_{j=1}^{m} \lambda_{t_{ij}} = \lambda_{t_i}$ whenever $V_{t_i} = \bigcup_{j=1}^{m} V_{t_{ij}}$ is a partition of V_{t_i}. Thus

$$\begin{aligned} u'\Big(\sum_{i=1}^{n} \mu_i \chi(V_{t_i}, e_i)\Big) &= \sum_{i=1}^{n} \mu_i \lambda_{t_i} e_i = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{t_{ij}} \mu_i e_i \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{t_{ij}} \nu_j r_j = u'\Big(\sum_{j=1}^{m} \nu_j \chi(V_{s_j} r_j)\Big). \end{aligned}$$

Next we prove that u' is linear. So let $f = \sum_{i=1}^{n} \mu_i \chi(V_{t_i}, e_i)$ and $g = \sum_{j=1}^{m} \nu_j \chi(V_{s_j}, r_j)$ be two functions in normal form in $C_{\mathrm{s}}([T], E)$, and let $\mu, \nu \in \mathbb{R}$. Setting $V_{t_{ij}} = V_{t_i} \cap V_{s_j}$ we see that

$$\mu f + \nu g = \sum_{i=1}^{n} \sum_{j=1}^{m} \chi(V_{t_{ij}}, \mu \mu_i e_i + \nu \nu_j r_j)$$

is a normal form for $\mu f + \nu g$. Therefore

$$\begin{aligned} u'(\mu f + \nu g) &= \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{t_{ij}} (\mu \mu_i e_i + \nu \nu_j r_j) \\ &= \sum_{i=1}^{n} \Big(\sum_{j=1}^{m} \lambda_{t_{ij}}\Big) \mu \mu_i e_i + \sum_{j=1}^{m} \Big(\sum_{i=1}^{n} \lambda_{t_{ij}}\Big) \nu \nu_j r_j \\ &= \mu \sum_{i=1}^{n} \lambda_{t_i} \mu_i e_i + \nu \sum_{j=1}^{m} \lambda_{t_j} \nu_j r_j = \mu u'(f) + \nu u'(g), \end{aligned}$$

since $V_{t_{ij}}$, $j = 1, \dots, m$, is a partition of V_{t_i}, and $V_{t_{ij}}$, $i = 1, \dots, n$, is a partition of V_{s_j}.

Clearly $u'(\chi(V_t, e)) = \lambda_t e$ by definition. So we have to prove that we can extend u' to a continuous $u : C([T], E) \to E$. Here continuity is meant with respect to the uniform topology of $C([T], E)$. In this topology, a neighborhood of 0 in $C([T], E)$ is of the form

$$U([T], E) = \{f \in C([T], E) : f(b) \in U \text{ for all } b \in [T]\}$$

where U is a neighborhood of 0 in E. Notice first that $C_{\rm s}([T], E)$ is dense in $C([T], E)$ in this topology. Indeed, let $f \in C([T], E)$ and U a neighborhood of 0 in E. Since f is continuous, for every $b \in [T]$ there exists a clopen neighborhood V_t of b such that $f(V_t) \subset f(b) + U$. Since T is finitely branching, $[T]$ is compact, so let $V_{t_1}, \ldots, V_{t_n}$ be a covering of $[T]$ such that $f(V_{t_i}) \subset f(b_i) + U$, $1 \le i \le n$. By dropping the requirement that $b_i \in V_{t_i}$, we may also assume that $V_{t_1}, \ldots, V_{t_n}$ are pairwise disjoint. So setting $g = \sum_{i=1}^n \chi(V_{t_i}, f(b_i)) \in C_{\rm s}([T], E)$, we easily see that $f(b) - g(b) \in U$ for all $b \in [T]$, and therefore $f \in g + U([T], E)$.

Next notice that u' is continuous, since if U is a convex neighborhood of 0 in E and $f = \sum_{i=1}^n \mu_i \chi(V_{t_i}, e_i) \in C_{\rm s}([T], E) \cap U([T], E)$ is in normal form, then $\mu_i e_i \in U$ by the definition of $U([T], E)$. Therefore also $\sum_{i=1}^n \lambda_{t_i} \mu_i e_i = u'(f) \in U$ since this is a convex combination of $\mu_i e_i$, $1 \le i \le n$.

