

C^1 stability of endomorphisms on two-dimensional manifolds

by

J. Iglesias, A. Portela and A. Rovella (Montevideo)

Abstract. A set of necessary conditions for C^1 stability of noninvertible maps is presented. It is proved that the conditions are sufficient for C^1 stability in compact oriented manifolds of dimension two. An example given by F. Przytycki in 1977 is shown to satisfy these conditions. It is the first example known of a C^1 stable map (noninvertible and nonexpanding) in a manifold of dimension two, while a wide class of examples are already known in every other dimension.

1. Introduction. After the work of many authors, it was proved by C. Robinson ([Rob]) and R. Mañé ([Ma]) that a diffeomorphism of a compact manifold is C^1 structurally stable if and only if it satisfies Axiom A and the strong transversality condition. It has also been known since M. Shub [Sh] that an expanding map is also C^1 stable. On the other hand, Anosov endomorphisms not falling in the above categories, fail to be stable ([Prz1] and [MP]).

The following is the definition of (strong) Axiom A:

DEFINITION 1. A C^1 map f satisfies *Axiom A* if:

- (A1) the nonwandering set Ω_f has a hyperbolic structure,
- (A2) the set of periodic points is dense in Ω_f , and
- (A3) each basic piece is either expanding or the restriction of f to it is injective.

In a compact manifold M , the following properties are necessary for a map $f \in C^1(M)$ to be C^1 structurally stable:

- (C1) The set of critical points of f is empty (a point x is *critical* or *singular* for f if the differential of f at x is noninvertible).
- (C2) The map f satisfies Axiom A without cycles.
- (C3) If the unstable set of a basic piece Λ intersects another basic set Λ' , $W^u(\Lambda) \cap \Lambda' \neq \emptyset$, then Λ is an expanding basic piece.

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Condition (C1) is obviously necessary for C^1 structural stability. Examples of C^r ($r \geq 2$) stable maps with critical points can be easily found for maps in manifolds of dimension one; some examples in dimension two were recently discovered (see [IPR1]). A theorem by P. Berger [Ber] gives sufficient conditions for C^∞ stability of maps having critical points.

F. Przytycki showed ([Prz2]) that if f is a C^1 Ω -stable map that satisfies conditions (A1) and (A2), then it also satisfies (A3). It follows that an Anosov endomorphism is not Ω -stable unless it is expanding or a diffeomorphism. Later, adapting the proof of the C^1 stability conjecture given by R. Mañé, it was proved by N. Aoki, K. Moriyasu and N. Sumi that if f is a C^1 Ω -stable map without critical points then $\Omega(f)$ has a hyperbolic structure ([AMS]). These two results imply that (C2) is a necessary condition for C^1 stability.

That (C3) is a necessary condition for C^1 stability was also shown by F. Przytycki in [Prz2, Theorem C].

F. Przytycki also gave there an example of a map on the two-torus satisfying the three conditions above, and asked if that map is structurally stable. The first objective of our work was to prove the stability of that map; we finally arrive at a characterization of C^1 stability in dimension two. Unfortunately, the techniques applied do not allow a generalization to higher dimensions. The same characterization of C^1 stability in higher dimensions stands as a conjecture.

Note that a self-map of a compact manifold must be a covering map if it has no singular points. It follows that the unique two-dimensional orientable manifold admitting a noninvertible C^1 stable map is the two-torus. There exist examples of C^1 stable maps on compact manifolds of dimension at least three. However, that construction cannot be carried out on manifolds of dimension two, because of the nature of the attractors (see [IPR2]). Indeed, the problem in dimension two is that it is not possible to find examples of C^1 stable maps without saddle type basic pieces, and for such basic pieces, unstable manifolds could have self-intersections.

If two (or more) unstable manifolds intersect at a point z , the *Strong Transversality Condition* requires that they intersect transversally and that the intersection is transverse to the stable manifold through z : see Definition 3 in the next section. Here we prove:

THEOREM 1. *The following conditions are necessary for an endomorphism f of a compact manifold M to be C^1 stable:*

- (1) *f has no critical points.*
- (2) *f satisfies Axiom A.*
- (3) *f satisfies the Strong Transversality Condition.*

If M is a two-dimensional manifold, then the above conditions are also sufficient for C^1 stability of f .

2. Definitions. Since the hypothesis of stability implies that the map satisfies Axiom A and has no critical points, these facts can be assumed throughout the whole article.

NOTATION. Let Ω denote the nonwandering set of f , $A = A(f)$ the union of the attracting basic pieces, $E = E(f)$ the union of the expanding basic pieces and $\Gamma = \Gamma(f) = \Omega \setminus (A \cup E)$.

Axiom A implies that the restriction of f to $\Gamma \cup A$ is injective. The union of the attracting periodic orbits will be denoted by A_{per} . Recall that a basic piece A is called *expanding* if there exist constants $c > 0$ and $\lambda > 1$ such that $|Df^n(v)| \geq c\lambda^n|v|$ for every $n \geq 0$ and $v \in T_A(M)$. It follows that a basic piece A is contained in Γ if and only if it has an unstable manifold of dimension less than the dimension of M , and this unstable manifold is not contained in A .

Definitions and basic properties of invariant manifolds associated to basic pieces of Axiom A maps are presented in [Prz1]. We will recall here some of these results.

Stable manifolds. The *stable set* of a point $x \in \Omega$ is denoted by $W^s(x)$ and defined as the set of points $y \in M$ such that $d(f^n(x), f^n(y)) \rightarrow 0$ as $n \rightarrow \infty$, where d is the distance induced by the Riemannian metric on M . For each $x \in \Gamma \cup A$ and $\epsilon > 0$ sufficiently small, the *local stable manifold* of x is defined as the set $W_\epsilon^s(x)$ of points $y \in M$ such that $d(f^n(x), f^n(y)) < \epsilon$ for every $n \geq 0$. It is known that $W_\epsilon^s(x)$ is tangent to the stable space at x and that $\{W_\epsilon^s(x) : x \in \Gamma \cup A\}$ is a family of C^1 embedded disks. The stable set of a basic piece A is defined as the union of the stable sets of points in A . The path connected component of $W^s(x)$ containing the point x will be denoted by $W_0^s(x)$.

Unstable manifolds. If $x \in \Gamma \cup A$, then there exists a unique preorbit $\{x_n\}$ of x contained in $\Gamma \cup A$. Provided $\epsilon > 0$ is sufficiently small, one can define the *local unstable manifold* of x as the set $W_\epsilon^u(x)$ of points y in M having a preorbit $\{y_n\}$ such that $d(x_n, y_n) < \epsilon$ for every $n \geq 0$. If $A \subset \Gamma$ is a basic piece then there is a well defined unstable space $E^u(x)$ for each $x \in A$ that is invariant and expanded by the differential. It is known that $W_\epsilon^u(x)$ is tangent to the unstable space at x and that $\{W_\epsilon^u(x) : x \in \Gamma \cup A\}$ is a continuous family of C^1 embedded disks. The *unstable set* of $x \in \Gamma \cup A$ is defined as $\bigcup_{n>0} f^n(W_\epsilon^u(x_n))$. An equivalent definition would be the following: $y \in W^u(x)$ if and only if there exists a preorbit $\{y_n\}$ of y such that $d(x_n, y_n) \rightarrow 0$, where $\{x_n\}$ is the preorbit of x contained in Γ .

