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Cycles of links and fixed points for orientation preserving homeomorphisms of the open unit disk

by

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Abstract. Michael Handel proved the existence of a fixed point for an orientation preserving homeomorphism of the open unit disk that can be extended to the closed disk, provided that it has points whose orbits form an *oriented cycle of links at infinity*. More recently, the author generalized Handel's theorem to a wider class of cycles of links . In this paper we complete this topic describing exactly which are all the cycles of links forcing the existence of a fixed point.

1. Introduction. Handel's fixed point theorem [7] has been of great importance for the study of surface homeomorphisms. It guarantees the existence of a fixed point for an orientation preserving homeomorphism f of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ provided that it can be extended to the boundary $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and that it has points whose orbits form an oriented cycle of links at infinity. More precisely, there exist n points $z_i \in \mathbb{D}$ such that

$$\lim_{k\to-\infty} f^k(z_i) = \alpha_i \in S^1, \qquad \lim_{k\to\infty} f^k(z_i) = \omega_i \in S^1,$$

$i = 1, \ldots, n$, where the $2n$ points $\{\alpha_i\}$, $\{\omega_i\}$ are different points in S^1 and have the following order property:

(∗) α_{i+1} is the only one among these points that lies in the open interval in the oriented circle S^1 from ω_{i-1} to ω_i.

(Although this is not Handel's original statement, it is an equivalent one as already pointed out in [9].)

Le Calvez gave an alternative proof of this theorem [9], relying only on Brouwer theory and plane topology, which allowed him to obtain a sharper

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result. Namely, he weakened the extension hypothesis by demanding the homeomorphism to extend just to $\mathbb{D}\cup\bigcup_{i\in\mathbb{Z}/n\mathbb{Z}}\{\alpha_i,\omega_i\}$ and he strengthened the conclusion by proving the existence of a simple closed curve of index 1.

The author generalized both Handel's and Le Calvez's results as follows [13]. Let $P\subset\mathbb{D}$ be a compact convex n-gon. Let $\{v_i : i\in\mathbb{Z}/n\mathbb{Z}\}$ be its set of vertices, and for each $i\in\mathbb{Z}/n\mathbb{Z}$, let e_i be the edge joining v_i and v_{i+1}. We suppose that each e_i is endowed with an orientation, so that we can tell whether P is to the right or to the left of e_i. We say that the orientations of e_i and e_j *coincide* if P is to the right (or to the left) of both e_i and e_j, $i,j\in\mathbb{Z}/n\mathbb{Z}$.

We define the *index* of P by

$$i(P)=1-\frac{1}{2}\sum_{i\in\mathbb{Z}/n\mathbb{Z}}\delta_i,$$

where $\delta_i=0$ if the orientations of e_{i-1} and e_i coincide, and $\delta_i=1$ otherwise.

We will denote by α_i and ω_i the first, and respectively the last, point where the straight line Δ_i containing e_i and inheriting its orientation intersects $\partial\mathbb{D}$.

We say that a homeomorphism $f:\mathbb{D}\to\mathbb{D}$ *realizes* P if there exists a family $(z_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ of points in $\mathbb{D}$ such that for all $i\in\mathbb{Z}/n\mathbb{Z}$,

$$\lim_{k\to-\infty} f^k(z_i)=\alpha_i,\qquad \lim_{k\to\infty} f^k(z_i)=\omega_i.$$

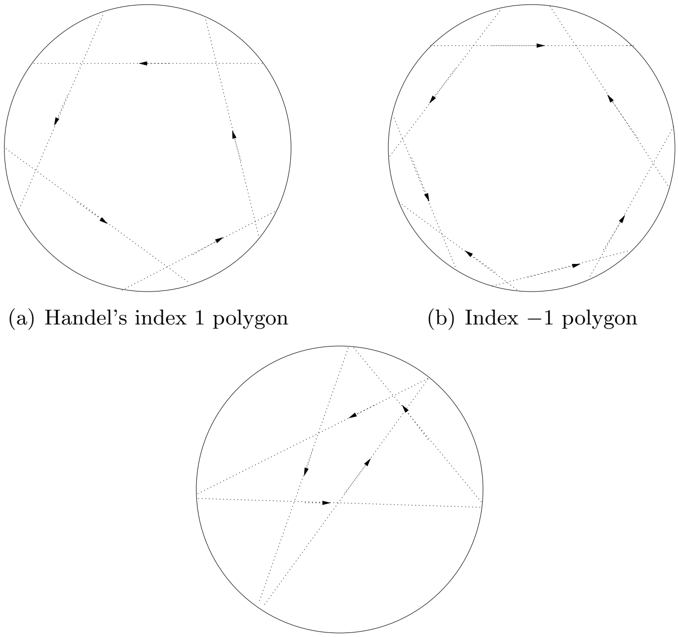
THEOREM 1.1 ([13]). *Let $f:\mathbb{D}\to\mathbb{D}$ be an orientation preserving homeomorphism which realizes a compact convex polygon $P\subset\mathbb{D}$ where the points α_i,ω_i, $i\in\mathbb{Z}/n\mathbb{Z}$, are all different. Suppose that f can be extended to a homeomorphism of $\mathbb{D}\cup\bigcup_{i\in\mathbb{Z}/n\mathbb{Z}}\{\alpha_i,\omega_i\}$. If $i(P)\neq 0$, then f has a fixed point. Furthermore, if $i(P)=1$, then there exists a simple closed curve $C\subset\mathbb{D}$ of index* 1.

The two polygons appearing in Figure 1(a)&(b) satisfy the hypothesis of this theorem. However, the polygon illustrated in (c) does not, as there are coincidences among the points $\{\alpha_i\}$, $\{\omega_i\}$, $i\in\mathbb{Z}/n\mathbb{Z}$.

The purpose of this paper is to complete this topic: we assume that there exist two families $(\alpha_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$, $(\omega_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ of points in S^1 and a family $(z_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ of points in $\mathbb{D}$ such that, for all $i\in\mathbb{Z}/n\mathbb{Z}$,

$$\lim_{k\to-\infty} f^k(z_i)=\alpha_i,\qquad \lim_{k\to\infty} f^k(z_i)=\omega_i,$$

and that f extends to a homeomorphism of $\mathbb{D}\cup\bigcup_{i\in\mathbb{Z}/n\mathbb{Z}}\{\alpha_i,\omega_i\}$, and we describe exactly which combinatorics of the points α_i,ω_i, $i\in\mathbb{Z}/n\mathbb{Z}$, force the existence of a fixed point.



(a) Handel's index 1 polygon (b) Index -1 polygon

(c) $\omega_i = \alpha_{i+2}$ $\forall i$

Fig. 1. The hypothesis of Theorem 1.1

A *cycle of links of order* $n \geq 3$ is a family of pairs of points on the circle S^1,

$$\mathcal{L} = ((\alpha_i, \omega_i))_{i \in \mathbb{Z}/n\mathbb{Z}},$$

such that for all $i \in \mathbb{Z}/n\mathbb{Z}$:

(1) $\alpha_i \neq \omega_i$,
(2) α_{i+1} and ω_{i+1} belong to different connected components of the set $S^1 \backslash \{\alpha_i, \omega_i\}$.

If $\mathcal{L}$ is a cycle of links, we define the set

$$\ell = \{\alpha_i, \omega_i : i \in \mathbb{Z}/n\mathbb{Z}\} \subset S^1$$

of points in the circle which belong to a pair in the cycle.

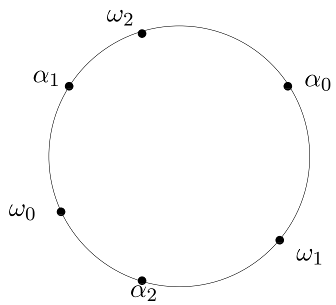
If $a, b \in \ell$, we write $a \to b$ if b follows a in the natural (positive) cyclic order on S^1, and $a \xrightarrow{=} b$ if either $a = b$ or $a \to b$.

We say that a cycle of links $\mathcal{L}$ is *elliptic* if for all $i \in \mathbb{Z}/n\mathbb{Z}$,

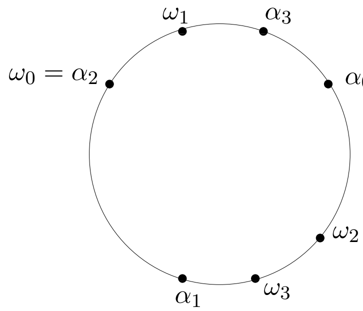
$$\omega_{i-1} \xrightarrow{=} \alpha_{i+1} \to \omega_i.$$

We say it is *hyperbolic* if $n = 2k$, $k \geq 2$, and for all $i \in \mathbb{Z}/n\mathbb{Z}$ with $i = 0 \mod 2$,

$$\alpha_i \to \alpha_{i-1} \xrightarrow{=} \omega_{i+1} \to \omega_i \xrightarrow{=} \alpha_{i+2}.$$



(a) An elliptic cycle of links of order 3



(b) A hyperbolic cycle of links of order 4

Finally, we say that $\mathcal{L}$ is *non-degenerate* if

$$(\alpha_i, \omega_i) \in \mathcal{L} \;\Rightarrow\; (\omega_i, \alpha_i) \notin \mathcal{L}.$$

Of course, we say it is *degenerate* if this condition is not satisfied. An example is illustrated in Figure 2.

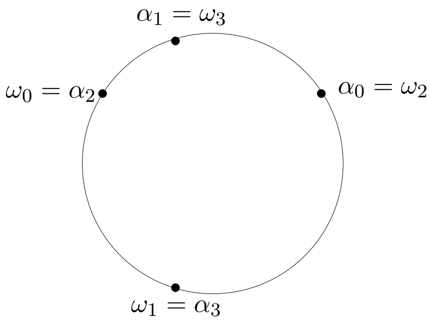


Fig. 2. A degenerate cycle of links

We say that a homeomorphism $f : \mathbb{D} \to \mathbb{D}$ *realizes* $\mathcal{L}$ if there exists a family $(z_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ of points in $\mathbb{D}$ such that, for all $i \in \mathbb{Z}/n\mathbb{Z}$,

$$\lim_{k \to -\infty} f^k(z_i) = \alpha_i, \qquad \lim_{k \to \infty} f^k(z_i) = \omega_i.$$

The following theorem is the main result of this article.

THEOREM 1.2. *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is an orientation preserving homeomorphism which realizes a cycle of links $\mathcal{L}$ and can be extended to a homeomorphism of $\mathbb{D} \cup \ell$. If $\mathcal{L}$ is either elliptic or hyperbolic, then f has a fixed point. Furthermore, if $\mathcal{L}$ is non-degenerate and elliptic, then there exists a simple closed curve $C \subset \mathbb{D}$ of index* 1.

It turns out that these results completely describe the combinatorics giving rise to fixed points:

Lemma 1.3. *Given a family* $((\alpha_i, \omega_i))_{i \in \mathbb{Z}/n\mathbb{Z}}$ *of pairs of points in* S^1, *one of the following is true:*

(1) *there exists a subfamily of* $((\alpha_i, \omega_i))_{i \in \mathbb{Z}/n\mathbb{Z}}$ *forming an elliptic or hyperbolic cycle of links,*
(2) *the oriented straight lines from* α_i *to* ω_i *bound a non-zero index polygon* $P \subset \mathbb{D}$,
(3) *there exists a fixed-point free orientation preserving homeomorphism* $f : \mathbb{D} \to \mathbb{D}$ *and a family of points* $(z_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ *in* $\mathbb{D}$ *such that for all* $i \in \mathbb{Z}/n\mathbb{Z}$,
$$\lim_{k \to -\infty} f^k(z_i) = \alpha_i, \qquad \lim_{k \to \infty} f^k(z_i) = \omega_i.$$

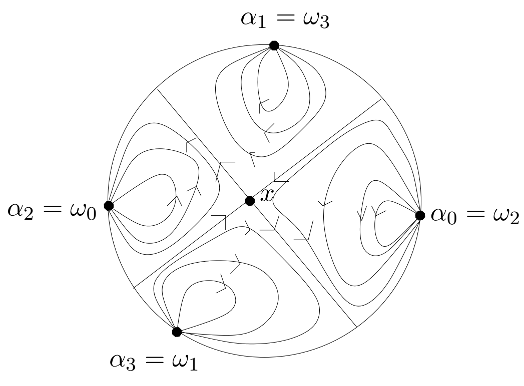
We finish this introduction with some remarks on Theorem 1.2.

The elliptic non-degenerate case contains Le Calvez's improvement of Handel's theorem. Indeed, if the points in ℓ are all different, then $\mathcal{L}$ is non-degenerate. As the example in Figure 1(c) shows, our theorem is more general even in this case.

The theorem contains the author's result on non-zero index polygons. Indeed, in [13] it is shown that if f realizes a non-zero index polygon where the points $\alpha_i, \omega_i, i \in \mathbb{Z}/n\mathbb{Z}$ are all different, then f realizes an elliptic or hyperbolic cycle of links. Again, as coincidences in ℓ are allowed, our theorem is more general even in this case.

The extension hypothesis is necessary. Indeed, if $f : \mathbb{D} \to \mathbb{D}$ is fixed-point free, one can easily construct a homeomorphism $h : \mathbb{D} \to \mathbb{D}$ such that hfh^{-1} realizes any prescribed cycle of links.

Non-degeneracy is necessary to obtain the index result. Let f_1 be the time-one map of the flow whose orbits are drawn in the figure below.



As we will explain below, one can perturb f_1 to a homeomorphism f such that:

- $\mathrm{Fix}(f) = \mathrm{Fix}(f_1) = \{x\}$,
- $f = f_1$ in a neighbourhood of x,
- f realizes $\mathcal{L} = ((\alpha_i, \omega_i))_{i\in\mathbb{Z}/4\mathbb{Z}}$.

We say that the set X is *free* if $f(X) \cap X = \emptyset$.

One can find (by means of a transverse foliation, for example) free and pairwise disjoint simple paths β_i and γ_i, $i \in \mathbb{Z}/4\mathbb{Z}$, such that:

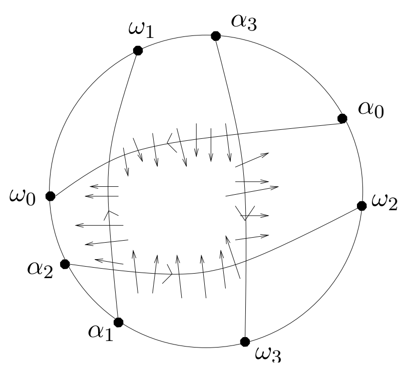
- β_i joins z_i and z_i', where
$$\lim_{k\to\infty} f_1^{-k}(z_i) = \alpha_i \quad \text{and} \quad \lim_{k\to\infty} f_1^{k}(z_i') = \alpha_{i^*},$$
$i^* = i + 1$ for even values of i, and $i^* = i - 1$ for odd values of i,
- γ_i joins $f_1^{p_i}(z_i')$ and z_i'', where $p_i > 0$ and $\lim_{k\to\infty} f_1^k(z_i'') = \omega_i$,
- γ_i and β_i are disjoint from the f_1-orbits of all z_j, z_j', z_j'' with $i \neq j$.

By thickening the paths $\{\beta_i\}$ and $\{\gamma_i\}$, one can find free, pairwise disjoint open disks $\{D_i'\}$ and $\{D_i''\}$ such that the disks D_i' and D_i'' are disjoint from the f_1-orbits of the points z_j, z_j', z_j'' for $i \neq j$.

