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Knots of (canonical) genus two

by

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Abstract. We give a description of all knot diagrams of canonical genus 2 and 3, and give applications to positive, alternating and homogeneous knots, including a classification of achiral genus 2 alternating knots, slice or achiral 2-almost positive knots, a proof of the 3- and 4-move conjectures, and the calculation of the maximal hyperbolic volume for canonical (weak) genus 2 knots. We also study the values of the link polynomials at roots of unity, extending denseness results of Jones. Using these values, examples of knots with non-sharp Morton (canonical genus) inequality are found. Several results are generalized to arbitrary canonical genus.

Contents

1. Introduction 2
2. Knot diagrams of canonical genus 2 5
3. Alternating genus 2 knots 15
4. Homogeneous genus 2 knots 19
5. Classifying positive diagrams of some positive genus 2 knots 22
6. Classifying all 2-almost positive diagrams of a slice or achiral knot 24
7. Almost positive knots 29
8. Unique and minimal positive diagrams 31
9. Some evaluations of the Jones and HOMFLY polynomials 34
 9.1. Roots of unity 34
 9.2. The Jones polynomial on the unit circle 37
 9.3. Jones's denseness result for knots 39
10. k-moves and the Brandt–Lickorish–Millett–Ho polynomial 44
 10.1. The minimal coefficients of Q 44
 10.2. Excluding weak genus 2 with the Q polynomial 46
 10.3. 16-crossing knots 49
 10.4. Unknotting numbers and the 3-move conjecture 50
 10.5. On the 4-move conjecture 51

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11. An asymptotical estimate for the Seifert algorithm 52
12. Estimates and applications of the hyperbolic volume 54
13. Genus 3 58
13.1. The homogeneity of 10_{151}, 10_{158} and 10_{160} 58
13.2. The complete classification 60
13.3. The achiral alternating knots 61
14. Questions 62
References 64

1. Introduction. The notion of a Seifert surface of a knot is classical [Se]. Seifert proved the existence of these surfaces by an algorithm constructing such a surface out of some diagram of the knot. Briefly, the procedure is as follows (see [Ad1, §4.3] or [Ro]): smooth out all crossings of the diagram, plug in discs into the resulting set of disjoint (Seifert) circles and connect the circles along the crossings by half-twisted bands. We will call the resulting surface the *canonical Seifert surface* (of this diagram) and its genus the *genus* of the diagram. The *canonical* (or *weak*) *genus* of a knot is the minimal genus of all its diagrams.

The weak genus appears in previous work of several authors, mainly in the context of showing it being equal to the classical Seifert genus for large classes of knots ([Cr1] and *loc. cit.*) However, Morton [Mo] showed that this is not true in general. Later, further examples have been constructed [Mr, Ko].

Motivated by Morton's striking observation, in [St4] we started the study of the weak genus in its own right. We gave a description of knot diagrams of genus 1 and made some statements about the general case.

The present paper is a continuation of [St4], and relies on similar ideas. Its motivation was the quest for more interesting phenomena occurring for knot diagrams of (canonical) genus higher than 1. The genus 1 diagrams, examined in [St4], revealed to be too narrow a class for such phenomena. In this paper we will study the weak genus in greater generality. We will prove several new results about properties of knots with arbitrary weak genus. In the cases of weak genus 2 and 3 we have obtained a complete description of diagrams. Using this description, we obtain computational examples and results, some of them (partly) solving several problems in previous papers of other authors.

For most practical applications, it is useful to consider weak genus 2. We therefore study it in detail. All methods should also work for higher genera, but applying them in practice seems hardly worthwhile, as the little qualitative novelty this project promises is counterbalanced by an extremely rapid increase of quantitative effort. Diagrams of genus 2 turned out to be attractive, because their variety is on the one hand sufficient to exhibit interesting phenomena and allows one to apply different types of combinatorial arguments to prove properties of them and of the knots they represent, but on the other hand not too great to make impossible arguments by hand, or with a reasonable amount of computer calculations. As we will see, many of the theorems we will prove for weak genus 2 cannot be any longer proved reasonably (at least with the same methods) for weak genus 3, if they remain true at all.

We briefly describe the structure of the paper. In §2 we prove our main result, Theorem 2.1, the description of diagrams of genus 2. It is based on a combination of computational and mathematical arguments. The subsequent sections are mostly devoted to applications of this description.

In §3 we give asymptotical estimates for the number of alternating and positive knots of genus 2 and given crossing number and classify the achiral alternating ones.

In §4 we show non-homogeneity of two of the undecided cases in [Cr1, appendix], following from the more general fact that homogeneous genus 2 knots are positive or alternating.

In §5 and §6 we use the Gauß sum inequalities of [St2] in combination with the result of §2 to show how to classify all positive diagrams of a positive genus 2 knot, on the simplest non-trivial examples 7_3 and 7_5, and classify all 2-almost positive unknot diagrams, recovering a result announced by Przytycki and Taniyama [PT] that the only non-trivial achiral (resp. slice) 2-almost positive knot is 4_1 (resp. 6_1).

In §7 we prove that there is no almost positive knot of genus 1, and in §8 that any positive knot of genus 2 has a positive diagram of minimal crossing number. We also give an example of a knot of genus 2 which has a single positive diagram.

Besides the results mentioned so far, which are direct applications of the description in Theorem 2.1, we develop several new theoretical tools, valid for arbitrary weak genus. Most of these tools can again be used to study the genus 2 case in further detail. As such a tool, most substantially we deal with behavior of the Jones and HOMFLY polynomials in §9. We show how unity root evaluations of the polynomials give information on the weak genus, and use this tool to exhibit the first examples of knots on which the weak genus inequality of Morton [Mo] is not sharp. We also give, as an aside, using some arguments from complex analysis and Lie group theory, generalizations of some denseness theorems of Jones in [J2] about the values at roots of unity of the Jones polynomial of knots of small braid index. Unity root evaluations of the Jones polynomial have recently become of interest because of a variety of relations to quantum physics, in particular the volume conjecture. (See [DLL, GL, MM].)

Since these unity root evaluations are closely related to the Nakanishi–Przytycki k-moves, we give several applications to these moves in §10, in particular the proof of the 3- and 4-move conjecture for weak genus 2 knots in §10.4 and §10.5. We also discuss how the criteria using the Jones and HOMFLY polynomials, and the examples they give rise to, can be complemented by applying the Brandt–Lickorish–Millett–Ho polynomial Q.

A further theoretical result is an asymptotical estimate for the quality of the Seifert algorithm in giving a minimal (genus) surface in §11.

In §12, we consider the hyperbolic volume. Brittenham [Br1] used a similar approach to ours to prove that the weak genus bounds the volume of a hyperbolic knot. We will slightly improve Brittenham's estimate of the maximal hyperbolic volume for given weak genus, and (numerically) determine the exact maximum for weak genus 1 and 2.

At the end of the paper we present the description for knot diagrams of genus 3 in §13, solving completely the knots undecided for homogeneity in Cromwell's tables [Cr1, appendix].

In §14 we conclude with some questions, and a counterexample to a conjecture of Cromwell [Cr2].

Although part of the material presented here (in particular the examples illustrating our theoretical results) uses some computer calculations, we hope that it has been obtained (and hence is verifiable) with reasonable effort. To facilitate this, we include some details of the calculations.

To further motivate our approach we outline applications given in several separate papers. For example, in [St7] the description of genus 2 diagrams is used to give a short proof of a result announced in [PT], that positive knots of genus at least 2 have $\sigma \geq 4$ (which builds on the result for genus 2 stated here in Corollary 3.2), and in [St3] we give a specific inequality between the Vassiliev invariant of degree 2 and the crossing number of almost positive knots of genus 2. In [St5] we generalize the classification of k-almost positive achiral knots for the case $k = 2$ (announced also in [PT] and given here as Proposition 6.1) for alternating knots to $k \leq 4$. We will use the present framework to develop a method for finding estimates on crossing numbers of semiadequate links in a subsequent paper.

In his recent book [Cr3] (Section 5.3) Cromwell gives an introductory exposition of the concepts and work which we give a rigorous account on in [St4] and here.

