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Supercompactness and failures of GCH

by

Sy-David Friedman (Wien) and **Radek Honzik** (Praha)

Abstract. Let $\kappa < \lambda$ be regular cardinals. We say that an embedding $j : V \to M$ with critical point κ is λ-*tall* if $\lambda < j(\kappa)$ and M is closed under κ-sequences in V.

Silver showed that GCH can fail at a measurable cardinal κ, starting with κ being κ^{++}-supercompact. Later, Woodin improved this result, starting from the optimal hypothesis of a κ^{++}-tall measurable cardinal κ. Now more generally, suppose that $\kappa \leq \lambda$ are regular and one wishes the GCH to fail at λ with κ being λ-supercompact. Silver's methods show that this can be done starting with κ being λ^{++}-supercompact (note that Silver's result above is the special case when $\kappa = \lambda$).

One can ask if there is an analogue of Woodin's result for λ-supercompactness. We answer this question in the following strong sense: starting with the GCH and κ being λ-supercompact and λ^{++}-tall, we preserve λ-supercompactness of κ and kill the GCH at λ by directly manipulating the size of 2^λ (i.e. we do not force the failure of GCH at λ as a consequence of having 2^κ large enough). The direct manipulation of 2^λ, where λ can be a successor cardinal, is the first step toward understanding which Easton functions can be realized as the continuum function on regular cardinals while preserving instances of λ-supercompactness.

1. Preliminaries

1.1. Tall supercompact cardinals. Let $\kappa \leq \lambda < \bar{\lambda}$ be regular cardinals and $j : V \to M$ an elementary embedding with critical point κ. We say that j is a $\bar{\lambda}$-*tall embedding with closure* λ if:

(i) ${}^{\lambda}M \subseteq M$,
(ii) $\bar{\lambda} < j(\kappa)$.

It is easy to see that if κ is λ-supercompact and $\bar{\lambda}$-tall, then we can witness this by a single embedding $j : V \to M$ which is $\bar{\lambda}$-tall with closure λ: just compose the supercompactness embedding $k : V \to N$ with the embedding $h : N \to M$ which witnesses the $j(\bar{\lambda})$-tallness of $j(\kappa)$ in N; then $j = h \circ k$ is as required.

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We say that a cardinal κ is *$\bar{\lambda}$-tall λ-supercompact* if κ is both $\bar{\lambda}$-tall and λ-supercompact. To keep our terminology consistent, we say that $j : V \to M$ is a $\bar{\lambda}$-tall λ-supercompact embedding if it is a $\bar{\lambda}$-tall embedding with closure λ.

Assume the GCH. If κ is $\bar{\lambda}$-tall λ-supercompact, then this fact can be witnessed by a $\bar{\lambda}$-tall λ-supercompact embedding of the extender type:

$$j : V \to M = \{j(f)(j''\lambda, \alpha) \mid f : P_\kappa(\lambda) \times \kappa \to V \ \&\ \alpha < \bar{\lambda}\}. \tag{1.1}$$

The Set Theory Handbook chapter [Cu] provides useful information relating to extender-type embeddings in general. For more details on tall cardinals, see [Ha].

In this paper, we will study the specific case of a λ^{++}-tall λ-supercompact cardinal κ. This notion generalizes that of a κ^{++}-tall measurable cardinal κ. Woodin showed that the existence of a κ^{++}-tall cardinal κ with the GCH is equiconsistent with the failure of GCH at the measurable cardinal. By results of Mitchell and Gitik [Mi, Gi], we also know that these two statements are equiconsistent with the existence of κ of Mitchell order κ^{++}. We generalize these results to a supercompactness setting: In the forward direction, starting with the GCH and a λ^{++}-tall λ-supercompact embedding we obtain the failure of GCH at λ preserving λ-supercompactness of κ, without changing the size of 2^μ for regular cardinals $\mu \in [\kappa, \lambda)$ (more patterns of the continuum function are possible by the methods in this paper; see the paragraph after the end of the proof of Theorem 2.3 and Problem 3.1).

In the converse direction, if the GCH fails at λ and κ is λ-supercompact, then any embedding witnessing this must be λ^{++}-tall. So it is just the forward direction that needs proving.

If we drop the requirement on the direct manipulation of the size of 2^λ, then the forward direction was already proved in [Co]. Cody constructs, under the same initial assumptions, a model where $2^\kappa = 2^\lambda = \lambda^{++}$ and κ is still λ-supercompact—failure of GCH at λ is thus obtained by adding new subsets to κ. See Problem 3.1 for more discussion of this point.

The precise statement of the main result of this paper is as follows (see also Theorem 2.3). Notice that achieving this result in full generality requires dealing with the situation when λ is a successor cardinal; this raises a host of interesting new challenges not present when λ equals the inaccessible κ.

THEOREM 1.1. (*GCH*) *Let $\kappa < \lambda$ be regular cardinals. Assume that κ is λ^{++}-tall λ-supercompact. Then there exists a forcing extension where κ is still λ^{++}-tall λ-supercompact and moreover the GCH fails at λ, while it holds in the interval $[\kappa, \lambda)$.*

The proof of the result makes essential use of the generalized Sacks forcing, which is reviewed in Section 1.3; a quick review of the facts related to lifting of embeddings is given in Section 1.4. The generalized Sacks forcing proved to be useful in the context of measurable cardinals in various settings; see for instance [FrHo2, FrHo1, FrMa, FrZd, FrHa].

1.2. The typical case: $\bar{\lambda} = \lambda^{++}$. In order to simplify notation and make the main idea of the proof more transparent, we will first work with the special case $\bar{\lambda} = \lambda^{++}$. Indeed, all the important ideas are present already in this case. Generalization to other $\bar{\lambda}$'s is quite natural—it suffices to modify the definition of f_λ in (2.1) of Section 2.1 in the obvious way.

Let $\kappa < \lambda$ be regular. Assume the GCH and let κ be a λ^{++}-tall λ-supercompact cardinal. Let $j : V \to M$, with $\lambda^{++} < j(\kappa) < \lambda^{+3}$, witness this fact:

$$M = \{j(f)(j''\lambda, \alpha) \mid f : P_\kappa(\lambda) \times \kappa \to V \ \&\ \alpha < \lambda^{++}\}. \tag{1.2}$$

We make the notational convention that λ^{++} always denotes the double successor of λ as calculated in V, i.e. the real λ^{++}. In contrast, $(\lambda^{++})^M$ denotes the double successor of λ as calculated in M. In (1.2) we do not assume $(\lambda^{++})^M = \lambda^{++}$.

If $j : V \to M$ is as in (1.2), then it has the following properties:

LEMMA 1.2. *Assume the GCH and let $j : V \to M$ be as in* (1.2).

(i) *The following inequalities hold:*

$$\begin{aligned} &\kappa < \lambda < \lambda^{++} < j(\kappa) < j(\kappa^{++}) \\ &\quad < \sup j''\lambda < j(\lambda) < (j(\lambda)^{++})^M < \lambda^{+3} = (j(\lambda)^{+3})^M. \end{aligned}$$

(ii) *All M-regular cardinals in the interval $(\lambda^{++}, j(\lambda^+)]$ have V-cofinality λ^+.*

Proof. (i) $\sup j''\lambda$ has cofinality λ in M, and since $\lambda < j(\kappa) < j(\lambda)$, and $j(\lambda)$ is regular in M, it must be the case that $\sup j''\lambda < j(\lambda)$.

Let us denote $\mu_0 = j(\lambda^{++}) = (j(\lambda)^{++})^M$. If $\gamma < \mu_0$, then γ is of the form $j(f)(j''\lambda, \alpha)$ for some $f : P_\kappa(\lambda) \times \kappa \to \lambda^{++}$ and some $\alpha < \lambda^{++}$. Since there are only λ^{++}-many such functions f, and ordinals $\alpha < \lambda^{++}$, the size of μ_0 in V is just λ^{++}. It follows that $\mu_0 < \lambda^{+3}$.

To see that $\lambda^{+3} = (j(\lambda)^{+3})^M$ is true, first notice that

$$\lambda^{+3} = \sup\{j(\alpha) \mid \alpha < \lambda^{+3}\}. \tag{1.3}$$

The identity (1.3) holds because given $\alpha < \lambda^{+3}$, any ordinal $\beta < j(\alpha)$ can be represented as $j(f)(j''\lambda, \bar{\beta})$ where f has its range included in α, and $\bar{\beta} < \lambda^{++}$; there are at most λ^{++} pairs $(f, \bar{\beta})$ like this, and therefore $j(\alpha)$ has size at most λ^{++} in V.

Further notice that

$$\sup\{j(\alpha) \mid \alpha < \lambda^{+3}\} = j(\lambda^{+3}). \tag{1.4}$$

The identity (1.4) holds because every $\alpha < j(\lambda^{+3})$ can be represented as $j(f)(j''\lambda, \bar{\alpha})$ where f has its range included in λ^{+3} and $\bar{\alpha} < \lambda^{++}$; since λ^{+3} is regular, the range of f (which has size at most $\lambda^{<\kappa}\kappa = \lambda$) is bounded by some $\beta < \lambda^{+3}$, and so $\alpha < j(\beta)$.

(ii) Each ordinal in the interval $(\lambda^{++}, j(\lambda^+)]$ can be written as $j(f)(j''\lambda, \alpha)$ for some $f : P_\kappa(\lambda) \times \kappa \to \lambda^+$ and $\alpha < \lambda^{++}$. If μ is an M-regular cardinal in $(\lambda^{++}, j(\lambda^+)]$, then its M-cofinality is greater than λ^{++}; it follows that for each such f, the intersection $X_f = \{j(f)(j''\lambda, \alpha) \mid \alpha < \lambda^{++}\} \cap \mu$ is bounded in μ. By the GCH, there are $(\lambda^+)^\lambda = \lambda^+$-many such f's. It follows that $\{\sup\ X_f \mid f : P_\kappa(\lambda) \times \kappa \to \lambda^+\}$ is a cofinal subset of μ of size λ^+. ∎

1.3. Sacks forcing at λ. This section presents the version of generalized Sacks forcing that will suit our purposes. For more details, consult [Ka].