Finally define $W^u(\Lambda)$ as the union of the unstable sets of points in Λ , and $W^u(\Gamma)$ as the union of the $W^u(\Lambda)$ for $\Lambda \subset \Gamma$. If Λ is an expanding basic piece, then there exists a neighborhood U of Λ such that the closure of U is contained in $f(U)$ and the intersection of the backward iterates of U is equal to Λ . The unstable set $W^u(\Lambda)$ of a basic piece $\Lambda \subset E$ is defined as the union of the future iterates of U .

Properties of stable and unstable sets

- (1) The path component $W_0^s(x)$ of $W^s(x)$ containing x is an injective immersion of a Euclidean space \mathbb{R}^n . Moreover, $W_0^s(x)$ can be obtained as the union, for $n \geq 0$, of the connected component of $f^{-n}(W_\epsilon^s(f^n(x)))$ that contains x . For every $x \in \Gamma \cup A$ we have

$$W^s(x) = \bigcup_{n \geq 0} f^{-n}(W_0^s(f^n(x))).$$

Therefore, each component of the stable set of a point x is always an injective immersed manifold. Moreover, different stable sets cannot intersect.

- (2) For each $z \in M$ there exists $x \in \Omega$ such that $z \in W^s(x)$. As stable sets are submanifolds, the stable space $E^s(z) = T_z(W^s(x))$ is well defined for every $z \in M$.
- (3) Unstable sets of different points can have nonempty intersection. Moreover, the unstable set of a point $x \in \Omega$ is a (not necessarily injective) immersed manifold.
- (4) The stable set of a basic piece Λ , defined as $W^s(\Lambda) = \bigcup_{x \in \Lambda} W^s(x)$, is backward invariant ($f^{-1}(W^s(\Lambda)) = W^s(\Lambda)$). The unstable set of a basic piece Λ is forward invariant but not necessarily backward invariant.

DEFINITION 2. Let A_1 and A_2 be different basic pieces of an Axiom A map f . Say that $A_1 > A_2$ if $W^u(A_1) \cap W^s(A_2) \neq \emptyset$.

If f is an Axiom A map that satisfies the transversality condition, then the relation above defines a partial order. Moreover, there exists an adapted filtration for this order (see [Prz2, Proposition 1.1]).

A closed set L is *attracting* if there exists an open neighborhood N of L such that the closure of $f(N)$ is contained in N , and the intersection of the forward iterates of N is equal to L .

The existence of filtrations for Axiom A maps with no cycles implies the following result:

LEMMA 1. *If f is an Axiom A map with no cycles, then $W^u(\Gamma) \cup A$ is a closed attracting set.*

For each point $z \in W^u(\Gamma)$ there exists a preorbit $\{z_n\}$ of z satisfying $d(x_n, z_n) \rightarrow 0$, where $\{x_n\}$ denotes the unique preorbit of x contained in Γ . It follows that there exists k_0 such that $z_k \in W_\epsilon^u(x_k)$ for every $k \geq k_0$, and therefore one has a well defined space

$$E^u(\{z_n\}) = Df_{z_k}^k(T_{z_k}(W_\epsilon^u(x_k))) \subset T_z(M).$$

But there may exist other preorbits of z converging to Γ . Let α_z be the collection of subspaces $E^u(\{z_n\}) \subset T_z M$ indexed by the different preorbits $\{z_n\}$ of z in $W^u(\Gamma)$.

Given a finite-dimensional vector space X , say that a collection of subspaces $\{X_i\}$ of X is *in general position* if the sum of the codimensions of the X_i is equal to the codimension of $\bigcap_i X_i$.

DEFINITION 3. An Axiom A map satisfies the *Strong Transversality Condition* if:

- (1) The collection α_z is in general position. If this is the case, the intersection of the $E^u(\{z_n\})$ is denoted by $E^u(z)$, and we have a well defined $E^u(z)$ for every $z \in M$.
- (2) If $z \in W^u(\Gamma)$, then $E^s(z) + E^u(z) = T_z(M)$.

This definition is intended to ensure not only the transversality of intersections between stable and unstable manifolds, but also between different unstable manifolds. This definition is more restrictive, even compared with the definition of transversality given in [BR, Definition 2.3] that was intended to characterize inverse stability.

If a point z has at least two different preorbits in the unstable set of Γ , then z is called an unstable intersection. This can also be defined as follows:

DEFINITION 4. A point $z \in M$ is an *unstable intersection* of f if there exist some positive integer k such that $f^{-k}(z)$ contains at least two points in $W^u(\Gamma)$. The set of unstable intersections is denoted by $I = I(f)$. Define also the *first unstable intersections* of f as the set $I_1 = I_1(f)$ of points z for which the above property holds with $k = 1$.

Note that I is a forward but not backward invariant set.

REMARK 1. The Strong Transversality condition implies Przytycki's necessary condition of stability (see (C3) in the introduction).

Proof. Assuming that the Strong Transversality Condition holds, that Λ is a basic piece and that there exists a point $x \in \Omega \setminus (E \cup \Lambda)$ such that $W^u(x) \cap \Lambda \neq \emptyset$, we must find a contradiction. As $x \notin E$, the codimension of $W^u(x)$ is positive. If $y \in W^u(x) \cap \Lambda$, then $y \in I(f)$, and the collection α_y contains, at least, two elements: one (denoted $\{\bar{y}\}$) is the preorbit of y contained in Λ , and another (denoted $\{\bar{y}_0\}$) converges to the basic piece

that contains x . On one hand, note that $E^u(\bar{y})$ and $E^s(y)$ are complementary subspaces because Λ is a hyperbolic basic piece. On the other hand, $E^u(\bar{y}_0)$ has positive codimension because the basic set that contains x is not expanding. It follows that the sum of the codimensions of $E^u(\bar{y})$, $E^u(\bar{y}_0)$ and $E^s(y)$ is greater than the dimension of M . This implies that f cannot satisfy both items of Definition 3.

3. Necessary conditions. Begin with a C^1 structurally stable map f . It is clear that f cannot have critical points. It is already known that f is an Axiom A map. It remains to prove that f satisfies the Strong Transversality Condition. Much of our proof rests on the techniques employed by J. Franks in [F] and on the Kupka–Smale Theorem.

Let \mathcal{U} be a neighborhood of f where every map is conjugate to f . For each $g \in \mathcal{U}$, $x \in \Gamma(g)$ and a positive integer R , define

$$W_R^u(x, g) = \bigcup_{k=0}^R g^k(W_\epsilon^u(x_k, g)),$$

where $\{x_k\}$ is the unique g -preorbit of x contained in $\Gamma(g)$. Define also $W_R^s(x, g)$ as the set of points at distance less than or equal to R in $W^s(x, g)$, where the distance is induced by the Riemannian metric of M in W^s : the distance between two points is the infimum of the length of curves joining the points within W^s .

Note that $\Omega \cap I = \emptyset$, ($I = I(f)$ of Definition 4): if $x \in I$ then $f^{-k}(x)$ has at least two points in $W^u(I)$, one of which cannot be contained in Ω , but this contradicts Theorem C of [Prz2] if $x \in \Omega$.

Proof of the first part of the Strong Transversality Condition

DEFINITION 5. Fix positive integers n and R . Given $g \in \mathcal{U}$ and $z \in W^u(\Gamma(g)) \cap I(g)$ denote by $P_z(n, R, g)$ the set of points $w \in W^u(\Gamma(g))$ such that the following conditions hold:

- (1) $w \in W_R^u(q_w, g)$ for some periodic point $q_w \in \Gamma(g)$ of period at most n .
- (2) $w \in g^{-1}(I_1(g)) \setminus I(g)$, and $g^k(w) = z$ for some positive $k = k_w$.