We construct a homeomorphism $h : \mathbb{D} \to \mathbb{D}$ such that:

- $h = \mathrm{Id}$ outside $\bigcup_{i\in\mathbb{Z}/4\mathbb{Z}} D_i' \cup D_i''$,
- $h(z_i) = z_i'$,
- $h(f_1^p(z_i')) = z_i''$.

So, if we define $f = h \circ f_1$, we obtain
$$\lim_{k\to\infty} f^{-k}(z_i) = \alpha_i, \qquad \lim_{k\to\infty} f^{k}(z_i) = \omega_i,$$
for all $i \in \mathbb{Z}/4\mathbb{Z}$. Clearly we can make this construction in such a way that $f = f_1$ in a neighbourhood of x. Moreover, as the disks $\{D_i'\}$ and $\{D_i''\}$ are free,
$$\mathrm{Fix}(f) = \mathrm{Fix}(f_1) = \{x\}.$$
So, f realizes the elliptic cycle $\mathcal{L}$, but there is no simple closed curve of index 1.

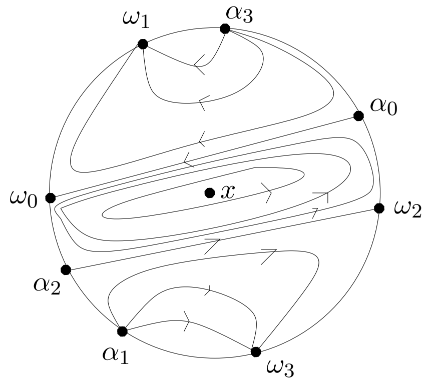


No negative-index fixed point is guaranteed by hyperbolicity. One could think that when $\mathcal{L}$ is hyperbolic, a negative-index fixed point should be obtained. For example, this would be the case if one had an oriented foliation $\mathcal{F}$ in $\mathbb{D} \setminus \mathrm{Fix}(f)$ whose leaves are Brouwer lines for f and simple paths γ_i, $i \in \mathbb{Z}/n\mathbb{Z}$, joining α_i and ω_i such that:

- each γ_i is positively transverse to $\mathcal{F}$,
- the paths $\{\gamma_i\}$ bound a compact disc in $\mathbb{D}$.

(See the figure above.) Indeed, in this case, the Poincaré–Hopf formula would give a singularity x of the foliation for which $i(\mathcal{F}, x) < 0$. So, $x \in \mathrm{Fix}(f)$ and by a result of Le Calvez ([10]) one has $i(f, x) = i(\mathcal{F}, x) < 0$.

However, this is not the case, as the following example shows. Let f_1 be the time-one map of the flow whose orbits are drawn in the figure below.



As we did in our preceding example, one can perturb f_1 to a homeomorphism f such that:

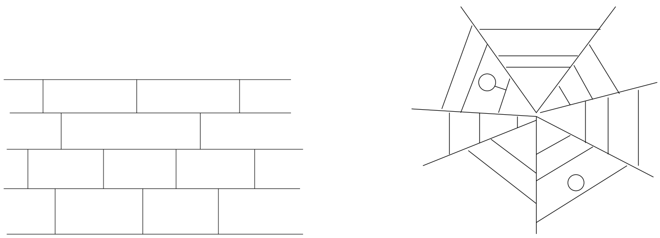
- $\mathrm{Fix}(f) = \mathrm{Fix}(f_1) = \{x\}$,
- $f = f_1$ in a neighbourhood of x,
- f realizes $\mathcal{L} = ((\alpha_i, \omega_i))_{i \in \mathbb{Z}/4\mathbb{Z}}$.

So, f realizes the hyperbolic cycle $\mathcal{L}$, but there is no fixed point of negative index.

The structure of this article is the following. In Section 2 we introduce the tools to be used (brick decompositions, Brouwer theory, repeller/attractor configurations [13]) and we sum up the results from [9] and [13] that will be used in the proofs. In Section 3 we state two lemmas that are key for the contradiction argument in the proof of Theorem 1.2, which is contained in Section 4. The last Section 5 is devoted to the proof of Lemma 1.3, which shows that our results are optimal.

2. Preliminaries

2.1. Brick decompositions. A *brick decomposition* $\mathcal{D}$ of an orientable surface M is a one-dimensional singular submanifold $\Sigma(\mathcal{D})$ (the *skeleton* of the decomposition), with the property that the set of singularities V is discrete and such that every $\sigma \in V$ has a neighbourhood U for which $U \cap (\Sigma(\mathcal{D}) \setminus V)$ has exactly three connected components. We have illustrated two brick decompositions in Figure 3. The *bricks* are the closures of the connected components of $M \setminus \Sigma(\mathcal{D})$ and the *edges* are the closures of the connected components of $\Sigma(\mathcal{D}) \setminus V$. We will write E for the set of edges, B for the set of bricks and finally $\mathcal{D} = (V, E, B)$ for a brick decomposition.



(a) $M = \mathbb{R}^2$ (b) $M = \mathbb{R}^2 \setminus \{0\}$

Fig. 3. Brick decompositions

Let $\mathcal{D} = (V, E, B)$ be a brick decomposition of M. We say that $X \subset B$ is *connected* if given two bricks $b, b' \in X$, there exists a sequence $(b_i)_{0 \le i \le n}$ with $b_0 = b$, $b_n = b'$ and such that b_i and b_{i+1} have non-empty intersection, $i \in \{0, \dots, n-1\}$. Whenever two bricks b and b' have non-empty intersection, we say that they are *adjacent*. Moreover, we say that a brick b is *adjacent to a subset* $X \subset B$ if $b \notin X$ but b is *adjacent* to one of the bricks in X. We say that $X \subset B$ is adjacent to $X' \subset B$ if X and X' have no common bricks but there exist $b \in X$ and $b' \in X'$ which are adjacent.

From now on we will identify a subset X of B with the closed subset of M formed by the union of the bricks in X. This may lead to ambiguities (for instance, two adjacent subsets of B have empty intersection in B and non-empty intersection in M), but we will point out such cases explicitly. We remark that ∂X is a one-dimensional topological manifold and that the connectedness of $X \subset B$ is equivalent to the connectedness of $X \subset M$ and to the connectedness of $\mathrm{Int}(X) \subset M$ as well. We say that a decomposition $\mathcal{D}'$ is a *subdecomposition* of $\mathcal{D}$ if $\Sigma(\mathcal{D}') \subset \Sigma(\mathcal{D})$.

If $f : M \to M$ is a homeomorphism, we define a map $\varphi : \mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{B})$ as follows:

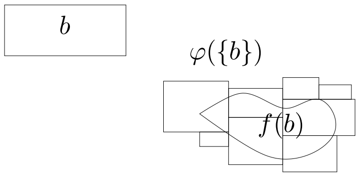
$$\varphi(X) = \{b \in B : f(X) \cap b \neq \emptyset\}.$$

We remark that $\varphi(X)$ is connected whenever X is. We define analogously a map $\varphi_- : \mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{B})$:

$$\varphi_-(X) = \{b \in B : f^{-1}(X) \cap b \neq \emptyset\}.$$

We define the *future* $[b]_\geq$ and the *past* $[b]_\leq$ of a brick b as follows:

$$[b]_\geq = \bigcup_{k\geq 0} \varphi^k(\{b\}), \qquad [b]_\leq = \bigcup_{k\geq 0} \varphi_-^k(\{b\}).$$



We also define the *strict future* $[b]_>$ and the *strict past* $[b]_<$ of a brick b:

$$[b]_> = \bigcup_{k>0} \varphi^k(\{b\}), \qquad [b]_< = \bigcup_{k>0} \varphi_-^k(\{b\}).$$

We say that a set $X \subset B$ is an *attractor* if it satisfies $\varphi(X) \subset X$; this is equivalent in M to the inclusion $f(X) \subset \mathrm{Int}(X)$. A *repeller* is any set X which satisfies $\varphi_-(X) \subset X$. In this way, the future of any brick is an attractor, and the past of any brick is a repeller. We observe that $X \subset B$ is a repeller if and only if $B \setminus X$ is an attractor.

Remark 2.1. The following properties can be deduced from the fact that $X \subset B$ is an attractor if and only if $f(X) \subset \mathrm{Int}(X)$:

(1) if $X \subset B$ is an attractor and $b \in X$, then $[b]_\geq \subset X$; if $X \subset B$ is a repeller and $b \in X$, then $[b]_\leq \subset X$,
(2) if $X \subset B$ is an attractor and $b \notin X$, then $[b]_\leq \cap X = \emptyset$; if $X \subset B$ is a repeller and $b \notin X$, then $[b]_\geq \cap X = \emptyset$,
(3) if $b \in B$ is adjacent to an attractor $X \subset B$, then $[b]_> \cap X \neq \emptyset$; if $b \in B$ is adjacent to a repeller $X \subset B$, then $[b]_< \cap X \neq \emptyset$,
(4) two attractors are disjoint as subsets of B if and only if they are disjoint as subsets of M; in other words, two disjoint (in B) attractors cannot be adjacent; also, two disjoint (in B) repellers cannot be adjacent.

The following conditions are equivalent:

$$b \in [b]_>, \ \ [b]_> = [b]_\geq, \ \ b \in [b]_<, \ \ [b]_< = [b]_\leq, \ \ [b]_< \cap [b]_\geq \neq \emptyset, \ \ [b]_\leq \cap [b]_> \neq \emptyset.$$

The existence of a brick $b \in B$ for which any of these conditions is satisfied is equivalent to the existence of a *closed chain of bricks*, i.e. a family $(b_i)_{i\in\mathbb{Z}/r\mathbb{Z}}$ of bricks such that for all $i \in \mathbb{Z}/r\mathbb{Z}$, $\bigcup_{k\geq 1} f^k(b_i) \cap b_{i+1} \neq \emptyset$.

In general, a *chain* for $f \in \mathrm{Homeo}(M)$ is a family $(X_i)_{0\le i\le r}$ of subsets of M such that $\bigcup_{k\ge 1} f^k(X_i)\cap X_{i+1}\neq\emptyset$ for all $0\le i\le r-1$. We say that the chain is *closed* if $X_r = X_0$.

We say that a subset $X\subset M$ is *free* if $f(X)\cap X=\emptyset$.

We say that a brick decomposition $\mathcal{D}=(V,E,B)$ is *free* if every $b\in B$ is a free subset of M. If f is fixed point free it is always possible, taking sufficiently small bricks, to construct a free brick decomposition.

We recall the definition of *maximal free decomposition*, which was introduced by Sauzet in his doctoral thesis [12]. Let f be a fixed point free homeomorphism of a surface M. We say that $\mathcal{D}$ is a maximal free decomposition if $\mathcal{D}$ is free and any strict subdecomposition is no longer free. Applying Zorn's lemma, it is always possible to prove the existence of a maximal free subdecomposition of a given brick decomposition $\mathcal{D}$.

2.2. Brouwer theory background. We say that $\Gamma:[0,1]\to\overline{\mathbb{D}}$ is an *arc* if it is continuous and injective. We say that an arc Γ *joins* $x\in\overline{\mathbb{D}}$ to $y\in\overline{\mathbb{D}}$ if $\Gamma(0)=x$ and $\Gamma(1)=y$. We say that an arc Γ joins $X\subset\overline{\mathbb{D}}$ to $Y\subset\overline{\mathbb{D}}$, if Γ joins some $x\in X$ to some $y\in Y$.

Fix an $f\in\mathrm{Homeo}^+(\mathbb{D})$. An arc γ joining $z\notin\mathrm{Fix}(f)$ to $f(z)$ such that $f(\gamma)\cap\gamma=\{z,f(z)\}$ if $f^2(z)=z$, and $f(\gamma)\cap\gamma=\{f(z)\}$ otherwise, is called a *translation arc.*

Proposition 2.2 (Brouwer's translation lemma, [1], [2], [4] or [6]). *If any of the following two hypotheses is satisfied:*

(1) *there exists a translation arc γ joining $z\in\mathrm{Fix}(f^2)\setminus\mathrm{Fix}(f)$ to $f(z)$,*
(2) *there exists a translation arc γ joining $z\notin\mathrm{Fix}(f^2)$ to $f(z)$ and an integer $k\ge 2$ such that $f^k(\gamma)\cap\gamma\neq\emptyset$,*

then there exists a simple closed curve of index 1.

If $z\notin\mathrm{Fix}(f)$, there exists a translation arc containing z; this is easy to prove once one knows that the connected components of the complement of $\mathrm{Fix}(f)$ are invariant. For this last fact, see [3] for a general proof in any dimension, or [8] for an easy proof in dimension 2.

We deduce:

Corollary 2.3. *If $\mathrm{Per}(f)\setminus\mathrm{Fix}(f)\neq\emptyset$, then there exists a simple closed curve of index* 1.

Proposition 2.4 (Franks' lemma [5]). *If there exists a closed chain of free, open and pairwise disjoint disks for f, then there exists a simple closed curve of index* 1.

Following Le Calvez [9], we will say that f is *recurrent* if there exists a closed chain of free, open and pairwise disjoint disks for f.

The following proposition is a refinement of Franks' lemma due to Guillou and Le Roux (see [11, p. 39]).

PROPOSITION 2.5. *Suppose there exists a closed chain $(X_i)_{i\in\mathbb{Z}/r\mathbb{Z}}$ for f of free subsets whose interiors are pairwise disjoint and which have the following property: given any two points $z, z' \in X_i$ there exists an arc γ joining z and z' such that $\gamma \setminus \{z, z'\} \subset \mathrm{Int}(X_i)$. Then f is recurrent.*

We deduce:

PROPOSITION 2.6. *Let $\mathcal{D} = (V, E, B)$ be a free brick decomposition of $\mathbb{D} \setminus \mathrm{Fix}(f)$. If there exists $b \in B$ such that $b \in [b]_>$, then f is recurrent.*

2.3. Little bricks at infinity. Fix $f \in \mathrm{Homeo}^+(\mathbb{D})$, different from the identity map and *non-recurrent*. We will make use of the following two propositions from [9] (both of them depend on the non-recurrent character of f). The first one (Proposition 2.2 in [9]) is a refinement of a result already appearing in [12]; the second one is Proposition 3.1 in [9].