DEFINITION 1.3. (GCH) For a regular cardinal λ, we say that $p \subseteq {}^{<\lambda}2$ is a cof ω_1*-splitting perfect tree at* λ if p is a tree of height λ closed under initial segments such that:

(i) For every s in p there is $s' \supseteq s$ in p such that s' splits, where s' *splits* if both $s'^\frown 0$ and $s'^\frown 1$ are in p.
(ii) If $\langle s_\xi \mid \xi < \delta\rangle$ for some limit ordinal $\delta < \lambda$ is an $\subseteq$-increasing chain of nodes in p, then the union $\bigcup_{\xi<\delta} s_\xi$ is also a node in p.
(iii) If s is a node in p and s is in ${}^\delta 2$ for some limit δ of cofinality ω_1, and moreover the set of nodes $s' \subsetneq s$ which split is unbounded in s, then s splits in p.
(iv) If s is a node in p and s is in ${}^\delta 2$ for some limit δ of cofinality other than ω_1, then s does not split in p.

REMARK 1.4. The use of ω_1 in the above definition is not essential. Any regular cardinal $\mu < \lambda$ other than ω, together with the related notion of a cof μ-splitting perfect tree at λ, can be used in the arguments which follow. The only point is that $\mathrm{cf}(\mu) \neq \omega$ so that the argument just before Definition 2.12 goes through. See also Remark 2.16.

First note that we do not demand that p is a λ-tree—a level of the tree may have size λ. Our definition differs from the one in [Ka] in that we control the splitting according to cofinalities; this is relevant to the lifting argument (see Lemma 2.15 and the following argument concluding the proof of Theorem 2.3).

We say that $s \in p$ is a *splitting node* if s splits or s is a limit of splitting nodes in p; by (iii) and (iv), a splitting node may not actually split in p. We will be careful about distinguishing the meaning of the phrase "s splits in p" (which means that $s^\frown 0 \in p$ and $s^\frown 1 \in p$) vs. "s is a splitting node". This convention is useful when dealing with the limit stages of constructions based on fusion (see for instance Lemma 2.7).

We write $\mathrm{Split}_\alpha(p)$ to denote the collection of the splitting nodes in p of rank α:

$$s \in \mathrm{Split}_\alpha(p) \;\leftrightarrow\; \mathrm{ot}(\{s' \subsetneq s \mid s' \text{ is a splitting node}\}) = \alpha. \tag{1.5}$$

The $\mathrm{cof}\,\omega_1$-splitting perfect trees at λ can be used to define a natural forcing notion which we simply denote by $\mathrm{Sacks}(\lambda, 1)$: A condition p is in this forcing if and only if p is a $\mathrm{cof}\,\omega_1$-splitting perfect tree at λ; the ordering is by inclusion. For $\alpha < \lambda$ we define

$$p \le_\alpha q \;\leftrightarrow\; p \le q \;\&\; p \cap {}^{\alpha+1}2 = q \cap {}^{\alpha+1}2. \tag{1.6}$$

It is a standard fact that if $\langle p_\alpha \mid \alpha < \lambda\rangle$ is a sequence of conditions in $\mathrm{Sacks}(\lambda, 1)$ and $p_{\alpha+1} \le_\alpha p_\alpha$ for each $\alpha < \lambda$, then the intersection $\bigcap\{p_\alpha \mid \alpha < \lambda\}$ is a condition and the greatest lower bound of $\{p_\alpha \mid \alpha < \lambda\}$. We call the sequence $\langle p_\alpha \mid \alpha < \lambda\rangle$ a *fusion sequence* and $\bigcap\{p_\alpha \mid \alpha < \lambda\}$ the *fusion limit*. It is easy to check that $\mathrm{Sacks}(\lambda, 1)$ is λ-closed. If the GCH holds, then $\mathrm{Sacks}(\lambda, 1)$ has size λ^+, and so preserves all cardinals $\ge \lambda^{++}$. The preservation of λ^+ follows by an easy fusion-type argument (the diamond-type argument, as in Lemma 2.7, is required only after we consider a product forcing with at least two components).

We write $\mathrm{Sacks}(\lambda, \alpha)$ for $\alpha \ge 1$ to denote the product with supports of size $\le \lambda$ of α-many copies of $\mathrm{Sacks}(\lambda, \alpha)$. We denote the support of a condition p by $\mathrm{supp}(p)$.

For p and q in $\mathrm{Sacks}(\lambda, \alpha)$, $F \subseteq \mathrm{supp}(p)$ and $|F| < \lambda$, we define

$$q \le_{F,\alpha} p \;\leftrightarrow\; q \le p \;\&\; \forall \xi \in F, q(\xi) \le_\alpha p(\xi). \tag{1.7}$$

We say that a pair $(\langle p_\alpha \mid \alpha < \lambda\rangle, \langle F_\alpha \mid \alpha < \lambda\rangle)$ is a *fusion sequence* in $\mathrm{Sacks}(\lambda, \alpha)$ if for each $\alpha < \lambda$, $F_\alpha \subseteq \mathrm{supp}(p_\alpha)$, $|F_\alpha| < \lambda$, $F_\alpha \subseteq F_{\alpha+1}$, $F_\delta = \bigcup_{\alpha<\delta} F_\alpha$ if δ is a limit ordinal, $\bigcup_{\alpha<\lambda} F_\alpha = \bigcup_{\alpha<\lambda} \mathrm{supp}(p_\alpha)$, and finally, for each $\alpha < \lambda$,

$$p_{\alpha+1} \le_{F_\alpha,\alpha} p_\alpha. \tag{1.8}$$

If $(\langle p_\alpha \mid \alpha < \lambda\rangle, \langle F_\alpha \mid \alpha < \lambda\rangle)$ is a fusion sequence, then the condition q given by $\mathrm{supp}(q) = \bigcup_{\alpha<\lambda} F_\alpha$ and $q(\xi) = \bigcap_{\alpha<\lambda} p_\alpha(\xi)$ for each $\xi \in \mathrm{supp}(q)$ is a condition which we call the *fusion limit*.

$\mathrm{Sacks}(\lambda, \alpha)$ is λ-closed and under the GCH it has the λ^{++}-cc. The preservation of λ^+ if λ is a successor cardinal and $\alpha > 1$ is not trivial; in [Ka], A. Kanamori used a fusion-type construction based on $\diamondsuit_\lambda$ (we review this argument in Lemma 2.7, rephrased in a way which fits our purpose). Let us note however that $\diamondsuit_\lambda$ is implied by the GCH below λ and so the GCH is enough to ensure that $\mathrm{Sacks}(\lambda, \alpha)$ is cofinality-preserving (see [Sh] for more).

1.4. Lifting of embeddings. For a general introduction to lifting of embeddings, see [Cu]. We state here several facts which we will use freely (and often tacitly) in our arguments.

Let $j : V \to M$ be an elementary embedding with critical point κ, P be a forcing notion and G be a P-generic filter over V.

FACT 1.5.

(i) (Silver) *If H is $j(P)$-generic over M and furthermore $j''G \subseteq H$, then j* lifts *to $j^* : V[G] \to M[H]$ in the sense that j^* is elementary and $j^*{\restriction}V = j$. Moreover, in this case $j^*(G) = H$.*

(ii) *If $j^* : V[G] \to M[H]$ is a lifting of j, and j is as in* (1.2)*, then*

(1.9)
$$M[H] = \{j^*(f)(j''\lambda, \alpha) \mid f \in V[G] \ \&\ f : (P_\kappa(\lambda))^V \times \kappa \to V[G] \ \&\ \alpha < \lambda^{++}\}.$$

(iii) *Assume the GCH. If j is as in* (1.2) *and P has the λ^+-cc, then $M[G]$ is still closed under λ-sequences in $V[G]$. The same conclusion holds when P is the forcing* $\mathrm{Sacks}(\lambda, \lambda^{++})$.

Proof. All claims follow from [Cu], except perhaps the last claim in (iii). By the fusion property of $\mathrm{Sacks}(\lambda, \lambda^{++})$, any λ-sequence of ordinals in $V[G]$ is covered by a λ-sequence back in V; M contains this covering sequence, and because $V[G]$ and $M[G]$ have the same subsets of λ, the original sequence can be decoded in $M[G]$. ■

2. The main theorem

2.1. Preparation of the universe. In the anticipation of the lifting argument as given in Section 2.2, we will need to have two special functions available in the universe—we will denote them as f_λ and f_Δ. Since their existence is not generally automatic, it may be necessary to force them.

First fix some canonical bijection π between κ and $\kappa \times \kappa$ (for instance, π may be given by the maximo-lexicographic ordering on $\kappa \times \kappa$).

Let $j : V \to M$ be as in (1.2). If there is some function

$$f_\lambda : \kappa \to \kappa \quad \text{such that} \quad j(f_\lambda)(\kappa) = j(\pi^{-1})(\langle \lambda, \lambda^{++} \rangle), \tag{2.1}$$

then we fix one and proceed to inquire about the function f_Δ; see below in (2.3). If there is no such f_λ, we are going to force it.

LEMMA 2.1. (*GCH*) *Assume $j : V \to M$ is as in* (1.2) *and the GCH holds. There is a forcing $P(f_\lambda)$ such if G_{f_λ} is $P(f_\lambda)$-generic over V, then:*

(i) *GCH holds in $V[G_{f_\lambda}]$.*

(ii) *j lifts in $V[G_{f_\lambda}]$ to $j^* : V[G_{f_\lambda}] \to M[j^*(G_{f_\lambda})]$ satisfying* (1.2).

(iii) *There exists in $V[G_{f_\lambda}]$ a function f_λ as in* (2.1)*; in fact, this f_λ is determined in a simple way by $\bigcup G_{f_\lambda}$.*

Proof. We will force f_λ using a *fast function* forcing due to Woodin. There are many variants of the forcing; we will use the following. A condition p in the forcing $P(f_\lambda)$ is a function from $\mathrm{dom}(p) \subseteq \kappa$ into κ such that $\mathrm{dom}(p)$

is an *Easton set*: for every inaccessible cardinal $\alpha < \kappa$, $|\mathrm{dom}(p) \cap \alpha| < \alpha$. We further require

$$\forall \gamma < \kappa, \gamma \in \mathrm{dom}(p) \to p''\gamma \subseteq \gamma. \tag{2.2}$$

The ordering is by reverse inclusion. Let us denote by G_{f_λ} the generic filter for $P(f_\lambda)$; $\bigcup G_{f_\lambda}$ is a function from a subset of κ into κ (which in $V[G_{f_\lambda}]$ can in some trivial fashion be extended to a function with domain equal to all of κ). Even with GCH, it may not in general be true that $P(f_\lambda)$ preserves all cardinals below κ. However, $P(f_\lambda)$ does preserve inaccessibility of cardinals, and moreover, the embedding $j : V \to M$ lifts to $j^* : V[G_{f_\lambda}] \to M^*$, where j^* satisfies (1.2). In particular Lemma 1.2 still holds for j^*.