Notation. For a knot K and a (knot) diagram D, $c(D)$ denotes the crossing number of D, $c(K)$ the crossing number of K (the minimal crossing number of all its diagrams), $w(D)$ the writhe of D, $w(K)$ the writhe of a reduced alternating diagram of K if K is alternating (this is an invariant of K, see [Ka1]), and $n(D)$ the number of Seifert circles of D. σ denotes the signature of a knot, u its unknotting number, $\widetilde{g}$ denotes its weak genus and g its classical (Seifert) genus. $!K$ denotes the obverse (mirror image) of a knot K. We will often assume a diagram to be reduced without pointing it out explicitly; this should always be clear from the context.

v_2 denotes the Vassiliev knot invariant of degree 2, normalized to be 0 on the unknot and 1 on the trefoil(s). v_3 denotes the primitive Vassiliev invariant of degree 3, normalized to be 4 resp. -4 on the positive (right-hand) resp. negative (left-hand) trefoil. As usual, V denotes the Jones [J1], Δ the Alexander [Al], ∇ the Conway [Co], Q the Brandt–Lickorish–Millett–Ho [BLM, Ho], and P the HOMFLY (or skein) [F&] polynomial. For the HOMFLY polynomial, we use the variable convention of [LM1].

For a polynomial Y and an integer k we denote by $[Y(x)]_{x^k}$ the coefficient of x^k in $Y(x)$. The minimal (resp. maximal) degree is defined to be the minimal (resp. maximal) k with $[Y(x)]_{x^k} \neq 0$ and is denoted by $\min\deg_x Y$ (resp. $\max\deg_x Y$). The *span* of Y is the difference between its maximal and minimal degrees. In case Y has only one variable, it will not be indicated in notation. The encoded notation for polynomials we use is the one of [St1]: if the absolute term occurs between the minimal and maximal degrees, then it is bracketed, else the minimal degree is recorded in braces before the coefficient list.

We use the notation of [Ro] for knots with up to 10 crossings, renumbering $10_{163}, \dots, 10_{166}$ by eliminating 10_{162}, the Perko duplication of 10_{161}, as has been done in the tables of [BZ]. The notation of [HT] is used for knots from 11 crossings on. (Note that for 11-crossing knots this notation *differs* from that of [Co] and [Pe].) We use the convention of the Rolfsen pictures to distinguish between a knot and its obverse whenever necessary.

For two sequences of positive integers $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ we say that a_n is $O(b_n)$, and write $a_n = O(b_n)$, iff $\limsup_{n\to\infty} a_n/b_n < \infty$. Likewise, we say a_n is $O^{\succeq}(b_n)$ iff $\liminf_{n\to\infty} a_n/b_n > 0$, and $a_n = O^{\asymp}(b_n)$ iff a_n is both $O(b_n)$ and $O^{\succeq}(b_n)$.

$\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{R}$ and $\mathbb{C}$ denote the integer, natural, positive natural, real and complex numbers respectively.

For a set S, the symbols $|S|$ and $\#S$ both denote the cardinality of S. The symbol $\subset$ denotes a not necessarily proper inclusion.

2. Knot diagrams of canonical genus 2. It is known that a Seifert surface obtained by applying Seifert's algorithm to a knot diagram D has genus

$$g(D) = \frac{c(D) - n(D) + 1}{2}.$$

This formula is shown by homotopy retracting the surface to a graph and determining its Euler characteristic by a simple vertex and edge count. The *weak* (or *canonical*) *genus* $\tilde{g}(K)$ of a knot K is defined as

$$\tilde{g}(K) := \min\{g(D) : D \text{ is a diagram of } K\}.$$

In the following we will describe all knot diagrams of genus 2 and deduce consequences for knots of weak genus 2 from this description.

As a preparation, we (re)introduce some terminology, recalling *inter alia* some of the definitions and facts from [St4]; more details may be found there.

First we need to introduce some transformations of diagrams which will be crucial later. In 1992, Menasco and Thistlethwaite [MT] proved the (long conjectured) statement that reduced alternating diagrams of the same knot (or link) must be transformable by *flypes*, where a flype is shown in Figure 1.

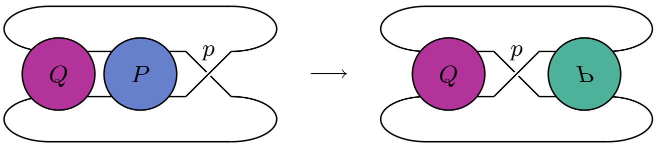


Fig. 1. A flype near the crossing p

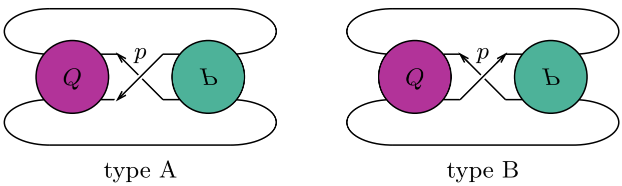
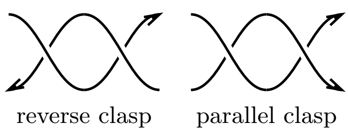


Fig. 2. A flype of type A and B

The tangle P in Figure 1 is called *flypable*, and we say that the crossing p *admits a flype* or that *the diagram admits a flype at* (*or near*) p. According to the orientation near p we distinguish two types of flypes as in Figure 2.

A *clasp* (or a *matched crossing pair*) is a tangle of the form



distinguished into *reverse* and *parallel* clasp depending on the strand orientation.

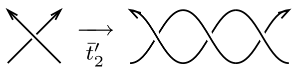
By switching one of the crossings in a clasp and applying a Reidemeister II move, one can eliminate both crossings. This procedure is called *resolving* a clasp. For the discussion below it is important to remark how resolving a clasp affects the genus of the diagram. It reduces the genus by 1 if the clasp is parallel, or if it is reverse and the Seifert circles on which the two clasped strands lie after the resolution are distinct. In this case we will call the clasp *genus reducing*. In contrast, a clasp resolution preserves the genus of the diagram if the clasp is reverse and the strands obtained after the resolution belong to the same Seifert circle (as for example in the $\bar{t}'_2$ move we will just introduce). Then we call the clasp *genus preserving.*

We will also need a class of diagram moves studied by Przytycki and Nakanishi.

DEFINITION 2.1 (see [Pr]). A t_k *move* is a local diagram move replacing a parallel pair of strands by k parallel half-twists. Similarly, a $\bar{t}_k$ *move* for k even is a replacement of a reversely oriented pair of strands by k reversely oriented half-twists. A k-*move* is the analogue of a t_k move in unoriented diagrams.

A $\bar{t}_2$ move is thus replacing a reversely oriented pair of strands by a reverse clasp. Of particular importance will be, as in [St4], a special instance of a $\bar{t}_2$ move.

DEFINITION 2.2. A $\bar{t}'_2$ *move* or *twist* is defined to be a $\bar{t}_2$ move [Pr] applied near a crossing



(together with the mirrored move), and a *reducing* $\bar{t}'_2$ *move* is the reverse operation to a $\bar{t}'_2$ move. We call a diagram $\bar{t}'_2$ *irreducible* if there is no sequence of type B flypes transforming it into a diagram on which a reducing $\bar{t}'_2$ move can be applied. Let c_g denote the maximal crossing number of an alternating $\bar{t}'_2$ irreducible genus g diagram.

A flype of type A never creates or destroys a fragment obtained from a crossing by a $\bar{t}'_2$ move and commutes with type B flypes, hence the applicability of a reducing $\bar{t}'_2$ move after type B flypes is independent of type A flypes. In terms of the associated Gauß diagram [FS, PV], a knot diagram is (modulo crossing changes) $\bar{t}'_2$ reducible after type B flypes iff it has three chords which do not mutually intersect and all intersect the same set of other chords.

In order to discard uninteresting cases, we will mainly consider only prime diagrams.

DEFINITION 2.3. A diagram D is called *composite* if there is a closed curve γ (transversely) intersecting the curve of D in two points, such that both the interior and exterior of γ contain crossings of D. Otherwise D is called *prime.*

It is a simple observation that $c_0 = 0$. Two results of [St4] were $c_1 = 4$ (independently observed by Lee Rudolph) and $c_g \leq 8c_{g-1} + 6$, so that $c_g = O(8^g)$. However, it was evident that this bound is far from sharp, and later we showed in [STV] that $c_g \leq 12g - 6$. The starting point for a significant part of the material that follows is to obtain a more precise description for $g = 2$.

Theorem 2.1. *Let K be a weak genus 2 knot. Then any prime genus 2 diagram of K is transformable by type B flypes into one which can be obtained by crossing changes and $\bar{t}'_2$ moves from an alternating diagram of one of the 24 knots in Figure* 3.