PROPOSITION 2.7 ([12], [9]). *Let $\mathcal{D} = (V, E, B)$ be a maximal free brick decomposition of $\mathbb{D} \setminus \mathrm{Fix}(f)$. Then the sets $[b]_\geq$, $[b]_>$, $[b]_\leq$ and $[b]_<$ are connected. In particular every connected component of an attractor is an attractor, and every connected component of a repeller is a repeller.*

PROPOSITION 2.8 ([9]). *If f satisfies the hypothesis of Theorem* 1.2, *then for all $i \in \mathbb{Z}/n\mathbb{Z}$ we can find a sequence $(\gamma_i^k)_{k\in\mathbb{Z}}$ of arcs such that:*

- *each γ_i^k is a translation arc from $f^k(z_i)$ to $f^{k+1}(z_i)$,*
- *$f(\gamma_i^k) \cap \gamma_i^{k'} = \emptyset$ if $k' < k$,*
- *the sequence $(\gamma_i^k)_{k\leq 0}$ converges to $\{\alpha_i\}$ in the Hausdorff topology,*
- *the sequence $(\gamma_i^k)_{k\geq 0}$ converges to $\{\omega_i\}$ in the Hausdorff topology.*

This result is a consequence of Brouwer's translation lemma and the hypothesis on the orbits of the points $(z_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$. In particular, the extension hypothesis of Theorem 1.2 is used. It allows us to construct a particular brick decomposition suitable for our purposes:

LEMMA 2.9. *For every $i \in \mathbb{Z}/n\mathbb{Z}$, take U_i^- a neighbourhood of α_i in $\overline{\mathbb{D}}$ and U_i^+ a neighbourhood of ω_i in $\overline{\mathbb{D}}$ such that $U_i^- \cap U_i^+ = \emptyset$. There exist two families $(b_i'^l)_{i\in\mathbb{Z}/n\mathbb{Z},\, l\geq 1}$ and $(b_i'^l)_{i\in\mathbb{Z}/n\mathbb{Z},\, l\leq -1}$ of closed disks in $\mathbb{D}$ and a family $(l_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ of integers such that:*

1. *each $b_i'^l$ is free and contained in U_i^- $(l \leq -1)$ or in U_i^+ $(l \geq 1)$,*
2. *$\mathrm{Int}(b_i'^l) \cap \mathrm{Int}(b_i'^{l'}) = \emptyset$ if $l \neq l'$,*
3. *for every $k > 1$ the sets $(b_i'^l)_{1\leq l\leq k}$ and $(b_i'^l)_{-k\leq l\leq -1}$ are connected,*
4. *for all $i \in \mathbb{Z}/n\mathbb{Z}$, $\partial \bigcup_{l\in\mathbb{Z}\setminus\{0\}} b_i'^l$ is a one-dimensional submanifold,*
5. *if $x \in \mathbb{D}$, then x belongs to at most two different disks in the family $(b_i'^l)_{l\in\mathbb{Z}\setminus\{0\},\, i\in\mathbb{Z}/n\mathbb{Z}}$,*

(6) *for all* $i \in \mathbb{Z}/n\mathbb{Z}$, $f^{l_i+l}(z_i) \in \mathrm{Int}(b_i'^{l+1})$ *for all* $l \geq 0$, *and* $f^{-l_i-l}(z_i) \in \mathrm{Int}(b_i'^{-l-1})$ *for all* $l \geq 0$,
(7) $f^k(z_j) \in b_i'^l$ *if and only if* $j = i$ *and* $k = l_i + l - 1$,
(8) *the sequence* $(b_i'^l)_{l\geq 1}$ *converges to* $\{\omega_i\}$ *in the Hausdorff topology and the sequence* $(b_i'^l)_{l\leq -1}$ *converges to* $\{\alpha_i\}$ *in the Hausdorff topology.*

The idea is to construct trees $T_i^- \subset U_i^-, T_i^+ \subset U_i^+$, $i \in \mathbb{Z}/n\mathbb{Z}$, by deleting the loops of the curves $\prod_{k\geq -1} \gamma_i^k \cap U_i^-$ and $\prod_{k\leq 1} \gamma_i^k \cap U_i^+$ respectively, and then thickening these trees to obtain the families $(b_i'^l)_{i\in\mathbb{Z}/n\mathbb{Z},\, l\geq 1}$ and $(b_i'^l)_{i\in\mathbb{Z}/n\mathbb{Z},\, l\leq -1}$. We refer the reader to [13] for a proof in English but we remark that these results are contained in [9]. We have illustrated these families in Figure 4.

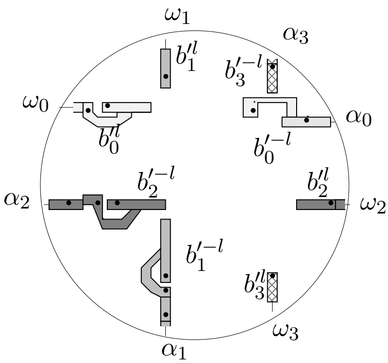


Fig. 4. The families $b_i'^l$

Remark 2.10. The fact that the sequence $(b_i'^l)_{l\geq 1}$ converges in the Hausdorff topology to ω_i, implies we can find an arc $\Gamma_i^+ : [0,1] \to \mathrm{Int}(\bigcup_{l\geq 0} b_i'^l) \cup \{\omega_i\}$ such that $\Gamma_i^+(1) = \omega_i$, $i \in \mathbb{Z}/n\mathbb{Z}$. Similarly, we can find an arc $\Gamma_i^- : [0,1] \to \mathrm{Int}(\bigcup_{l\geq 0} b_i'^{-l}) \cup \{\alpha_i\}$ such that $\Gamma_i^-(1) = \alpha_i$, $i \in \mathbb{Z}/n\mathbb{Z}$.

2.4. Repeller/attractor configurations

2.4.1. *Cyclic order at infinity.* Let $(a_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ be a family of non-empty, pairwise disjoint, closed, connected subsets of $\mathbb{D}$, such that $\overline{a}_i \cap \partial\mathbb{D} \neq \emptyset$ and $U = \mathbb{D} \setminus \bigcup_{i\in\mathbb{Z}/n\mathbb{Z}} a_i$ is a connected open set. As U is connected, and its complementary set in $\mathbb{C}$, namely

$$\{z \in \mathbb{C} : |z| \geq 1\} \cup \bigcup_{i\in\mathbb{Z}/n\mathbb{Z}} a_i,$$

is also connected, U is simply connected.

With these hypotheses, there is a natural cyclic order on the sets $\{a_i\}$. Indeed, U is conformally isomorphic to the unit disc via the Riemann map

$\varphi : U \to \mathbb{D}$, and one can consider the Carathéodory extension of φ,

$$\hat{\varphi} : \hat{U} \to \overline{\mathbb{D}},$$

which is a homeomorphism between the prime ends completion $\hat{U}$ of U and the closed unit disk $\overline{\mathbb{D}}$. The set $\hat{J}_i$ of prime ends whose impression is contained in a_i is open and connected. It follows that the images $J_i = \hat{\varphi}(\hat{J}_i)$ are pairwise disjoint open intervals in S^1, and are therefore cyclically ordered following the positive orientation of the circle.

2.4.2. *Repeller/attractor configurations.* We recall the definition of repeller/attractor configuration that was introduced in [13].

We fix $f \in \mathrm{Homeo}^+(\mathbb{D})$ together with a maximal free brick decomposition $\mathcal{D} = (V, E, B)$ of $\mathbb{D} \setminus \mathrm{Fix}(f)$.

Let $(R_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ and $(A_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ be two families of connected, pairwise disjoint subsets of B such that:

(1) For all $i \in \mathbb{Z}/n\mathbb{Z}$,

  (a) R_i is a repeller and A_i is an attractor,
  (b) there exists non-empty, closed, connected subsets $r_i \subset \mathrm{Int}(R_i)$, $a_i \subset \mathrm{Int}(A_i)$ of $\mathbb{D}$ such that $\overline{r_i} \cap \partial\mathbb{D} \neq \emptyset$ and $\overline{a_i} \cap \partial\mathbb{D} \neq \emptyset$,

(2) $\mathbb{D} \setminus \bigcup_{i\in\mathbb{Z}/n\mathbb{Z}}(a_i \cup r_i)$ is a connected open set.

We say that the pair $((R_i)_{i\in\mathbb{Z}/n\mathbb{Z}}, (A_i)_{i\in\mathbb{Z}/n\mathbb{Z}})$ is a *repeller/attractor configuration of order* n. We will write

$$\mathcal{E} = \{R_i, A_i : i \in \mathbb{Z}/n\mathbb{Z}\}.$$

Property (2) in the previous definition allows us to give a cyclic order to the sets r_i, a_i, $i \in \mathbb{Z}/n\mathbb{Z}$ (see the beginning of this section).

We say that a repeller/attractor configuration of order $n \geq 3$ is an *elliptic configuration* if:

(1) the cyclic order of the sets r_i, a_i, $i \in \mathbb{Z}/n\mathbb{Z}$, has the *elliptic order property*:

$$a_0 \to r_2 \to a_1 \to \cdots \to a_i \to r_{i+2} \to a_{i+1} \to \cdots \to a_{n-1} \to r_1 \to a_0,$$

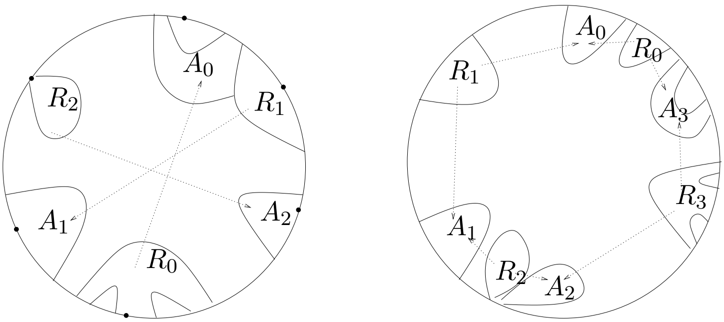
(2) for all $i \in \mathbb{Z}/n\mathbb{Z}$ there exists a brick $b_i \in R_i$ such that $[b_i]_{\geq} \cap A_i \neq \emptyset$.

We say that a repeller/attractor configuration is a *hyperbolic configuration* if:

(1) the cyclic order of the sets r_i, a_i, $i \in \mathbb{Z}/n\mathbb{Z}$, has the *hyperbolic order property*:

$$r_0 \to a_0 \to r_1 \to a_1 \to \cdots \to r_i \to a_i \\ \to r_{i+1} \to a_{i+1} \to \cdots \to r_{n-1} \to a_{n-1} \to r_0,$$

(2) for all $i \in \mathbb{Z}/n\mathbb{Z}$ there exist two bricks $b_i^i, b_i^{i-1} \in R_i$ such that $[b_i^i]_> \cap A_i \neq \emptyset$ and $[b_i^{i-1}]_> \cap A_{i-1} \neq \emptyset$.



(a) An elliptic configuration (b) A hyperbolic configuration

We will make use of the following results from [13]:

PROPOSITION 2.11 ([13]). *If there exists an elliptic configuration of order $n \geq 3$, then f is recurrent.*

PROPOSITION 2.12 ([13]). *If there exists a hyperbolic configuration of order $n \geq 2$, then* $\mathrm{Fix}(f) \neq \emptyset$.

3. Two technical lemmas. In this section we give applications of Propositions 2.11 and 2.12 that will be used in the proof of Theorem 1.2.

We fix $f \in \mathrm{Homeo}^+(\mathbb{D})$ together with a maximal free brick decomposition $\mathcal{D} = (V, E, B)$ of $\mathbb{D} \setminus \mathrm{Fix}(f)$, and we suppose that f is non-recurrent.

Let a_i, $i \in \mathbb{Z}/n\mathbb{Z}$, be non-empty, pairwise disjoint, closed, connected subsets of $\mathbb{D}$ such that $\overline{a}_i \cap \partial\mathbb{D} \neq \emptyset$ for all $i \in \mathbb{Z}/n\mathbb{Z}$, and $U = \mathbb{D} \setminus \bigcup_{i \in \mathbb{Z}/n\mathbb{Z}} a_i$ is a connected open set. We consider the Riemann map $\varphi : U \to \mathbb{D}$, and the open intervals on the circle J_i, $i \in \mathbb{Z}/n\mathbb{Z}$, defined in 2.4.1. We recall that the interval J_i corresponds to the prime ends in U whose impression is contained in a_i.

Let $(I_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ be the connected components of $S^1 \setminus \bigcup_{i \in \mathbb{Z}/n\mathbb{Z}} J_i$. So, each I_i is a closed interval, which may be reduced to a point.

REMARK 3.1. One can cyclically order the sets $(a_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$, $(r_j)_{i \in \mathbb{Z}/m\mathbb{Z}}$, where $(r_j)_{i \in \mathbb{Z}/m\mathbb{Z}}$ is any family of closed, connected and pairwise disjoint subsets of U satisfying:

- $\overline{r_j} \cap \partial U \neq \emptyset$, $j \in \mathbb{Z}/m\mathbb{Z}$,
- for all $j \in \mathbb{Z}/m\mathbb{Z}$, there exists $i_j \in \mathbb{Z}/n\mathbb{Z}$ such that $\overline{\varphi(r_j)} \cap S^1 \subset I_{i_j}$,
- the correspondence $j \mapsto i_j$ is injective.

Lemma 3.2. *Suppose that:*

(1) *the cyclic order of the sets a_i, $i \in \mathbb{Z}/n\mathbb{Z}$, is*

$$a_0 \to a_1 \to \cdots \to a_i \to a_{i+1} \to \cdots \to a_{n-1} \to a_0,$$

(2) *for all $i \in \mathbb{Z}/n\mathbb{Z}$ there exists $b_i^+ \in B$ such that $a_i \subset [b_i^+]_>$,*

(3) *there exist three bricks $(b_s^-)_{s\in\mathbb{Z}/3\mathbb{Z}}$ such that*

   (a) *for all $s \in \mathbb{Z}/3\mathbb{Z}$ and all $i \in \mathbb{Z}/n\mathbb{Z}$, one has $b_s^- \subset [b_i^+]_<$ (and so $\overline{[b_s^-]_<} \subset U$),*

   (b) *$\overline{[b_s^-]_<} \cap \partial U \neq \emptyset$ for all $s \in \mathbb{Z}/3\mathbb{Z}$,*

   (c) *for every $s \in \mathbb{Z}/3\mathbb{Z}$ there exists $i_s \in \mathbb{Z}/n\mathbb{Z}$ such that*

$$\overline{\varphi([b_s^-]_<)} \cap S^1 \subset I_{i_s},$$

Then the correspondence $s \mapsto i_s$ is not injective.

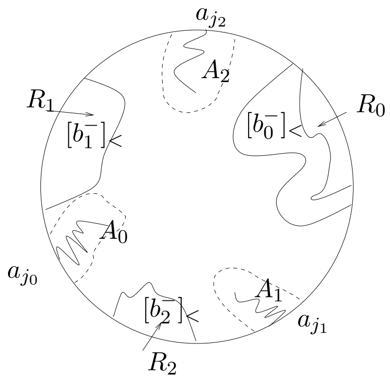


Fig. 5. Lemma 3.2

Proof. We will prove that if the correspondence $s \mapsto i_s$ is injective, then we can construct an elliptic configuration of order 3. As we are assuming f is not recurrent, this is not possible by Proposition 2.11.

We begin by proving that $[b_s^-]_< \cap [b_r^-]_< \neq \emptyset$ implies $i_s = i_r$. Indeed, if $[b_s^-]_< \cap [b_r^-]_< \neq \emptyset$, then $[b_s^-]_< \cup [b_r^-]_<$ is a connected set and $\overline{\varphi([b_s^-]_< \cup [b_r^-]_<)}$ intersects both I_{i_s} and I_{i_r}. If $i_s \neq i_r$, then there exist $j_0, j_1 \in \mathbb{Z}/n\mathbb{Z}$ such that any arc joining J_{j_0} and J_{j_1} separates I_{i_r} from I_{i_s} in $\overline{\mathbb{D}}$. Our hypothesis (3)(a) allows us to take a crosscut γ from a_{j_0} to a_{j_1} such that $\gamma \cap U \subset [b_s^-]_>$. So, $\overline{\varphi(\gamma \cap U)}$ is an arc joining J_{j_0} and J_{j_1}, and

$$\overline{\varphi(\gamma \cap U)} \cap \varphi([b_s^-]_< \cup [b_r^-]_<) \neq \emptyset.$$

This gives us

$$([b_s^-]_< \cup [b_r^-]_<) \cap [b_s^-]_> \neq \emptyset,$$

and as we are supposing that f is not recurrent,

$$[b_r^-]_< \cap [b_s^-]_> \neq \emptyset.$$

So,

$$[b_s^-]_< \subset [b_r^-]_<,$$

which implies

$$\overline{\varphi([b_s^-]_<)} \cap S^1 \subset I_{i_s} \cap I_{i_r},$$

a contradiction.