We may now define $u : C([T], E) \to E$ as follows: For $f \in C([T], E)$ there is a net $(f_\gamma)_{\gamma \in \Gamma}$ in $C_{\rm s}([T], E)$ converging to f. Then it is easy to check that $(u'(f_\gamma))_{\gamma \in \Gamma}$ is a Cauchy net in E, so we can define $u(f)$ to be $\lim_{\gamma \in \Gamma} u'(f_\gamma)$. It is also easy to check that this definition is independent of the choice of the net $(f_\gamma)_{\gamma \in \Gamma}$ and that u is indeed linear and continuous. ∎

The next proposition is Theorem 1.1 for the case $Y = [T]$, where T is a tree of height ω (not necessarily finitely branching). Here *height* ω means that the set of all predecessors of each $t \in T$ is finite. As in Lemma 2.1, we denote by V_t the basic clopen neighborhoods of $[T]$, i.e. $V_t = \{b \in [T] : t \in b\}$.

PROPOSITION 2.2. *Let X be a paracompact k-space, T a tree of height ω, E a complete locally convex linear space and $\Phi : X \to 2^{[T]} \setminus \{\emptyset\}$ a lower semicontinuous set-valued function. Then there is a continuous linear operator $S : C([T], E) \to C(X, E)$ such that for any $f \in C([T], E)$ and $x \in X$,*

$$S(f)(x) \in \overline{\rm conv}\, f(\Phi(x)).$$

Furthermore, S is continuous with respect to the topologies of uniform convergence on compact subsets.

The proof is in several steps. The main argument is that using Lemma 2.1 we associate to every $x \in X$ a continuous linear function $\nu_x : C([T], E) \to E$. Having done this in an appropriate manner, we may define $S(f)(x) = \nu_x(f)$ and then prove that due to some properties of the function $x \mapsto \nu_x$, we have the required properties for S.

Proof of Proposition 2.2. We first define $U_t = \{x \in X : \Phi(x) \cap V_t \neq \emptyset\}$ for $t \in T$. Since Φ is lower semicontinuous, U_t is open for any $t \in T$. Roughly speaking, U_t is the set of all $x \in X$ such that ν_x may take account of all functions of the form $\chi(V_t, r)$ in $C([T], E)$, i.e. $\nu_x(\chi(V_t, r))$ may be not zero. Since Φ has non-empty values and $V_t = \bigcup\{V_s : s \in S(t)\}$ for every $t \in T$, we have the following properties for the family U_t, $t \in T$:

- $U_{r(T)} = X$. (Recall that $r(T)$ denotes the root of T.)
- $U_t = \bigcup\{U_s : s \in S(t)\}$ for every $t \in T$.

We use these properties of U_t to define, recursively on t, closed subsets F_t of X with the following properties:

(1) $F_{r(T)} = X$.
(2) $F_t = \bigcup\{F_s : s \in S(t)\} = \bigcup\{\mathrm{int}_{F_t}(F_s) : s \in S(t)\}$ for every $t \in T$. (Here $\mathrm{int}_{F_t}(F_s)$ is the relative interior of F_s in F_t.)
(3) The family $\{F_s : s \in S(t)\}$ is a neighborhood finite covering of F_t. (Recall (see for example [5]) that a *neighborhood finite covering* is a covering such that every x has a neighborhood that meets only finitely many members of the covering.)
(4) $F_t \subset U_t$ for all $t \in T$.

For $t = r(T)$, we set $F_{r(T)} = X$. Since $U_{r(T)} = X$, condition (4) is satisfied.

Assuming we have defined F_t for some $t \in T$, we define simultaneously F_s, $s \in S(t)$: First, since $F_t \subset U_t = \bigcup\{U_s : s \in S(t)\}$, we deduce that $F_t = \bigcup\{F_t \cap U_s : s \in S(t)\}$. Since F_t is itself a paracompact space as a closed subspace of X, we may easily find $F_s \subset F_t \cap U_s$, $s \in S(t)$, satisfying (2) and (3), and the induction step has been done.

Next, using (2) and the fact that F_t, $t \in T$, is paracompact, we define a partition of unity in F_t, $\{f_s : s \in S(t)\}$, subordinate to $\{\mathrm{int}_{F_t}(F_s) : s \in S(t)\}$. Every f_s is extended by 0 on $X \setminus F_s$ and the resulting map, which is not necessarily continuous, will also be called f_s. For $t \in T$ we now define a map $\lambda_t : X \to [0,1]$ by

$$\lambda_t = \prod\{f_s : s \preceq t\}.$$

We have the following:

Claim. *For every $t \in T$, λ_t is a continuous map.*

It suffices to prove inductively on t that

$$\overline{\{x \in X : \lambda(x) \neq 0\}} = \mathrm{supp}(\lambda_t) \subset \mathrm{int}_X(F_t), \tag{1}$$

since in this case $X = (X \setminus \mathrm{supp}(\lambda_t)) \cup \mathrm{int}_X(F_t)$ and λ_t is continuous by definition on both open sets: on $X \setminus \mathrm{supp}(\lambda_t)$ being constantly 0, and on $\mathrm{int}_X(F_t)$ being the product of continuous functions.