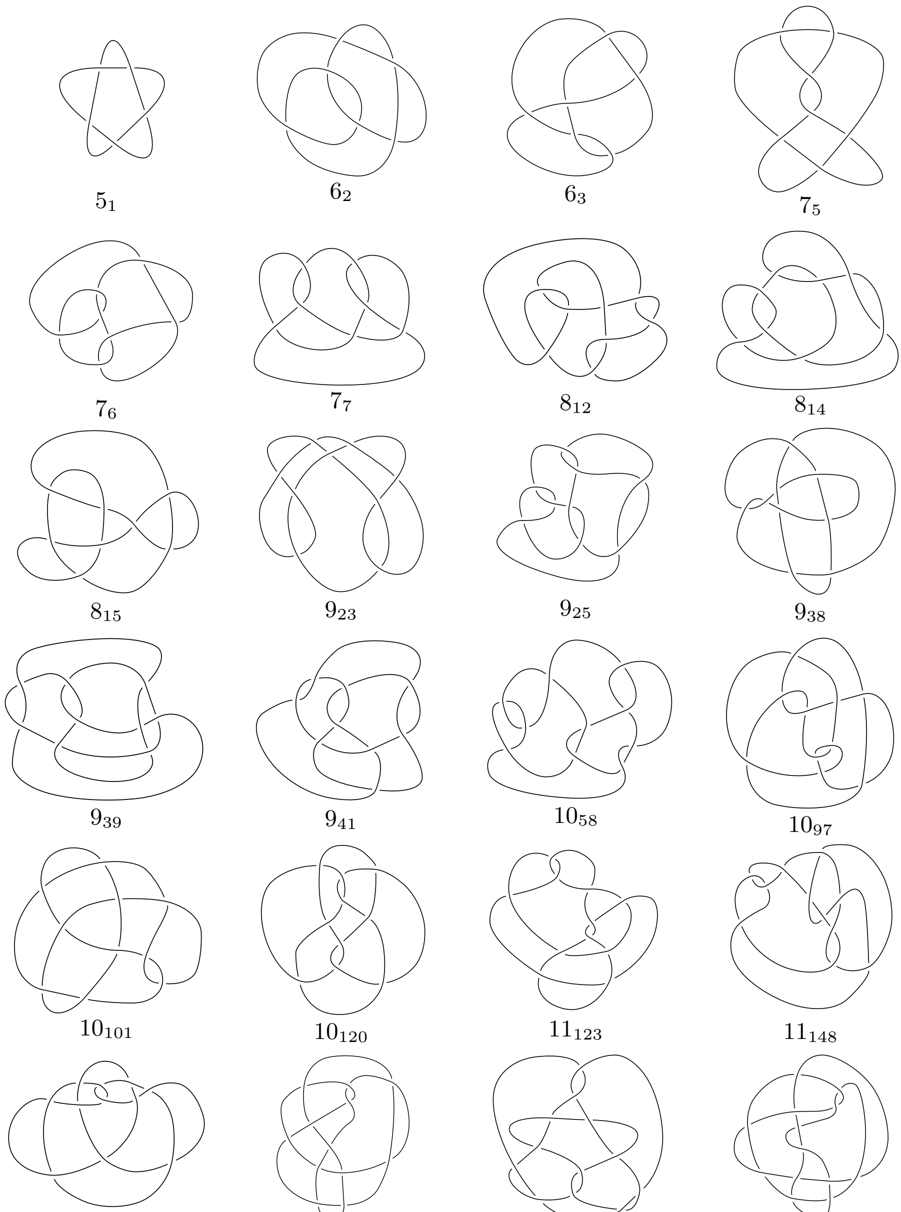


Fig. 3. The 24 alternating genus 2 knots without an alternating $\bar{t}'_2$ reducible diagram

We will say that a diagram *generates a series* or a $\bar{t}'_2$ *twist sequence* of diagrams by crossing changes and $\bar{t}'_2$ moves (so that a $\bar{t}'_2$ twist sequence is a special case of what was called in [St6] a “braiding sequence”). In this terminology the description of genus 1 diagrams in [St4] says that the only genus 1 generators are (the reduced alternating diagrams of) 3_1 and 4_1. Although we point out that some knots of Figure 1 occur in multiple diagrams, it will be sometimes possible and convenient to identify the series generated by all diagrams of a knot and call them the series *generated by the knot.*

It is convenient to use an alternating knot as a generating knot. Note that an alternating diagram which does not admit reducing $\bar{t}'_2$ moves does not admit such moves after crossing changes either. It is also important to note that for each alternating knot either all or no alternating diagrams are $\bar{t}'_2$ irreducible modulo flypes. This follows from the Menasco–Thistlethwaite flyping theorem [MT], from the fact that the applicability of a reducing $\bar{t}'_2$ move is preserved by type A flypes, and from the commuting of type A and type B flypes (i.e., if we can apply a type A flype and then a type B flype, we can do so in reverse order with the same result). Hence it suffices to check the one specific diagram included in the tables to figure out whether the knot has a $\bar{t}'_2$ irreducible diagram.

For technical reasons (to have a numbering of the crossings) it will turn out useful to record and fix a Dowker notation [DT] for each of these knots. (This is the notation in the tables of [HT].)

5_1:	6	8	10	2	4								
6_2:	4	8	10	12	2	6							
6_3:	4	8	10	2	12	6							
7_5:	4	10	12	14	2	8	6						
7_6:	4	8	12	2	14	6	10						
7_7:	4	8	10	12	2	14	6						
8_{12}:	4	8	14	10	2	16	6	12					
8_{14}:	4	8	10	14	2	16	6	12					
8_{15}:	4	8	12	2	14	6	16	10					
9_{23}:	4	10	12	16	2	8	18	6	14				
9_{25}:	4	8	12	2	16	6	18	10	14				
9_{38}:	6	10	14	18	4	16	2	8	12				
9_{39}:	6	10	14	18	16	2	8	4	12				
9_{41}:	6	10	14	12	16	2	18	4	8				
10_{58}:	4	8	14	10	2	18	6	20	12	16			
10_{97}:	4	8	12	18	2	16	20	6	10	14			
10_{101}:	4	10	14	18	2	16	6	20	8	12			
10_{120}:	6	10	18	12	4	16	20	8	2	14			
11_{123}:	4	10	14	20	2	8	18	22	6	12	16		
11_{148}:	4	10	16	20	12	2	18	6	22	8	14		
11_{329}:	6	12	18	22	14	4	20	8	2	10	16		
12_{1097}:	6	12	20	14	22	4	18	24	8	2	10	16	
12_{1202}:	6	20	10	24	14	4	18	8	22	12	2	16	
13_{4233}:	6	12	22	26	16	4	20	24	8	14	2	10	18

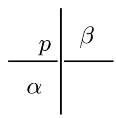
Proof of Theorem 2.1. By [STV] any genus 2 diagram of a weak genus 2 knot can be obtained modulo type B flypes by crossing changes and $\bar{t}_2'$ moves from an alternating diagram with at most 18 crossings. Now the 24 knots in Figure 3 have been obtained by checking Thistlethwaite's tables of $\leq$15-crossing knots for $\bar{t}_2'$ irreducible alternating genus 2 diagrams.

It would be in principle possible to deal with the crossing numbers 16 to 18 also by computer, but these tables are not yet available to me (those of 16 crossings at least at the time of the original writing), and to save a fair amount of electronic capacity, it is preferable to use mathematical arguments instead. Let us give the following

LEMMA 2.1. *If there is a $\bar{t}_2'$ irreducible alternating genus 2 diagram D of c crossings with a matched crossing pair (clasp), then there is a $\bar{t}_2'$ irreducible genus 2 diagram of $c-2$ crossings, or $c \leq 12$.*

For the proof we need to make some definitions.

DEFINITION 2.4. A *region* of a knot diagram is a connected component of the complement of its underlying curve in the plane. Every crossing p is bordered by four (not necessarily distinct) regions. We call two of them, α and β, *opposite* at p, notationally $\alpha \underset{p}{\rightleftharpoons} \beta$, if they do not bound a common line segment (edge) in a neighborhood of p.



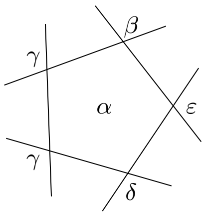
One can see that if two of the four regions bordering a crossing are equal, then they are opposite. In this case we call the crossing *reducible* or *nugatory*, or an *isthmus*.

DEFINITION 2.5. We call two crossings p and q of a knot diagram *linked*, notationally $p \cap q$, if the crossing strands are passed in cyclic order $pqpq$ along the solid line, and *unlinked* if the cyclic order is $ppqq$. Call two crossings p and q *equivalent* if they are linked with the same set of other crossings, that is, $\forall c \neq p, q : c \cap p \Leftrightarrow c \cap q$. Call p and q $\sim$*-equivalent* ($p \sim q$) if they are equivalent and unlinked, and $\underset{*}{\sim}$*-equivalent* ($p \underset{*}{\sim} q$) if they are equivalent and linked.