Define $\mathcal{K}_1(n, R)$ as the set of maps $g \in \mathcal{U}$ such that the collection of subspaces

$$\{Dg_w^{k_w}(T_w(W^u(w))) : w \in P_z(n, R, g)\}$$

is in general position.

Note that $T_w(W^u(w))$ is well defined since $w \notin I(g)$.

Using the same techniques applied to prove the Kupka–Smale Theorem in the case of diffeomorphisms, one can conclude that for every n and R positive, $\mathcal{K}_1(n, R)$ is open and dense in \mathcal{U} .

It follows that the set \mathcal{K}_1 , the intersection of the sets $\mathcal{K}_1(n, R)$ for n and R positive integers, is a residual set in \mathcal{U} .

Let f be a C^1 structurally stable map, and assume that the first condition of Definition 3 does not hold. Then there exists $z \in I$ such that the collection α_z is not in general position. This means that a finite subset $E^u(\{z_n^1\}), \dots, E^u(\{z_n^r\})$ is not in general position. As the preimages of z are wandering and the periodic points are dense in Ω , there exists a perturbation g of f such that for each $1 \leq i \leq r$, the sequence $\{z_n^i\}$ belongs to the unstable set of a periodic point p_i of g . As g is conjugate to a map in \mathcal{K}_1 , the sum of the codimensions of the subspaces $E^u(\{z_n^i\})$ is less than or equal to the dimension of M . In addition, as the subspaces $E^u(\{z_n^i\})$ are not in general position, the arguments of Franks [F, Lemma 2.1] imply that there exists a perturbation g_1 of g such that the unstable manifolds of the periodic points p_i of g_1 intersect in a submanifold of codimension less than the sum of their codimensions; but this contradicts the fact that g_1 must be conjugate to a map in \mathcal{K}_1 .

Proof of the second part of the Strong Transversality Condition

DEFINITION 6. Given positive integers n and R , let $\mathcal{K}_2(n, R)$ be the set of maps $g \in \mathcal{K}_1(n, R) \cap \mathcal{U}$ such that the intersection of the subspaces $Dg_w^{k_w}(T_w(W^u(w)))$ for $w \in P_z(n, R, g)$ is transverse to $E^s(z)$ whenever z belongs to $W_R^s(p, g)$ and p is a periodic point of g whose period is at most n .

LEMMA 2. For every n and R , $\mathcal{K}_2(n, R)$ is open and dense in \mathcal{U} .

Proof. It is clear that the property is open and that the collection of subspaces $Dg_w^{k_w}(T_w(W^u(w)))$ for $w \in P_z(n, R, g)$ is transverse if $g \in \mathcal{K}_1(n, R)$. A perturbation supported in a neighborhood of z will produce no changes in unstable manifolds but will make the stable manifold through z transverse to the collection of subspaces. ■

It follows that the intersection \mathcal{K}_2 of the $\mathcal{K}_2(n, N)$ is a residual subset of \mathcal{U} .

To prove the second assertion of the Strong Transversality Condition, assume on the contrary that there are $x \in \Omega(f)$ and $y \in W^s(x) \cap W^u(I)$ such that $E^s(y)$ and $E^u(y)$ are not transverse. We can assume, as above, that x is a periodic point of f , and also that every $w \in P_y$ belongs to the unstable manifold of a periodic point. If the sum of the dimensions of $E^s(x)$ and $E^u(y)$ is greater than or equal to the dimension of the ambient manifold, one can produce a perturbation g such that, if $W^u(x)$ is defined as

$$W^u(x) = \bigcap_{w \in P_y} Dg_w^{n_w}(E^u(w)),$$

then $W^u(x) \cap W^s(x)$ contains a disc of codimension less than the sum of the

codimensions of $W^s(x)$ and $W^u(x)$, which constitutes an obstruction to the equivalence with a map of \mathcal{K}_2 . If the sum of the dimensions is less than the dimension of M , then a perturbation in \mathcal{K}_2 will not be equivalent to f .

4. Example. The product of two C^1 stable maps cannot be C^1 Ω -stable, unless both maps are diffeomorphisms or both are expanding. Indeed, if a stable map is not a diffeomorphism, then it has a noninjective expanding basic piece, and if a stable map is not expanding, then it has a nontrivial attractor. The product of an expanding basic piece times an attractor is a basic piece that is neither expanding nor injective. The example given by Przytycki (see the final section of [Prz2]) is a C^0 perturbation of the product map $(s, t) = (f_1(s), f_2(t))$, where f_1 and f_2 are maps of the circle whose graphs are shown in Fig. 1. We will consider the two-torus as the product $\{(s, t) : s \in [-\pi, \pi], t \in [-\pi, \pi]\}$, where $-\pi$ and π are identified.

The map f_1 is a diffeomorphism having an attractor at $s = 0$ and a repeller at $s = \pi$. The map f_2 is a C^0 perturbation of $z \mapsto z^2$ (*derived from expanding map*). The nonwandering set of f_2 is the union of an attracting fixed point at $t = 0$ and an expanding Cantor set K_1 . The set K defined as $\{0\} \times K_1$ is a hyperbolic isolated transitive saddle type set, but the restriction of the map to this set is not injective.

Przytycki proposed to perturb the product map as follows:

$$f(s, t) = (f_1(s) + \sin(t)\varphi(s), f_2(t)),$$

where φ is a smooth function satisfying two more assumptions. For every s we have $0 \leq \varphi(s) \leq \varepsilon$. There exists a constant s_0 such that $\varphi(s) = \varepsilon$ for $|s| < s_0$, and $\varphi(s) = 0$ if $|s - \pi| < s_0$. The absolute value of the derivative of φ is also smaller than ε . At each s , $|\varphi'(s)| < |f_1'(s)|$.

Other conditions are assumed for the maps f_1 and f_2 . Both f_1 and f_2 are odd functions on $[-\pi, \pi]$. In a neighborhood of 0, the function f_1 is equal to the map $s \mapsto \lambda s$, where $0 < \lambda < 1$ is very small. The map f_2 has fixed points at 0, δ and $-\delta$. Moreover f_2 is taken such that the preimage of δ is $-\pi + \delta/3$ (so the preimage of $-\delta$ is $\pi - \delta/3$). The derivative of f_2 in $|t| > \delta$ is greater than two. The constant ε is taken sufficiently small; and once ε is chosen, the constant λ is taken still smaller.

PROPOSITION 1. *The map f satisfies the necessary conditions for C^1 stability stated in Theorem 1.*

Proof. (1) f is a covering. If ε is small enough, then the derivative of the first coordinate with respect to s never vanishes. This implies that f is a local homeomorphism. As every point has exactly two preimages, the claim follows.

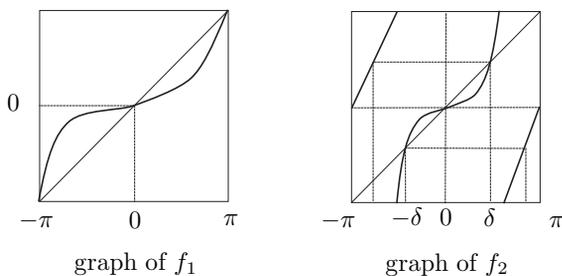


Fig. 1

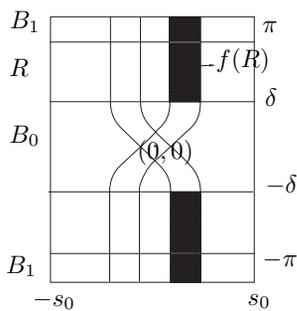


Fig. 2. The picture shows $U = [-s_0, s_0] \times [-\pi, \pi]$ and $f(U)$. The shadowed region is $f(R)$.