So, if the correspondence $s \mapsto i_s$ is injective, the sets $[b_s^-]_<$ are pairwise disjoint, and one can cyclically order the $n+3$ sets $a_i, [b_s^-]_<$, $i \in \mathbb{Z}/n\mathbb{Z}$, $s \in \mathbb{Z}/3\mathbb{Z}$ (see Remark 3.1). We may suppose without loss of generality that

$$[b_0^-]_< \to [b_1^-]_< \to [b_2^-]_< \to [b_0^-]_<.$$

For all $s \in \mathbb{Z}/3\mathbb{Z}$, we can take $j_s \in \mathbb{Z}/3\mathbb{Z}$ such that

$$[b_0]_<^- \to a_{j_2} \to [b_1^-]_< \to a_{j_0} \to [b_2^-]_< \to a_{j_1} \to [b_0]_<^-$$

(see Figure 9 below).

For all $s \in \mathbb{Z}/3\mathbb{Z}$, we define

$$R_s = [b_s^-]_<, \qquad A_s = [b_{j_s}^+]_>.$$

We want to show that

$$((R_s)_{s\in\mathbb{Z}/3\mathbb{Z}}), (A_s)_{s\in\mathbb{Z}/3\mathbb{Z}})$$

is an elliptic configuration. It is enough to show that the sets A_s, R_s, $s \in \mathbb{Z}/3\mathbb{Z}$, are pairwise disjoint, because of the cyclic order of these sets , and our hypothesis (3)(a). We already know that the sets R_s, $s \in \mathbb{Z}/3\mathbb{Z}$, are pairwise disjoint. As we are supposing that f is not recurrent, and $b_{j_s}^+ \in [b_{s'}^-]_>$ for any s, s' in $\mathbb{Z}/3\mathbb{Z}$ (see (3)(a)), we know that

$$[b_{j_s}^+]_> \cap [b_{s'}^-]_< = \emptyset$$

for all s, s' in $\mathbb{Z}/3\mathbb{Z}$. So, the sets $\{A_s\}$ are disjoint from the sets $\{R_s\}$, and we just have to show that the sets $\{A_s\}$ are pairwise disjoint to finish the proof of the lemma.

Because of the symmetry of the problem it is enough to show that

$$A_0 \cap A_1 = \emptyset.$$

If this is not so, then

$$A_0 \cup A_1 = [b_{j_0}^+]_> \cup [b_{j_1}^+]_>$$

would be a connected set containing both a_{j_1} and a_{j_0}, and the cyclic order would imply that

$$([b_{j_0}^+]_> \cup [b_{j_1}^+]_>) \cap [b_{j_0}^+]_< \neq \emptyset,$$

by our hypothesis (3)(a). As we are supposing that f is not recurrent, we have

$$[b_{j_1}^+]_> \cap [b_{j_0}^+]_< \neq \emptyset.$$

But this implies that $[b_{j_1}^+]_>$ is a connected set containing both a_{j_1} and a_{j_0}. Once again our hypothesis (3)(a) and the cyclic order give us

$$[b_{j_1}^+]_> \cap [b_{j_1}^+]_< \neq \emptyset,$$

and we are done. ∎

For our next lemma, we keep the assumption on the cyclic order of the sets a_i, $i \in \mathbb{Z}/n\mathbb{Z}$:

$$a_0 \to a_1 \to \cdots \to a_i \to a_{i+1} \to \cdots \to a_{n-1} \to a_0.$$

We define I_i to be the connected component of $S^1 \setminus \bigcup_{j\in\mathbb{Z}/n\mathbb{Z}} J_j$ that follows J_{i-1} in the natural cyclic order on S^1, so that we have

$$J_{i-1} \to I_i \to J_i$$

for all $i \in \mathbb{Z}/n\mathbb{Z}$.

Lemma 3.3. *If for all $i \in \mathbb{Z}/n\mathbb{Z}$:*

(1) *there exists $b_i^+ \in B$ such that $a_i \subset [b_i^+]_>$,*
(2) *there exists $b_i^- \in B$ such that $b_i^- \subset [b_j^+]_<$, $j \in \{i-1, i\}$,*
(3) *$[b_i^-]_< \subset U$, and $\overline{[b_i^-]_<} \cap \partial U \neq \emptyset$,*
(4) *$\overline{\varphi([b_i^-]_<)} \cap S^1 \subset I_i$,*

then $\mathrm{Fix}(f) \neq \emptyset$.

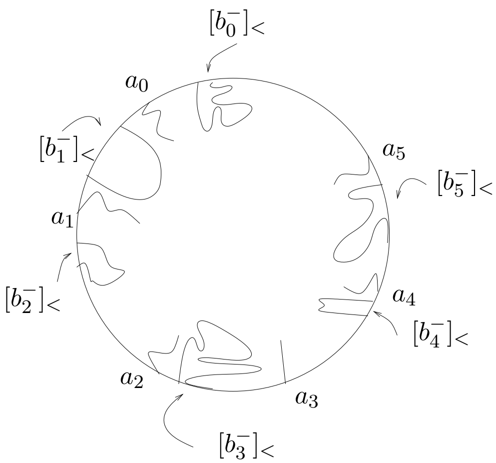


Fig. 6. Lemma 3.3 with $n = 6$

Proof. By Proposition 2.12 it is enough to show that we can construct a hyperbolic configuration.

We begin by proving that the sets $\{[b_i^-]_<\}$ are pairwise disjoint. Otherwise, there exist $i \neq j$ such that

$$[b_i^-]_< \cap [b_j^-]_< \neq \emptyset.$$

Then $[b_i^-]_< \cup [b_j^-]_<$ is a connected set and $\overline{\varphi([b_i^-]_< \cup [b_j^-]_<)}$ intersects both I_i and I_j. The cyclic order implies that any arc joining J_{i-1} and J_i separates I_i from I_j, $i \neq j$.

Our hypothesis (2) allows us to take a crosscut γ from a_{i-1} to a_i such that

$$\gamma \cap U \subset [b_i^-]_>.$$

So, $\overline{\varphi(\gamma \cap U)}$ is an arc joining J_{i-1} and J_i, and

$$\overline{\varphi(\gamma \cap U)} \cap \varphi([b_i^-]_< \cup [b_j^-]_<) \neq \emptyset.$$

This gives us

$$([b_i^-]_< \cup [b_j^-]_<) \cap [b_i^-]_> \neq \emptyset,$$

and as we are supposing that f is not recurrent,

$$[b_j^-]_< \cap [b_i^-]_> \neq \emptyset.$$

So, $[b_i^-]_< \subset [b_j^-]_<$, which implies

$$\overline{\varphi([b_i^-]_<)} \cap S^1 \subset I_i \cap I_j,$$

a contradiction.

So, we can cyclically order the $2n$ sets a_i, $[b_i^-]_<$, $i \in \mathbb{Z}/n\mathbb{Z}$ (see Remark 3.1). Moreover, for all $i \in \mathbb{Z}/n\mathbb{Z}$,

$$a_{i-1} \to [b_i^-]_< \to a_i.$$

Define $A_i = [b_i^+]_>$ and $R_i = [b_i^-]_<$ for $i \in \mathbb{Z}/n\mathbb{Z}$. To finish the proof of the lemma, it is enough to show that the sets R_i, A_i, $i \in \mathbb{Z}/n\mathbb{Z}$, are pairwise disjoint. Indeed, if this is true, our previous remark on the cyclic order, and our hypothesis (2) imply that $((R_i)_{i\in\mathbb{Z}/n\mathbb{Z}}, (A_i)_{i\in\mathbb{Z}/n\mathbb{Z}})$ is a hyperbolic configuration.

We have already proved that the sets R_i, $i \in \mathbb{Z}/n\mathbb{Z}$, are pairwise disjoint. We will also show that $[b_i^-]_< \cap [b_j^+]_> = \emptyset$ for any $j \in \mathbb{Z}/n\mathbb{Z}$. By hypothesis (2), $[b_i^-]_< \cap [b_i^+]_> = \emptyset$, as we are supposing that f is not recurrent. If $[b_i^-]_< \cap [b_j^+]_> \neq \emptyset$ for some $j \neq i$, then $[b_j^+]_< \subset [b_i^-]_<$, $j \neq i$. Therefore, $\overline{\varphi([b_j^+]_<)} \cap S^1 \subset I_i$, $j \neq i$, which contradicts our hypothesis (4)

We have proved that the sets R_i are disjoint from the sets A_i, $i \in \mathbb{Z}/n\mathbb{Z}$. So, in order to finish, we only have to prove that the sets A_i, $i \in \mathbb{Z}/n\mathbb{Z}$, are pairwise disjoint.

If this is not the case, there would exist $i \neq j$ such that $[b_i^+]_> \cap [b_j^+]_> \neq \emptyset$. So, $[b_i^+]_> \cup [b_j^+]_>$ is a connected set containing $a_i \cup a_j$, and must therefore intersect $[b_i^+]_<$, because of the cyclic order and hypothesis (2). We may of course assume that $[b_j^+]_> \cap [b_i^+]_< \neq \emptyset$. Now, we see that $[b_j^+]_>$ is a connected set containing $a_j \cup a_i$ and must therefore intersect $[b_j^+]_<$. This contradiction proves our claim. ∎

4. Proof of the main result. This section is devoted to the proof of Theorem 1.2.

We fix an orientation preserving homeomorphism $f : \mathbb{D} \to \mathbb{D}$ which realizes a cycle of links $\mathcal{L} = ((\alpha_i, \omega_i))_{i \in \mathbb{Z}/n\mathbb{Z}}$. We recall that this means that there exists a family $(z_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ of points in $\mathbb{D}$ such that for all $i \in \mathbb{Z}/n\mathbb{Z}$,

$$\lim_{k \to -\infty} f^k(z_i) = \alpha_i, \qquad \lim_{k \to \infty} f^k(z_i) = \omega_i.$$

We also recall that

$$\ell = \{\alpha_i, \omega_i : i \in \mathbb{Z}/n\mathbb{Z}\} \subset S^1,$$

and that we supppose that f can be extended to a homeomorphism of $\mathbb{D} \cup \ell$.

4.1. The elliptic case. Let us state our first proposition:

PROPOSITION 4.1. *If $\mathcal{L}$ is elliptic, then* $\mathrm{Fix}(f) \neq \emptyset$. *Moreover, one of the following holds:*

(1) *f is recurrent,*
(2) *$\mathcal{L}$ is a degenerate cycle.*

As the proof is long, we will first describe our strategy. The first part of the work consists in constructing a brick decomposition which is suitable for our purposes. Once this is done, we show that if f is not recurrent, then the elliptic order property gives rise to constraints on the order of the cycle of links $\mathcal{L}$. We will show (as a consequence of Lemma 3.2) that the only possibility for the order of $\mathcal{L}$ is $n = 4$. The case $n = 4$ is special, as degeneracies may occur (see Figure 2, and the introduction, where we explain that non-degeneracy is necessary to obtain the index result). For $n = 4$ we prove that $\mathrm{Fix}(f) \neq \emptyset$, and that if f is not recurrent, then $\mathcal{L}$ is degenerate.

I. *Construction of the brick decomposition.* We first note that we may assume that $n > 3$: if $n = 3$, the definition of cycle of links implies automatically that the points $\{\alpha_i\}, \{\omega_i\}$ are all different, and the proof follows from Le Calvez's improvement to Handel's theorem. As we are dealing with the elliptic case, the only possible coincidences among the points $\{\alpha_i\}$, $\{\omega_i\}$ are of the form $\omega_{i-2} = \alpha_i$. In particular, the points $\{\omega_i\}$ are all different and for all $i \in \mathbb{Z}/n\mathbb{Z}$ we can take a neighbourhood U_i^+ of ω_i in $\overline{\mathbb{D}}$ in such a way that $U_i^+ \cap U_j^+ = \emptyset$ if $i \neq j$. We define $U_i^- = U_{i-2}^+$ if $\alpha_i = \omega_{i-2}$, and for all $i \in \mathbb{Z}/n\mathbb{Z}$ such that $\alpha_i \neq \omega_{i-2}$ we take a neighbourhood U_i^- of α_i in $\overline{\mathbb{D}}$ in such a way that $U_i^- \cap U_j^+ = \emptyset$ for all $j \in \mathbb{Z}/n\mathbb{Z}$, and $U_i^- \cap U_j^- = \emptyset$ for all $i \neq j$.

We suppose from now on that f is not recurrent.

We apply Lemma 2.9 and obtain families $(b_i'^l)_{l\in\mathbb{Z}\setminus\{0\},\, i\in\mathbb{Z}/n\mathbb{Z}}$ of closed disks. So, the disks in $(b_i'^l)_{l\geq 1,\, i\in\mathbb{Z}/n\mathbb{Z}}$ have pairwise disjoint interiors.

Let I_{reg} be the set of $i \in \mathbb{Z}/n\mathbb{Z}$ such that $\alpha_i \neq \omega_{i-2}$, or such that $\alpha_i = \omega_{i-2}$ but there exists $K > 0$ such that

$$\bigcup_{k>K} \mathrm{Int}(b_{i-2}'^k) \cap \bigcup_{k>K} \mathrm{Int}(b_i'^{-k}) = \emptyset.$$

Let I_{sing} be the complement of I_{reg} in $\mathbb{Z}/n\mathbb{Z}$.

After discarding a finite number of disks, we can suppose that the disks $b_i'^l$ with $l \geq 1$, $i \in \mathbb{Z}/n\mathbb{Z}$, and $b_i'^{-l}$ with $l \geq 1$, $i \in I_{\mathrm{reg}}$, have pairwise disjoint interiors.

If $i \in I_{\mathrm{sing}}$, then $\alpha_i = \omega_{i-2}$ and for all $k > 0$ there exist $k' > k$, $j' > k$ such that $\mathrm{Int}(b_{i-2}'^{k'}) \cap \mathrm{Int}(b_i'^{-j'}) \neq \emptyset$.

In the following lemma we refer to the family of integers $(l_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ constructed in Lemma 2.9.