We will not go into details as regards the preservation inaccessibility. The argument uses the usual factorization of $P(f_\lambda)$ below a given condition p into $P_0 \times P_1$, where P_0 has small size (below an inaccessible) and P_1 is sufficiently closed.

We show in some detail that j lifts (we use tacitly all the facts stated in Section 1.4). Fix a "master condition" $p_0 = \{\langle \kappa, j(\pi^{-1})(\langle \lambda, \lambda^{++}\rangle)\rangle\}$. We will construct in $V[G_{f_\lambda}]$ a $j(P(f_\lambda))$-generic filter H over M such that

(i) $p_0 \in H$,
(ii) $j''G_{f_\lambda} \subseteq H$.

By Section 1.4, this is enough to conclude that j lifts to $V[G_{f_\lambda}]$, and $V[G_{f_\lambda}]$ contains the desired function f_λ. Now we argue that we can ensure (i) and (ii). First notice that $j(P(f_\lambda))$ factors below p_0 as $P(f_\lambda) \times P^*$, and it easily follows that G_{f_λ} is $P(f_\lambda)$-generic over M. Since we work below p_0, P^* is λ^{+++}-directed closed in M. Every maximal antichain in P^* which lies in M can be represented as $j(f)(j''\lambda, \alpha)$ for some $\alpha < \lambda^{++}$ and $f : P_\kappa(\lambda) \times \kappa \to H(\kappa^+)$. By the GCH in V, there are only λ^+-many such f's. By the λ^{+++}-closure of P^*, it is possible to diagonalize over all relevant maximal antichains in M in λ^+-many steps and construct an H_0, a P^*-generic filter over M which contains p_0. Since $P(f_\lambda)$ has the κ-cc, and P^* is κ-closed, the usual mutual genericity argument ensures that $H = G(f_\lambda) \times H_0$ is as required. ∎

Let us rename $j^* : V[G_{f_\lambda}] \to M^*$ back to $j : V \to M$ in order to keep the notation as simple as possible. This j still satisfies (1.2) and has the properties as stated in Lemma 1.2. Let us denote $\Delta = \sup j''\lambda$ for the rest of the paper. Let us also fix a bijection $c : P_\kappa(\lambda) \to \lambda$. We are now interested in the existence of a function f_Δ in V which satisfies

$$f_\Delta : \lambda \to \lambda \text{ is such that } j(f_\Delta)(\Delta) > j(c)(j''\lambda). \tag{2.3}$$

Such a function will be useful to us in Lemma 2.15. If λ is inaccessible or a successor of a cardinal λ' such that $\mathrm{cf}(\lambda') \geq \kappa$, then the existence of such

f_Δ is automatic. Indeed, let us call $\xi < \lambda$ a *closure point* of c if for every bounded subset $x \subseteq \xi$ in $P_\kappa(\lambda)$, $c(x) < \xi$. Let us denote by C_c the set of closure points of c; then C_c is closed and $\mathrm{cf}(\lambda') \geq \kappa$ implies that it is also unbounded (as in this case $\lambda'^{<\kappa} = \lambda'$). We can define f_Δ as follows: for each $\xi < \lambda$, let $f_\Delta(\xi)$ be the least element of C_c strictly above ξ. Then $j(f_\Delta)(\Delta)$ is the least element of $j(C_c)$ strictly above Δ; since $j''\lambda \subseteq \Delta$, $j(f_\Delta)(\Delta)$ must be greater than $j(c)(j''\lambda)$.

If $\mathrm{cf}(\lambda') < \kappa$, then we are going to force f_Δ. We can proceed as above where we forced f_λ; however, since we already have f_λ in our universe, we can force f_Δ in a more gentle way—namely, without collapsing cardinals.

Lemma 2.2. (*GCH*) *Assume* $j : V \to M$ *is as in* (1.2), *the GCH holds, and* V *contains a function* f_λ *as in* (2.1). *There is a cofinality-preserving forcing* $P(f_\Delta)$ *such that if* $G * g$ *is* $P(f_\Delta)$*-generic over* V, *then:*

(i) *GCH holds in* $V[G * g]$.
(ii) j *lifts in* $V[G * g]$ *to* $j^* : V[G * g] \to M[j^*(G * g)]$ *satisfying* (1.2).
(iii) *There exists in* $V[G * g]$ *a function* f_Δ *as in* (2.3); *in fact, this* f_Δ *is equal to* $\bigcup g$.

Proof. Given $\xi < \kappa$, we can view ξ as coding a unique pair of ordinals $\langle \zeta, \zeta' \rangle$ modulo the bijection π fixed above: $\pi(\xi) = \langle \zeta, \zeta' \rangle$. We write $(\xi)_0$ for ζ and $(\xi)_1$ for ζ'; by elementarity, $j(f_\lambda)(\kappa)_0 = \lambda$ and $j(f_\lambda)(\kappa)_1 = \lambda^{++}$. We say that $\xi < \kappa$ is a *closure point of* f_λ if

$$f_\lambda(\zeta)_0 < \xi \quad \text{and} \quad f_\lambda(\zeta)_1 < \xi \quad \text{for all } \zeta < \xi. \tag{2.4}$$

Let us denote by $C(f_\lambda)$ the closed unbounded set of closure points of f_λ.

Next, $P(f_\Delta)$, the forcing notion to add f_Δ, is defined as a two-stage iteration $P(f_\Delta)^0 * \dot{\mathrm{A}}\mathrm{dd}(\lambda, 1)$, where $P(f_\Delta)^0$ is an iteration with Easton support:

(*) $P(f_\Delta)^0 = \langle (P(f_\Delta)^0_\alpha, \dot{Q}_\alpha) \mid \alpha < \kappa \rangle$, where $\dot{Q}_\alpha$ is the name for the trivial forcing unless α is in $C(f_\lambda)$, $\alpha < f_\lambda(\alpha)_0$, and $f_\lambda(\alpha)_0$ is a regular cardinal, in which case $\dot{Q}_\alpha$ is a name for $\mathrm{Add}(f_\lambda(\alpha)_0, 1)$ (the Cohen forcing which adds a subset of $f_\lambda(\alpha)_0$).

By standard arguments, one can easily show that $P(f_\Delta)$ is cofinality-preserving. We show now that $j : V \to M$ lifts to $P(f_\Delta)$. Let $G * g$ be $P(f_\Delta)^0 * \dot{\mathrm{A}}\mathrm{dd}(\lambda, 1)$-generic over V. Again, we will be tacitly using all the facts stated in Section 1.4. We are going to build a $j(P(f_\Delta))$-generic filter $H * h$ over M such that $j''(G * g) \subseteq H * h$.

The forcing $j(P(f_\Delta))$ is a two-stage iteration $j(P(f_\Delta)^0) * \dot{\mathrm{A}}\mathrm{dd}(j(\lambda), 1)$. The first part $j(P(f_\Delta)^0)$ is an Easton-supported iteration of length $j(\kappa)$ which coincides with $P(f_\Delta)^0$ when the former is restricted to κ; it follows that G is $j(P(f_\Delta)^0)_\kappa$-generic over M. Since κ is a closure point of $j(f_\lambda)$, $\lambda = j(f_\lambda)(\kappa)_0 > \kappa$ and λ is a regular cardinal, the next forcing in

$j(P(f_\Delta)^0)$ is by elementarity equal to $\mathrm{Add}(\lambda,1)$ of $M[G]$; it follows that $G*g$ is $j(P(f_\Delta)^0)_{\kappa+1}$-generic over M. Since the next closure point of $j(f_\lambda)$ strictly above κ must be strictly greater than $j(f_\lambda)(\kappa)_1=\lambda^{++}$, we can conclude that the iteration $j(P(f_\lambda)^0)$ in the interval $(\kappa,j(\kappa))$ is λ^{+++}-closed in $M[G*g]$. As in the argument for $P(f_\lambda)$ above, we can diagonalize over all relevant maximal antichains in $j(P(f_\lambda)^0)$ in the interval $(\kappa,j(\kappa))$ existing in $M[G*g]$ in λ^+-many steps, obtaining a generic filter $\tilde{H}$ over $M[G*g]$. It follows we can partially lift in $V[G*g]$ to $j:V[G]\to M[H]$, where $H=G*g*\tilde{H}$.

It remains to lift $\mathrm{Add}(\lambda,1)$ of $V[G]$ to $\mathrm{Add}(j(\lambda),1)$ of $M[H]$. We will construct in $V[G*g]$ an $\mathrm{Add}(j(\lambda),1)$-generic over $M[H]$ and denote it h; we will moreover ensure that

(i) $j''g$ is contained in h,
(ii) $(\bigcup h)(\Delta)>j(c)(j''\lambda)$.

By the λ-closure of $M[H]$ in $V[G*g]$, $\bigcup j''g$ is a condition in $\mathrm{Add}(j(\lambda),1)$ of $M[H]$ whose domain is included (and cofinal) in Δ. Set $p_0=\bigcup j''g\cup\langle\Delta,\delta\rangle$, where δ is any ordinal greater than $j(c)(j''\lambda)$. We will build h above this "master condition" p_0. Every maximal antichain in $\mathrm{Add}(j(\lambda),1)$ of $M[H]$ which exists in $M[H]$ can be represented as $j(f)(j''\lambda,\alpha)$ for some $f:P_\kappa(\lambda)^V\to H(\lambda^+)^{V[G]}$, and there are just λ^+-many of such f's. It follows that we can diagonalize over all relevant maximal antichains and construct h as required.