(2) f satisfies Axiom A. The proof below will show that the nonwandering set of f is the union of the following basic pieces:

- An attracting fixed point $p = (0, 0)$.
- A saddle type fixed point $C = (\pi, 0)$.
- An expanding set $K = \{\pi\} \times K_1$.
- A saddle type basic piece Γ_0 close to $\{0\} \times K_1$.

That the first three are basic pieces is clear. Towards the proof of the existence, hyperbolicity and injectivity of Γ_0 , it will be necessary to construct invariant cone fields in the region $R = \{(s, t) : |s| < s_0, |t| \in (\delta, \pi - \delta)\}$. Note first that $f_1(\sigma) + \varphi(\sigma) \sin \delta = \sigma$ has the solution $\sigma = \epsilon \sin \delta / (1 - \lambda) < s_0$. It follows that the points $A = (\sigma, \delta)$ and $-A = (-\sigma, -\delta)$ are fixed for f .

Assume that $v = (v_1, v_2)$ is a vector satisfying $|v_1/v_2| < \rho$. If $(w_1, w_2) = Df_X(v)$, where X belongs to the region R , then $|w_1/w_2| < \lambda\rho/2 + \epsilon/2$, where we have used that the derivative of f_2 is greater than 2 and that $\varphi = \epsilon$ in that region. If ρ is taken greater than $\epsilon/(2 - \lambda)$ then the region R admits an expanded almost vertical invariant cone field (because s_0 is greater than $\sigma + \rho\pi$ if ϵ is sufficiently small). The same holds for the region $R' = \{(s, t) : s - t \in R\}$. The horizontal lines form a contracting invariant foliation in $R \cup R'$. Therefore the set Γ_0 of points having the whole orbit

contained in $R \cup R'$ has a hyperbolic structure; it follows also that Ω is hyperbolic and that Γ_0 is a basic piece. To obtain Axiom A it remains to prove the injectivity in Γ_0 . Note that the intersection of the future images of R is a region bounded by segments of the unstable manifolds of the fixed points in Γ_0 , A and $-A$. These are curves of the form $(\gamma(t), t)$ for $t \in [\delta, \pi - \delta]$, where $\gamma(t)$ is at a distance less than $\rho\pi$ from the first coordinate of A or $-A$. In any case it is easily seen that the image of a curve like that must be contained in $\{(s, t) : s > 0\}$, if λ is taken sufficiently small. Analogously, the image of the region R' is contained in $\{s < 0\}$ by symmetry. It follows that the restriction of f to $R \cup R'$ is injective.

(3) *Strong Transversality Condition.* Obviously, stable and unstable manifolds intersect transversally. Moreover, the intersections of unstable manifolds occur in B_0 , the immediate basin of $(0, 0)$. It remains to show that these intersections are transverse. It is sufficient to prove that the intersections in I_1 (see Section 2, Definition 4) are transverse. A point z belongs to I_1 if and only if one of the preimages of z belongs to B_0 and the other belongs to B_1 , where B_1 denotes the component of $f^{-1}(B_0)$ that is not B_0 . The unstable manifolds of points in Γ_0 are almost vertical in $R \cup R' \cup B_1$. Note that if $\gamma(t) = (\alpha(t), t)$ is almost vertical, where $|t \pm \pi| < \delta$, then the image under f of γ is a curve contained in the immediate basin of $(0, 0)$ and its tangent vector is $W_1 = (\lambda\alpha'(t) + \varepsilon \cos t, f_2'(t))$. It can be parametrized as $\tilde{\gamma}(t') = (\beta(t'), t')$, with t' now varying in $[-\delta, \delta]$. The image under f of a curve of the form of $\tilde{\gamma}$ has tangent vector $W_2 = (\lambda\beta'(t') + \varepsilon \cos t', f_2'(t'))$. As t is close to π and t' close to 0, the signs of the first coordinates of W_1 and W_2 are different if λ is sufficiently small. But as the second coordinates are positive, we conclude that these curves intersect transversally. ■

5. Sufficient conditions. From now on, it is assumed that f has properties (1) to (3) of Theorem 1.

We will first construct the conjugacy h in fundamental domains of the periodic attractors.

5.1. Controlling unstable intersections. To begin this section we simplify the wording of the Strong Transversality Condition in dimension two. Unstable manifolds of basic pieces in Γ have dimension one. This implies that for every positive k , and $z \in M$, the intersection of $f^{-k}(z)$ with $W^u(\Gamma)$ contains at most two points. Then $E^u(z)$ (see Definition 3) has dimension zero whenever $z \in I$. Moreover, as unstable intersections must be transverse to stable manifolds, it follows that I is contained in the basin of periodic attractors. The next result implies that I_1 (the set of first intersections, see Definition 4) is compactly contained in the basin of A_{per} , the set of attracting periodic orbits.

LEMMA 3. *If f satisfies the Strong Transversality Condition, then I_1 is a closed set.*

Proof. Let $\{z_n\}$ be a sequence of points in I_1 that converges to z , and for each n let $z_n^1 \neq z_n^2$ be points in $W^u(\Gamma)$ such that $f(z_n^1) = f(z_n^2) = z_n$. Assume that for $i = 1, 2$ the sequences $\{z_n^i\}$ converge to points z^1 and z^2 , which must be different since f is locally invertible. Then z^1 and z^2 belong to $W^u(\Gamma) \cup A$, as this is a closed set. But the Strong Transversality Condition implies that neither z^1 nor z^2 can belong to A . Thus $z = f(z^1) = f(z^2)$ belongs to I , and again by the Strong Transversality Condition, $z \in I_1$. ■

LEMMA 4. *Given a neighborhood U of $I_1(f)$ and $\delta > 0$, there exists a neighborhood \mathcal{U} of f , and a positive integer k , such that, for every $g \in \mathcal{U}$, the following properties hold:*

- (1) *The set of intersections $I_1(g)$ is contained in U and $g^{-k}(I_1(g)) \cap W^u(\Gamma(g)) \subset W_\delta^u(\Gamma(g))$.*
- (2) *There exists $\rho > 0$ such that, if x and y are different points in $\Gamma(g)$, then the distance between two different points in $g^k(W_\delta^u(x, g)) \cap g^k(W_\delta^u(y, g))$ is greater than ρ .*

Proof. Note that the inclusion in (1) is satisfied by the map f for some positive k , because each point in $I_1(f)$ has exactly two preorbits converging to Γ and $I_1(f)$ is compact. By the local stability of basic pieces it follows that local unstable manifolds of size δ for the map g are C^1 close to those of f ; moreover, as k is fixed, the assertion also holds for g .

(2) As the Strong Transversality Condition is open, $I_1(g)$ is also compact. Assume towards a contradiction that there exist sequences $\{x_n\}$ and $\{y_n\}$ contained in $\Gamma(g)$ such that for each $n > 0$ there exist points $z_n^1 \neq z_n^2$ with $d(z_n^1, z_n^2) < 1/n$ and $z_n^i \in g^k(W_\delta^u(x_n, g)) \cap g^k(W_\delta^u(y_n, g))$. Assuming that the sequences $\{z_n^i\}$ converge to a point z , a contradiction arises because $z \in I(g)$ turns out to be a point of nontransverse intersection between unstable manifolds. ■

Let $B_0(A)$ denote the immediate basin of the attractor A . To deal with the unstable intersections in $B_0 = B_0(A)$, let

$$L = L(f, p) = \bigcup_{\ell \geq 0} f^\ell(I_1(f) \cap B_\ell(f, p)),$$

where $B_\ell = B_\ell(f, p)$ is defined inductively as follows: $B_1 = f^{-1}(B_0) \setminus B_0$ and $B_\ell = f^{-1}(B_{\ell-1})$ for every $\ell > 1$.