LEMMA 4.2. *For $i \in I_{\mathrm{sing}}$, we can find sequences $(c_i^m)_{m\geq 0}$ of free closed disks such that:*

1. $c_i^m \subset U_{i-2}^+ = U_i^-$,
2. *there exists an increasing sequence $(k_i^m)_{m\geq 0}$ such that $b_{i-2}'^{k_i^m} \cap c_i^m \neq \emptyset$ for all $m \geq 0$,*
3. $(b_{i-2}'^{k_i^p} \cup c_i^p) \cap (b_{i-2}'^{k_i^m} \cup c_i^m) = \emptyset$ *for all $p \neq m$,*
4. *there exists an increasing sequence $(j_i^m)_{m\geq 0}$ such that $f^{-l_i - j_i^m + 1}(z_i) \in c_i^m$ for all $m \geq 0$,*
5. *the sequence $(c_i^m)_{m\geq 0}$ converges to $\omega_{i-2} = \alpha_i$ in the Hausdorff topology,*
6. $b_{i-2}'^{k_i^m} \cap c_i^m$ *is an arc for all $m \geq 0$ (so, $c_i^m \cup b_{i-2}'^{k_i^m}$ is a topological closed disk),*
7. $\partial(\bigcup_{k\geq 1} b_{i-2}'^k \cup \bigcup_{m\geq 0} c_i^m)$ *is a one-dimensional submanifold,*
8. *if $x \in \mathbb{D}$, then x belongs to at most two different disks in the family $\{b_{i-2}'^k, c_i^m : k \geq 1,\, m \geq 0\}$.*

Proof. Take $i \in I_{\mathrm{sing}}$ and consider the family $(b_{i-2}'^k)_{k\geq 1} \subset U_{i-2}^+$ of closed disks. As $i \in I_{\mathrm{sing}}$, there exists $j_i^0 > 1$ such that

$$\mathrm{Int}\Big(\bigcup_{k\geq 1} b_{i-2}'^k\Big) \cap \mathrm{Int}(b_i'^{-j_i^0}) \neq \emptyset.$$

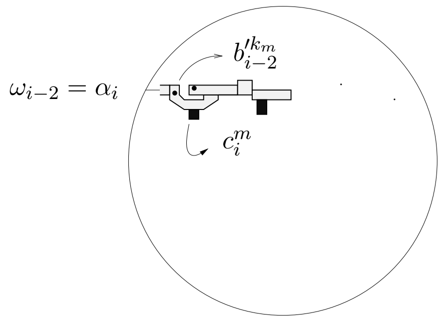


Fig. 7. The disks $b'^{k_m}_{i-2}$ and c_i^m

By Lemma 2.9(7), $f^{(-l_i-j_i^0+1)}(z_i) \in \mathrm{Int}(b'^{-j_i^0}_i) \setminus \bigcup_{l\geq 1} b'^{l}_{i-2}$. We take an arc

$$\gamma_i^0 \subset \mathrm{Int}(b'^{-j_i^0}_i) \setminus \mathrm{Int}\Big(\bigcup_{l\geq 1} b'^{l}_{i-2}\Big)$$

joining $f^{(-l_i-j_i^0+1)}(z_i)$ and a point $x_i^0 \in \partial \bigcup_{l\geq 1} b'^{l}_{i-2}$. We define $k_i^0 \geq 1$ by

$$x_i^0 \in b'^{k_i^0}_{i-2}.$$

We define inductively for $m \geq 0$:

- $U_m \subset U^+_{i-2} = U^-_i$, a neighbourhood of $\omega_{i-2} = \alpha_i$ in $\overline{\mathbb{D}}$ such that
$$\overline{U_m} \cap (\mathrm{Int}(b'^{k_i^m}_{i-2}) \cup \mathrm{Int}(b'^{-j_i^m}_i)) = \emptyset,$$
- $K_m > 0$ such that for all $k \geq K_m$, $b'^{k}_{i-2} \cup b'^{-k}_i \subset U_m$,
- $j_i^{m+1} > K_m$ such that $\mathrm{Int}(\bigcup_{k\geq K_m} b'^{k}_{i-2}) \cap \mathrm{Int}(b'^{-j_i^{m+1}}_i) \neq \emptyset$,
- $\gamma_i^{m+1} \subset \mathrm{Int}(b'^{-j_i^{m+1}}_i) \setminus \bigcup_{l\geq K_m} b'^{l}_{i-2}$, an arc joining $f^{(-l_i-j_i^{m+1}+1)}(z_i)$ and a point $x_i^{m+1} \in \partial \bigcup_{k\geq K_m} b'^{k}_{i-2}$,
- $k_i^{m+1} > K_m$ by
$$x_i^{m+1} \in b'^{k_i^{m+1}}_{i-2}.$$

The existence of K_m comes from the fact that both sequences $(b'^{-l}_i)_{l\geq 1}$ and $(b'^{l}_{i-2})_{l\geq 1}$ converge to $\alpha_i = \omega_{i-2}$ in the Hausdorff topology; that of j_i^{m+1} from the fact that $i \in I_{\mathrm{sing}}$; that of γ_i^{m+1} from the choice of j_i^{m+1} and the fact that $f^{(-l_i-j_i^{m+1}+1)}(z_i) \in \mathrm{Int}(b'^{-j_i^{m+1}}_i) \setminus \bigcup_{l\geq K_m} b'^{l}_{i-2}$, and that of x_i^{m+1} and k_i^{m+1} follows from the choice of j_i^{m+1}.

By thickening these arcs $\{\gamma_i^m\}$, we can construct disks $\{c_i^m\}$ satisfying all the conditions of the lemma. ∎

The proposition above allows us to construct a maximal free brick decomposition (V, E, B) such that:

- for all $i \in \mathbb{Z}/n\mathbb{Z}$ and $l \geq 1$, there exists $b_i^l \in B$ such that $b_i'^l \subset b_i^l$,
- for all $i \in I_{\mathrm{reg}}$ and $l \geq 1$, there exists $b_i^{-l} \in B$ such that $b_i'^{-l} \subset b_i^{-l}$,
- for all $m \geq 0$ and $i \in I_{\mathrm{sing}}$, there exists $b_i^{-j_i^m} \in B$ such that $c_i^m \subset b_i^{-j_i^m}$.

II. *The "domino effect" of the elliptic order property*

LEMMA 4.3. *Take two indices i, j in $\mathbb{Z}/n\mathbb{Z}$, and two integers k and N. If b_j^k and b_{j+2}^k are contained in $[b_i^N]_>$, then there exists $k' \in \mathbb{Z}$ such that $b_l^{k'}$ is contained in $[b_i^N]_>$ for all $l \in \mathbb{Z}/n\mathbb{Z}$.*

Proof. We will show that if b_j^k and b_{j+2}^k are contained in $[b_i^N]_>$, then there exists k'' such that both $b_{j+1}^{k''}$ and $b_{j+3}^{k''}$ are contained in $[b_i^N]_>$. If b_j^k and b_{j+2}^k are contained in $[b_i^N]_>$, then b_j^l and b_{j+2}^l are contained in $[b_i^N]_>$ for all $l \geq k$. By Remark 2.10, we can find an arc

$$\gamma : [0,1] \to [b_i^N]_> \cup \{\omega_j, \omega_{j+2}\}$$

joining ω_j and ω_{j+2}. As $n > 3$, and the coincidences are of the form $\alpha_i = \omega_{i-2}$, we know that the points $\alpha_{j+1}, \omega_j, \alpha_{j+3}, \omega_{j+2}$ are all different. So, γ separates both α_{j+1} from ω_{j+1} and α_{j+3} from ω_{j+3}. Hence, there exists $k'' > 0$ such that $[b_{j+1}^{k''}]_\leq \cap [b_i^N]_> \neq \emptyset$ and $[b_{j+3}^{k''}]_\leq \cap [b_i^N]_> \neq \emptyset$. We are done by induction, and by taking k' large enough. ■

In the following lemma we make reference to the sequences $(k_i^m)_{m\geq 0}$ and $(j_i^m)_{m\geq 0}$ defined in Lemma 4.2.

LEMMA 4.4. *For every $i \in I_{\mathrm{sing}}$, there exists $N > 0$ such that $[b_i^{-j_i^N}]_\geq$ contains $b_{i-2}^{k_i^N}$.*

Proof. We will prove the following stronger statement which implies immediately that $[b_i^{-j_i^N}]_\geq$ contains $b_{i-2}^{k_i^N}$: there exists $N > 0$ such that $f(c_i^N) \cap b_{i-2}'^{k_i^N} \neq \emptyset$.

I. Let us begin by studying the local dynamics of the brick decomposition at $\alpha_i = \omega_{i-2}$, $i \in I_{\mathrm{sing}}$. We define, for all $m \geq 0$,

$$X_m = b_{i-2}'^{k_i^m} \cup c_i^m,$$

and we recall that every X_m is a closed disk (see Lemma 4.2). Then, for all $m \geq 0$,

$$f^{l_{i-2}+k_i^m-1}(z_{i-2}) \cup f^{-l_i-j_i^m-j_i^m}(z_i) \in X_m.$$

So, given any two positive integers $m > p$, one has

$$\bigcup_{k\geq 1} f^k(X_p) \cap X_m \neq \emptyset \quad \text{and} \quad \bigcup_{k\geq 1} f^k(X_m) \cap X_p \neq \emptyset.$$

Moreover, $X_m \cap X_p = \emptyset$ and X_m and X_p are topological closed disks. Therefore, if we can find $m > p \geq 0$ such that both X_p and X_m are free sets, then f would be recurrent by Proposition 2.5. Hence, we can suppose that for all $m \geq 0$ the set X_m is not free. So, as for all $m \geq 0$ both $b_i'^{k_m}$ and c_i^m are free sets, we see that either $f(b'^{k_i^m}_{i-2}) \cap c_i^m \neq \emptyset$, or $f(c_i^m) \cap b'^{k_i^m}_{i-2} \neq \emptyset$. If there exists $m > 0$ such that $f(c_i^m) \cap b'^{k_i^m}_{i-2} \neq \emptyset$, we are done. So, we may assume that for all $m \geq 0$, $f(b'^{k_i^m}_{i-2}) \cap c_i^m \neq \emptyset$. Then $f(b^{k_i^m}_{i-2}) \cap b_i^{-j_i^m} \neq \emptyset$ for all $m \geq 0$. In particular, $[b^{k_i^m}_{i-2}]_>$ contains b_i^l for all $l > 0$ and for all $m \geq 0$.

II. We will show that this implies that f is recurrent. As $[b^{k_i^m}_{i-2}]_>$ contains b_i^k and b_{i-2}^k for $k > k_i^m$, Lemma 4.3 implies that for all $m \geq 0$ there exists $l_m > 0$ such that $[b^{k_i^m}_{i-2}]_>$ contains b_j^l for all $j \in \mathbb{Z}/n\mathbb{Z}$ and for all $l \geq l_m$.

In particular, Remark 2.10 tells us that for all $m \geq 0$ there exists an arc

$$\Gamma_m : [0,1] \to [b^{k_i^m}_{i-2}]_> \cup \{\omega_{i-2}, \omega_{i-4}\}$$

joining ω_{i-2} and ω_{i-4}, which implies that Γ_m separates α_{i-1} from α_{i-3} in $\overline{\mathbb{D}}$ (see Figure 8(a) and observe that as $n > 3$ the points $\alpha_{i-3}, \omega_{i-4}, \alpha_{i-1}, \omega_{i-2}$ are all different). Since we are assuming that f is not recurrent, we deduce that the closure of $[b^{k_i^m}_{i-2}]_\leq$ cannot contain both points α_{i-1} and α_{i-3}.

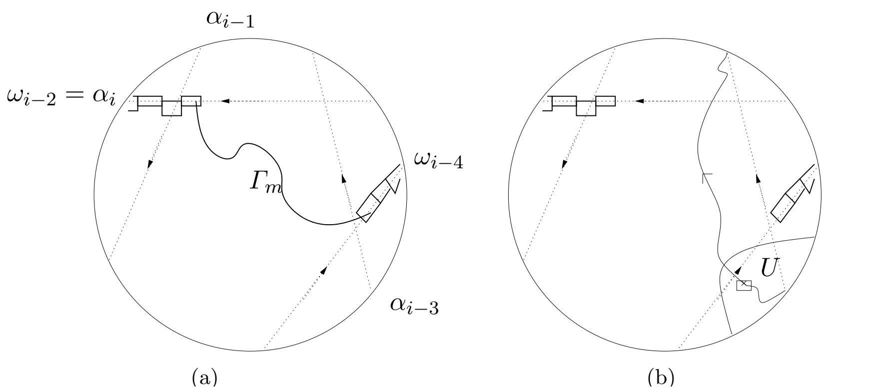


Fig. 8. The proof of Lemma 4.4

We will suppose that for all $m \geq 0$, the closure of $[b^{k_i^m}_{i-2}]_\leq$ does not contain one of the points α_{i-1} and α_{i-3}, and obtain a contradiction. As $m > p$ implies

$$[b^{k_i^p}_{i-2}]_\leq \subset [b^{k_i^m}_{i-2}]_\leq,$$

one of the points α_{i-1}, α_{i-3} is not contained in the closure of any of the sets $[b^{k_i^m}_{i-2}]_\leq$, $m \geq 0$. Suppose that α_{i-3} is not contained in $\overline{[b^{k_i^m}_{i-2}]_\leq}$ for any $m \geq 0$ (the other case is analogous). In particular, for all $m \geq 0$, $[b^{k_i^m}_{i-2}]_\leq$ does not contain any of the bricks containing the orbit of z_{i-3}. We take a neighbourhood U of α_{i-3} in $\overline{\mathbb{D}}$ such that $U \cap [b_{i-2}^{k_i^0}]_{\leq} = \emptyset$ and such that $U \cap \bigcup_{l>k_i^0} b_{i-2}^l = \emptyset$. We also take $j > 0$ such that $f^{-j}(z_{i-3}) \in U$, and an arc $\beta : [0,1] \to U$ joining α_{i-3} and $f^{-j}(z_{i-3})$. Finally, we take a brick $b \in B$ such that $f^{-j}(z_{i-3}) \in b$. As $\bigcup_{l\geq 1} b_{i-3}^{\prime l} \subset [b]_{\geq}$, Remark 2.10 allows us to take an arc $\gamma : [0,1] \to [b]_{\geq} \cup \omega_{i-3}$ joining $f^{-j}(z_{i-3})$ and ω_{i-3}.

So, $\beta.\gamma$ separates α_{i-2} from ω_{i-2} in $\overline{\mathbb{D}}$ and

$$\beta.\gamma \cap \Big(\bigcup_{l>k_0} b_{i-2}^l \cup [b_{i-2}^{k_i^0}]_{\leq}\Big) \neq \emptyset,$$

which implies

$$\gamma \cap \Big(\bigcup_{l>k_0} b_{i-2}^l \cup [b_{i-2}^{k_i^0}]_{\leq}\Big) \neq \emptyset,$$

because of our choice of U (see Figure 8(b)). Hence,

$$b_{\geq} \cap \bigcup_{l>0} [b_{i-2}^l]_{<} \neq \emptyset,$$

which implies that for some $m \geq 0$,

$$[b]_{\geq} \cap [b_{i-2}^m]_{<} \neq \emptyset.$$

Therefore, $b \in [b_{i-2}^{k_i^m}]_{\leq}$, and $[b_{i-2}^{k_i^m}]_{\leq}$ contains a brick containing one point of the orbit of z_{i-3}. This contradiction finishes the proof of the lemma. ∎

Lemma 4.5. *There exists $k > 0$ such that for any pair of indices i, j in $\mathbb{Z}/n\mathbb{Z}$, the attractor $[b_i^{-k}]_{>}$ contains b_j^k.*

Proof. For all $i \in I_{\mathrm{reg}}$, we know that $\bigcup_{l\geq 1} b_i^{\prime -l} \subset \bigcup_{l>0} [b_i^{-l}]_{>}$ (note that this is not necessarily the case if $i \in I_{\mathrm{sing}}$). So, by Remark 2.10, there exists an arc

$$\Gamma_i : [0,1] \to \bigcup_{l>0} [b_i^{-l}]_{>} \cup \{\alpha_i, \omega_i\}$$

joining α_i and ω_i. Hence, Γ_i separates both α_{i-1} from ω_{i-1} and α_{i+1} from ω_{i+1} in $\overline{\mathbb{D}}$. Therefore, there exists $m > 0$ such that $[b_i^{-m}]_{>}$ contains both b_{i+1}^m and b_{i-1}^m. By Lemma 4.3, $[b_i^{-m}]_{>}$ contains b_j^l for all $j \in \mathbb{Z}/n\mathbb{Z}$, and l large enough.