It follows that we can lift to $j:V[G*g]\to M[H*h]$, where $V[G*g]$ contains the desired function f_Δ. ∎

After renaming j^* back to j, we will from now on assume without loss of generality that if $j:V\to M$ is as in (1.2) and the GCH holds, then j has the properties in Lemma 1.2, and moreover contains the functions f_λ and f_Δ satisfying (2.1) and (2.3), respectively.

2.2. The lifting argument. The main result of this paper is the formulation and verification of the lifting argument for a λ^{++}-tall λ-supercompact cardinal κ. We present the argument in the proof of the following theorem.

THEOREM 2.3. (*GCH*) *Let $\kappa<\lambda$ be regular and let κ be λ^{++}-tall λ-supercompact. Assume that we have in the ground model V the functions f_λ and f_Δ fixed in Section* 2.1. *Then there exists a cofinality-preserving forcing notion $\mathbb{P}$ such that whenever G is $\mathbb{P}$-generic, κ is still λ^{++}-tall λ-supercompact in $V[G]$, GCH holds in the interval $[\kappa,\lambda)$ and moreover $2^\lambda=\lambda^{++}$ in $V[G]$.*

Proof. The proof is through a sequence of lemmas and remarks. First we make the following assumption which streamlines the presentation of the proof. We will prove the theorem under the assumption that λ is a successor cardinal, say $\lambda = \lambda'^{+}$, where $\mathrm{cf}(\lambda') > \omega$. If $\mathrm{cf}(\lambda') = \omega$, then some details in the proof—in particular in the definition of *rich reduction*—must be changed but the proof is otherwise the same; see Remark 2.16 for more about this. If λ is inaccessible, then a much easier argument can be used (avoiding the use of $\diamondsuit_\lambda$); see Remark 2.18.

Let us fix a function f_λ as in (2.1) and let $C(f_\lambda)$ be the closed unbounded set of its closure points as in (2.4).

Let $\mathbb{P}^0$ be the reverse-Easton iteration defined as follows:

(**) $\mathbb{P}^0 = \langle (\mathbb{P}^0_\alpha, \dot{Q}_\alpha) \mid \alpha < \kappa \rangle$, where $\dot{Q}_\alpha$ is the trivial forcing unless α is in $C(f_\lambda)$, $\alpha < f_\lambda(\alpha)_0$, and $f_\lambda(\alpha)_0$ is a regular cardinal, in which case $\dot{Q}_\alpha$ is the name for the forcing $\mathrm{Sacks}(f_\lambda(\alpha)_0, f_\lambda(\alpha)_1)$.

Let us define $\mathbb{P} = \mathbb{P}^0 * \dot{\mathrm{Sacks}}(\lambda, \lambda^{++})$.

Lemma 2.4. *(GCH) $\mathbb{P}$ is cofinality-preserving.*

Proof. This can be shown by standard arguments, invoking [Ka] for the Sacks forcing. ■

Let $G * g$ be $\mathbb{P}$-generic, where g is $\mathrm{Sacks}(\lambda, \lambda^{++})$-generic over $V[G]$. In order to prove Theorem 2.3, we will lift $j : V \to M$ to $j^* : V[G * g] \to M[j^*(G * g)]$. See Section 1.4 for more information about lifting.

Lemma 2.5. *In $V[G * g]$, j lifts to $j^* : V[G] \to M[G * g * H]$, where H is a generic filter for the iteration $j(\mathbb{P}^0)$ in the interval $(\kappa, j(\kappa))$.*

Proof. The argument is quite standard, and is in fact quite similar to the analogous argument for $P(f_\Delta)$ in Section 2.1. First, G is $j(\mathbb{P}^0)_\kappa$-generic over M. Next, from the definition of the forcing, $j(\mathbb{P}^0)$ at κ is equal to $\mathrm{Sacks}(j(f_\lambda)(\kappa)_0, j(f_\lambda)(\kappa)_1) = \mathrm{Sacks}(\lambda, \lambda^{++})$ of $M[G]$, which is the same forcing as $\mathrm{Sacks}(\lambda, \lambda^{++})$ of $V[G]$. Finally, the forcing $j(\mathbb{P}^0)$ in the interval $(\kappa, j(\kappa))$ is λ^{+++}-closed in $M[G * g]$. It follows that the generic H can be constructed in $V[G * g]$. ■

By Section 1.4, j^* has the extender representation:

$$\begin{aligned} (2.5)\qquad & M[G * g * H] \\ &= \{ j^*(f)(j''\lambda, \alpha) \mid f \in V[G] \ \& \ f : (P_\kappa(\lambda))^V \times \kappa \to V[G] \ \& \ \alpha < \lambda^{++} \}. \end{aligned}$$

Remark 2.6. Since it can cause no confusion, we will use the letter j for the lifted embedding $j : V[G] \to M[G * g * H]$.

Let us denote the forcing $\mathrm{Sacks}(\lambda, \lambda^{++})$ of $V[G]$ as $\mathbb{Q}$. In order to complete the lifting, we show that the λ-closure h of $j[g]$ is $j(\mathbb{Q})$-generic over $M[G * g * H]$, where $j(\mathbb{Q}) = \mathrm{Sacks}(j(\lambda), j(\lambda^{++}))^{M[G*g*H]}$ and where the λ-closure of $j[g]$ is generated by limits of $\leq$-decreasing sequences of elements in $j[g]$ of length λ:

$$h = \Big\{ q \in j(\mathbb{Q}) \ \Big| \ \text{there is a } \le\text{-decreasing sequence } \langle q_\alpha \mid \alpha < \lambda \rangle \text{ of conditions in } g \text{ and } q \ge \bigwedge_{\alpha<\lambda} j(q_\alpha) \Big\}, \tag{2.6}$$

where $\bigwedge_{\alpha<\lambda} j(q_\alpha)$ is the greatest lower bound of $\{j(q_\alpha) \mid \alpha < \lambda\}$ in $j(\mathbb{Q})$.

Note that by the closure of $M[G*g*H]$ under λ-sequences in $V[G*g]$ and the λ^+-closure of $j(\mathbb{Q})$ in $M[G*g*H]$, the limits $\bigwedge_{\alpha<\lambda} j(q_\alpha)$ are well-defined. It is easy to show that h is a filter. So it remains to show that it meets every dense open set of $j(\mathbb{Q})$ which lies in $M[G*g*H]$.

Since $2^{\lambda'} = \lambda$ in $V[G]$, we can by [Sh] fix a diamond sequence $\diamondsuit_\lambda = \langle S_\alpha \mid \alpha < \lambda \rangle$ such that for every $X \subseteq \lambda \times \lambda$, $\{\alpha < \lambda \mid X \cap (\alpha \times \alpha) = S_\alpha\}$ is stationary.

The following Lemma 2.7 provides an argument for a "simple reduction" (see the statement of the lemma for a precise definition) of a dense open set E in $\mathbb{Q}$. Lemma 2.7 is a version of Theorem 2.2 in [Ka], generalized for our needs. An easy argument based on Lemmas 2.7 and 2.9 can be used to show that $\mathbb{Q}$ preserves λ^+ (which we stated above without proof, having referred directly to [Ka]). While simple reduction is enough to show that λ^+ is preserved, it does not seem good enough to show that h is a generic filter. We will therefore introduce a stronger form of reduction, "rich reduction", in Definition 2.12.

In preparation for the statement of Lemma 2.7, we make the following notational conventions. If p is a condition in $\mathrm{Sacks}(\lambda,1)$ and s is in p, we write $p|s$ to denote the restriction of p to s: $p|s = \{s' \in p \mid s' \subseteq s \text{ or } s \subseteq s'\}$. If $r \le p|s$, then by the *amalgamation* of r and p we mean the tree r' which is obtained by thinning $p|s$ to r while keeping the rest of p intact, i.e. $r' = r \cup (p \setminus p|s)$.

If p is a condition in $\mathrm{Sacks}(\lambda,1)$ and s is in $\mathrm{Split}_\alpha(p)$, we write $p|^l s$ (l stands for *left*) for the restriction of p to the leftmost continuation of s in p (1):

$$p|^l s = \begin{cases} p|s^\frown 0, & \alpha \text{ a successor, or } \mathrm{cf}(\alpha) = \omega_1, \\ p|s^\frown i, & \alpha \text{ a limit, } \mathrm{cf}(\alpha) \ne \omega_1;\ i \in \{0,1\} \text{ unique such that } s^\frown i \in p. \end{cases}$$

The necessity to distinguish different cofinalities in defining $p|^l s$ follows from the fact that the limits of splitting nodes may not actually split themselves (see Definition 1.3 for more about this).

The notation $p|^l \vec{s}$ is straightforwardly generalized to $p \in \mathrm{Sacks}(\lambda, \lambda^{++})$ and a sequence $\vec{s} : \mathrm{supp}(p) \to {}^{<\lambda}2$, where for each $\beta \in \mathrm{supp}(p)$, $\vec{s}(\beta)$ is a

(1) We identify 0 with "left", and 1 with "right".

splitting node in $p(\beta)$. Similarly we generalize the notion of amalgamation to sequences of trees.

Let us further, for each $\alpha < \lambda$ and $\xi < \alpha$, denote by $S_\alpha(\xi)$ the characteristic function of the projection of S_α to ξ: $S_\alpha(\xi)(\zeta) = 1 \Leftrightarrow \langle \xi, \zeta \rangle \in S_\alpha$.

Before reading Lemma 2.7, recall the notational conventions as spelled out in Section 1.3, in particular in (1.8).

LEMMA 2.7. *Let E be a dense open set in $\mathbb{Q}$. For every $p \in \mathbb{Q}$ there are*

(i) *a decreasing fusion sequence $(\langle p_\alpha \mid \alpha < \lambda \rangle, \langle F_\alpha \mid \alpha < \lambda \rangle)$ with $p_0 = p$,*
(ii) *a condition $q \le p$, the fusion limit of the sequence $\langle p_\alpha \mid \alpha < \lambda \rangle$,*
(iii) *a sequence of bijections $\langle \pi_\alpha \mid \alpha < \lambda \rangle$, $\pi_\alpha : F_\alpha \to \eta_\alpha$ for some $\eta_\alpha < \lambda$, such that if $\alpha < \beta$, then $\pi_\alpha \subseteq \pi_\beta$, and $\pi_\gamma = \bigcup_{\alpha<\gamma} \pi_\alpha$ for γ a limit,*
(iv) *a bijection $\pi : \mathrm{supp}(q) \to \lambda$, the union of π_α's: $\pi = \bigcup_{\alpha<\lambda} \pi_\alpha$,*

such that whenever $t \le q$, then there is an $\alpha < \lambda$ such that

(2.7) $\alpha = \eta_\alpha$,

(2.8) *for each $\beta \in F_\alpha$, $S_\alpha(\pi_\alpha(\beta)) \in \mathrm{Split}_\alpha(q(\beta))$,*
and the restriction $t|^l \langle S_\alpha(\pi_\alpha(\beta)) \mid \beta \in F_\alpha \rangle$ is defined and is in E.