It is an exercise to check that $\sim$-equivalence and $\underset{*}{\sim}$-equivalence are indeed equivalence relations and that two crossings are $\sim$- (resp. $\underset{*}{\sim}$-) equivalent if and only if after a sequence of flypes they can be made to form a reverse (resp. parallel) clasp.

DEFINITION 2.6. If $(a_1, \dots, a_n)$ is a finite sequence of objects, then $(a_{k_1}, \dots, a_{k_l})$ is a *subsequence* if $k_i \geq k_{i-1}+1$, $k_1 \geq 1$ and $k_l \leq n$, that is, the a_{k_l}'s do not need to appear immediately one after the other in $(a_1, \dots, a_n)$.

DEFINITION 2.7. Let α be a region of D, i.e. a connected component of the complement of the plane curve of D in the plane. The sequence of regions opposite to α at the crossings that α borders, taken in counterclockwise order and modulo cyclic permutation, is called the *bordering sequence* for α in D.



$\longrightarrow \quad (\beta\gamma\gamma\delta\varepsilon)$

Note that by connecting crossings with the same region γ opposite to α by arcs in γ we see that the bordering sequence for α has no subsequence of the kind $\beta\gamma\beta\gamma$.

DEFINITION 2.8. Call a set of crossings $\alpha_1, \dots, \alpha_n$ *mutually enclosed* with respect to α if $\alpha_1, \dots, \alpha_n$ belong to the bordering sequence for α and this bordering sequence can be cyclically permuted so as to have the subsequence $\alpha_1\alpha_2 \dots \alpha_n\alpha_n \dots \alpha_2\alpha_1$.

The *enclosing index* $\varepsilon_{\alpha,D}$ of α in D is the maximum size of a mutually enclosed set of non-nugatory crossings with respect to α. The enclosing index ε_D of D is the maximum of the enclosing indices of all its regions.

To explain our argument for Lemma 2.1 in more detail, we first need a further lemma.

LEMMA 2.2. *If we have a genus reducing clasp resolution $D \to D'$, joining regions β_1 and β_2 of D to β of D', and reduce D' to D'' by Reidemeister I moves, flypes and reverse $\bar{t}_2'$ moves, then*

$$c(D) - c(D'') \le 4 + 4\varepsilon_{D'}.$$

Proof. In the absolute term '4', two of the crossings come from the clasp, and two from the (Reidemeister I) reducible crossings in D'.

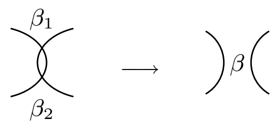
If there were three reducible crossings a, b, c in D' not reducible in D, then $\beta_1 \underset{p}{\rightleftharpoons} \beta_2$ in D for any $p \in \{a, b, c\}$, and $a \sim b \sim c$ in D (and not $a \underset{*}{\sim} b \underset{*}{\sim} c$, as we can see from (1)), a contradiction to its $\bar{t}_2'$ irreducibility (see the remark after Definition 2.5).

Separating β in D' into β_1 and β_2 in D by reversing the clasp resolution enables us to add one $\bar{t}_2'$ twist to crossings participating in two mutually enclosed sets with respect to β in D', leading to the term involving $\varepsilon_{D'}$. ∎

Proof of Lemma 2.1. We distinguish two cases for the matched crossing pair in D.

(i) Strands are reverse and belong to distinct Seifert circles. Then annihilating the matched crossing pair gives a $c-2$-crossing alternating diagram D' of genus 2.

We claim that D' has no $\bar{t}'_2$ reducible crossings. The reason is that creating a situation of being able to perform a $\bar{t}'_2$ move after elimination of the matched pair always forces the strands in the matched pair to belong to the same Seifert circle (see Figure 4). Namely, if after resolving the clasp, three crossings a, b and c become $\sim$-equivalent, then there are two regions α and β of D such that $\alpha \underset{p}{\leftharpoondown\!\rightharpoonup} \beta$ for some $p \in \{a, b, c\}$. Resolving the clasp joins two regions β_1 and β_2 of D to one region β of D':



Therefore, as a, b, and c are not all $\sim$-equivalent in D, we can assume that $\alpha \underset{a}{\leftharpoondown\!\rightharpoonup} \beta_1$ and $\alpha \underset{b}{\leftharpoondown\!\rightharpoonup} \beta_2$ in D. But then there exists in D a dashed arc γ as in Figure 4. Then all Seifert circles on D different from k, the Seifert circle in the clasp, intersect the dashed curve γ in two points in total. Thus both these crossings must belong to the same Seifert circle, and hence resolving the clasp would be genus reducing.

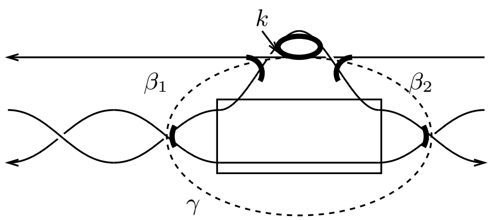
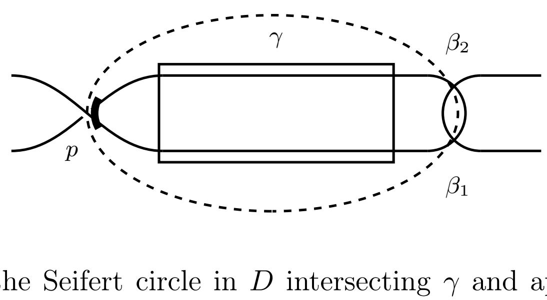


Fig. 4. When resolving a clasp makes a reducing $\bar{t}'_2$ move applicable, the segments of the resolved clasp always belong to the same Seifert circle.

Moreover, D' has no reducible crossings. Assume that p were such. Then for some region α of D' we have $\alpha \underset{p}{\leftharpoondown\!\rightharpoonup} \alpha$. But then either p is reducible in D, or $\alpha = \beta$ and $\beta_1 \underset{p}{\leftharpoondown\!\rightharpoonup} \beta_2$. Then we have a dashed curve γ like

(1) 

Then consider the Seifert circle in D intersecting γ and apply exactly the same argument as before to see that the clasp resolution must be genus reducing.

(ii) Strands are parallel or belong to the same Seifert circle and are reverse. Then annihilating the matched crossing pair reduces the canonical genus of the diagram and we obtain a genus 1 diagram D'.

We will now apply Lemma 2.2. For any genus 1 diagram D' we have $\varepsilon_{D'} = 1$, and using $c(D'') \leq 4$ we obtain from the lemma $c(D) \leq 12$, concluding the second case of the proof of Lemma 2.1. ∎

Proof of Theorem 2.1 (continued). To show that there are no $\bar{t}'_2$ irreducible genus 2 diagrams with > 13 crossings we proceed by induction on the crossing number.

The cases of 14 and 15 crossings were excluded using Thistlethwaite's tables (as I mentioned above). Then the cases of 16 and 17 crossings can be (significantly) reduced, with the use of Lemma 2.1, to the cases with no matched pair.

The latter cases are excluded as follows. Let D be such a diagram (that is, a genus 2 diagram with no matched pair). Smoothing out a crossing augments the number of 2-gon components of the diagram complement in the plane (or equivalently, the number of matched crossing pairs) by at most 2. Thus after smoothing out a linked pair of crossings in D we obtain a diagram D' of genus 1 with at most four matched pairs. Then D' is modulo its reducible crossings either a diagram obtained from 3_1 by at most two $\bar{t}'_2$ moves or a diagram obtained from 4_1 by at most one $\bar{t}'_2$ move.

Thus D' has at most seven non-reducible crossings. Now we count the reducible crossings of D' (cf. the proof of [St4, Theorem 3.1] or of Lemma 2.2 above). Smoothing out two crossings in D identifies either two pairs or one triple of regions. If p is reducible in D', then $\beta_1 \underset{p}{\rightleftharpoons} \beta_2$ in D, where $\beta_{1,2}$ are among the identified regions. There are two or three possible (unordered) pairs (β_1, β_2) of identified regions in D, and so there are at most four or six crossings p as above. Since two of these crossings must be those smoothed out, D' cannot have more than four reducible crossings.

We conclude that D' must have at most 11 crossings, so D has at most 13 crossings.

The same argument inductively excludes all higher crossing numbers, and Theorem 2.1 is now proved. ∎

Corollary 2.1. *With c_g as in Definition* 2.2, *we have* $c_2 = 13$. ∎

Remark 2.1. Note that some of the 24 knots may have alternating diagrams differing by a type A flype and twisting at them gives mutated diagrams, the mutations being type A "flypes" at a $\bar{t}'_2$ twisted crossing as shown in Figure 5. However, we can often ignore these mutations, since they will be mostly irrelevant for our work.