For all $i \in I_{\mathrm{sing}}$, the previous lemma tells us that there exists $N > 0$ such that $[b_i^{-j_i^N}]_{\geq}$ contains $b_{i-2}^{k_i^N}$. Clearly, $[b_i^{-j_i^N}]_{\geq}$ also contains $b_i^{k_i^N}$ and so once again, Lemma 4.3 implies that $[b_i^{-j_i^N}]_{\geq}$ contains b_j^l for all $j \in \mathbb{Z}/n\mathbb{Z}$ and l large enough. We finish by taking k sufficiently large. ∎

III. *Constraints on the order of the cycle of links $\mathcal{L}$.* We fix $k > 0$ such that for any i, j in $\mathbb{Z}/n\mathbb{Z}$, $[b_i^{-k}]_>$ contains b_j^k. We define

$$a_i = \Big(\bigcup_{m\geq k} b_i^m\Big) \cap \Gamma_i^+, \qquad i \in \mathbb{Z}/n\mathbb{Z}$$

(see Remark 2.10 for the definition of Γ_i^+). We may suppose that

$$U = \mathbb{D} \setminus \bigcup_{i\in\mathbb{Z}/n\mathbb{Z}} a_i$$

is simply connected. As $a_i \subset \bigcup_{m\geq k} b_i^m$, and we are supposing that f is not recurrent, we know that $[b_i^{-k}]_< \subset U$ for all $i \in \mathbb{Z}/n\mathbb{Z}$.

Let $\varphi : U \to \mathbb{D}$ be the Riemann map and consider the intervals J_i, $i \in \mathbb{Z}/n\mathbb{Z}$, defined in 3.1. We define I_i as to be the connected component of $S^1 \setminus \bigcup_{l\in\mathbb{Z}/n\mathbb{Z}} J_l$ following J_{i-2} in the natural (positive) cyclic order on S^1. So, each I_i is a closed interval, and we have

$$J_{i-2} \to I_i \to J_{i-1}$$

for all $i \in \mathbb{Z}/n\mathbb{Z}$.

Lemma 4.6. *For all $i \in \mathbb{Z}/n\mathbb{Z}$:*

1. *there exists $j_i \in \mathbb{Z}/n\mathbb{Z}$ such that $\overline{\varphi([b_i^{-k}]_<)} \cap S^1 \subset I_{j_i}$,*
2. *$j_i \in \{i-1, i\}$,*
3. *if $\alpha_i \neq \omega_{i-2}$, then $j_i = i$.*

Proof. (1) If there exists $x \in \overline{\varphi([b_i^{-k}]_<)} \cap J_j$ for some $j \in \mathbb{Z}/n\mathbb{Z}$, then $\overline{[b_i^{-k}]_<} \cap a_j \neq \emptyset$. As $[b_i^{-k}]_<$ is closed in $\mathbb{D}$, and as $a_j \subset \mathbb{D}$, we obtain $[b_i^{-k}]_< \cap a_j \neq \emptyset$, a contradiction. So, $\overline{\varphi([b_i^{-k}]_<)} \subset \bigcup_{j\in\mathbb{Z}/n\mathbb{Z}} I_j$. If $\overline{\varphi([b_i^{-k}]_<)}$ intersects I_j and I_k, $k \neq j$, then there exist two different indices i_0 and i_1 in $\mathbb{Z}/n\mathbb{Z}$ such that any arc joining J_{i_0} and J_{i_1} separates I_j from I_k. We take a crosscut γ from a_{i_0} to a_{i_1} such that $\gamma \subset [b_i^{-k}]_>$. So,

$$\varphi(\gamma \cap U) \cap \varphi([b_i^{-k}]_<) \neq \emptyset,$$

and consequently

$$[b_i^{-k}]_> \cap [b_i^{-k}]_< \neq \emptyset,$$

which contradicts our assumption that f is not recurrent.

(2) Take a crosscut $\gamma \subset [b_i^{-k}]_>$ from a_{i-3} to a_{i-1}. Then the elliptic order property implies that α_i belongs to the closure of only one of the two connected components of $U \setminus \gamma$: the one to the right of γ. We use here the fact that $\alpha_i \notin \{\omega_{i-3}, \omega_{i-1}\}$. So, $[b_i^{-k}]_<$ also belongs to the connected component of $U \setminus \gamma$ which is to the right of γ. Consequently, $\varphi([b_i^{-k}]_<)$ belongs to the connected component of $\mathbb{D} \setminus \varphi(\gamma \cap U)$ which is to the right

of $\varphi(\gamma \cap U)$. As $\overline{\varphi(\gamma \cap U)}$ is an arc from J_{i-3} to J_{i-1}, the closure of this connected component only contains I_i and I_{i-1}. So, we obtain $j_i \in \{i-1, i\}$.

(3) If $\alpha_i \neq \omega_{i-2}$, we can apply exactly the same argument as in the preceding item, but using a crosscut γ from a_{i-2} to a_{i-1}, obtaining $j_i = i$. ∎

REMARK 4.7. If we set $b_i^- = b_i^{-k}$ and $b_i^+ = b_i^k$, then the bricks b_i^-, $i \in \{i_0, i_1, i_2\}$, satisfy all the hypotheses of Lemma 3.2, where i_0, i_1, i_2 are any three different indices in $\mathbb{Z}/n\mathbb{Z}$. Indeed, k is chosen so that (2) and (3)(a) hold, (3)(b) is granted since $\alpha_i \subset \overline{[b_i^-]_<}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$, and (3)(c) is the content of item (1) in Lemma 4.6.

The second item in Lemma 4.6 gives us:

COROLLARY 4.8. *If* $|i-l| \geq 2$, *then* $j_i \neq j_l$.

The constraint on the order $\mathcal{L}$ follows:

LEMMA 4.9. *The order of* $\mathcal{L}$ *is either* 4 *or* 5.

Proof. If $n \geq 6$, the sets $\{i, i-1\}$, $i \in \{0, 2, 4\}$, are pairwise disjoint, and so the three indices j_0, j_2, j_4 given by Lemma 4.6 are different. This contradicts Lemma 3.2. ∎

LEMMA 4.10. *We have* $n = 4$.

Proof. We show that $n = 5$ also contradicts Lemma 3.2. If j_0, j_2, j_3 are all different, we are done because of Lemma 3.2. Otherwise, the only possibility is that $j_2 = j_3 = 2$ (see Lemma 4.6). But then j_1, j_3 and j_4 are different. ∎

LEMMA 4.11. $\mathcal{L}$ *is degenerate.*

Proof. We will show that if $n = 4$ and $\mathcal{L}$ is non-degenerate, we can also find a triplet i_0, i_1, i_2 in $\mathbb{Z}/n\mathbb{Z}$ such that the corresponding j_{i_s}, $s \in \{0, 1, 2\}$, are different.

For a non-degenerate cycle of links, there can be at most two coincidences of the type $\alpha_i = \omega_{i-2}$. Furthermore, if $\alpha_i = \omega_{i-2}$ and $\alpha_j = \omega_{j-2}$ for some $i \neq j$, then $|i-j| = 1$. Indeed, the points in ℓ are ordered as follows:

$$\omega_0 \xrightarrow{=} \alpha_2 \to \omega_1 \xrightarrow{=} \alpha_3 \to \omega_2 \xrightarrow{=} \alpha_0 \to \omega_3 \xrightarrow{=} \alpha_1 \to \omega_0,$$

and non-degeneracy means that we cannot have both $\omega_i = \alpha_{i+2}$ and $\omega_{i+2} = \alpha_i$, for some $i \in \mathbb{Z}/4\mathbb{Z}$. So, there exists $l \in \mathbb{Z}/4\mathbb{Z}$ such that $\alpha_l \neq \omega_{l-2}$ and $\alpha_{l+1} \neq \omega_{l-1}$. We can suppose without loss of generality that $\alpha_0 \neq \omega_2$, and $\alpha_1 \neq \omega_3$ (see Figure 9). Items (2) and (3) in Lemma 4.6 imply that j_0, j_1, and j_3 are different, and we are done. ∎

The following lemma finishes the proof of Proposition 4.1.

LEMMA 4.12. *If* $n = 4$, *then* $\mathrm{Fix}(f) \neq \emptyset$.

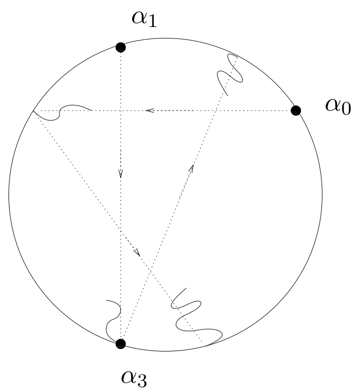


Fig. 9. The case $n = 4$

Proof. We will be done by constructing a hyperbolic repeller/attractor configuration of order 2. We define

$$R_0 = [b_0^{-k}]_<, \quad R_1 = [b_2^{-k}]_<, \quad A_0 = [b_3^k]_>, \quad A_1 = [b_1^k]_>.$$

By the choice of k, there exist two bricks c_i^i, c_i^{i-1}, contained in R_i, $i \in \mathbb{Z}/2\mathbb{Z}$, such that $[c_i^j]_> \cap A_j \neq \emptyset$ if $j \in \{i, i-1\}$.

Moreover, the cyclic order of these sets is the following:

$$R_0 \to A_0 \to R_1 \to A_1 \to R_0.$$

Indeed, we know that $j_0 \in \{0,3\}$, $j_2 \in \{2,1\}$, and the cyclic order of the intervals J_i, I_i, $i \in \mathbb{Z}/4\mathbb{Z}$, is

$$I_0 \to J_3 \to I_1 \to J_0 \to I_2 \to J_1 \to I_3 \to J_2 \to I_0.$$

So, we just have to show that the sets R_i, A_i, $i \in \mathbb{Z}/2\mathbb{Z}$, are pairwise disjoint. The choice of k implies that $[b_i^{-k}]_< \cap [b_j^k]_> = \emptyset$ for all i, j in $\mathbb{Z}/4\mathbb{Z}$. As a consequence, we just have to check $R_0 \cap R_1 = \emptyset$, and $A_0 \cap A_1 = \emptyset$.

If this is not the case, $[b_0^{-k}]_< \cup [b_2^{-k}]_<$ is a connected set separating $[b_1^k]_>$ and $[b_3^k]_>$. Again by the choice of k we have

$$([b_0^{-k}]_< \cup [b_2^{-k}]_<) \cap [b_0^{-k}]_> \neq \emptyset,$$

and as we are supposing that f is not recurrent,

$$[b_2^{-k}]_< \cap [b_0^{-k}]_> \neq \emptyset.$$

But then

$$[b_2^{-k}]_< \cap [b_2^{-k}]_> \neq \emptyset,$$

because $[b_2^{-k}]_<$ contains $[b_0^{-k}]_<$ and therefore separates $[b_1^k]_>$ and $[b_3^k]_>$, both of which are contained in $[b_2^{-k}]_>$.

Analogously, if $A_0 \cap A_1 \neq \emptyset$, then $[b_3^k]_> \cup [b_1^k]_>$ is a connected set separating $[b_2^{-k}]_<$ and $[b_0^{-k}]_<$. Again by the choice of k we have

$$([b_3^k]_> \cup [b_1^k]_>) \cap [b_3^k]_< \neq \emptyset,$$

and as we are supposing that f is not recurrent,

$$[b_1^k]_> \cap [b_3^k]_< \neq \emptyset.$$

But then

$$[b_1^k]_> \cap [b_1^k]_< \neq \emptyset,$$

because $[b_1^k]_>$ contains $[b_3^k]_>$ and therefore separates $[b_0^{-k}]_<$ and $[b_2^{-k}]_<$, both of which are contained in $[b_1^k]_<$. ∎

4.2. The hyperbolic case. Our next proposition finishes the proof of Theorem 1.2:

PROPOSITION 4.13. *If $\mathcal{L}$ is hyperbolic, then* $\mathrm{Fix}(f) \neq \emptyset$.

We recall that the order of a hyperbolic cycle of links is an even number. That is, from now on $n = 2m$, $m \geq 2$. The hyperbolic order property implies that the only possible coincidences among the points α_i, ω_i, $i \in \mathbb{Z}/n\mathbb{Z}$, are of the form $\omega_{i-2} = \alpha_i$ for even values of i, or $\omega_{i+2} = \alpha_i$ for odd values of i.

As the points $\{\omega_i\}$ are all different, we can take a neighbourhood U_i^+ of ω_i in $\overline{\mathbb{D}}$ in such a way that that $U_i^+ \cap U_j^+ = \emptyset$ if $i \neq j$. For even values of i, we define $U_i^- = U_{i-2}^+$ if $\alpha_i = \omega_{i-2}$, and if $\alpha_i \neq \omega_{i-2}$ we take a neighbourhood U_i^- of α_i in $\overline{\mathbb{D}}$ in such a way that $U_i^- \cap U_j^+ = \emptyset$ for any j, and $U_i^- \cap U_j^- = \emptyset$ if $j \neq i$. Similarly, for odd values of i, we define $U_i^- = U_{i+2}^+$ if $\alpha_i = \omega_{i+2}$, and if $\alpha_i \neq \omega_{i+2}$ we take a neighbourhood U_i^- of α_i in $\overline{\mathbb{D}}$ in such a way that $U_i^- \cap U_j^+ = \emptyset$ for any j, and $U_i^- \cap U_j^- = \emptyset$ if $j \neq i$.

We keep the assumption that f is not recurrent.

We apply Lemma 2.9 and obtain families $(b_i'^l)_{l \in \mathbb{Z}\setminus\{0\},\, i \in \mathbb{Z}/2m\mathbb{Z}}$ of closed disks. So, the disks in $(b_i'^l)_{l \geq 1,\, i \in \mathbb{Z}/2m\mathbb{Z}}$ have pairwise disjoint interiors.

Let I_{reg} be the set of even $i \in \mathbb{Z}/2m\mathbb{Z}$ such that $\alpha_i \neq \omega_{i-2}$, or such that $\alpha_i = \omega_{i-2}$ but there exists $K > 0$ such that $\bigcup_{k>K} b_{i-2}'^k \cap \bigcup_{k>K} b_i'^{-k} = \emptyset$, together with the set of odd $i \in \mathbb{Z}/2m\mathbb{Z}$ such that $\alpha_i \neq \omega_{i+2}$, or such that $\alpha_i = \omega_{i+2}$ but there exists $K > 0$ such that $\bigcup_{k>K} b_{i+2}'^k \cap \bigcup_{k>K} b_i'^{-k} = \emptyset$. Let I_{sing} be the complement of I_{reg} in $\mathbb{Z}/2m\mathbb{Z}$.

We can suppose that all the disks in the families $(b_i'^l)_{l \geq 1,\, i \in \mathbb{Z}/2m\mathbb{Z}}$, $(b_i'^{-l})_{l \geq 1,\, i \in I_{\mathrm{reg}}}$ have disjoint interiors.

We define $i^* = i - 2$ if i is even, and $i^* = i + 2$ if i is odd.