Relation (1) is obviously valid in the case where $t = r(T)$, since $\lambda_{r(T)} = f_{r(T)} = 1$ and $F_{r(T)} = X$. Assuming that it is valid for t, we observe that for $s \in S(t)$,

$$\{x \in X : \lambda_s(x) \neq 0\} = \{x \in X : \lambda_t(x) \neq 0\} \cap \{x \in X : f_s(x) \neq 0\},$$

and consequently

$$\operatorname{supp}(\lambda_s) \subset \operatorname{supp} \lambda_t \cap \operatorname{supp} f_s \subset \operatorname{int}_X(F_t) \cap \operatorname{int}_{F_t}(F_s) \subset \operatorname{int}_X(F_s).$$

For any $A \subset X$ we define the tree $T_A = \{t \in T : A \cap F_t \neq \emptyset\}$. Notice that if it is non-empty, T_A is rooted. (We also denote $T_{\{x\}}$ by T_x for simplicity.) To avoid confusion we continue to denote by $S(t)$ the immediate successors of t in T. Next we argue that if $A \neq \emptyset$ is a compact subspace of X, then T_A is finitely branching and rooted.

So, let $t \in T_A$. By definition $A \cap F_t$ is non-empty. Since $\{F_s : s \in S(t)\}$ is a neighborhood finite covering of F_t, for every $x \in A \cap F_t$ we can find an open $G_x \ni x$ such that $G_x \cap F_s$ is non-empty for finitely many $s \in S(t)$ only. Using now the compactness of $A \cap F_t$, let $G_{x_1}, \ldots, G_{x_n}$ be a covering of $A \cap F_t$. Then if $s \in S(t)$,

$$\begin{aligned} &s \text{ is an immediate successor of } t \text{ in } T_A \\ &\qquad \Leftrightarrow F_s \cap A \text{ is non-empty} \\ &\qquad \Leftrightarrow \text{for some } i = 1, \ldots, n,\ G_{x_i} \cap F_s \text{ is non-empty}, \end{aligned}$$

and for any particular $i = 1, \ldots, n$, there are only finitely many $s \in S(t)$ satisfying this last condition. Therefore indeed T_A is finitely branching, and obviously rooted since $A \neq \emptyset$.

Observe that since $\operatorname{supp}(\lambda_t) \subset F_t$ by (1), we see that $\lambda_t(x) = 0$ whenever $x \notin F_t$. Therefore if $x \in A$,

$$\begin{aligned} &\sum \{\lambda_s(x) : s \in S(t)\} \\ &\qquad = \sum \{\lambda_s(x) : s \text{ is an immediate successor of } t \text{ in } T_A\}. \end{aligned}$$

Furthermore, since f_s, $s \in S(t)$, has been chosen to be a partition of unity in F_t, we get $\sum \{\lambda_s(x) : s \in S(t)\} = \lambda_t(x)$. Combining these two equalities we find that if $x \in A$, then

$$\sum \{\lambda_s(x) : s \text{ is an immediate successor of } t \text{ in } T_A\} = \lambda_t(x). \tag{2}$$

Since moreover $r(T) = r(T_A)$ whenever $A \neq \emptyset$, we may use Lemma 2.1. So for every $x \in X$ and every $A \subset X$ containing x, there exists $\nu_x^A : C([T_A], E) \to E$ linear and continuous such that $\nu_x^A(\chi(V_t^A, r)) = \lambda_t(x) r$. Here V_t^A denotes the basic clopen neighborhood of $[T_A]$, $\{b \in [T_A] : t \in b\}$.

Observe that if $B \supset A \ni x$, both maps $C([T_B], E) \ni f \mapsto \nu_x^B(f)$ and $C([T_B], E) \ni f \mapsto \nu_x^A(f|[T_A])$ agree on simple functions, since $\nu_x^B(\chi(V_t^B, r)) = \lambda_t(x)r$ and also $\nu_x^A(\chi(V_t^B, r)|[T_A]) = \nu_x^A(\chi(V_t^A, r)) = \lambda_t(x)r$. Therefore, $\nu_x^B(f) = \nu_x^A(f|[T_A])$ for every $f \in C([T_B], E)$. For any $x \in X$, we define

$$\mu_x : C([T], E) \ni f \mapsto \nu_x^{\{x\}}(f|[T_x]) \in E.$$

It is easy to check that also $\mu_x(\chi(V_t, r)) = \lambda_t(x)r$ for any $t \in T$ and $r \in E$. Using μ_x, we define $S : C([T], E) \to C(X, E)$ by $S(f)(x) = \mu_x(f)$ and it suffices to check that S satisfies all conditions in the proposition.