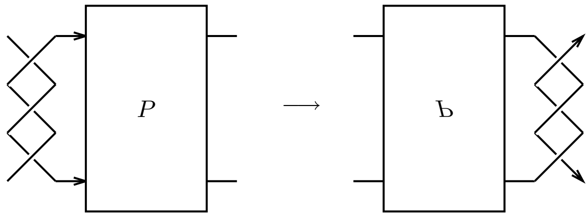


Fig. 5. A "flype" near a $\bar{t}_2'$ twisted crossing is an iterated mutation.

For example, whenever we make use of the Vassiliev invariants, signature and knot polynomials in our proofs, the arguments apply for all mutated diagrams as well, as these invariants are preserved under mutation. (For Vassiliev invariants mutation invariance holds at least up to degree 10, and all the invariants we will use are of such degree.) This is relevant for Sections 4 to 8. Also mutations do not occur in ≤ 10 crossings (relevant for §5 and §7; the few cases remaining can be checked directly), and rational knots and the unknot have no mutants [HR] (relevant for §6).

Thus we consider only one diagram for each of the 24 knots given in Figure 3.

Remark 2.2. The present description will be used later to prove non-existence of minimal canonical Seifert surfaces for some knots of genus 2 when both the obstruction of Morton [Mo] and of the Seifert genus fail. (See Remark 9.1.) An explicit computer check gave minimal canonical Seifert surfaces for all knots up to 12 crossings (not only those of genus 2), although minimal crossing number diagrams do not always suffice to give such a surface. (Among the Rolfsen knots examples are the genus 3 knots 10_{155}, 10_{157} and 10_{159} and the genus 2 knots 10_{162} and 10_{164}, where I found only 11 crossing diagrams doing the job; many more such examples exist.)

In [St4] I showed that the number of knot diagrams of given genus g is polynomially bounded in the crossing number. One sees that the maximal exponent in this polynomial is $d_g - 1$, where d_g is the maximal number of $\sim$-equivalence classes in all diagrams of genus g. For genus 1 we had $d_1 - 1 = 2$ and for genus 2 we obtain $d_2 - 1 = 8$ for this maximal exponent. The numbers d_n seem not less important than c_n and will occur several times later.

Corollary 2.2. *The number of diagrams of genus* 2 *and crossing number n is $O^{\asymp}(n^8)$. Hence there are $O^{\asymp}(n^8)$ alternating genus* 2 *knots of crossing number n and $O(n^9)$ positive knots of genus* 2 *or unknotting number* 2 *and crossing number at most n.*

Proof (to be continued). For the alternating case the only non-obvious point is to show that there are $O^{\asymp}(n^8)$ alternating knots and not only $O(n^8)$. I will give an argument for this at the end of §3.

The positive case is somewhat more involved as we do not have the result of [Ka2, Mu, Th1] on minimality (in crossing number) of alternating diagrams. Therefore we have a result only for bounded but not fixed crossing number. We also need to use the fact that a positive genus 2 knot has a positive diagram of minimal crossing number. This is again not straightforward and will be proved in Theorem 8.1. The result for the unknotting number and positive knots follows from the inequality $u \geq g$ (see [St2, Corollary 4.3]). ■

3. Alternating genus 2 knots. The $\bar{t}'_2$ twist sequences of some of the 24 knots contain those of some others as a subfamily. This happens when resolving a clasp. The relations are given in Figure 6. Therein the knots encircled are those whose twist sequences are not contained in any other (we will call them *main*), and for the others not all of the sequences containing them are indicated (but at least one is).

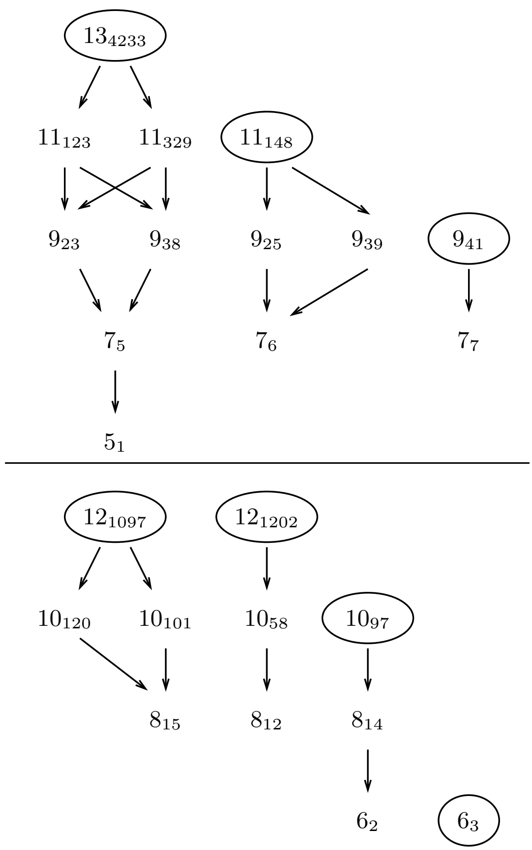


Fig. 6. Some of the inclusion relations, under resolving clasps, between the twist sequences of the 24 generating knots, and the indication (by encircling) of the main twist sequences

Remark 3.1. It is striking and suggested by the figure that inclusions of series occur only between generators of the same parity of the crossing number. This will be so for higher genera diagrams, too. As already remarked, whenever resolving a clasp simplifies the diagram by more than two crossings (by removing nugatory crossings), the resulting diagram must already have smaller genus.

We record two minor consequences. First note that 6_3 is simple but main. Some reason for this is that it is the only knot among the 24 where the numbers of positive and negative crossings in the alternating diagram are both odd. Therefore, we have

Proposition 3.1. *Let K be an alternating genus 2 knot such that $\{c(K), w(K)\} \bmod 4 = \{0, 2\}$. Then K is an arborescent knot with Conway notation $(p, q)rs(t, u)$ with $p, q, r, s, t, u > 0$ all odd.* ■

Another interesting aspect is to consider the achiral knots among the alternating genus 2 knots. First we obtain

Proposition 3.2. *A prime alternating genus 2 knot K has zero signature if and only if a diagram of K can be obtained from a diagram of 6_3, 7_7, 8_{12}, 9_{41}, 10_{58} or 12_{1202} by (repeated) $\bar{t}'_2$ moves.*

Proof. One direction follows from computing the signatures of the 24 knots and the fact that a $\bar{t}'_2$ move in an alternating diagram does not change the signature (which follows from the Traczyk–Murasugi formula, see e.g. [Tr] or [Ka1, p. 437]). For the converse, note that by a result of Menasco [Me] the primeness of an alternating knot is equivalent to the primeness of (any)one of its (reduced) alternating diagrams. ■

Corollary 3.1. *Let K be a prime achiral alternating genus 2 knot. Then a diagram of K can be obtained from a diagram of 6_3, 8_{12}, 10_{58} or 12_{1202} by (repeated) $\bar{t}'_2$ moves.*

Proof. This follows from the preceding proposition by excluding the odd crossing number knots. ■

It is, however, much more interesting to have an exact classification of all such knots. This is obtained by applying the flyping theorem of Menasco and Thistlethwaite. (Here for completeness I include the composite case.)

Theorem 3.1. *Let K be an achiral alternating genus 2 knot. Then a diagram of K is either*

(i) *a composite diagram*
   (a) *$C(q, q) \# C(p, p)$ with $p, q > 0$ even or*
   (b) *$K \# !K$ with $K \in \{C(p, q) : p, q > 0$ even$\} \cup \{P(p, q, r) : p, q, r > 0$ odd$\}$;*

(ii) *an arborescent diagram with Conway notation* $(a,b)cc(a',b')$ *with* $a,b,c,a',b' > 0$ *odd and* $\{a,b\} = \{a',b'\}$ *(in which case the knot is* $+$*achiral if* $a = a'$ *and* $-$*achiral if* $a = b'$*),*

(iii) *a rational diagram* $C(a,b,b,a)$ *with* $a,b > 0$ *even (which is invertible so the knot is* $+-$*achiral), or*

(iv) *a diagram in the* $\bar{t}'_2$ *twist sequence of* 12_{1202} *with* a,b,c $\bar{t}'_2$ *twists at the three positive clasps and* a',b',c' *twists at the three negative clasps, such that* $a,b,c \geq 0$ *and* $\{a,b,c\} = \{a',b',c'\}$ *(in which case the knot is* $+$*achiral or* $-$*achiral depending on whether the cyclic orderings of* (a,b,c) *and* (a',b',c') *along the knot are the same or reverse).*

Proof. If the knot K is composite it must have two prime factors of genus 1 and by a result of Menasco [Me] both are alternating. By the uniqueness of the decomposition into prime factors, if K is achiral both factors must be so, or mutually obverse. Now use the classification of alternating genus 1 knots in [St4]. It is an easy consequence of that classification that the only achiral knots among them are the rational knots $C(q,q)$ with $q > 0$ even. Then one obtains the above characterization.