We say that q is a simple reduction *of the dense open set E with parameters $\langle p_\alpha \mid \alpha < \lambda \rangle$, $\langle F_\alpha \mid \alpha < \lambda \rangle$ and $\langle \pi_\alpha \mid \alpha < \lambda \rangle$.*

Proof. Use some standard fixed strategy to keep the sequences $\langle F_\alpha \mid \alpha < \lambda \rangle$ and $\langle \pi_\alpha \mid \alpha < \lambda \rangle$ continuous and extending under inclusion. It suffices to show how to construct $p_{\alpha+1}$ if we have p_α, F_α and π_α.

Consider the following property of F_α and π_α:

($*$) π_α is a function from F_α onto α, i.e. $\eta_\alpha = \alpha$.

If ($*$) does not hold, set $p_{\alpha+1} = p_\alpha$, $F_{\alpha+1} = F_\alpha$ and $\pi_{\alpha+1} = \pi_\alpha$. If ($*$) holds, consider the following property of p_α, F_α, and π_α:

($**$) For each $\beta \in F_\alpha$, $S_\alpha(\pi_\alpha(\beta))$ is a splitting node in $p_\alpha(\beta)$.

If ($**$) does not hold, set $p_{\alpha+1} = p_\alpha$, $F_{\alpha+1} = F_\alpha$ and $\pi_{\alpha+1} = \pi_\alpha$.

If both ($*$) and ($**$) hold, consider the restriction $p_\alpha|^l \langle S_\alpha(\pi_\alpha(\beta)) \mid \beta \in F_\alpha \rangle$, which we will denote as p^*_α. Extend p^*_α to some $r_\alpha \le p^*_\alpha$ with r_α in E, and define $p_{\alpha+1}$ to be the amalgamation of r_α and p_α.

Define $q = \bigwedge_{\alpha<\lambda} p_\alpha$, $\pi = \bigcup_{\alpha<\lambda} \pi_\alpha$. We can assume without loss of generality that π is a bijection from $\mathrm{supp}(q)$ onto λ. Let $t \le q$ be arbitrary. For each $\beta \in \mathrm{supp}(q)$, let $\mathrm{lbranch}(t(\beta)) \in {}^\lambda 2$ denote the leftmost branch of $t(\beta)$. Denote by $L(t)$ the sequence of the leftmost branches in t on $\mathrm{supp}(t) \cap \mathrm{supp}(q)$ mod π: $L(t) = \langle b(\beta) \mid \beta < \lambda \rangle$, where $b(\beta) \in {}^\lambda 2$ and $b(\beta)(\delta) =$

lbranch$(t(\pi^{-1}(\beta)))(\delta)$ for every $\delta < \lambda$. Then $L(t)$ can be viewed as a subset of $\lambda \times \lambda$.

For each $\alpha < \lambda$, consider the sequence $\langle s^\alpha(\beta) \mid \beta \in F_\alpha \rangle$ such that $s^\alpha(\beta) : \rho^\alpha(\beta) \to 2$ for some $\rho^\alpha(\beta) \geq \alpha$ defined as follows:

(2.9) For each $\beta \in F_\alpha$, $s^\alpha(\beta) \subseteq$ lbranch$(t(\beta))$ and
$s^\alpha(\beta)$ is an αth splitting node in $t(\beta)$.

SUBLEMMA 2.8. *The sets* $C_1 = \{\alpha < \lambda \mid \alpha = \eta_\alpha\}$ *and* $C_2 = \{\alpha < \lambda \mid \rho^\alpha(\beta) = \alpha$ *for every* $\beta \in F_\alpha\}$ *are both closed unbounded in* λ.

Proof. A standard Löwenheim–Skolem type argument using the fact that F_α's and π_α's are continuous and also the splitting nodes in the trees are continuous. ■

By $\diamondsuit_\lambda$, there exists some $\alpha \in C_1 \cap C_2$ such that $S_\alpha = L(t) \cap (\alpha \times \alpha)$. It follows that the construction step $p_{\alpha+1}$ was non-trivial, $t|^l \langle S_\alpha(\pi(\beta)) \mid \beta \in F_\alpha \rangle \leq r_\alpha$ and hence $t|^l \langle S_\alpha(\pi(\beta)) \mid \beta \in F_\alpha \rangle$ is in E as required. ■

We now generalize Lemma 2.7 to take care of λ-many dense open sets in $\mathbb{Q}$.

LEMMA 2.9. *Let* $\langle E_\alpha \mid \alpha < \lambda \rangle$ *be a sequence of dense open sets in* $\mathbb{Q}$ *and let* $p \in \mathbb{Q}$. *Then there is a fusion sequence* $(\langle q_\alpha \mid \alpha < \lambda \rangle, \langle \tilde{F}_\alpha \mid \alpha < \lambda \rangle)$ *with* $p = q_0$ *and fusion limit* q *such that for each* $\alpha < \lambda$ *the condition* $q_{\alpha+1}$ *is a simple reduction of* E_α *below* q_α *as in Lemma* 2.7.

We say that q *is* a simple reduction *of the sequence* $\langle E_\alpha \mid \alpha < \lambda \rangle$ *with parameters* $\langle q_\alpha \mid \alpha < \lambda \rangle$ *and* $\langle \tilde{F}_\alpha \mid \alpha < \lambda \rangle$.

Proof. Notice that in Lemma 2.7, we could have started with another two parameters $\gamma < \lambda$ and $F \subseteq \mathrm{supp}(p)$, $|F| < \lambda$. With these, modify the proof of Lemma 2.7 to set $F_0 = F$ and demand $p_{\alpha+1} \leq_{F_\alpha, \gamma+\alpha} p_\alpha$, which can be ensured by starting the construction at some indecomposable ordinal $\alpha > \gamma$ and setting $p_\beta = p_0$ for $\beta < \alpha$. Thus, in order to construct $q_{\alpha+1}$, start the construction in Lemma 2.7 with $\tilde{F}_\alpha \subseteq \mathrm{supp}(q_\alpha)$ and α as the two additional parameters. This ensures that $q_{\alpha+1}$ is a simple reduction of E_α below q_α (with respect to $\langle F_\alpha \mid \alpha < \lambda \rangle$, where $F_0 = \tilde{F}_\alpha$, and $\langle \pi_\alpha \mid \alpha < \lambda \rangle$ as detailed in the construction in the proof of Lemma 2.7). ■

In order to be able to show that h is a generic filter, we need to introduce a stronger form of reduction—the *rich reduction.*

Consider now the following modification of the construction of the simple reduction q below p in Lemma 2.7. Again, q will be the fusion limit of a sequence $(\langle p_\alpha \mid \alpha < \lambda \rangle, \langle F_\alpha \mid \alpha < \lambda \rangle)$, with respect to the sequence of mappings $\langle \pi_\alpha \mid \alpha < \lambda \rangle$. If α is a successor ordinal, or a limit ordinal of cofinality other than ω, we construct $p_{\alpha+1} \leq_{F_\alpha, \alpha} p_\alpha$ exactly as in Lemma 2.7. However, if α is a limit ordinal of cofinality ω and $(*)$ in Lemma 2.7 holds we will do more. In order to explain what we do, we first introduce some notation.

Definition 2.10. Assume α has cofinality ω and $\alpha = \eta_\alpha$. We say that a sequence $\vec{\alpha} = \langle \alpha_n \mid n < \omega \rangle$ is *suitable* for α if it is strictly increasing and cofinal in α, and moreover for each n, the construction of p_{α_n+1} was non-trivial; i.e. α_n (with parameters p_{α_n}, F_{α_n}, and π_{α_n}) satisfies both $(*)$ and $(**)$ in Lemma 2.7.

Let $\vec{\alpha}$ be suitable for α. For each δ in F_α, let $n(\delta) \in \omega$ be the least natural number n such that $\delta \in F_{\alpha_n}$; then we can meaningfully consider $S_{\alpha_m}(\delta) \restriction \alpha_{m-1}$, the restriction of $S_{\alpha_m}(\delta)$ to the initial segment of length α_{m-1}, for every $m > n(\delta)$. For every $\delta \in F_\alpha$, we define a family of initial segments $I^{\vec{\alpha}}(\delta)$ as follows:

$$I^{\vec{\alpha}}(\delta) = \{S_{\alpha_m}(\delta) \restriction \alpha_{m-1} \mid m > n(\delta)\}. \tag{2.10}$$

Definition 2.11. We say that $I^{\vec{\alpha}}(\delta)$ is *coherent* if the union $\bigcup I^{\vec{\alpha}}(\delta)$ determines a function in ${}^{\alpha}2$.

Notice that the fact that $I^{\vec{\alpha}}(\delta)$ is coherent is equivalently expressed by demanding $S_{\alpha_m}(\delta) \restriction \alpha_{m-1} \subseteq S_{\alpha_k}(\delta) \restriction \alpha_{k-1}$ whenever $n(\delta) < m \leq k$.

So assume $p_\alpha, F_\alpha, \pi_\alpha$ are given, α has cofinality ω, and $\alpha = \eta_\alpha$ (i.e. $(*)$ holds). In order to construct $p_{\alpha+1}$, first check if $(**)$ holds. If it does, construct r_α as in Lemma 2.7 and denote by p_α^+ the amalgamation of r_α and p_α. If $(**)$ does not hold, set $p_\alpha^+ = p_\alpha$.