Step 1. *For each $f \in C([T], E)$, $S(f) : X \to E$ is continuous.*

Since X is a k-space, it suffices to check the continuity on a given compact $A \subset X$. So let $(x_\delta)_{\delta \in \Delta}$ be a net in A converging to some point $x \in A$. We have to show that in E, $S(f)(x_\delta) \xrightarrow{\delta \in \Delta} S(f)(x)$ or equivalently that $\mu_{x_\delta}(f)$ converges to $\mu_x(f)$. Now, for every $x \in A$ we have

$$\mu_x(f) = \nu_x^{\{x\}}(f|[T_x]) = \nu_x^A(f|[T_A]).$$

Therefore it suffices to show that the net $(\nu_{x_\delta}^A(f|[T_A]))_{\delta \in \Delta}$ converges to $\nu_x^A(f|[T_A])$. So, let U be an open neighborhood of 0 in E, and find an open convex neighborhood G of 0 such that $\bar{G} + \bar{G} + \bar{G} \subset U$. Notice first that for any $h \in G([T_A], E)$ and $z \in A$ we have $\nu_z^A(h) \in \bar{G}$. This is so since if $h = \sum_{i=1}^n a_i \chi(V_{t_i}^A, r_i)$ is a simple function in a normal form in $G([T_A], E)$, then $a_i r_i \in G$ for every $i = 1, \ldots, n$, because these are values taken by h. Therefore $\nu_z^A(h) = \sum_{i=1}^n \lambda_{t_i}(z) a_i r_i \in G$ as a convex combination of $a_i r_i$.

In the general case, h is the limit point of some net $(h_\gamma)_{\gamma \in \Gamma}$ of simple functions in $G([T_A], E)$. Since ν_z^A is continuous, we have $\nu_z^A(h_\gamma) \xrightarrow{\gamma \in \Gamma} \nu_z^A(h)$, and therefore $\nu_z^A(h) \in \bar{G}$.

Next, since ν_x^A is continuous and simple maps are dense in $C([T_A], E)$, there is a simple map g such that both

$$f|[T_A] - g \in G([T_A], E) \quad \text{and} \quad \nu_x^A(f|[T_A] - g) \in G.$$

Let $g = \sum_{i=1}^n a_i \chi(V_{t_i}^A, r_i)$ be a normal form for g. Observe that

$$\nu_{x_\delta}^A(g) = \sum_{i=1}^n \lambda_{t_i}(x_\delta) \mu_i r_i \xrightarrow{\delta \in \Delta} \sum_{i=1}^n \lambda_{t_i}(x) \mu_i r_i = \nu_x^A(g),$$

as the maps $z \mapsto \lambda_{t_i}(z)$ are all continuous. So, there is a $\delta_0 \in \Delta$ such that for all $\delta \ge \delta_0$, we have $\nu_{x_\delta}^A(g) - \nu_x^A(g) \in G$. Then for all $\delta \ge \delta_0$,

$$\begin{aligned} \nu_{x_\delta}^A(f|[T_A]) &= \nu_{x_\delta}^A(f|[T_A] - g) + \nu_{x_\delta}^A(g) \in \bar{G} + \nu_x^A(g) + G \\ &\subset \bar{G} + G + \nu_x^A(f|[T_A]) + G \subset \nu_x^A(f|[T_A]) + U \end{aligned}$$

and the proof of Step 1 is complete.

Clearly S is a linear map. Next we prove

STEP 2. *For every* $f \in C([T],E)$ *and* $x \in X$,

$$S(f)(x) \in \overline{\mathrm{conv}}\, f(\Phi(x)).$$

Let $y \in [T_x]$. If V_t is a basic clopen neighborhood of y in $[T]$, then clearly $t \in T_x$. Therefore $x \in F_t \subset U_t = \{z \in X : \Phi(z) \cap V_t \neq \emptyset\}$. Thus $\Phi(x) \cap V_t \neq \emptyset$, and since V_t was an arbitrary neighborhood of y, we deduce that $y \in \overline{\Phi(x)}$. It follows that $[T_x] \subset \overline{\Phi(x)}$, and consequently

$$f([T_x]) \subset f(\overline{\Phi(x)}) \subset \overline{f(\Phi(x))} \subset \overline{\mathrm{conv}}\, f(\Phi(x)).$$

Therefore

$$\overline{\mathrm{conv}}\, f([T_x]) \subset \overline{\mathrm{conv}}\, f(\Phi(x)),$$

so it suffices to prove that $S(f)(x) \in \overline{\mathrm{conv}}\, f([T_x])$.