LEMMA 4.14. *If $i \in I_{\mathrm{sing}}$, then we can find sequences of free closed disks $(c_i^n)_{n \geq 0}$ satisfying:*

(1) $c_i^n \subset U_{i^*}^+ = U_i^-$,

(2) *there exists an increasing sequence* $(k_i^n)_{n\ge 0}$ *such that* $b_{i^*}^{\prime k_i^n} \cap c_i^n \neq \emptyset$ *for all* $n \ge 0$,
(3) $(b_{i^*}^{\prime k_i^n} \cup c_i^n) \cap (b_{i^*}^{\prime k_i^p} \cup c_i^p) = \emptyset$ *for all* $n \neq p$,
(4) *there exists an increasing sequence* $(j_i^n)_{n\ge 0}$ *such that* $f^{-j_i^n}(z_i) \in c_i^n$,
(5) *the sequence* $(c_i^n)_{n\ge 0}$ *converges to* $\omega_{i^*} = \alpha_i$ *in the Hausdorff topology*,
(6) $b_{i^*}^{\prime k_i^n} \cap c_i^n$ *is an arc for all* $n \ge 0$,
(7) $\partial(\bigcup_{k\ge 1} b_{i^*}^{\prime k} \cup \bigcup_{n\ge 0} c_i^n)$ *is a one-dimensional submanifold*,
(8) *if* $x \in \mathbb{D}$, *then* x *belongs to at most two different disks in the family* $\{b_{i^*}^{\prime k}, c_i^n : k \ge 1,\, n \ge 0\}$.

Proof. Note that the local dynamics in a neighbourhood of a point α_i, $i \in I_{\mathrm{sing}}$, is exactly the same as that in the elliptic case. So, the same proof we did for Lemma 4.2 works here as well. ∎

We construct a maximal free brick decomposition (V, E, B) such that:

- for all $i \in \mathbb{Z}/2m\mathbb{Z}$ and for all $l \ge 1$, there exists $b_i^l \in B$ such that $b_i^{\prime l} \subset b_i^l$,
- for all $i \in I_{\mathrm{reg}}$ and for all $l \ge 1$, there exists $b_i^{-l} \in B$ such that $b_i^{\prime -l} \subset b_i^{-l}$,
- for all $n \ge 0$ and for all $i \in I_{\mathrm{sing}}$, there exists $b_i^{-j_i^n} \in B$ such that $c_i^n \subset b_i^{-j_i^n}$.

Lemma 4.15. *If* $i \in I_{\mathrm{sing}}$, *then there exists* $N > 0$ *such that* $[b_i^{-j_i^N}]_{\ge}$ *contains* $b_{i^*}^{k_i^N}$.

Proof. Fix an even index $i \in I_{\mathrm{sing}}$ (the proof for odd indices is analogous). The first part of the proof is identical to part I in the proof of Lemma 4.4. Indeed, this proof is local, that is, it does not depend on how the rest of the points in ℓ are ordered. So, there are two possibilities: either $f(c_i^N) \cap b_{i-2}^{\prime k_i^N} \neq \emptyset$ or $f(b_{i-2}^{\prime k_i^N}) \cap c_i^N \neq \emptyset$. In the first case we are done, as it implies immediately the statement of the lemma. As a consequence, we may assume that for all $n \ge 0$, $[b_{i-2}^{k_i^n}]_{>}$ contains b_i^l for all $l > 0$. We will show that this contradicts the fact that f is not recurrent.

With this last assumption, for all $n \ge 0$ there exists an arc

$$\Gamma_n : [0,1] \to [b_{i-2}^{k_i^n}]_{>} \cup \{\omega_{i-2}, \omega_i\}$$

joining ω_{i-2} and ω_i (see Remark 2.10). So, the arc Γ_n separates α_{i-1} from α_{i-3} in $\overline{\mathbb{D}}$ for all $n > 0$ (see Figure 10, and note that the points α_{i-1}, α_{i-3}, ω_{i-2}, ω_i are all different).

We deduce (as we are supposing that f is not recurrent) that for any $n > 0$, $\overline{[b_{i-2}^{k_i^n}]_{\le}}$ cannot contain both α_{i-1} and α_{i-3}. So, one of the points α_{i-1}

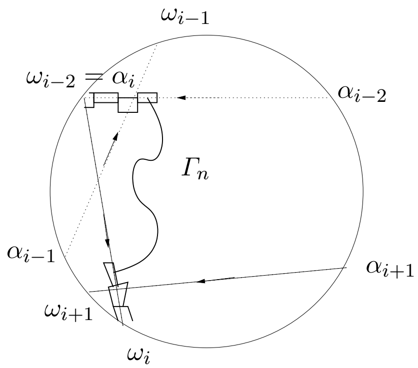


Fig. 10. The proof of Lemma 4.15

or α_{i-3} is not contained in any of the sets $\overline{[b_{i-2}^{k_i^n}]_\leq}$, $n > 0$. We will suppose that for all $n > 0$, $\alpha_{i-1} \notin \overline{[b_{i-2}^{k_i^n}]_\leq}$ (the proof is analogous in the other case). We fix $n > 0$ and consider the connected set

$$K = \bigcup_{l \geq k_i^n} b_{i-2}^l \cup [b_{i-2}^{k_i^n}]_\leq.$$

We choose a neighbourhood U of α_{i-1} in $\overline{\mathbb{D}}$ such that $U \cap K = \emptyset$. Then we take $j > 0$ such that $f^{-j}(z_{i-1}) \in U$, and $b \in B$ such that $f^{-j}(z_{i-1}) \in b$. We take an arc $\gamma \subset U$ joining α_{i-1} and $f^{-j}(z_{i-1})$, and an arc $\beta \subset [b]_\geq \cup \omega_{i-1}$ joining $f^{-j}(z_{i-1})$ and ω_{i-1}. We deduce that $\gamma.\beta \cap K \neq \emptyset$, and as $\gamma \subset U$, we have $\beta \cap K \neq \emptyset$. So, there exists $l \geq k_i^n$ such that $b \in [b_{i-2}^l]_\leq$, and consequently $\alpha_{i-1} \in \overline{[b_{i-2}^l]_\leq}$. This contradiction finishes the proof of the lemma. ∎

Lemma 4.16. *There exists $k > 0$ such that for all even values of $i \in \mathbb{Z}/2m\mathbb{Z}$, both attractors $[b_i^{-k}]_>$ and $[b_{i-1}^{-k}]_>$ contain b_l^k for all $l \in \{i-2, i-1, i, i+1\}$.*

Proof. If $i \in I_{\mathrm{sing}}$, the previous lemma tells us that there exists $N > 0$ such that $[b_i^{-j_i^N}]_\geq$ contains $b_{i-2}^{k_i^N}$. So, we can find an arc

$$\Gamma : [0,1] \to [b_i^{-j_i^N}]_> \cup \{\omega_{i-2}, \omega_i\}$$

joining ω_{i-2} and ω_i. This arc separates both α_{i-1} from ω_{i-1} and α_{i+1} from ω_{i+1} in $\overline{\mathbb{D}}$ (see Figure 10). As a consequence, both $\bigcup_{k\geq 1}[b_{i-1}^k]_\leq$ and $\bigcup_{k\geq 1}[b_{i+1}^k]_\leq$ intersect Γ, and so there exists $k > 0$ such that b_{i-1}^k and b_{i+1}^k belong to $[b_i^{-j_i^N}]_>$. If $i-1 \in I_{\mathrm{sing}}$, we can show analogously that $[b_{i-1}^{-j_{i-1}^N}]_>$ contains b_l^k for all $l \in \{i-2, i-1, i, i+1\}$ and some $k > 0$.

If $i \in I_{\mathrm{reg}}$, we can find an arc

$$\Gamma : [0,1] \to \bigcup_{l>0} [b_i^{-l}]_> \cup \{\alpha_i, \omega_i\}$$

joining α_i and ω_i. So, Γ separates (in $\overline{\mathbb{D}}$) both α_{i+1} from ω_{i+1} and α_{i-1} from ω_{i-1}. Therefore, both $\bigcup_{k\geq 1}[b_{i-1}^k]_\leq$ and $\bigcup_{k\geq 1}[b_{i+1}^k]_\leq$ intersect Γ, and there exist $k, N > 0$ such that $[b_i^{-N}]_> \cap [b_{i-1}^k]_\leq \neq \emptyset$ and $[b_i^{-N}]_> \cap [b_{i+1}^k]_\leq \neq \emptyset$. Once b_{i-1}^l and b_{i+1}^l belong to $[b_i^{-N}]_>$, we can find an arc

$$\Gamma' : [0,1] \to [b_i^{-N}]_> \cup \{\omega_{i-1}, \omega_{i+1}\}$$

joining ω_{i-1} and ω_{i+1}. So, Γ' separates α_{i-2} from ω_{i-2} in $\overline{\mathbb{D}}$, and one obtains $b_{i-2}^k \in [b_i^{-N}]_>$ for some $k > 0$. We obtain the result by sufficiently enlarging k. ∎

We fix $k > 0$ as in Lemma 4.16.

Lemma 4.17. *There exists $p > k$ such that $[b_i^{-k}]_< \cap b_j'^l = \emptyset$ for all i, j in $\mathbb{Z}/2m\mathbb{Z}$ and $l \geq p$.*

Proof. Fix $i \in \mathbb{Z}/2m\mathbb{Z}$ even. There exists an arc

$$\gamma_i : [0,1] \to [b_i^{-k}]_> \cup \{\omega_{i+1}, \omega_{i-1}\}$$

joining ω_{i+1} and ω_{i-1}. As the three points $\alpha_i, \omega_{i+1}, \omega_{i-1}$ are different, γ_i separates α_i from any ω_j for $j \notin \{i-2, i-1, i+1\}$ (in $\overline{\mathbb{D}}$).

So, there exists $l_i > k$ such that γ_i separates $[b_i^{-k}]_<$ from any $b_j'^l$ with $l > l_i$ and $j \notin \{i-2, i-1, i+1\}$. Moreover, we already know that $[b_i^{-l_i}]_< \cap [b_j^{l_i}]_> = \emptyset$ if $j \in \{i-2, i-1, i+1\}$, because $[b_i^{-l_i}]_>$ contains $b_j^{l_i}$. In particular, $[b_i^{-l_i}]_< \cap b_j'^l = \emptyset$ for $l \geq l_i$ and $j \in \{i-2, i-1, i+1\}$.

If i is odd, we can use the same argument with an arc

$$\gamma_{i-1} : [0,1] \to [b_i^{-k}]_> \cup \{\omega_i, \omega_{i-2}\}$$

joining ω_i and ω_{i-2}.

We finish by taking $p = \max\{l_i : i \in \mathbb{Z}/2m\mathbb{Z}\}$. ∎

Thanks to the two preceding lemmas we may fix $k > 0$ such that:

- both attractors $[b_i^{-k}]_>$ and $[b_{i-1}^{-k}]_>$ contain b_l^k for all even values of i and for all $l \in \{i-2, i-1, i, i+1\}$,
- $[b_i^{-k}]_< \cap b_j'^l = \emptyset$ for all i, j in $\mathbb{Z}/2m\mathbb{Z}$ and $l \geq k$.

We define

$$a_i = \Gamma_i^+ \cap \bigcup_{l\geq k} b_i'^l$$

for all $i \in \mathbb{Z}/2m\mathbb{Z}$. The cyclic order of the sets a_i satisfies

$$a_{i-2} \to a_{i+1} \to a_i$$

for all even values of i. We may suppose that each a_i is an arc, and so $U = \mathbb{D} \setminus \bigcup_{i\in\mathbb{Z}/2m\mathbb{Z}} a_i$ is simply connected. Let $\varphi : U \to \mathbb{D}$ be the Riemann map and consider the intervals J_i defined in 2.4.1.

For all even i, we define I_i to be the connected component of $S^1 \setminus \bigcup_{l\in\mathbb{Z}/2m\mathbb{Z}} J_l$ following J_{i-2} in the natural (positive) cyclic order on S^1. We define I_{i+1} to be the connected component of $S^1 \setminus \bigcup_{l\in\mathbb{Z}/2m\mathbb{Z}} J_l$ following I_i. So, for all even i we have

$$J_{i-2} \to I_i \to J_{i+1} \to I_{i+1} \to J_i.$$

LEMMA 4.18. *For all $i \in \mathbb{Z}/2m\mathbb{Z}$:*

(1) $[b_i^{-k}]_< \subset U$,

(2) *if i is even, then* $\overline{\varphi([b_i^{-k}]_<)} \cap S^1 \subset I_i \cup I_{i-1}$, *and* $\overline{\varphi(b_{i-1<}^{-k})} \cap S^1 \subset I_i \cup I_{i+1}$,

(3) *there exists j_i such that* $\overline{\varphi([b_i^{-k}]_<)} \cap S^1 \subset I_{j_i}$ *(so, for i even, $j_i \in \{i, i-1\}$, $j_{i-1} \in \{i, i+1\}$).*

Proof. (1) This is trivial because of the choice of $k > 0$.

(2) First, we show that $\overline{\varphi([b_i^{-k}]_<)} \subset \bigcup_{j\in\mathbb{Z}/2m\mathbb{Z}} I_j$. Otherwise, there exists $x \in \overline{\varphi([b_i^{-k}]_<)} \cap J_j$ for some $j \in \mathbb{Z}/2m\mathbb{Z}$. So, $\overline{[b_i^{-k}]_<}$ contains a point in a_j. As $[b_i^{-k}]_<$ is a closed subset of $\mathbb{D}$, and $a_j \subset \mathbb{D}$, we obtain $[b_i^{-k}]_< \cap a_j \neq \emptyset$, contradicting the previous item.

Fix $i \in \mathbb{Z}/2m\mathbb{Z}$ even. Take a crosscut $\gamma \subset [b_i^{-k}]_>$ from ω_{i-1} to ω_{i+1}. So, α_i belongs to the closure of only one of the connected components of $\overline{\mathbb{D}} \setminus \gamma$: the one to the right of γ. Hence, $\varphi([b_i^{-k}]_<)$ belongs to the connected component of $\mathbb{D}\setminus\varphi(\gamma\cap U)$ which is to the right of $\varphi(\gamma\cap U)$. As $\overline{\varphi(\gamma \cap U)}$ is an arc joining J_{i-1} and J_{i+1}, the cyclic order implies that $\overline{\varphi([b_i^{-k}]_<)} \cap S^1 \subset I_i \cup I_{i-1}$.

The statement for $i-1$ is proved analogously.

(3) Suppose i is even (as before, the other case is analogous). The previous item implies that if $\overline{\varphi([b_i^{-k}]_<)}$ intersects I_j and I_l, $j \neq l$, then $\{j, l\} = \{i, i-1\}$.

Take a crosscut $\gamma \subset [b_i^{-k}]_>$ from ω_{i-1} to ω_{i-2}. Then $\overline{\varphi(\gamma \cap U)}$ separates I_{i-1} from I_i in $\overline{\mathbb{D}}$. This gives us

$$[b_i^{-k}]_< \cap [b_i^{-k}]_> \neq \emptyset,$$

a contradiction. ∎

REMARK 4.19. If we set $a'_i = a_{2i}$, $b_i^- = b_{2i}^{-k}$, and $b_i^+ = b_{2i}^k$ for all $i \in \mathbb{Z}/m\mathbb{Z}$, then a'_i, b_i^-, b_i^+, $i \in \mathbb{Z}/m\mathbb{Z}$, satisfy hypotheses (1)–(3) of Lemma 3.3. So, if we prove that $j_{2i} = 2i$ for all $i \in \mathbb{Z}/m\mathbb{Z}$, then $\mathrm{Fix}(f) \neq \emptyset$. Indeed, the sets a'_i, $i \in \mathbb{Z}/m\mathbb{Z}$, are cyclically ordered as follows:

$$a'_0 \to a'_1 \to a'_2 \to \cdots \to a'_{m-2} \to a'_{m-1} \to a'_0.$$

Moreover, if we set $J'_i = J_{2i}$ for all $i \in \mathbb{Z}/m\mathbb{Z}$, we have

$$J'_{i-1} \to I_{2i} \to J'_i$$

for all $i \in \mathbb{Z}/2m\mathbb{Z}$, and so $j_{2i} = 2i$ is exactly hypothesis (4) of Lemma 3.3.