LEMMA 5. *L is compact and $f(L) \cap L = \emptyset$.*

Proof. Note first that as I is contained in B , we have $I_1 \cap \partial B_\ell = \emptyset$ for each $\ell \geq 0$. So $I_1 \cap B_\ell$ is compact for every ℓ . Moreover, by Lemma 3 there

exist at most finitely many values of ℓ for which the intersection of I_1 with B_ℓ is nonempty. This implies the first assertion.

Notice that $x \in M$ implies $\sharp(o^-(x) \cap I_1) \leq 1$, where $o^-(x)$ denotes the union of all the preimages of x . If $x \in L$, then there exists $y \in I_1 \cap B_\ell$ such that $f^\ell(y) = x$. It follows that y is the unique point in $o^-(f(x)) \cap I_1$, hence $f(x) \notin L$ by definition of L . ■

5.2. Fundamental domains. Let A be a periodic attractor such that the set I of unstable intersections intersects the basin of A . We can assume that A is a fixed point and take K equal to the closure of a fundamental domain such that K is an annulus and L is contained in the interior of K . A fundamental domain with these properties can be constructed as in the proof of Theorem C in [IPR1]: there it was shown how to construct a fundamental domain containing in its interior a compact set L , provided $f(L) \cap L = \emptyset$. Let $\partial_- K$ be the connected component of ∂K that is closer to p , and $\partial_+ K$ be the other component of the boundary of K ($\partial_- K$ separates p from $\partial_+ K$ in B_0). Let Q be the component of $B_0 \setminus K$ that contains p . It can also be assumed that for every $x \in Q \cap W^u(\Gamma)$ there exist $j > 0$ and $y \in K \cap W^u(\Gamma)$ such that $f^j(y) = x$. This follows from the fact that the transversality condition implies that the preimage of p cannot belong to $W^u(\Gamma)$, so it is possible to take a neighborhood V_0 of p where just the points coming through K can enter V_0 ; then one can take K contained in V_0 , and replace L by a homeomorphic image of it. Given $\delta > 0$, let $k > 0$ be such that

$$(1) \quad f^{-k}(K) \cap W^u(\Gamma) \subset W_\delta^u(\Gamma).$$

Observe that k can be taken so as to also satisfy the conditions imposed on k in Lemma 4. For each basic piece Λ in Γ let ϵ_s be a small positive number and define a fundamental domain for Λ :

$$D(\Lambda) = D_{\epsilon_s}(\Lambda) = W_{\epsilon_s}^s(\Lambda) \setminus f(W_{\epsilon_s}^s(\Lambda)).$$

5.3. Local perturbations. It is classical that a perturbation can be performed as a finite sequence of perturbations with small supports. Indeed, if g is a perturbation of f then there exists a diffeomorphism t , C^1 close to the identity, such that $g = ft$ (take $t(x)$ as the point in $f^{-1}(g(x))$ that is closest to x). Given a finite covering $\{W_1, \dots, W_s\}$ of M , and a small perturbation t of the identity, there exist $\{t_1, \dots, t_s\}$ such that $t = t_1 \dots t_s$ and the support of t_i is contained in W_i (the support of t_i is the closure of the set of points where $t_i(x) \neq x$). For the proof of this, see [PS].

We begin by taking an appropriate covering of M . If $x \in W^u(\Lambda)$ for a basic piece $\Lambda \subset \Gamma$, then let W be a neighborhood of x compactly contained in the interior of

$$\bigcup_{n \geq 0} f^n(V(\Lambda)) \cup W^u(\Lambda),$$

where $V(\Lambda)$ is a small neighborhood of $D(\Lambda)$. Moreover we require that the closure of W does not intersect $\Omega \setminus \Lambda$. If $x \in \Lambda$ then take W compactly contained in the immediate basin of Λ if Λ is a nonperiodic attractor, and compactly contained in the open set Q defined in 5.2 if Λ is a periodic attractor. If $x \notin W^u(\Gamma) \cup \Lambda$, then W will be a neighborhood of x such that the closure of W does not intersect $W^u(\Gamma) \cup \Lambda$ (recall that the latter is closed by Lemma 1).

From now on it is assumed that g is a perturbation of f and that the set of points where f and g are not equal is contained in an open set W as constructed above. On the other hand, it is clear that t_i converges to the identity when g converges to f . We note that in any case, each attractor of f has a fundamental domain K and each basic piece has a fundamental domain $D(\Lambda)$ whose closure does not intersect that of W .

5.4. Construction of unstable foliations. Let Λ be a basic piece of Γ . We will follow the proof of de Melo [dM]. A foliation \mathcal{F}_f will be defined in a small neighborhood $V(\Lambda)$ of $D(\Lambda)$. Note that $D(\Lambda)$ has no intersections with the unstable set of Λ , but will certainly intersect unstable leaves of basic pieces that are greater than Λ in the order of Definition 2. Then an induction argument is applied: the foliation constructed in $V(\Lambda)$ must contain the leaves that are iterates of those unstable foliations constructed in previous steps and intersecting $D(\Lambda)$.

Once this foliation is defined in $V(\Lambda)$, the union of its forward iterates with $W_\delta^u(\Lambda)$ will contain a neighborhood S_f of Λ . We will fix some positive number δ such that $S_f \supset W_\delta^u$. This procedure can be repeated with all the basic pieces in Γ , giving a foliation \mathcal{F}_f defined in S_f .

By Lemma 3, this neighborhood S_f can be required to have the following properties:

- (1) The restriction of f to S_f is injective.
- (2) The intersection of S_f with a basic piece of f not contained in Γ is empty.
- (3) The intersection of S_f with I is empty.

And as these properties are open, there exists a neighborhood \mathcal{U} of f such that for every $g \in \mathcal{U}$ there exists a neighborhood S_g of $\Gamma(g)$ with the above properties.

LEMMA 6. *If g is a perturbation of f that coincides with f in a neighborhood of a fundamental domain of Γ , then there exists a foliation \mathcal{F}_g defined in S_g and map $H_g : \mathcal{F}_f \rightarrow \mathcal{F}_g$ with the following properties:*

- (1) *Each leaf of \mathcal{F}_g is an injectively immersed one-dimensional manifold, which is contained in $W^u(x, g)$ whenever $x \in \Gamma(g)$ and is transverse*

to $W^s(\Gamma(g))$. Moreover, the tangent spaces to the leaves vary continuously.

- (2) For $x \in S_g$, denote by $F_x(g)$ the leaf through x . If $y \in S_g$ and $g^n(x) = y$, then the connected component of $g^n(F_x(g)) \cap S_g$ that contains y is contained in $F_y(g)$.
- (3) If F_1 is a leaf of \mathcal{F}_f , then $H_g(F_1)$ is a leaf of \mathcal{F}_g that is close to F_1 in the C^0 topology.
- (4) $g(H_g(F_x(f))) \supset H_g(F_{f(x)}(f))$, whenever x and $f(x)$ belong to S_f .