We fix an arbitrary neighborhood U of 0 in E and prove that

$$S(f)(x) \in U + \mathrm{conv}\, f([T_x]) \neq \emptyset.$$

First, let G be an open convex and symmetric neighborhood of 0 in E such that $\bar{G} + \bar{G} \subset U$. Let moreover $g = \sum_{i=1}^{n} a_i \chi(V_{t_i}^{\{x\}}, r_i) \in C([T_x],E)$ be a simple function in normal form such that

$$g - f|[T_x] \in G([T_x],E). \tag{3}$$

Since g is in normal form, we know that $\{V_{t_i}^{\{x\}} : 1 \leq i \leq n\}$ is a partition of $[T_x]$, and in this case,

$$\sum_{i=1}^{n} \lambda_{t_i}(x) = \lambda_{r(T)}(x) = 1. \tag{4}$$

Thus,

$$\begin{aligned} S(f)(x) &= \mu_x(f) = \nu_x(f|[T_x]) \\ &= \nu_x(f|[T_x] - g) + \nu_x(g) \\ &\in \bar{G} + \sum_{i=1}^{n} \lambda_{t_i}(x) a_i r_i, \end{aligned} \tag{5}$$

where the membership relation follows from the fact that $f|[T_x] - g \in G([T_x],E)$ as we have argued in Step 1. Since $t_i \in T_x$, we have $x \in F_{t_i}$, $1 \leq i \leq n$. Furthermore by the definition of T_x and the property $F_t = \bigcup\{F_s : s \in S(t)\}$, it is easy to see that for any $t \in T_x$, $V_t^{\{x\}}$ is non-empty. Therefore we may choose $b_i \in V_{t_i}^{\{x\}}$ so that $g(b_i) = a_i r_i$ and by (3), $f(b_i) - a_i r_i \in G$.

Using (5) and (4) now gives

$$\begin{aligned} S(f)(x) &\in \bar{G} + \sum_{i=1}^{n} \lambda_{t_i}(x) a_i r_i \\ &= \overline{G} + \sum_{i=1}^{n} \lambda_{t_i}(x)(a_i r_i - f(b_i)) + \sum_{i=1}^{n} \lambda_{t_i}(x) f(b_i) \\ &\subset \overline{G} + G + \operatorname{conv} f([T_x]) \subset U + \operatorname{conv} f([T_x]), \end{aligned}$$

which is exactly what we needed to prove.

In the next two steps we study the continuity properties of S.

Step 3. *S is continuous with respect to the uniform topology.*

It suffices to prove that for any convex neighborhood U of 0 in E, $S(U([T], E)) \subset \bar{U}(X, E)$. This is true since for any $f \in U([T], E)$ and $x \in X$, $S(f)(x) \in \overline{\operatorname{conv}}\, f(\Phi(x))$, $f(\Phi(x)) \subset U$ and U is convex.

Step 4. *S is continuous with respect to the topologies of uniform convergence on compact subsets.*

Here, it suffices to find, for any convex neighborhood U of 0 in E and any compact $A \subset X$, a compact $B \subset [T]$ such that $S(U(B, E)) \subset \bar{U}(A, E)$.

Set $B = [T_A] \subset Y$. Then B is compact since we have proved that if A is compact, then T_A is finitely branching and rooted. Observe that if $x \in A$, then $T_x \subset T_A$ and therefore $[T_x] \subset [T_A]$. Consequently, if $f \in U([T_A], E)$, then as proved in Step 2,

$$S(f)(x) \in \overline{\operatorname{conv}}\, f([T_x]) \subset \overline{\operatorname{conv}}\, f([T_A]) \subset \bar{U},$$

since U is convex. Therefore $S(f) \in \bar{U}(A, E)$. ■

The next lemma will be used to generalize Proposition 2.2 in the case where instead of $[T]$ we have an arbitrary complete metric space Y.

Lemma 2.3. *Assume that X is a normal topological space and Y a complete metric space. Let moreover $\{U_i : i \in I\}$ be a neighborhood basis for Y containing Y, and $\{G_i : i \in I\}$ open subsets of X, such that:*

1. *If $U_{i_0} = Y$, then $G_{i_0} = X$.*
2. *If for some $J \subset I$ and some $i \in I$, $U_i = \bigcup_{j \in J} U_j$, then also $G_i = \bigcup_{j \in J} G_j$.*

Then the set-valued map $\Phi : X \to 2^Y$ defined by

$$\Phi(x) = \{y \in Y : \forall i \; y \in U_i \Rightarrow x \in G_i\}$$

is lower semicontinuous and takes non-empty values.