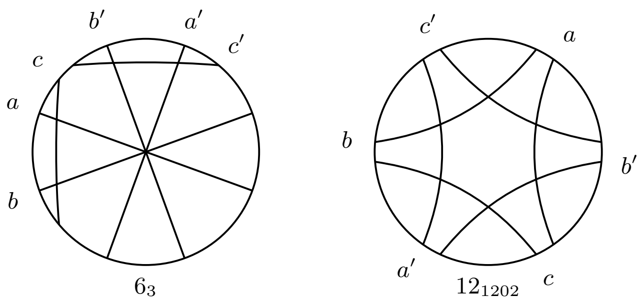


Fig. 7. Schematic drawing of the Gauß diagrams in the $\bar{t}'_2$ twist sequence of 12_{1202} and 6_3. Orientation of the arrows is ignored. A number like a at each chord denotes that it stands for a family of a neighboring non-intersecting chords. The crossings are negative for the groups labeled by a',b' and c', and positive for the groups labeled by a,b and c. For 12_{1202} all six numbers are even, and for 6_3 odd.

If the knot K is prime, using Corollary 3.1, we need to discuss four cases.

12_{1202}: It is easy to see (e.g., by looking at the Gauß diagram [FS, PV] shown in Figure 7) that neither the diagram of 12_{1202} nor any other diagram in its $\bar{t}'_2$ twist sequence admits a flype. Hence the knot is achiral if and only if the Gauß diagram is isomorphic to itself (or its mirror image) with the signs of the crossings switched, which happens exactly in the cases recorded above.

10_{58}: To show that we have no achiral knot here we use the intersection graph of the Gauß diagram. Its vertices correspond to the arrows in the Gauß diagram and are equipped with the sign of the crossing in the knot diagram. Two vertices a and b are connected by an edge if and only if the arrows in the Gauß diagram intersect (or the crossings are linked in the sense of Definition 2.5). A flype preserves the intersection graph and hence the intersection graph of an achiral alternating knot diagram must have an automorphism reversing the signs of all vertices. To see that no diagram in the $\bar{t}_2'$ twist sequence of 10_{58} has such an automorphism, consider the equivalence relation between vertices from Definition 2.5. Then the number of $\sim$-equivalence classes of positive resp. negative crossings in each such diagram is 2 resp. 3, and hence there can be no automorphism of the desired kind.

8_{12}: Use again the intersection graph. Looking at the number of positive and negative arrows intersecting only one $\sim$- or $\underset{*}{\sim}$-equivalence class of arrows, we find that in the form $C(a,b,c,d)$ we must have $a=d$. Then $b=c$ follows by looking at the total number of positive and negative arrows (or the writhe). One can also use known general arguments about rational knots.

6_3: The Gauß diagram is shown schematically in Figure 7. Looking at the number of positive and negative arrows intersecting only ones of the same sign we find $c=c'$, and hence by the writhe argument $a+b=a'+b'$. Then counting the number of intersections between arrows of the same sign we find $ab=a'b'$, whence $\{a,b\}=\{a',b'\}$. ∎

Remark 3.2. As far as orientation goes for the composite case, the non-invertible genus 1 alternating knots are $P(p,q,r)$ with $3\le p<q<r$ (see [T]). So, if K is one of these knots, the knot $K\#!K$ is $+$achiral and $K\#-!K$ is $-$achiral. The other knots are invertible and so $+-$achiral.

Using the intersection graph arguments we can now easily complete the proof of Corollary 2.2 in the alternating case.

Proof of Corollary 2.2 (continued). The only point is to convince oneself that the $O^{\asymp}(n^8)$ alternating diagrams remain in that quantity after modding out by flypes. For this consider just the diagrams where the number of $\bar{t}_2'$ moves applied to any $\sim$-equivalence class of crossings in the diagram generating the series is different. Then there can be no isomorphism of any two of the intersection graphs (just because the sets of cardinalities of the $\sim$-equivalence classes are never the same). But the number of compositions of length k of some number n into strictly ascending parts is the same as the number of compositions of $n-\binom{k}{2}$ into k non-strictly ascending parts (or the number of partitions of $n-\binom{k}{2}$ of length k), which is $O^{\asymp}(n^{k-1})$.

The proof of Corollary 2.2 is now complete modulo Theorem 8.1. ∎

Considering the signature σ, we mention a final consequence of Theorem 2.1 for positive knots, which also follows from [PT].

COROLLARY 3.2. *A positive genus* 2 *knot has* $\sigma = 4$.

Proof. It is clear that $\sigma \leq 4$. To show $\sigma = 4$ it suffices to check it on the (positively crossing switched) generating diagrams in Figure 3, as a $\bar{t}'_2$ move never reduces σ. ∎

4. Homogeneous genus 2 knots. Let us stipulate, throughout this section, that diagrams and knots are taken *up to mirror image*. In [Cr1], Cromwell introduced a certain class of link diagrams he called homogeneous, which possess minimal (genus) canonical Seifert surfaces. Roughly, a diagram is homogeneous if the connected components, called *blocks*, of the complement of its Seifert picture (set of all Seifert circles lying in the projection plane) contain only crossings of the same sign. Letting this sign always remain the same or always change when passing through a Seifert circle, we obtain the positive (or negative) and alternating diagrams as special cases. For five 10-crossing knots Cromwell could not decide about the existence of a homogeneous diagram: 10_{144}, 10_{151}, 10_{158}, 10_{160} and 10_{165}. Two of them have genus 2: 10_{144} and 10_{165}. The present discussion enables us to handle these cases.

THEOREM 4.1. *Any homogeneous genus* 2 *knot* K *is alternating or positive.*

Note that this is no longer true for genus 3, as shown by Cromwell's example 9_{43}.

COROLLARY 4.1. *The knots* 10_{144} *and* 10_{165} *are non-homogeneous.*

Proof. The knots 10_{144} and 10_{165} violate obstructions to being positive (e.g. [Cr1, Theorem 4(b)] or [St2]) or alternating (one edge coefficient of the Jones polynomial is not ± 1, see [Ka2, Mu, Th1]), hence cannot be homogeneous. ∎

Before we start with the proof of Theorem 4.1, we need one more definition.

DEFINITION 4.1. The *interior* of a Seifert circle is the bounded component of its complement in the plane, and its *exterior* is the unbounded one. The Seifert circle is called *separating* if both its interior and exterior contain at least one other Seifert circle (or equivalently, at least one crossing), and *non-separating* otherwise.

First we record a statement we will use later to reduce the number of cases to discuss.

Lemma 4.1. *Let D be an alternating diagram with* (i) *exactly three negative crossings*, *all connecting a non-separating Seifert circle*, *or* (ii) *with exactly two negative crossings. Assume furthermore that*, *whichever case* (i) *or* (ii) *we are in*, *no flype can be performed at any one of these two or three crossings. Then any homogeneous diagram in the series of all diagrams obtained from D by flypes is either positive or alternating.*

Proof. Assume a knot has in (all) its alternating reduced diagram(s) at most three negative (or positive) crossings. Then the fact that alternating diagrams are homogeneous shows that any Seifert circle must be connected from the same side by crossings of the same sign, and then by the non-existence of isthmus crossings any Seifert circle is connected by either no or at least two negative crossings. So, if they are at most three, all the negative crossings connect the same pair of Seifert circles (there cannot be three Seifert circles, each connected with the other two, for orientation reasons). Then they belong to the same block.

If the crossings are three, by assumption one of the two Seifert circles to which they connect has an empty interior (or exterior), and the diagram does not admit a flype near one of these crossings. Then (e.g. by looking at the chords of the three crossings in the Gauß diagram) one convinces himself that the triple of crossings is preserved by flypes, and so the Seifert circle stays empty after any flype. Thus any alternating diagram of the knot has at most one separating Seifert circle, and then each homogeneous diagram in the series of this diagram is either positive or alternating.

If the negative crossings are two and the diagram has two separating Seifert circles, then these are exactly the Seifert circles connected by the two negative crossings, and both inside the inner one and outside the outer one (or inside both if one does not contain the other) there are crossings. But then these negative crossings admit a flype. ∎

Proof of Theorem 4.1. With $g(K) = 2$, a homogeneous diagram of K, if any, must lie in one of the 24 series (the composite diagrams are connected sums of alternating pretzel diagrams, so the claim is trivial for such diagrams).