Proof. Until now we have shown the existence of a foliation \mathcal{F}_f of S_f that satisfies (1) and (2) for the map f . Let $\Lambda_1(g)$ be a maximal basic piece and $V(\Lambda_1(g))$ a neighborhood of $D(\Lambda_1(g))$, where f and g coincide. Define $\mathcal{F}_g = \mathcal{F}_f$ in $V(\Lambda_1(g))$. Then a neighborhood of $\Lambda_1(g)$ will be covered by the union of the future iterates of the leaves of \mathcal{F}_g and $W_\delta^u(\Lambda_1(g))$. After this, proceed by induction. Assume that \mathcal{F}_g is defined in $\Lambda_i(g)$ for $1 \leq i \leq j-1$ and satisfies (1) and (2). Then define \mathcal{F}_g in a neighborhood $V(\Lambda_j(g))$ of $D(\Lambda_j(g))$ in such a way that it coincides with leaves that are forward iterates of leaves of \mathcal{F}_g defined at previous stages and coincides with \mathcal{F}_f elsewhere. Again we cover a neighborhood of $\Lambda_j(g)$ by taking the union of the future iterates of these foliations with $W_\delta^u(\Lambda_j(g))$. The continuity of the foliation follows as in [dM].

The map H_g is defined to be the identity for leaves of \mathcal{F}_f contained in $V(\Lambda_1)$. Then, preserving conjugacy and continuity, extend H_g to all the leaves of \mathcal{F}_f that are contained in a neighborhood of Λ_1 . Proceed in the same way to cover neighborhoods of the remaining basic pieces.

It is standard that \mathcal{F}_g and H_g defined in this way have the asserted properties. ■

Let \mathcal{U} , ρ and k be as in Lemma 4.

REMARK 2. (a) If $g \in \mathcal{U}$, $F_x(g)$ and $F_y(g)$ are distinct leaves of \mathcal{F}_g and $g^k(F_x(g)) \cap g^k(F_y(g)) \neq \emptyset$, then the intersection is transverse and the distance between two points in this intersection is at least $\rho/2$.

(b) For every $x \in S_g$, the cardinality of $g^{-k}(g^k(x)) \cap S_g$ is at most two.

REMARK 3. Let W be an open set as constructed at Subsection 5.3. The perturbation g of f coincides with f outside W . Once W is known, one can choose fundamental domains K of the attractors and $D(\Lambda)$ of the basic pieces in Γ so that:

- (1) W does not intersect $\bigcup_{j=0}^k \partial f^{-j}(K)$.
- (2) $W \cap V(\Lambda) = \emptyset$ for a neighborhood $V(\Lambda)$ of the fundamental domain of the basic piece Λ (this validates the hypothesis of Lemma 6).
- (3) The set W intersects neither $O := \partial(\bigcup_{i=0}^k f^i(S_f))$, nor $\bigcup_{j=0}^k f^{-j}(O)$.

From $f(W) = g(W)$ and item (3) above, it follows that

$$\bigcup_{i=0}^k f^i(S_f) = \bigcup_{i=0}^k g^i(S_g).$$

5.5. Construction of the conjugacy on the basins of the attractors. The reader may keep in mind that the intersections of unstable manifolds occur in the basins of attracting periodic orbits, and never in the basin of a nontrivial attractor (those having unstable manifold of dimension one).

Let K be a fundamental domain of an attracting fixed point and assume that L intersects this basin. The map f is perturbed to a map g in an open set W that satisfies the requirements of the above subsection. The construction of the conjugacy begins in this fundamental domain. We can now state the fundamental step.

LEMMA 7. *There exists a homeomorphism h defined in K such that:*

- (1) *If $x \in \partial_+ K$, then $g(h(x)) = h(f(x))$.*
- (2) *If $x \in S_f$ and $f^j(x) \in K$ for some $j > 0$, then $g^j(H(F_x(f)))$ contains $h(f^j(x))$.*

Before beginning the proof of the lemma, we will construct a field of directions in a preimage of K where the intersections of unstable leaves have not occurred yet.

Let $k > 0$ be as defined in Lemma 4 and in equation (1) of Subsection 5.2:

$$(2) \quad f^{-k}(K) \cap W^u(\Gamma) \subset W_\delta^u(\Gamma) \subset S_f.$$

Note that $f^{-k}(K)$ does not contain future iterates of $I(f)$ and each component of $f^{-k}(K)$ is diffeomorphic to K . Denote by K' the unique component of $f^{-k}(K)$ that is contained in B_0 , and by $\partial_\pm K'$ the component of the boundary of K' whose image under f^k is equal to $\partial_\pm K$.

By Remark 3, it is clear that $g^{-k}(K) = f^{-k}(K)$ and that the set

$$S' = \{x \in g^{-k}(K) \cap S_g : \text{there exists } y \in S_g \text{ such that } y \neq x \text{ and } g^k(x) = g^k(y)\}$$

does not depend on the perturbation made (although f^k and g^k do not necessarily coincide in S').

Given g as above, we will now construct a continuous field of directions χ_g defined in a neighborhood V_1 of $g^{-k}(K) \cap S_g$. We will first construct χ_f :

If $x \in S'$, then $\chi_f(x)$ is the unique direction such that $Df_x^k(\chi_f(x)) = Df_y^k(v_y)$, where v_y is the direction tangent to $F_y(f)$ at y , and y is as in the definition of S' . Note first that the transversality condition and the

construction of S_f imply that there exists at most one point $y \neq x$, $y \in S_f$, such that $f^k(x) = f^k(y)$, so χ_f is well defined.

Note that χ_f is continuous in S' and that the transversality condition implies that χ_f is transverse to the leaves of the foliation \mathcal{F}_f .

Now take points x and $f(x)$, both contained in $f^{-k}(K) \cap S_f$, and define $\chi_f(x)$ transverse to the leaf of \mathcal{F}_f at x . Next define $\chi_f(f(x)) = Df_x(\chi_f(x))$. Note that as K is a fundamental domain, then x and $f(x)$ belong to the immediate basin of the attractor, so x belongs to $\partial_+ K'$. We have thus defined χ_f on three closed disjoint sets: $S' \cap S_f$, $\partial_+ K' \cap S_f$ and $f(\partial_+ K') \cap S_f$. It is continuous and transverse to the foliations. To extend this field of directions to a neighborhood V_1 of $f^{-k}(K) \cap S_f$, one can use diverse techniques, for example the averaging method employed in [dM, Lemma 1.1].

The objective now is to define a similar field of directions for g close to f ; we proceed as above, defining first χ_g at a point x in S' as the preimage under Dg_x^k of the image under Dg_y^k of the direction of the leaf $F_g(y)$, where y is the (unique) point in S_f such that $g^k(y) = g^k(x)$. Recall that as was explained above, the set S' does not depend on the perturbation. It is also known that S_f does not depend on the perturbation either. Moreover, as L is interior to K , it follows that S' is a neighborhood of $g^{-k}(L)$ contained in the interior of $g^{-k}(K)$. It follows by the requirements made in Remark 3 that χ_f and χ_g coincide close to the boundary of S' . Thus χ_g can be extended to V_1 (where χ_f is defined), by imposing that $\chi_g = \chi_f$ outside S' .

For future reference, we state as a claim what we have established about the fields of directions.