Furthermore if T is a rooted tree of height ω, $Y = [T]$ and $\{U_i : i \in I\} = \{V_t : t \in T\}$, where as usual $V_t = \{b \in [T] : t \in b\}$, we may replace (1) *and* (2) *by:*

(1′) $G_{r(T)} = X$,

(2′) $G_t = \bigcup\{G_s : s \in S(t)\}$,

where $r(T)$ is the root of T and $S(t)$ is the set of immediate successors of t in T. In this case

$$\Phi(x) = \Big\{b \in [T] : x \in \bigcap_{t\in b} G_t\Big\}.$$

Proof. We first prove that

$$x \in G_i \;\Leftrightarrow\; \Phi(x) \cap U_i \neq \emptyset. \tag{6}$$

For the easy implication, let $\Phi(x) \cap U_i \neq \emptyset$ and $y \in \Phi(x) \cap U_i$. Then by definition, $x \in G_i$. For the converse direction, let $x \in G_i$. We recursively define a sequence $U_i = U_{i_1} \supset U_{i_2} \supset \cdots$ of open subsets of Y such that

- $\operatorname{diam}(\overline{U}_{i_n}) < 1/2^n$,
- $\overline{U}_{i_{n+1}} \subset U_{i_n}$,
- $x \in G_{i_n}$ for all n.

We set $U_{i_1} = U_i$ and assume that $U_{i_1}, \ldots, U_{i_n}$ have been defined. Let

$$J = \{j \in I : \operatorname{diam}(\overline{U}_j) < 1/2^{n+1} \text{ and } \overline{U}_j \subset U_{i_n}\}$$

and observe that $U_{i_n} = \bigcup_{j\in J} U_j$. Therefore, by our assumption also $G_{i_n} = \bigcup_{j\in J} G_j$, and since by the inductive hypothesis we have $x \in G_{i_n}$, there must exist an $i_{n+1} \in J$ such that $x \in G_{i_{n+1}}$. So the induction step is finished.

Let now y be the unique element of $\bigcap_{n=1}^{\infty} \overline{U}_{i_n} = \bigcap_{n=1}^{\infty} U_{i_n}$. Obviously $y \in U_i = U_{i_1}$. It suffices to prove moreover that $y \in \Phi(x)$. To this end, we have to prove that if for some j, $y \in U_j$, then $x \in G_j$. So let $y \in U_j$. Since $\{U_{i_n} : n \in \mathbb{N}\}$ is a neighborhood basis at y, for some $n \in \mathbb{N}$ we must have $y \in U_{i_n} \subset U_j$. Therefore if $K = \{k \in I : U_k \subset U_j\}$ then both $i_n \in K$ and $U_j = \bigcup_{k\in K} U_k$. Consequently, $G_j = \bigcup_{k\in K} G_k$, and since $x \in G_{i_n}$, it follows that $x \in G_j$, which is what we needed to prove.

For the particular i_0 for which $U_{i_0} = Y$, we have $x \in G_{i_0} = X$ and therefore by (6), $\Phi(x)$ is non-empty. The same relation shows that for any i,

$$G_i = \{x \in X : \Phi(x) \cap U_i \neq \emptyset\}.$$

Therefore if J is an arbitrary subset of I, we have

$$\Big\{x \in X : \Phi(x) \cap \bigcup_{j\in J} U_j \neq \emptyset\Big\} = \bigcup_{j\in J} \{x \in X : \Phi(x) \cap U_j \neq \emptyset\} = \bigcup_{j\in J} G_j,$$

and this last equality shows that Φ is indeed lower semicontinuous.

The second part of the lemma follows even more easily using exactly the same method as the first one. ∎

Lemma 2.4. *Let* (*as in Theorem* 1.1) X *be a paracompact* k*-space,* Y *a complete metric space, and* $\Phi : X \to 2^Y \setminus \{\emptyset\}$ *a lower semicontinuous set-valued map with non-empty values. Then there exists a tree* T *of height* ω, *a continuous surjection* $\varphi : [T] \to Y$ *and a lower semicontinuous set-valued map* $\Psi : X \to 2^{[T]} \setminus \{\emptyset\}$ *such that* $\Psi(x) \subseteq \varphi^{-1}(\overline{\Phi(x)})$.