Next, for each suitable $\vec{\alpha}$ such that $I^{\vec{\alpha}}(\delta)$ is coherent for each $\delta \in F_\alpha$, check the following property:

$(***)_{\vec{\alpha}}$ For each $\delta \in F_\alpha$, $\bigcup I^{\vec{\alpha}}(\delta)$ is a splitting node in $p_\alpha(\delta)$.

Enumerate all suitable and coherent sequences $\vec{\alpha}$ satisfying $(***)_{\vec{\alpha}}$ as $\langle a_\xi \mid \xi < \zeta \rangle$. Note that since λ' has cofinality $> \omega$, we have $\zeta \leq \lambda'$. Now build a decreasing sequence $p_\alpha^+ \geq_{F_\alpha,\alpha} p_\alpha^0 \geq_{F_\alpha,\alpha} p_\alpha^1 \geq \cdots$ such that the greatest lower bounds are taken at limits, and for each p_α^ξ, the condition $p_\alpha^{\xi+1}$ is obtained as an amalgamation of p_α^ξ and a condition r^ξ which satisfies $r^\xi \leq p_\alpha^\xi | \langle \bigcup I^{a_\xi}(\delta) \mid \delta \in F_\alpha \rangle$ and $r^\xi \in E$. Notice that $p_\alpha^\xi | \langle \bigcup I^{a_\xi}(\delta) \mid \delta \in F_\alpha \rangle$ is the same condition as $p_\alpha^\xi |^l \langle \bigcup I^{a_\xi}(\delta) \mid \delta \in F_\alpha \rangle$ since $\mathrm{cf}(\alpha) = \omega$ and therefore by our Definition 1.3, there is no actual splitting at nodes of length α. Set $p_{\alpha+1}$ to be the greatest lower bound of the sequence $p_\alpha^+ \geq_{F_\alpha,\alpha} p_\alpha^0 \geq_{F_\alpha,\alpha} p_\alpha^1 \geq \cdots$ (note that if there are no $\vec{\alpha}$'s satisfying $(***)_{\vec{\alpha}}$, then $p_{\alpha+1} = p_\alpha^+$). It follows that $p_{\alpha+1} \leq_{F_\alpha,\alpha} p_\alpha$ as desired.

Definition 2.12. Let E be a dense open set in $\mathbb{Q}$ and p a condition in $\mathbb{Q}$. We say that q is a *rich reduction* of E with parameters $\langle p_\alpha \mid \alpha < \lambda \rangle$, $\langle F_\alpha \mid \alpha < \lambda \rangle$, and $\langle \pi_\alpha \mid \alpha < \lambda \rangle$ if q is constructed exactly as described in the previous paragraphs, starting with the paragraph just before Definition 2.10.

First notice that if q is a rich reduction, it is in particular a simple reduction, and hence the conclusion of Lemma 2.7 applies to q. Intuitively, with rich reductions we give ourselves up to λ'-many options to thin out to E at stages of cofinality ω; in contrast, in the simple reduction, we give ourselves at most one option to thin out to E (this one option is determined by the relevant set S_α). The key property of the rich reduction is that the options to which we can thin out are rich enough to argue that h is indeed generic.

For completeness, we state Lemma 2.9 formulated for the rich reduction.

LEMMA 2.13. *Let $\langle E_\alpha \mid \alpha < \lambda\rangle$ be a sequence of dense open sets in $\mathbb{Q}$ and let $p \in \mathbb{Q}$. Then there is a fusion sequence $(\langle q_\alpha \mid \alpha < \lambda\rangle, \langle \tilde{F}_\alpha \mid \alpha < \lambda\rangle)$ with $p = q_0$ and fusion limit q such that for each $\alpha < \lambda$ the condition $q_{\alpha+1}$ is a rich reduction of E_α below q_α.*

We say that q is a rich reduction *of the sequence $\langle E_\alpha \mid \alpha < \lambda\rangle$ with parameters $\langle q_\alpha \mid \alpha < \lambda\rangle$ and $\langle \tilde{F}_\alpha \mid \alpha < \lambda\rangle$.*

Proof. This is just like the proof of Lemma 2.9, except that $q_{\alpha+1}$ as in that proof is constructed using the notion of rich reduction. ∎

Let us now return to the lifting argument.

Recall that we denote $\sup j''\lambda$ as Δ. We now show that for each $\gamma < j(\lambda)^{++}$, $j[g]$ determines a unique element of ${}^{\Delta}2$, which we denote g_γ.

LEMMA 2.14. *For each $\gamma < j(\lambda)^{++}$,*

$$\bigcap\{j(p)(\gamma) \mid p \in g \;\&\; \gamma \in \operatorname{supp}(j(p))\} \cap {}^{\Delta}2 = \{g_\gamma\} \quad \textit{for some } g_\gamma.$$

Proof. Let γ be given. There is some function $e : (P_\kappa(\lambda) \times \kappa)^V \to \lambda^{++}$ such that $j(e)(j''\lambda, \alpha) = \gamma$ for some $\alpha < \lambda^{++}$. Since $(P_\kappa(\lambda) \times \kappa)^V$ has size λ, we can view e as a function from λ to λ^{++}, and so $\operatorname{rng}(e)$ can play the role of a support of a condition in $\mathbb{Q}$. For each $\alpha < \lambda$, there is by density a condition p_α in g such that $\operatorname{supp}(p_\alpha)$ contains $\operatorname{rng}(e)$ and for every $\xi \in \operatorname{supp}(p_\alpha)$, $p_\alpha(\xi)$ has stem of length at least α; by elementarity the stem of $j(p_\alpha)$ has at every element of its domain, in particular at γ, length at least $j(\alpha)$. It follows that the intersection $\bigcap_{\alpha<\lambda} j(p_\alpha)(\gamma)$ determines a unique subset g_γ of Δ. ∎

A similar argument implies that if q is a condition in g, then $j(q)|\langle g_\gamma \mid \gamma \in \operatorname{supp}(j(q))\rangle$ is well-defined and a condition in $j(\mathbb{Q})$. Indeed, for each $\alpha < \lambda$ there is a condition $q_\alpha \le q$ such that q_α is in g, $\operatorname{supp}(q_\alpha) = \operatorname{supp}(q)$, and at each $\xi \in \operatorname{supp}(q_\alpha)$, $q_\alpha(\xi)$ has stem of length at least α. Then the greatest lower bound of the sequence $\langle j(q_\alpha) \mid \alpha < \lambda\rangle$ is some condition $r \in j(\mathbb{Q})$ with $\operatorname{supp}(r) = \operatorname{supp}(j(q))$, and for each $\xi \in \operatorname{supp}(r)$, the stem of $r(\xi)$ has length at least Δ. It follows that the sequence $\langle g_\gamma \mid \gamma \in \operatorname{supp}(j(q))\rangle$ is in $M[G * g * H]$, and so $j(q)|\langle g_\gamma \mid \gamma \in \operatorname{supp}(j(q))\rangle$ is well-defined and a condition in $j(\mathbb{Q})$.

If $p \in \mathrm{Sacks}(\lambda, 1)$ and $\xi < \zeta < \lambda$, we say that p *does not split between ξ and ζ* if there is no node $s \in p$ such that both $s^\frown 0$ and $s^\frown 1$ are in p, and the height of s is an ordinal in the interval $[\xi, \zeta]$.

LEMMA 2.15. *Let p be a condition in $\mathbb{Q}$ and $\langle F_\alpha \mid \alpha < \lambda\rangle$ a continuous and increasing sequence of subsets of λ^{++}, each with size $< \lambda$, such that $\bigcup\{F_\alpha \mid \alpha < \lambda\} = \mathrm{supp}(p)$. Let $\langle F^*_\alpha \mid \alpha < j(\lambda)\rangle = j(\langle F_\alpha \mid \alpha < \lambda\rangle)$. Then for each $\Delta \le \gamma < j(\lambda)$ and $q \le p$ there exists some $r \le q$ such that $j(r)$ does not split between Δ and γ on coordinates in F^*_γ.*

Proof. We have formulated the lemma with some care to anticipate a possible extension of the present technique to the *iteration* of the generalized Sacks forcing at λ; moreover, the lemma as it stands is sufficient for our purposes to prove Theorem 2.3. However, since $\mathrm{Sacks}(\lambda, \lambda^{++})$ is a *product*, we can prove a substantially stronger statement:

(†) Let p be a condition in $\mathbb{Q}$ and $\Delta < \gamma < j(\lambda)$. Then for each $q \le p$ there is some $r \le q$ such that $\mathrm{supp}(r) = \mathrm{supp}(q)$ and for each $\delta \in \mathrm{supp}(j(r))$, $j(r)(\delta)$ does not split between Δ and γ.

Let $d : P_\kappa(\lambda)^V \times \kappa \to \lambda$ be such that $j(d)(j''\lambda, \bar\gamma) = \gamma$ for some $\bar\gamma < \lambda^{++}$. Recall that we assume that there exists in V a function f_Δ (and a function c) as in (2.3) in Section 2.1. Let us define a certain closed unbounded set C_d as follows: we say that $\xi < \lambda$ is a *closure point for the pair* (f_Δ, d) if

(i) $f_\Delta(\zeta) < \xi$ whenever $\zeta < \xi$,
(ii) $d(x, \zeta) < \xi$ whenever $\zeta < \xi$ and $c(x) < \xi$.

Let C_d be the closed unbounded set of the closure points for the pair (f_Δ, d).

In order to prove (†), we thin out $q(\delta)$ for each $\delta \in \mathrm{supp}(q)$ according to C_d as follows (note that we will consider each coordinate δ in $\mathrm{supp}(q)$ separately; this is possible as we work with a product). Fix $\delta \in \mathrm{supp}(q)$. We will build a fusion sequence $q(\delta) = q^0(\delta) \ge_0 q^1(\delta) \ge_1 \cdots$ for this δ. If α is a limit ordinal, set $q^\alpha(\delta) = \bigwedge_{\beta<\alpha} q^\beta(\delta)$. In order to construct $q^{\alpha+1}(\delta)$ from $q^\alpha(\delta)$, consider each node $s \in {}^{\alpha+1}2 \cap q^\alpha(\delta)$ and thin out $q^\alpha(\delta)|s$ to some $r(s)$ such that $r(s)$ has stem of length at least that of the next element of C_d strictly above α. Set $q^{\alpha+1}(\delta)$ to be the amalgamation of the trees $r(s)$'s; then $q^{\alpha+1}(\delta) \le_\alpha q^\alpha(\delta)$. Let $\tilde q(\delta)$ be the fusion limit of the sequence $\langle q^\alpha(\delta) \mid \alpha < \lambda\rangle$. Finally, set $r(\delta) = \tilde q(\delta)$ for each $\delta \in \mathrm{supp}(q)$, and $r(\delta') = 1_{\mathrm{Sacks}(\lambda,1)}$ on $\delta' \notin \mathrm{supp}(q)$.