CLAIM. There exists an open set V_1 , a neighborhood of $S_f \cap f^{-k}(K)$, such that, for each perturbation g of f that coincides with f outside a set W satisfying the conditions stated in Remark 3, there exists a continuous field of directions χ_g with the following properties:

- (1) *If x belongs to S' then $\chi_g(x) = Dg_x^{-k}(Dg_y^k(v_y))$, where v_y is tangent to the leaf of \mathcal{F}_g through y , and y is as in the definition of S' .*
- (2) *For every $x \in V_1$, $\chi_g(x)$ is transverse to $F_x(g)$.*
- (3) *If x and $g(x)$ belong to V_1 , then $Dg_x(\chi_g(x)) = \chi_g(g(x))$.*
- (4) *$\chi_f = \chi_g$ outside S' .*

The integral line of χ_g through the point x will be denoted by $\tilde{\chi}_g(x)$. Note that the leaves of $\tilde{\chi}$ and the leaves of the foliation \mathcal{F}_g give a local product structure at $g^{-k}(K) \cap S_g$, precisely:

There exists a neighborhood V_1 of $g^{-k}(K) \cap S_g$ and a positive number δ_0 such that, for every pair of points x and y in $g^{-k}(K) \cap S_g$ at a distance less than δ_0 , the intersection of the leaf of $\tilde{\chi}_g$ through x with the leaf of $\mathcal{F}(g)$

through y consists of exactly one point contained in V_1 . This intersection varies continuously with x and y .

Proof of Lemma 7. Recall that if $x \in K$, then, according to item (b) in Remark 2, the cardinality of $f^{-k}(x) \cap S_f$ is equal to 0, 1 or 2; define h as follows:

- 1) If the cardinality is equal to zero, then $h(x) = x$.
- 2) If it is equal to one, let y be the unique point in $f^{-k}(x) \cap S_f$. Let y' be the unique point of intersection of the integral line of χ_g ($\chi_f(y) = \chi_g(y)$ because $y \notin S'$) through y with $H_g(F_y(f))$. Then let $h(x) = g^k(y')$.
- 3) If it is equal to two, and $f^{-k}(x) \cap S_f = \{y_1, y_2\}$, then $h(x)$ is defined as the intersection of the leaves $g^k(H(F_{y_1}))$ and $g^k(H(F_{y_2}))$ with the ball of radius $\rho/4$ and centered at x .

To prove that h is well defined, one must consider the third case. The intersection defining h contains exactly one point. If the perturbation g of f is sufficiently small, then property (1) in Lemma 6 implies that F_y and $H(F_y)$ are close as C^1 embeddings, so their g^k -iterates are close as well. Moreover, by (a) in Remark 2, the intersection defining h is transverse and is unique at a distance less than $\rho/4$ from x .

Now we will prove the continuity of h . It depends on the location of the support of the perturbation. Assume first that W is contained in the exterior of $\bigcup_{i \geq 0} f^i(S_f)$ (see Remark 3). In this case $\mathcal{F}_f = \mathcal{F}_g$, H_g is the identity, and consequently h is equal to the identity. The same happens whenever W intersects an attractor.

It remains to consider the case where the closure of W is contained in the interior of $\bigcup_{i \geq 0} f^i(S_f)$.

Note that h is continuous in the interior of $\bigcup_{i \geq 0} f^i(S_f)$ by the continuity of the foliations. On the other hand, h is the identity in the complement of $\bigcup_{i=0}^k f^i(S_f)$, so it remains to prove the continuity in the boundary. Note that as $f(W) = g(W)$, then f^k and g^k coincide in

$$\left(\bigcup_{n \geq 0} f^n(S_f) \setminus \bigcup_{n \geq 0} f^n(W) \right) \cap K.$$

Let x be a point in the boundary of $(\bigcup_{i \geq 0} f^i(S_f)) \cap K$, and y a point in $f^{-k}(x) \cap S_f$. Then y belongs to the boundary of S_f , so $F_y(f) = F_y(g)$ and $\chi_f(y) = \chi_g(y)$. Moreover, as $f^j(y) \notin W$ for every $0 \leq j \leq k$, there exists a neighborhood U_y of y such that $f^j(z) = g^j(z)$ for every $z \in U_y$. This implies that $h(x) = x$.

Note that h is injective because if $x \in K$, then the cardinalities of $f^{-k}(x) \cap S_f$ and $g^{-k}(h(x)) \cap S_f$ coincide, since $f = g$ outside the future iterates of S_f .

To prove assertion (1) of the lemma, take $x \in \partial_+ K$. If x does not belong to the union of the future iterates of S_f then $h(x) = x$ and $h(f(x)) = f(x)$, and (1) holds because g and f coincide at both points. In the remaining case, there exists a unique $y \in S_f$ such that $f^k(y) = x$ and we fall into case (2) of the definition of h . Moreover, $f(y)$ is the unique f^k -preimage of $f(x)$ contained in S_f . If y' is the point of intersection of $\tilde{\chi}_g(y) = \tilde{\chi}_f(y)$ with $H_g(F_y(f))$, then by definition, $h(x) = g^k(y')$. By the third property of the fields of directions χ , we have $g(\tilde{\chi}_g(y)) = \tilde{\chi}_g(g(y))$. In addition, $g(y')$ belongs to $g(H_g(F_y(f)))$ and therefore $g^k(g(y')) = h(f(x))$.

The second assertion of the lemma follows by construction. ■

This homeomorphism h can be extended to the whole immediate basin of p_f as follows: if $x \in B_0(p_f)$, then there exists a unique $j \in \mathbb{Z}$ such that $f^j_{|B_0(p_f)}(x) \in K$; then define $h(x) = g^{-j}_{|B_0(p_g)}(h(f^j(x)))$. This new extension of h conjugates the restrictions of f and g to the corresponding immediate basins of the attractors. It is injective and open, because h is. Moreover, h carries p_f to p_g and so its image contains a neighborhood of p ; it follows that it is onto $B_0(p_g)$.

When A is a nonperiodic attractor, the restriction of f to a neighborhood of A is injective. Moreover, as there cannot be unstable intersections in its basin, the argument applied for diffeomorphisms allows one to construct a local conjugacy C^0 close to the identity defined in the whole immediate basin of the attractor A and satisfying the properties in the statement of Lemma 7. Summing up, there exists a conjugacy h from $B_0(f)$ (the union of the immediate basins of the attractors of f) onto $B_0(g)$.

Finally, we claim that for every point $x \in S_f$ and every nonnegative integer j such that $f^j(x) \in B_0(f)$, we have $h(f^j(x)) \in g^j(H(F_x(f)))$. Indeed, let $J(x)$ be minimum such that $f^J(x) \in B_0(f)$. Let $i \in \mathbb{Z}$ be such that $f^{J+i}(x) \in K$. Note that i is greater than or equal to zero, by the construction of K (see Subsection 5.2). Then the claim follows by the second assertion in the previous lemma.

5.6. Extension of h to the whole manifold. For each $x \in f^{-\ell}(B_0)$, one has precisely d^ℓ points in $g^{-\ell}h f^\ell(x)$ to choose $h(x)$. Our arguments will show that there exists one of these points closest to x .

Let $A_j \subset \Gamma$, $1 \leq j \leq n$, U_j be a collection of disjoint neighborhoods and denote by U' the union of the U_j . Let U be a neighborhood of the attractors. The proof of the following lemma is straightforward.