Proof. Let $\{U_i : i \in I\}$ be a basis for the topology of Y consisting of open non-empty sets. For every $i \in I$, we define

$$I(i) = \{j \in I : \overline{U}_j \subset U_i \text{ and } \operatorname{diam}(\overline{U}_j) \le \min\{1, \operatorname{diam}(\overline{U}_i)/2\}\}.$$

We also define a tree T of finite sequences of I as follows:

$$(i_1, \ldots, i_n) \in T \;\Leftrightarrow\; i_{k+1} \in I(i_k) \text{ for every } k = 1, \ldots, n-1.$$

Due to the completeness of Y, for every $b \in [T]$, the set $\bigcap_{i \in b} U_i = \bigcap_{i \in b} \overline{U}_i$ is a singleton. Identifying it with its unique element we may thus define a map

$$\varphi : [T] \ni b \mapsto \bigcap_{i \in b} U_i \in Y.$$

We first prove that if $t = (i_1, \ldots, i_n) \in T$, then $\varphi(V_t) = U_{i_n}$, and consequently φ is continuous and onto.

For the easy direction, if $b \in V_t$, then $\varphi(b) = \bigcap_{i \in b} U_i \in U_{i_n}$ since $i_n \in b$. For the other direction, it is easy to see that for any $i \in I$, $U_i = \bigcup_{j \in I(i)} U_j$. Therefore if $y \in U_{i_n}$ we may inductively define a sequence $U_{i_n} \supset U_{i_{n+1}} \supset \cdots$ such that $i_{n+k+1} \in I(i_{n+k})$ and $y \in U_{i_{n+k}}$ for all $k = 0, 1, \ldots$. Setting $b = (i_1, \ldots, i_n, i_{n+1}, \ldots) \in [T]$ we conclude that $b \in V_t$ and $\varphi(b) = y$.

Now, for $t = (i_1, \ldots, i_n) \in T$, we set $U_t = U_{i_n}$ and $U_\emptyset = Y$. Observe that as mentioned before, $U_t = \bigcup\{U_s : s \in S(t)\}$. For $t \in T$ we also set $G_t = \{x \in X : \Phi(x) \cap U_t\} \ne \emptyset$. Since for every x we have $\Phi(x) \ne \emptyset$, we infer that $G_\emptyset = X$. Since moreover $U_t = \bigcup\{U_s : s \in S(t)\}$, it follows that also $G_t = \bigcup\{G_s : s \in S(t)\}$. Thus the map

$$\Psi : X \ni x \mapsto \Big\{b \in [T] : x \in \bigcap_{t \in b} G_t\Big\} \in 2^{[T]}$$

is lower semicontinuous and its values are non-empty according to Lemma 2.3. Observe that if $b \in \Psi(x)$, then for every $t \in b$, $x \in G_t$ and therefore $\Phi(x) \cap U_t \ne \emptyset$. Since $\{U_t : t \in b\}$ is a neighborhood basis at $\varphi(b)$, we find that $\varphi(b) \in \overline{\Phi(x)}$. It follows that for any $x \in X$,

$$\Psi(x) \subset \varphi^{-1}(\overline{\Phi(x)}), \tag{7}$$

which is what we needed to prove. ∎

We now use Proposition 2.2 and Lemmata 2.3 and 2.4 to prove Theorem 1.1.

Proof of Theorem 1.1. Keeping the notation from the proof of the previous lemma, define

$$S_1 : C(Y,E) \ni f \mapsto f \circ \varphi \in C([T],E).$$

It is easy to check that S_1 is linear and continuous with respect to both topologies of uniform convergence and of uniform convergence on compact subspaces.

Also, Proposition 2.2 ensures the existence of a linear continuous (in both topologies) operator $S_2 : C([T],E) \to C(X,E)$ such that $S_2(h)(x) \in \overline{\mathrm{conv}}\, h(\Psi(x))$. Set now $S = S_2 \circ S_1$ and observe that

$$\begin{aligned} S(f)(x) &= S_2(S_1(f))(x) \in \overline{\mathrm{conv}}\, S_1(f)(\Psi(x)) = \overline{\mathrm{conv}}\, f(\varphi(\Psi(x))) \\ &\subset \overline{\mathrm{conv}}\, f(\overline{\Phi(x)}) \qquad \text{by (7)} \\ &= \overline{\mathrm{conv}}\, f(\Phi(x)). \ \blacksquare \end{aligned}$$

3. Proof of Theorem 1.2. By Lemma 2.4 for $X = Y = K$ and Φ being the (single-valued) identity map, we construct T, Ψ and φ. Then $\varphi(\Psi(x)) = \{x\}$ for any $x \in K$. Since K is compact and metric, T is countable, so $[T]$ is homeomorphic to a compact subset of the Cantor set $\mathcal{C}$. By identifying $[T]$ with its image, we see that $[T]$ is a retract of $\mathcal{C}$ and thus φ and Ψ can be extended so that $\varphi : \mathcal{C} \to K$ is continuous onto, $\Psi : K \to 2^{\mathcal{C}} \setminus \{\emptyset\}$ is lower semicontinuous and moreover $\Psi(x) \subseteq \varphi^{-1}(x)$ for all $x \in K$.