We show that r satisfies (†). By elementarity, for every $\delta \in \mathrm{supp}(j(r))$, the tree $j(r)(\delta)$ is the fusion limit according to the j-version of the construction in the previous paragraph starting with $j(q)(\delta)$. In particular, at stage Δ, the appropriate stage of the fusion construction ensures that $j(r)(\delta)$ does not split between Δ (as $\mathrm{cf}(\Delta) \ne \omega_1$, and therefore by Definition 1.3, there is no actual splitting at Δ) and the next element of $j(C_d)$, the closed unbounded set of closure point for the pair $(j(f_\Delta), j(d))$. However, since $\gamma = j(d)(j''\lambda, \bar\gamma)$, and crucially $j(f_\Delta)(\Delta) > j(c)(j''\lambda)$, the next closure point in $j(C_d)$ must be strictly greater than γ. ∎

Note that Lemmas 2.14 and 2.15 together imply that for each $\gamma < j(\lambda)^{++}$, the intersection $\bigcap_{p\in h} p(\gamma)$ determines a unique element of ${}^{j(\lambda)}2 \cap V[G*g]$. This is not yet enough to conclude that h is a generic filter over $M[G*g*H]$; however, it is a necessary condition for h being generic. We now use the notion of rich reduction to argue that h meets all dense open sets in $M[G*g*H]$, and is therefore a generic filter for $j(\mathbb{Q})$ over $M[G*g*H]$.

For the rest of the argument, fix a dense open set D in $j(\mathbb{Q})$, where

$$D = j(f)(j''\lambda, \nu) \quad \text{for some } \nu < \lambda^{++} \text{ and } f : (P_\kappa(\lambda)\times\kappa)^V \to V[G], \tag{2.11}$$

where we can assume that the range of f consists only of dense open sets in $\mathbb{Q}$. Since $(P_\kappa(\lambda)\times\kappa)^V$ has size λ in $V[G]$, we can enumerate the range of f as some sequence $\langle E_\alpha \mid \alpha < \lambda\rangle$.

By Lemma 2.13, we can choose a $q \in g$ which is a rich reduction of the sequence $\langle E_\alpha \mid \alpha < \lambda\rangle$ with some parameters $\langle q_\alpha \mid \alpha < \lambda\rangle$, $\langle \tilde F_\alpha \mid \alpha < \lambda\rangle$, and $\langle p^\alpha_\beta \mid \beta < \lambda\rangle$, $\langle F^\alpha_\beta \mid \beta < \lambda\rangle$, and $\langle \pi^\alpha_\beta \mid \beta < \lambda\rangle$ (where the last three sequences witness the construction of $q_{\alpha+1}$ which richly reduces E_α). Let us for $\alpha < j(\lambda)$ denote by the star "$*$" the j-version of the parameters: for instance $\langle \pi^{*\alpha}_\beta \mid \beta < j(\lambda)\rangle$, and so on. By elementarity, $j(q)$ is a rich reduction of the sequence $\langle E^*_\alpha \mid \alpha < j(\lambda)\rangle$ (with the *-version of the relevant parameters), so in particular it is a rich reduction of our dense open set D. Let us fix μ such that

$$E^*_\mu = D. \tag{2.12}$$

Furthermore, let $\langle S^*_\alpha \mid \alpha < j(\lambda)\rangle$ denote the j-version of the diamond sequence $\diamondsuit_\lambda$ in $V[G]$ which we have fixed above.

Consider the condition $j(q)|\langle g_\delta \mid \delta \in \mathrm{supp}(j(q))\rangle \le q$.

By Lemma 2.7 and elementarity of j, there is some $\alpha_0 > \Delta$ such that $\pi^{*\mu}_{\alpha_0} : F^{*\mu}_{\alpha_0} \to \alpha_0$, and the restriction of $j(q)|\langle g_\delta \mid \delta \in \mathrm{supp}(j(q))\rangle$ to the sequence $\langle S^*_{\alpha_0}(\pi^{*\mu}_{\alpha_0}(\delta)) \mid \delta \in F^{*\mu}_{\alpha_0}\rangle$ is defined. In particular, for each such δ, $g_\delta \subseteq S^*_{\alpha_0}(\pi^{*\mu}_{\alpha_0}(\delta))$.

Choose some $\beta_0 \ge \alpha_0$ such that $\tilde F^*_{\beta_0} \supseteq F^{*\mu}_{\alpha_0}$ (there is always some such β_0 because the supports in $\langle \tilde F^*_\alpha \mid \alpha < j(\lambda)\rangle$ must eventually cover $F^{*\mu}_{\alpha_0}$). By Lemma 2.15 and elementarity of j, there is some $q_0 \le q$, $q_0 \in g$, such that $j(q_0)$ does not split between Δ and β_0 on $\tilde F^*_{\beta_0}$. In particular, $j(q_0)$ does not split between Δ and α_0 on $F^{*\mu}_{\alpha_0}$. Let us denote by $\vec g^{\,0} = \langle g^0_\delta \mid \delta \in F^{*\mu}_{\alpha_0}\rangle$ the unique nodes in $j(q_0)(\delta)\cap {}^{\alpha_0}2$, $\delta \in F^{*\mu}_{\alpha_0}$, such that $g^0_\delta\restriction\Delta = g_\delta$ for each δ in $F^{*\mu}_{\alpha_0}$.

Consider now the condition $j(q)|\vec g^{\,0} \le j(q)$. By invoking Lemma 2.7 and elementarity again, there is some $\alpha_1 > \alpha_0$ such that $\pi^{*\mu}_{\alpha_1} : F^{*\mu}_{\alpha_1} \to \alpha_1$, and the restriction $j(q)|\vec{g}^0$ to $\langle S^*_{\alpha_1}(\pi^{*\mu}_{\alpha_1}(\delta)) \mid \delta \in F^{*\mu}_{\alpha_1}\rangle$ is defined. In particular, for each $\delta \in F^{*\mu}_{\alpha_0}$, $g^0_\delta \subseteq S^*_{\alpha_1}(\pi^{*\mu}_{\alpha_1}(\delta))$.

We can repeat this argument ω-many times, obtaining an increasing sequence $\langle \alpha_n \mid n < \omega\rangle$ of ordinals, a decreasing sequence of conditions $\langle q_n \mid n < \omega\rangle$ in g and a sequence $\langle \vec{g}^n \mid n < \omega\rangle$. Let $\alpha = \sup\{\alpha_n \mid n < \omega\}$, $\vec{\alpha} = \langle \alpha_n \mid n < \omega\rangle$ and $\tilde{q} = \bigwedge_{n<\omega} q_n$ (note that $\tilde{q} \in g$). By the construction, $\vec{\alpha}$ is a suitable and coherent sequence in the sense of $(***)_{\vec{\alpha}}$, and so the j-version of the construction of the rich reduction $j(q)$ was non-trivial at stage α, and hence

$$(2.13) \qquad j(\tilde{q})|\langle g_\delta \mid \delta \in F^{*\mu}_\alpha\rangle \text{ is in } D \cap h \text{ as required.}$$

Note that (2.13) crucially uses the fact that $j(q)|\langle \bigcup I^{\vec{\alpha}}(\delta) \mid \delta \in F^{*\mu}_\alpha\rangle$ is the same condition as $j(q)|^l\langle \bigcup I^{\vec{\alpha}}(\delta) \mid \delta \in F^{*\mu}_\alpha\rangle$ since $\mathrm{cf}(\alpha) \neq \omega_1$, and so by Definition 1.3, there is no actual splitting for nodes of length α.

This ends the proof of Theorem 2.3 when λ is a successor of a cardinal λ' and $\mathrm{cf}(\lambda') > \omega$. See the following remarks for the remaining cases.

REMARK 2.16 (The case of $\mathrm{cf}(\lambda') = \omega$). We need to modify the description of rich reduction in Definition 2.12 because when discussing $(***)_{\vec{\alpha}}$ in the paragraph preceding Definition 2.12, we have used the assumption that $\mathrm{cf}(\lambda') > \omega$ to argue that all suitable and coherent sequences (see Definitions 2.10 and 2.11, respectively) can be enumerated in at most λ'-many steps (in $V[G]$, $\mathrm{cf}(\lambda') > \omega$ implies $\lambda'^\omega = \lambda'$). To overcome this technical problem, consider the following modifications to the proof:

(i) In Definition 1.3, change the conditions (iii) and (iv) to refer to cofinality ω instead of cofinality ω_1, obtaining a *cof ω-splitting perfect tree at λ*. Define the Sacks forcing at λ with these trees. Since ordinals with cofinality ω are stationary in λ, the forcing behaves essentially in the same way as the one composed of the cof ω_1-splitting perfect trees.

(ii) Lemma 2.7 remains exactly the same, and so does the notion of simple reduction.

(iii) The notion of rich reduction will now concern ordinals $\alpha = \eta_\alpha$, where $\mathrm{cf}(\alpha) = \omega_1$ (note that by using cof ω-splitting perfect trees now, the nodes of length α do not actually split). In particular, in the definition of a suitable sequence in Definition 2.10, assume now that $\vec{\alpha} = \langle \alpha_i \mid i < \omega_1\rangle$ is a closed unbounded sequence in α. The notion of coherence in Definition 2.11 remains exactly the same, with the exception that the indices now range over ω_1. However, there are still possibly λ-many such suitable and coherent sequences. To repair this, we will argue that without loss of generality only λ'-many suitable sequences can be considered. Let f_α be a fixed injection from α to λ'.