LEMMA 8. *There exists $N > 0$ such that for each $x \in M$, the set $\{j \geq 0 : f^j(x) \notin U \cup U'\}$ contains at most N elements.*

Let α be the expansivity constant of the restriction of f to $U \cup U'$ (recall that f is a diffeomorphism from U_j to $f(U_j)$ for $U_j \subset \Gamma$). Let $\epsilon_0 > 0$ be such that $f(x) = f(y)$ implies $x = y$ or $d(x, y) > \epsilon_0$. Let $\epsilon < \min\{\alpha/2, \epsilon_0/2\}$.

If U' is sufficiently small and \mathcal{F} is the foliation obtained in Lemma 6, then the following additional property holds: There exist constants $\delta > 0$, $C > 0$ and $\lambda > 1$ such that $d(f^n(x), f^n(y)) \geq C\lambda^n d(x, y)$ whenever $f^j(x)$ and $f^j(y)$ belong to the same leaf of \mathcal{F} and $d(f^j(x), f^j(y)) \leq \delta$ for every $0 \leq j \leq n - 1$. By changing the metric one can obtain $C = 1$.

In what follows, \mathcal{U} will be a neighborhood of f such that the same properties hold for g in \mathcal{U} .

LEMMA 9. *Let x and y be points in U' such that $f(y) = x$. If h is defined in x and $d(h(x), x) < \epsilon$, then h can be defined in y in such a way that $d(h(y), y) < \epsilon$.*

Proof. By the choice of ϵ_0 and \mathcal{U} , it follows that $g^{-1}(h(x)) \cap B(y; \epsilon_0/2)$ contains just one point, denoted y' . If $h(y)$ is defined as equal to y' , then we must prove that $d(y, y') < \epsilon$.

The proof that follows is inspired by [IPR2, Lemma 2]. Assume first that $A_j \subset \Gamma$. Given a positive constant ρ there exists a neighborhood \mathcal{U} of f such that the leaves $F_y(f)$ and $H(F_y(f))$ are ρ -close in the C^1 topology. Then there exists a point $z \in H(F_y(f))$ such that $d(z, h(y)) < \rho$. It follows that

$$\begin{aligned} d(g(h(y)), g(y)) &\geq d(g(h(y)), g(z)) - d(g(z), g(y)) \\ &\geq \lambda d(h(y), z) - K\rho \geq \lambda d(h(y), y) - \lambda\rho - K\rho, \end{aligned}$$

where K is taken so that $d(g(z_1), g(z_2)) \leq Kd(z_1, z_2)$ for every $g \in \mathcal{U}$ and z_1, z_2 in M .

On the other hand,

$$\begin{aligned} d(g(h(y)), g(y)) &= d(h(f(y)), g(y)) \\ &\leq d(h(x), x) + d(f(y), g(y)) < \epsilon + \delta, \end{aligned}$$

where δ is the C^0 distance between f and g . These inequalities imply that $d(h(y), y) \leq (\epsilon + \delta + \rho(\lambda + K))\lambda^{-1}$. This is less than ϵ if δ and ρ are sufficiently small, which is obtained by diminishing \mathcal{U} .

It remains to consider the case where $A_j \subset E$, but here the same argument (simplified because the basic piece is expanding) applies. ■

Once a small $\epsilon > 0$ and an integer N are fixed, there exists $\epsilon' > 0$ such that $d(h(f^j(x)), f^j(x)) < \epsilon'$ for some $j \leq N$ implies $d(h(x), x) < \epsilon$.

On the other hand, one can make the restriction of h to a neighborhood U of the attractors as close to the identity as wished, say $d(h(y), y) < \epsilon'$ whenever $y \in U$. If $x \in U' \cap B$, it is known by Lemma 8 that $f^k(x) \notin U' \cap U$ for at most N iterates. Using Lemma 9 we conclude:

LEMMA 10. *Given $\epsilon > 0$ there exist a neighborhood \mathcal{U} of f and a number $\epsilon' > 0$ such that $d(h(y), y) < \epsilon'$ for every $y \in U$ implies that $d(h(x), x) < \epsilon$ for every $x \in U' \cap B$.*

Consequently, $d(h(x), x) < \epsilon$ for every $x \in B$: indeed, this follows by the previous lemma since $x \in B$ stays at most N iterates in $B \setminus (U \cup U')$. It remains to prove that h can be extended to the closure of B , which equals M . If $x \notin B$, then there exists q such that $f^n(x) \in U'$ for every $n \geq q$. If $y = f^q(x)$ then there exists j such that either $y \in \Lambda_j$ or $y \in W^s(\Lambda_j) \setminus \Lambda_j$ for some j .

Assume $y \in \Lambda_j$, and let $\{z_n\}$ be a sequence in B convergent to y . Assume for contradiction that there exist two subsequences of $\{h(z_n)\}$ converging to different points. Say $h(z'_n) \rightarrow z$ and $h(z''_n) \rightarrow w$. Note that as z and w belong to $\Lambda_j(g)$, one can use the expansivity of g in Λ_j to ensure that there exists an integer m such that

$$(3) \quad d(g^m(z), g^m(w)) > 2\epsilon.$$

Once m is fixed, we can choose n large and points $f^m(z'_n)$ and $f^m(z''_n)$ that are arbitrarily close (if m was negative, then $f^m(z'_n)$ is the preimage of y that is closest to $f^m(y) \cap \Lambda_j$). As h is ϵ -close to the identity in B , it follows that

$$d(h(f^m(z'_n)), h(f^m(z''_n))) < 2\epsilon.$$

But $h(f^m(z_n)) = g^m(h(z_n))$, so for n large the above contradicts (3).

Consider now the case where $y \in W^s(\Lambda_j) \setminus \Lambda_j$. Assume for contradiction that there exist two sequences z'_n and z''_n in B both converging to y but such that $h(z'_n)$ and $h(z''_n)$ converge to different points z and w . Clearly z and w belong to $H(F_y(f))$. It follows now that there exists a positive m such that $d(g^m(z), g^m(w)) > 2\epsilon$. Reasoning as above provides a contradiction.

6. More examples and questions. The construction of the example in Section 4 allows some generalizations. For instance, one can replace the fixed attractor by a nonperiodic attractor. The result is a map that is C^1 Ω -stable but all its perturbations are not C^1 stable, because the unstable intersections occur in the basin of an attractor that is not periodic, thus contradicting the Strong Transversality Condition.

Every covering map f of the two-torus induces a linear map on the homology $H^1(T^2) = \mathbb{R}^2$. At the same time, a linear map A in the plane with integer coefficients induces a covering of the two-torus whose induced map is A . Moreover, the map f belongs to the same homotopy class as the associated linear map.

The example of Section 4 induces the linear map $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. It is easy to imitate the idea to produce a C^1 stable map whose induced map is the expanding matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

In both cases, however, the covering is semiconjugate to an expanding map. Indeed, Przytycki's example is semiconjugate to $z \mapsto z^2$ in S^1 .

QUESTION 1. Is every C^1 stable map in the two-torus semiconjugate to an expanding map?

An affirmative answer to this question would be very useful towards a classification of C^1 stable maps of the two-torus.

The homotopy class of a diffeomorphism always contains a C^1 stable diffeomorphism; also, the class of an expanding map contains a C^1 stable map.

QUESTION 2. Does every homotopy class contain a C^1 stable map?

If the answer to the second question is negative when the manifold is the two-torus, then the answer to the first question is affirmative.

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J. Iglesias, A. Portela, A. Rovella
IMERL
Facultad de Ingeniería
Universidad de La República
Julio Herrera y Reissig 565
C.P. 11300, Montevideo, Uruguay
E-mail: jorgei@fing.edu.uy
aldo@fing.edu.uy
leva@cmat.edu.uy

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