Let S be the map given by Theorem 1.1. Then for any $f \in C(\mathcal{C},X)$, $S(f)(k) \in \overline{\mathrm{conv}}\, f(\varphi^{-1}(k))$. Setting

$$L : C(K,X) \ni g \mapsto g \circ \varphi \in C(\mathcal{C},X),$$

we see that for any $g \in C(K,X)$ and $k \in K$,

$$S \circ L(g)(k) = S(g \circ \varphi)(k) \in \overline{\mathrm{conv}}\, g \circ \varphi(\varphi^{-1}(k)) = \overline{\mathrm{conv}}\{g(k)\}.$$

It follows that $S \circ L(g) = g$, and therefore $C(K,X)$ is a complemented subspace of $C(\mathcal{C},X)$.

Let $e : \mathcal{C} \to K$ be any embedding. Define $\Phi : K \to 2^{\mathcal{C}}$ by

$$\Phi(k) = \begin{cases} \{e^{-1}(k)\} & \text{if } k \in e(\mathcal{C}), \\ \mathcal{C} & \text{otherwise}, \end{cases}$$

and check directly that Φ is lower semicontinuous. Let $S : C(\mathcal{C},X) \to C(K,X)$ be the operator given by Theorem 1.1. Thus $S(f)(k) \in \overline{\mathrm{conv}}\, f(\Phi(k))$ for any $f \in C(\mathcal{C},X)$ and $k \in K$. Let also $L : C(K,X) \to C(\mathcal{C},X)$ be given by $L(g) = g \circ e$. As above we check that $L \circ S$ is the identity on $C(\mathcal{C},X)$ and hence $C(\mathcal{C},X)$ embeds as a complemented subspace of $C(K,X)$.

The proof can now be completed using the decomposition method as done by Pełczyński in the classical case [12].

Acknowledgments. I would like to thank the referee for valuable crucial suggestions regarding the structure and proofs of this paper.

This research was partially supported by EPEAEK program "PYTHAGORAS".

References

[1] S. A. Argyros and A. Arvanitakis, *A characterization of regular averaging operators and its consequences*, Studia Math. 151 (2002), 207–226.

[2] S. Ditor, *On a lemma of Milutin concerning averaging operators in continuous function spaces*, Trans. Amer. Math. Soc. 149 (1970), 443–452.

[3] S. Ditor, *Averaging operators in* $C(S)$ *and lower semicontinuous sections of continuous maps*, Trans. Amer. Math. Soc. 175 (1973), 195–208.

[4] S. Ditor and R. Haydon, *On absolute retracts,* $P(S)$*, and complemented subspaces of* $\mathcal{C}(D^{\omega_1})$, Studia Math. 56 (1976), 243–251.

[5] J. Dugundji, *Topology*, UBS, New Delhi, 1989.

[6] R. Engelking, *General Topology*, PWN–Polish Sci. Publ., Warszawa, 1977.

[7] A. Etcheberry, *Isomorphism of spaces of bounded continuous functions*, Studia Math. 53 (1975), 103–127.

[8] R. Haydon, *On a problem of Pełczyński: Milutin spaces, Dugundji spaces and* AE(0-dim), Studia Math. 52 (1974), 23–31.

[9] R. Haydon, *Embedding* D^τ *in Dugundji spaces, with an application to linear topological classification of spaces of continuous functions*, Studia Math. 56 (1976), 229–242.

[10] E. Michael, *Continuous selections I*, Ann. of Math. 63 (1956), 361–382.

[11] A. A. Milyutin, *On spaces of continuous functions*, PhD thesis, Moscow State Univ., 1952 (in Russian).

[12] A. Pełczyński, *Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions*, Dissertationes Math. (Rozprawy Mat). 58 (1968), 92 pp.

[13] D. Repovš, P. V. Semenov and E. V. Shchepin, *Milyutin mappings and their applications*, Topology Proc. 22 (1997), 385–401.

[14] D. Repovš, P. V. Semenov and E. V. Shchepin, *On zero-dimensional Milutin maps and Michael selection theorems*, Topology Appl. 54 (1993), 77–83.

[15] V. Valov, *Milyutin mappings and* AE(0)*-spaces*, C. R. Acad. Bulg. Sci. 40 (1987), 9–11.

[16] V. Valov, *On a theorem of Arvanitakis*, arXiv:1108.2695.

[17] T. Yamauchi, *On a simultaneous selection theorem*, preprint.

Alexander D. Arvanitakis
Department of Mathematics
National Technical University of Athens
15780 Athens, Greece
E-mail: aarva@math.ntua.gr

Received 5 August 2004;
in revised form 29 December 2004 and 23 August 2012