We are now ready to prove Proposition 4.13:

Proof of Proposition 4.13. Because of the previous remark, it is enough to show that $j_{2i} = 2i$ for all $i \in \mathbb{Z}/m\mathbb{Z}$. We will show that if this is not the case, we contradict Lemma 3.2.

Lemma 4.18 tells us that $j_{2i} \in \{2i, 2i-1\}$. Let us assume that $j_{2i} = 2i-1$. This implies that j_{2i-2}, j_{2i-1}, and j_{2i} are different. Indeed, by Lemma 4.18, $j_{2i-2} \in \{2i-3, 2i-2\}$, $j_{2i-1} \in \{2i, 2i+1\}$, and by assumption $j_{2i} = 2i-1$. Moreover:

- $[b^{-k}_{2i}]_>$ contains b^k_{2i}, b^k_{2i-1}, and b^k_{2i-2},
- $[b^{-k}_{2i-1}]_>$ contains b^k_{2i}, b^k_{2i-1}, and b^k_{2i-2},
- $[b^{-k}_{2i-2}]_>$ contains both b^k_{2i-2} and b^k_{2i-1}.

So, as j_{2i-2}, j_{2i-1}, and j_{2i} are different, if we show that $[b^{-k}_{2i-2}]_>$ also contains b^k_{2i}, we contradict Lemma 3.2. Take a crosscut $\gamma \subset [b^{-k}_{2i-2}]_>$ from a_{2i-2} to a_{2i-4}. Then $\overline{\varphi(\gamma \cap U)}$ separates I_{2i-1} from J_{2i}. On the other hand, $\overline{\varphi([b^k_{2i}]_<)}$ joins both these sets, as we are assuming $j_{2i} = 2i-1$, and by the definition of J_{2i}. So,

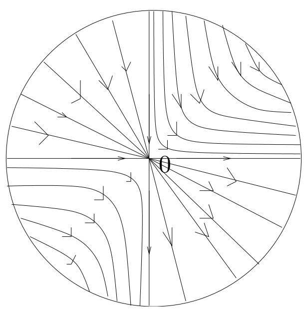
$$\varphi([b^k_{2i}]_<) \cap \varphi(\gamma \cap U) \neq \emptyset,$$

and we are done. ∎

5. Proof of Lemma 1.3. We finish by proving Lemma 1.3, which shows that our theorem is optimal.

We begin with a perturbation lemma.

Let $(\phi_t)_{t\in\mathbb{R}}$ be the flow in $\mathbb{D}$ whose orbits are drawn in the figure below:



We say that a flow $(\varphi_t)_{t\in\mathbb{R}}$ in $\mathbb{D}$ is *locally conjugate to* $(\phi_t)_{t\in\mathbb{R}}$ *at* z_0 if there exist an open neighbourhood U of z_0 and a homeomorphism $h : \mathbb{D} \to U$ such that $h(0) = z_0$ and $h^{-1}\varphi_t h = \phi_t$ for all $t \in \mathbb{R}$.

If $\varphi : \mathbb{D} \to \mathbb{D}$ is a homeomorphism, we write $\alpha(x, \varphi)$ for the set of accumulation points of the backward φ-orbit of x, and $\omega(x, \varphi)$ for the set of accumulation points of the forward φ-orbit of x.

LEMMA 5.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be the time-one map of the flow which is locally conjugate to $(\phi_t)_{t\in\mathbb{R}}$ at z_0, and U an open neighbourhood of z_0 where $h^{-1}\varphi h = \phi_1$. Then, for any $x, y \in U$ such that $\omega(x, \varphi) = z_0 = \alpha(y, \varphi)$, there exists an orientation preserving homeomorphism $g : \mathbb{D} \to \mathbb{D}$ supported in the union of two free and disjoint open disks such that*

$$\alpha(x, \varphi \circ g) = \alpha(x, \varphi), \qquad \omega(x, \varphi \circ g) = \omega(y, \varphi).$$

Proof. Let $\Delta \subset \mathbb{D}$ be the oriented straight line through 0 with tangent unit vector $e^{i\pi/4}$, and let L (resp. R) be the connected component of $U \setminus h(\Delta)$ which is to the left (resp. to the right) of $h(\Delta)$.

Note that given two points z_1, z_2 in the same connected component C of $U \setminus h(\Delta)$ that do not belong to the same orbit of $(\varphi_t)_{t\in\mathbb{R}}$, there exists an arc $\delta \subset C$ joining z_0 and z_1 such that $\varphi(\delta) \cap \delta = \emptyset$. Moreover, any $x \in U$ such that $\omega(x, \varphi) = z_0$ belongs to L, and any $y \in U$ such that $\alpha(y, \varphi) = z_0$ belongs to R. Moreover, there exist $z \in L$ and $n > 0$ such that $\varphi^n(z) \in R$.

So, we can take a free arc $\delta_1 \subset L$ joining x and z, and a free arc $\delta_2 \subset R$ joining $\varphi^n(z)$ and $\varphi^{-1}(y)$. Moreover, we may suppose that

$$\begin{aligned} \delta_1 \cap \{\varphi^{-k}(x) : k > 0\} &= \delta_2 \cap \{\varphi^k(y) : k \geq 0\} \\ &= (\delta_1 \cup \delta_2) \cap \{\varphi^k(z) : 0 < k < n\} = \emptyset. \end{aligned}$$

We thicken the δ_i's to open free and disjoint disks $D_1 \subset L$, $D_2 \subset R$ such that

$$\begin{aligned} D_1 \cap \{\varphi^{-k}(x) : k > 0\} &= D_2 \cap \{\varphi^k(y) : k \geq 0\} \\ &= (D_1 \cup D_2) \cap \{\varphi^k(z) : 0 < k < n\} = \emptyset. \end{aligned}$$

Finally, we construct an orientation preserving homeomorphism $g : \mathbb{D} \to \mathbb{D}$ supported in $D_1 \cup D_2$ such that $g(x) = z$ and $g(\varphi^n(z)) = \varphi^{-1}(y)$. Then we obtain

$$\alpha(x, \varphi \circ g) = \alpha(x, \varphi), \qquad \omega(x, \varphi \circ g) = \omega(y, \varphi),$$

as desired. ∎

REMARK 5.2. In fact, given a finite set of points $x_i, y_i \in U$, $i = 1, \ldots, n$, which belong to different orbits of $(\varphi_t)_{t\in\mathbb{R}}$ and such that $\omega(x_i) = z_0 = \alpha(y_i)$, $i = 1, \ldots, n$, there exists an orientation preserving homeomorphism $g : \mathbb{D} \to \mathbb{D}$ supported in a finite union of free and disjoint open disks such that

$$\alpha(x_i, \varphi \circ g) = \alpha(x_i, \varphi), \qquad \omega(x_i, \varphi \circ g) = \omega(y_i, \varphi),$$

$i = 1, \ldots, n$. Indeed, we choose different points $z_i \in L$ and positive integers $n_i > 0$ such that $\varphi^{n_i}(z_i) \in R$. Then we take pairwise disjoint arcs δ_i^1 joining

x_i and z_i, and δ_i^2 joining $\varphi^{n_i}(z_i)$ and $\varphi^{-1}(y_i)$ in such a way that all these arcs are disjoint from the backward φ-orbit of x_i, the forward φ-orbit of y_i and the transitional orbits $\varphi(z_i), \dots, \varphi^{n_i-1}(z_i)$. This allows us to construct the desired perturbation g.

Given a family $\mathcal{K} = ((\alpha_i, \omega_i))_{i\in\mathbb{Z}/n\mathbb{Z}}$ of pairs of points in S^1, we denote by Δ_i the oriented segment joining α_i and ω_i. We say that $z \in \mathbb{D}$ is a *multiple point* if z belongs to at least two different Δ_i's. Let z be a multiple point, and let $I = \{i \in \mathbb{Z}/n\mathbb{Z} : z \in \Delta_i\}$. We say that a multiple point $z \in \mathbb{D}$ has *zero index* if there exists an oriented straight line Δ containing z such that the algebraic intersection number $\Delta \wedge \Delta_i$ equals 1 for all $i \in I$. Note that this is the case for any multiple point such that $\#I = 2$.

We say that a pair $(\alpha_k, \omega_k) \in \mathcal{K}$ is *i-separated* if α_k and ω_k belong to different connected components of $S^1 \setminus \{\alpha_i, \omega_i\}$.

A *degeneracy* of $\mathcal{K}$ is a pair of elements of the family, (α_i, ω_i) and (α_j, ω_j), such that $\alpha_j = \omega_i$ and $\alpha_i = \omega_j$. We say that a degeneracy is *trivial* if the connected component of $S^1 \setminus \{\alpha_i, \omega_i\}$ containing α_k is independent of the i-separated pair $(\alpha_k, \omega_k) \in \mathcal{K}$.

We will deduce Lemma 1.3 from the following lemma.

LEMMA 5.3. *Let $\mathcal{K} = ((\alpha_i, \omega_i))_{i\in\mathbb{Z}/n\mathbb{Z}}$ be a family of pairs of points in S^1. Suppose that:*

(1) *every multiple point is of zero index,*
(2) *every polygon $P \subset \mathbb{D}$ whose boundary is contained in $\bigcup_{i\in\mathbb{Z}/n\mathbb{Z}} \Delta_i$ has zero index,*
(3) *every degeneracy is trivial.*

Then there exists a flow $(\varphi_t)_{t\in\mathbb{R}}$ in $\mathbb{D}$ such that:

(i) *$(\varphi_t)_{t\in\mathbb{R}}$ is locally conjugate to $(\phi_t)_{t\in\mathbb{R}}$ at every singularity z_0,*
(ii) *for all $i \in \mathbb{Z}/n\mathbb{Z}$ there exist two points $z_i^-, z_i^+ \in \mathbb{D}$ such that $\alpha(z_i^-) = \alpha_i$ and $\omega(z_i^+) = \omega_i$,*
(iii) *the $2n$ points z_i^-, z_i^+, $i \in \mathbb{Z}/n\mathbb{Z}$, are different.*

Proof. First suppose that there are no degeneracies in $\mathcal{K}$. In this case, the orientations of the Δ_i's induce a flow $(\varphi_t)_{t\in\mathbb{R}}$ on $\bigcup_{i\in\mathbb{Z}/n\mathbb{Z}} \Delta_i$ with a singularity at each multiple point. By (1), we may extend this flow to a neighbourhood of every multiple point in such a way that it is locally conjugate to $(\phi_t)_{t\in\mathbb{R}}$. Moreover, by (2) we may extend $(\varphi_t)_{t\in\mathbb{R}}$ to the rest of $\mathbb{D}$ without singularities, and we are done.

If $\mathcal{K}$ contains one degeneracy $(\alpha_i, \omega_i) = (\omega_j, \alpha_j)$, we "open it up" as follows. We consider the family of segments $\bigcup_{k\in\mathbb{Z}/n\mathbb{Z},\, k\neq j} \Delta_k$ and a simple curve γ_j joining α_j and ω_j such that:

(a) $\gamma_j \cap \Delta_i = \{\alpha_i, \omega_i\}$,
(b) $\gamma_j \cap \Delta_k \cap \mathbb{D} \neq \emptyset$ if and only if (α_k, ω_k) is j-separated, and in this case $\#\{\gamma_j \cap \Delta_k \cap \mathbb{D}\} = 1$,
(c) γ_j does not intersect any multiple point.

Now, the orientations of the Δ_i's, $i \neq j$, and the orientation of γ_j induce a flow $(\varphi_t)_{t\in\mathbb{R}}$ on $\bigcup_{i\in\mathbb{Z}/n\mathbb{Z},\, i\neq j} \Delta_i \cup \gamma_j$ with a singularity at each multiple point of $\bigcup_{i\in\mathbb{Z}/n\mathbb{Z},\, i\neq j} \Delta_i$ and also at the intersection points of γ_j with the Δ_i's, $i \neq j$.

Note that as γ_j does not intersect any multiple point, we may extend $(\varphi_t)_{t\in\mathbb{R}}$ to a neighbourhood of every multiple point of $\bigcup_{k\in\mathbb{Z}/n\mathbb{Z},\, k\neq j} \Delta_k$ in such a way that it is locally conjugate to $(\phi_t)_{t\in\mathbb{R}}$. Moreover, a point $z_0 \in \gamma_j$ belongs to at most one Δ_k, $k \neq j$, and the intersection is transversal by item (b) above. So, we may as well extend $(\varphi_t)_{t\in\mathbb{R}}$ to a neighbourhood of z_0 so as to have local conjugation with $(\phi_t)_{t\in\mathbb{R}}$ as well. As the degeneracies considered are trivial, we can extend $(\varphi_t)_{t\in\mathbb{R}}$ to the rest of $\mathbb{D}$ without singularities.

If more than one degeneracy occurs, triviality implies that they are disjoint. That is, if $(\alpha_i, \omega_i) = (\omega_j, \alpha_j)$, and $(\alpha_k, \omega_k) = (\omega_l, \alpha_l)$, then (α_i, ω_i) is not k-separated. So, we can "open up" both degeneracies in such a way that $\gamma_j \cap \gamma_l = \emptyset$, and construct our flow $(\varphi_t)_{t\in\mathbb{R}}$ analogously. ∎

We deduce:

Corollary 5.4. *With the same hypothesis of the preceding lemma, there exists a fixed-point free orientation preserving homeomorphism* $f : \mathbb{D} \to \mathbb{D}$ *that realizes* $\mathcal{K}$.

Proof. Let φ be the time-one map of the flow given by the preceding lemma. By simultaneous applications of Lemma 5.1, we can construct an orientation preserving homeomorphism $g : \mathbb{D} \to \mathbb{D}$ supported in disjoint open free disks such that

$$\lim_{k\to-\infty} (\varphi \circ g)^k (z_i^-) = \alpha_i, \qquad \lim_{k\to\infty} (\varphi \circ g)^k (z_i^-) = \omega_i$$

(see also Remark 5.2).

Then the homeomorphism $\varphi \circ g$ realizes $\mathcal{K}$. Moreover, as we have local conjugation to the flow $(\phi_t)_{t\in R}$ at every singularity of φ, and $\varphi \circ g = \varphi$ in a neighbourhood of each singularity, we can further perturb $\varphi \circ g$ into a homeomorphism $f : \mathbb{D} \to \mathbb{D}$ realizing $\mathcal{K}$ and which is fixed point free. ∎

This last lemma finishes the proof of Lemma 1.3:

Lemma 5.5. *If a multiple point has non-zero index, then there exists a subfamily of* $\mathcal{K}$ *forming an elliptic cycle of links.*

Proof. Let x be a multiple point of non-zero index, and let

$$I = \{i \in \mathbb{Z}/n\mathbb{Z} : x \in \Delta_i\}.$$

As x has non-zero index, there exists indices $i, j \in I$ such that the oriented interval in S^1 joining α_i and α_j contains ω_k, $k \in I$. Then $\mathcal{L} = (\alpha'_l, \omega'_l)_{l\in\mathbb{Z}/3\mathbb{Z}}$ is an elliptic cycle of links, where

$$(\alpha'_0, \omega'_0) = (\alpha_i, \omega_i), \quad (\alpha'_1, \omega'_1) = (\alpha_j, \omega_j), \quad (\alpha'_2, \omega'_2) = (\alpha_k, \omega_k). \ \blacksquare$$

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