The series of 9_{38}, 10_{101}, 10_{120}, 11_{123}, 11_{329}, 12_{1097} and 13_{4233} are excluded by positivity (their alternating diagrams are positive, and hence so is any homogeneous diagram in their series).

Consider the series of 9_{39}, 9_{41}, 10_{97}, 11_{148} and 12_{1202}. The diagram of 12_{1202} does not admit a flype (hence it is the only alternating diagram of 12_{1202}) and it has exactly one separating Seifert circle. 9_{39}, 10_{97} and 11_{148} have two negative crossings which do not admit a flype. Finally, 9_{41} has three negative crossings, none of which admits a flype and which together bound an empty Seifert circle. Then by the lemma each homogeneous diagram in the series of all five knots is either positive or alternating.

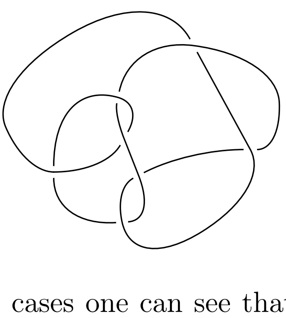
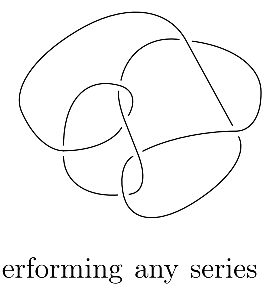
There remain the 12 arborescent generating knots $5_1, \dots, 9_{25}$ and 10_{58}. To handle these series, use the ≤ 3 negative (or positive) crossing argument of Lemma 4.1. It works except for 6_3, 7_6, 7_7, 8_{12} and 10_{58}. (Note that in most cases of two negative crossings they form a flypable clasp and hence cannot admit a flype themselves.)

6_3 is excluded because it has only three Seifert circles, hence it cannot have two separating ones.

8_{12} is excluded because it admits only type B flypes and so the series of all its diagrams are equivalent, but the one of $C(2,2,2,2)$ contains only rational knots, and such knots are alternating.

Now, 10_{58} has an alternating diagram with five clasps, two of them negative (say, modulo mirroring). We find that the only possibility to flype is to flype the tangles of these clasps, giving us (modulo symmetries) a total of four alternating diagrams of 10_{58}. The only way to make them homogeneous, but not positive and not alternating, is to switch exactly one of the clasps in three of these diagrams, and then possibly to perform $\bar{t}_2'$ moves. As 10_{58}'s alternating diagrams differ only by type B flypes, it suffices to consider one of these three diagrams. But it is easy to see that the diagram simplifies to an alternating one of one crossing less.

7_6 is excluded similarly. We have the two negative crossings admitting a flype, the flypable tangle being a positive clasp. The proof of Lemma 4.1 shows that the possibility to obtain, modulo flypes, a homogeneous diagram is to switch or not the negative and/or the flypable positive clasp. From the four cases only the two where the flypable clasp is switched are neither alternating nor positive. We end up with

 and .

But in both cases one can see that after performing any series of $\bar{t}_2'$ moves the diagram can be simplified to an alternating one.

7_7 is excluded the same way. The only way to obtain a homogeneous non-positive and non-alternating diagram is to switch exactly one of the two positive flypable clasps, but all diagrams in this series simplify to an alternating one.

In fact, one should be even a little more careful. Theorem 2.1 just said that one obtains a diagram in the series modulo type B flypes (and a type B flype may change the homogeneity of the diagram). But one can find that the only cases where the flype is necessary are to have 2 and 2 (for 7_6 and 10_{58}) and 2 and 1 (for 7_7) flype-admitting crossings on both sides of the flypable negative clasp(s), and these cases can be handled exactly as above. ■

5. Classifying positive diagrams of some positive genus 2 knots. The strict increase of v_2 and v_3 under $\bar{t}'_2$ moves at a positive diagram enables us to classify with reasonable effort all positive diagrams of positive knots of genus 2 (or higher, if an analogue of Theorem 2.1 is worked out), if they are not too complicated. We describe this procedure for the examples 7_3 and 7_5 (for which the use of v_2 suffices). The result is a special case of a more general procedure, so the discussion aims to show how in principle such a task can be solved.

For an alternating knot K, denote by $\overline{K}$ the diagram obtained from an alternating diagram of K by making it positive by crossing changes (this is defined up to flypes).

PROPOSITION 5.1. *The positive diagrams of* 7_3 *are* (*up to flypes*): $\overline{7}_3$, $\overline{8}_4$, $\overline{8}_{11}$, $\overline{8}_{13}$, $\overline{9}_{12}$, $\overline{9}_{14}$, $\overline{9}_{21}$, $\overline{9}_{37}$, *and* $\overline{10}_{13}$. *The positive diagrams of* 7_5 *are*: $\overline{7}_5$, $\overline{8}_6$, $\overline{8}_8$, $\overline{8}_{14}$, $\overline{9}_8$, $\overline{9}_{15}$, $\overline{9}_{19}$, $\overline{10}_{35}$.

Proof. We have $v_2(7_5) = 4$ and $v_2(7_3) = 5$. Let D be a positive diagram of 7_3 or 7_5. Then D belongs to the twist sequence of one of the 24 knots above. In the case of 8_{15}, 9_{23}, 9_{38}, 10_{101}, 10_{120}, 11_{123}, 11_{329}, 12_{1097} and 13_{4233} the alternating diagrams are positive, and since $\bar{t}'_2$ moves preserve alternation, all positive diagrams of their twist sequence are alternating diagrams with at least eight crossings, and hence by [Ka2, Mu, Th1] never belong to 7_3 or 7_5. The same is true for the twist sequence of 7_5, with the exception that in it exactly the diagram of 7_5 belongs to itself and no one belongs to 7_3.

By an analogous argument the only diagram in the twist sequence of 5_1 belonging to 7_3 is 7_3's usual $(1,1,1,1,3)$ pretzel diagram, and no diagram belongs to 7_5.

In the series of 9_{39}, 9_{41}, 10_{97}, 11_{148} and 12_{1202} the positive diagram obtained by crossing changes from the alternating one has $v_2 > 5$, and as v_2 is (strictly) augmented by applying $\bar{t}'_2$ moves to a positive diagram (by the Polyak–Viro formula, see [St2, Exercise 4.3]), 7_3 and 7_5 do not occur here.

We are left with 9_{25}, 10_{58}, 8_{14}, 8_{12}, 7_7, 7_6, 6_3 and 6_2. We briefly discuss these series separately.

6_2: Making 6_2's diagram positive by crossing changes, we obtain 5_1. The (positive) diagram has Dowker notation $4\ {-8}\ 10\ 12\ {-2}\ 6$, the alternating one the same notation only without minus signs. By increase of v_2 under $\bar{t}'_2$ moves on a positive diagram we need to apply twists on the positive generator diagram only as long as $v_2 \le 5$. Twisting at crossings 2 to 6, we obtain the diagrams $\overline{8}_4$ and $\overline{8}_{11}$ of 7_3 (the $P(1,-4,3)$ and $P(1,-2,5)$ pretzel diagrams), and at crossing 1 the diagram $\overline{8}_6$ of 7_5. In the case of the diagrams of 7_3, further twists can be excluded, since $v_2 = 5$, but for 7_5 (with $v_2 = 4$), we must also consider a double twist at crossing 1. This gives a diagram of 9_7 (with $v_2 = 5$), which finishes the case distinction for the series of 6_2.

6_3: $-4\ {-8}\ 10\ {-2}\ 12\ 6$.

Since v_2 attains the value 5, $\bar{t}'_2$ moves at crossings $2,\ 3,\ 4$ or 6 cannot appear with another $\bar{t}'_2$ move. These twists yield the diagram $\overline{8}_{13}$ of 7_3. Twists at crossings 1 and 5 yield the diagram $\overline{8}_8$ of 7_5. For two twists we thus need to consider only those two crossings. Twisting twice at one of them gives 9_7, and twisting once at each gives 9_{23}. Both 9_7 and 9_{23} have $v_2 = 5$, and so we see that there are no more relevant diagrams.

7_6: $4\ 8\ 12\ 2\ {-14}\ 6\ {-10}$.

To save work, note that we have $5 \sim 7$ in the sense of Definition 2.5. Thus crossing 7 can be excluded from twisting. Without twists, this is a diagram of 5_1. The twists at crossings 2, 3, 4 or 6 give the diagrams $\overline{9}_{12}$, $\overline{9}_{21}$ of 7_3. The twists at crossings 1 and 5 result in the diagrams $\overline{9}_8$ and $\overline{9}_{15}$ of 7_5. Two twists at crossing 1 or 5 give 9_7, and one twist at each of the two gives 9_{23}, and so we are done.