CLAIM 2.17. *Every suitable sequence $\vec{\alpha} = \langle \alpha_i \mid i < \omega_1 \rangle$ contains a subsequence $\langle \alpha_i^* \mid i < \omega_1 \rangle$ cofinal in α such that $f''_\alpha\{\alpha_i^* \mid i < \omega_1\}$ is bounded in λ'.*

Proof. This follows from the fact that λ' has cofinality ω, and $\vec{\alpha}$ is a sequence of length ω_1. ∎

By the GCH, there are only λ'-many bounded subsets of λ', and so there are only λ'-many such subsequences. Therefore, in Definition 2.12 of rich reduction consider only suitable and coherent sequences $\vec{\alpha}$ such that $\vec{\alpha}$ is the closure of a sequence $\langle \alpha_i^* \mid i < \omega_1 \rangle$ cofinal in α such that $f''_\alpha\{\alpha_i^* \mid i < \omega_1\}$ is bounded in λ'. Claim 2.17 shows that this is still sufficient.

(i) The final argument, which starts at (2.11) and which shows that $D \cap h$ is non-empty for every relevant dense open set D, is a simple modification to the proof above: instead of constructing sequences $\langle \alpha_n \mid n < \omega \rangle$, $\langle q_n \mid n < \omega \rangle$ and $\langle \vec{g}^n \mid n < \omega \rangle$, construct analogous sequences $\langle \alpha_i \mid i < \omega_1 \rangle$, $\langle q_i \mid i < \omega_1 \rangle$, and $\langle \vec{g}^i \mid i < \omega_1 \rangle$. By Claim 2.17, the sequence $\langle \alpha_i \mid i < \omega_1 \rangle$ can be thinned into a closed cofinal subsequence which is considered in the j-version of the rich reduction step at α.

REMARK 2.18 (The case of an inaccessible λ). If λ is inaccessible, then the use of a $\diamondsuit_\lambda$-style argument can be completely avoided. In the case of an inaccessible λ, redefine $p \leq_\alpha q$ for p and q in $\mathrm{Sacks}(\lambda, 1)$ (as in Definition 1.3) and $\alpha < \lambda$ as $p \leq q$ and $\mathrm{Split}_\alpha(p) = \mathrm{Split}_\alpha(q)$. Let p be a condition and $F \subseteq \mathrm{supp}(p)$, $|F| < \lambda$; we say that $\vec{s} = \langle s(\delta) \mid \delta \in F \rangle$ is a *selection sequence for* (p, F) *at* α if for every $\delta \in F$, $s(\delta)$ is in $\mathrm{Succ}_\alpha(p(\delta))$, where $\mathrm{Succ}_\alpha(p(\delta))$ is the collection of all nodes $t^\frown i$ in $p(\delta)$ with $t \in \mathrm{Split}_\alpha(p(\delta))$ and $i \in \{0, 1\}$.

The appropriate version of Lemma 2.13 is now as follows. If $\langle E_\alpha \mid \alpha < \lambda \rangle$ is a sequence of dense open sets in $\mathbb{Q}$, then one can construct below each $p \in \mathbb{Q}$ a decreasing sequence of conditions $p = p_0 \geq_{F_0,0} p_1 \geq_{F_1,1} \cdots$ such that at each $\alpha < \lambda$, $p_{\alpha+1}$ reduces E_α in the following strong sense. For each selection sequence $\vec{s}$ for (p_α, F_α) at α, the restriction $p_{\alpha+1}|\vec{s}$ is in E_α. Such $p_{\alpha+1}$ can be constructed because by the inaccessibility of λ, there are strictly less than λ-many selection sequences for each (p_α, F_α).

The lifting argument is analogous to the one given above but simpler because now we can assume that given $D = E^*_\mu$ as in (2.12), there exists some γ with $\varDelta < \gamma < j(\lambda)$ and $q \in g$ such that *any* thinning of $j(q)$ to stems of length γ hits D. Such a thinning is easily obtained by application of Lemma 2.15.

The impossibility of considering all relevant selection sequences when reducing a sequence $\langle E_\alpha \mid \alpha < \lambda \rangle$ for a successor λ is the main (and only) reason why the more complicated construction using $\diamondsuit_\lambda$ was used in the proof of Theorem 2.3.

This ends the proof of Theorem 2.3. ■

The technique in this paper allows one to control the continuum function on regular cardinals in many ways while preserving the desired strength of the given embedding j. In Theorem 2.3, we have limited ourselves to enlarging 2^λ to λ^{++} while keeping GCH in the interval $[\kappa, \lambda)$. More patterns are possible, using the ideas in [FrHo2]; in fact any "reasonable" pattern of the continuum function can be realized, with the exception of the configuration in Problem 3.1 which is currently open.

We close this paper with a final remark on the notions of suitable and coherent sequences presented in Definitions 2.10 and 2.11. We can consider a version of diamond which by definition already contains all these sequences (which are needed for the lifting argument). Indeed, assume $\lambda = (\lambda'^+)$; let us denote by $\diamondsuit'_\lambda$ the following combinatorial principle:

$\diamondsuit'_\lambda$: There is a sequence $\langle \vec{S}_\alpha \mid \alpha < \lambda\rangle$ such that for every $\alpha < \lambda$,
(i) $|\vec{S}_\alpha| \le \lambda'$,
(ii) each $S \in \vec{S}_\alpha$ is a subset of $\alpha \times \alpha$.
It is seen that for each $X \subseteq \lambda \times \lambda$, the set $\{\alpha < \lambda \mid X \cap (\alpha \times \alpha) \in \vec{S}_\alpha\}$ is stationary in λ.

The principle $\diamondsuit'_\lambda$ is a consequence of $\diamondsuit_\lambda$ because every $\diamondsuit_\lambda$-sequence is by definition also a $\diamondsuit'_\lambda$-sequence. However, the utility of $\diamondsuit'_\lambda$ comes from the fact if $\langle \vec{S}_\alpha \mid \alpha < \lambda\rangle$ is a $\diamondsuit'_\lambda$-sequence, then any sequence $\langle \vec{S}^0_\alpha \mid \alpha < \lambda\rangle$ such that $\vec{S}_\alpha \subseteq \vec{S}^0_\alpha$ and $|\vec{S}^0_\alpha| \le \lambda'$ is a also a $\diamondsuit'_\lambda$-sequence. It is easy to enlarge a given sequence so that it contains all suitable and coherent sequences used in this proof for each α such that $\mathrm{cf}(\alpha) = \omega$ (when $\mathrm{cf}(\lambda') > \omega$) or $\mathrm{cf}(\alpha) = \omega_1$ (when $\mathrm{cf}(\lambda') = \omega$). The presentation of the proof of Theorem 2.3 would be notationally (though not conceptually) simpler if we attempted to thin at each α not just to the single element S_α in the $\diamondsuit_\lambda$-sequence, but to all elements in $\vec{S}_\alpha$ in the $\diamondsuit'_\lambda$-sequence. This would enable us to work with a single type of reduction and not with two reductions (simple and rich) as we have done.

3. Open problems

Problem 3.1. *With the assumptions as in Theorem* 2.3, *can one obtain a model where* $2^\kappa < \lambda^{++}$, $2^\mu = 2^\lambda = \lambda^{++}$, *where* μ *is a regular cardinal in the open interval* (κ, λ), *and* κ *is still* λ*-supercompact?*

We do not know at the moment if the statement in Problem 3.1 can be proved using the methods introduced in this paper. Notice that the formulation in Problem 3.1 explicitly excludes the border cases $\mu = \kappa$ and $\mu = \lambda$. The reason is that the case of $\mu = \kappa$ is included in [Co], and $\mu = \lambda$ is solved by the present paper.

The main problem in generalizing the methods in this paper to solve Problem 3.1 concerns the length of fusion sequences. If μ is a regular cardinal in the interval $[\kappa, \lambda)$, the forcing Sacks(μ, λ^{++}) has fusion for sequences of length only μ, while fusion of length λ is necessary to carry out the reduction argument ([2]). It seems that one needs a forcing notion which adds new subsets of μ, but supports a genuine fusion construction of length up to λ.

Instead of the Sacks forcing, one can attempt to use the Cohen forcing with the "surgery argument" introduced by Woodin (see [Cu]) to prove results similar to those in the present paper. This is the approach originally adopted by Cody in [Co] who generalized the surgery argument of Woodin to supercompacts and proved in particular the following: starting with GCH and a λ^{++}-tall λ-supercompact κ, there is a forcing extension with $2^\kappa = 2^\lambda = \lambda^{++}$ and with κ still λ-supercompact. However, as is stated in [Co], this method does not seem sufficient to handle the direct manipulation of the size of 2^μ for a regular cardinal $\mu \in (\kappa, \lambda]$ because it relies on κ being the critical point of the given embedding.

Although it is hard to ascertain the exact limits of the surgery argument, once we move to iterations rather than products, as in [FrHo1], the surgery argument seems insufficient. It seems therefore worthwhile to attempt to find a generalization of the Sacks forcing which could be used to solve Problem 3.1.

We now turn to another problem. It is known that κ being measurable is strictly weaker in terms of consistency strength than κ being κ^{++}-tall (which is equiconsistent with κ being $H(\kappa^{++})$-strong) ([3]).

PROBLEM 3.2. *Consider the following concepts, where $\kappa < \lambda$ are regular: κ is λ-supercompact, κ is λ^{++}-tall λ-supercompact, and κ is λ-supercompact and $H(\lambda^{++})$-strong. Obviously, in terms of consistency, the first concept is less than or equal to the second, which is in turn less than or equal to the third. Is any of these inequalities strict?*

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([2]) A dense open set of Sacks$(j(\mu), j(\lambda^{++}))$ in the target model is represented as $j(f)(j''\lambda, \alpha)$, where f has domain $P_\kappa\lambda \times \kappa$ which has size λ under our assumptions.

([3]) κ is $H(\kappa^{++})$-strong if there is an embedding $j : V \to M$ with $H(\kappa^{++})$ included in M; this concept also goes under the names of $\kappa + 2$-strong or $P_2\kappa$-hypermeasurable.

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Sy-David Friedman
Kurt Gödel Research Center
for Mathematical Logic
Währinger Strasse 25
1090 Wien, Austria
E-mail: sdf@logic.univie.ac.at

Radek Honzik
Department of Logic
Charles University
Celetná 20
Praha 1, 116 42, Czech Republic
E-mail: radek.honzik@ff.cuni.cz

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