7_7: $-4\ 8\ {-10}\ 12\ 2\ {-14}\ 6$.

Without twists, this is a diagram of 5_1. The twists at crossings 2, 4, 5 or 7 give the diagram $\overline{9}_{14}$ of 7_3. The twist at crossing 3 gives its diagram $\overline{9}_{37}$. The twists at crossings 1 or 6 give the diagram $\overline{9}_{19}$ of 7_5. Two twists at the latter crossings again give 9_7 and 9_{23}.

8_{12}: $4\ {-8}\ 14\ 10\ {-2}\ {-16}\ 6\ {-12}$.

This is a diagram of 5_1. We have three reverse clasps, $(2,5)$, $(3,7)$ and $(6,8)$, and also $1 \sim 4$. Thus consider only crossings $1,\ 2,\ 3$ and 6. Twisting once at 1 or 6, we obtain 7_5 ($\overline{10}_{35}$) and at 2 or 3, 7_3 ($\overline{10}_{13}$) with $v_2 = 5$. For two twists we need to consider only crossings 1 and 6. Then one obtains diagrams of 9_7 and 9_{23} with $v_2 = 5$. Thus more twists cannot give any diagram of interest.

8_{14}: $4\ -8\ 10\ 14\ -2\ 16\ 6\ 12$.

Without a twist this is a diagram of 7_5 ($\overline{8}_{14}$). The alternating diagram has a negative clasp $(2,5)$. Applying a $\bar{t}'_2$ move at a crossing outside this clasp gives a diagram of an alternating knot of ≥ 9 crossings, which is excluded. Thus consider twists at a crossing in the clasp (both crossings are equivalent with respect to twists). A twist gives 9_{18} with $v_2 > 5$, which is excluded, so there are no more diagrams of 7_3 and 7_5.

9_{25}: $4\ 8\ 12\ 2\ -16\ 6\ 18\ -10\ 14$.

Again there is a negative clasp $(5,8)$. Use the above argument (for 8_{14}). Without twists it is 8_{15}, and with one twist near a crossing in the clasp it is 9_{18} with $v_2 > 5$, so there are no diagrams.

10_{58}: $4\ -8\ 14\ 10\ -2\ -18\ 6\ 20\ -12\ 16$.

This is a diagram of 8_{15}. With one twist we obtain diagrams of 10_{55} and 10_{63} with $v_2 \geq 5$, so there are no diagrams we seek.

By this exhaustive case distinction we have the desired description. ■

Besides the diagrams we were interested in, we came across many others used to exclude further possibilities. From this we also obtain the following useful

Example 5.1. The knot $!10_{145}$ is not positive. It is obviously almost positive as shown by its Rolfsen diagram [Ro, appendix]. This is the reason for the difficulties in showing its non-positivity by obstructions based on skein arguments (see e.g. [CM]), as skein arguments apply for almost positive knots in the same way as for positive ones. The first non-positivity proof is due to Cromwell [Cr1, Corollary 5.1] and uses the fact that the Alexander polynomial is monic. In our context the non-positivity follows from the proof of Proposition 5.1. We have $v_2(!10_{145}) = 5$ and $g(!10_{145}) = 2$, and so if $!10_{145}$ were positive, it would have appeared in the above case distinction, but it did not.

6. Classifying all 2-almost positive diagrams of a slice or achiral knot. In this section we give a proof of the classification, announced by Przytycki and Taniyama in [PT], of 2-almost positive achiral and slice knots. Our proof will actually also describe all 2-almost positive diagrams of such knots.

Proposition 6.1. *The only non-trivial achiral* 2*-almost positive knot is* 4_1 (*the figure eight knot*), *and the only non-trivial slice* 2*-almost positive knot is* 6_1 (*stevedore's knot*). *Each of them has only the two obvious* 2*-almost positive* (*twist knot*) *diagrams.*

Our arguments will also apply to the unknot. We thereby extend the result, announced by Przytycki and Taniyama and proved in [St3], determining all almost positive unknot diagrams. Our work can also be considered a partial extension, for special diagrams, of the description of almost alternating unknot diagrams given recently by Tsukamoto [Ts]. Although for the unknot the full description of 2-almost positive (and 2-almost special alternating) diagrams is not short enough to be nicely formulable, we have the following more self-contained statement, closer in spirit to Tsukamoto's result.

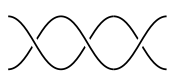
PROPOSITION 6.2. *All* 2*-almost special alternating unknot diagrams have an unknotted clasp. All* 2*-almost positive unknot diagrams are trivializable by crossing number reducing Reidemeister* I, II, *and Reidemeister* III *moves.*

This proposition will not be proved separately, since it can be checked as a consequence of the list of unknot diagrams obtained while proving Proposition 6.1. We will, however, give a shorter proof in subsequent work, where we will obtain extensions to 3- and 4-almost positive unknot diagrams. Here we focus on the proof of Proposition 6.1. The procedure is similar to the one in the previous section, with the difference that it is better now to use the signature instead of Vassiliev invariants.

Proof of Proposition 6.1. By the slice Bennequin inequality (see [Ru]), 2-almost positive diagrams of achiral or slice knots have canonical genus $\widetilde{g} \le 2$ and $\sigma = 0$. For simplicity we restrict ourselves to the (interesting) case where the diagram is prime, as the composite case reduces to it and to the almost positive diagram case.

$\widetilde{g} = 0$: A prime diagram of canonical genus zero has no crossings, and hence is not 2-almost positive.

$\widetilde{g} = 1$: If we have a subdiagram like



then the diagram D reduces to a prime almost positive diagram and so D belongs to a positive or almost positive knot. If such a knot is slice or achiral, then it is the unknot. Let p and q be odd and even positive integers. All prime almost positive diagrams of the unknot are unknotted twist knot diagrams [St3] (that is, twist knot diagrams with one of the crossings in the clasp changed). Hence D is either an unknotted twist knot diagram with one of the crossings in the twist changed, a pretzel diagram $P(3, -1, p)$ with one of the crossings in the 3-crossing group changed, or a rational diagram $C(4, -q)$ with two of the crossings in the 4-crossing group changed.

If we do not have a subdiagram like the one above, then the classification of diagrams of canonical genus 1 [St3] shows that we have either a $C(-2,q)$ or $P(p,-1,-1)$ diagram, which are the even and odd crossing number diagrams of the (negative clasp) even crossing number twist knots. The only achiral twist knot is 4_1 (a fact almost trivial to prove using knot polynomials) and the only slice twist knot is 6_1 (less trivial to prove, see Casson and Gordon [CG], and [Ka3, p. 215, bottom]). After discussing the case $\tilde{g} = 2$ below, in which only the unknot occurs, we will conclude that each of these knots has only the two 2-almost positive diagrams we just found.

$\tilde{g} = 2$: Again we discuss the 24 cases separately. Consider all diagrams D_0 obtained by switching the crossings of the generators so that exactly two are negative. Then apply $\bar{t}_2'$ moves at some of the positive crossings of D_0. Using the fact that σ does not decrease when a $\bar{t}_2'$ move is applied to a positive crossing in any diagram, we can exclude any diagrams obtained by $\bar{t}_2'$ moves (at positive crossings) from D if $\sigma(D) > 0$. (Here D will be obtained by some $\bar{t}_2'$ moves from D_0.)

Moreover, some symmetries reduce the number of cases to be checked. When fixing the crossings to be switched to become negative, only one choice of crossing(s) in each $\sim$- and $\underset{*}{\sim}$-equivalence class has to be considered. The diagrams for the other choices are obtained (even after $\bar{t}_2'$ twists) by flypes from the choice made. Also, when applying twists, it has to be done only at one choice of crossing(s) in a $\sim$-equivalence class. (In a $\underset{*}{\sim}$-equivalence class, *a priori* all crossings must be discussed, if more than one crossing is involved in the twisting, and we would like to take care of mutations. However, signature and unknottedness are invariant under mutations, so that the outcome of our calculation *a posteriori* also justifies symmetry reduction in $\underset{*}{\sim}$-equivalence classes.)

For 5_1, using signature and symmetry arguments, and that $\sigma(P(-1,-1,1,3,3)) > 0$, we see that the only diagrams with $\sigma = 0$ are $P(-1,-1,1,1,p)$ with p odd and up to permutation of the entries, and they are all unknotted. Considering the remaining 23 series, a complete distinction of the cases was done using KnotScape and is given in the three tables on the following pages. By explicit computation of σ we find that $\sigma(D_0) > 0$ except for the choices of negative crossings given in the tables below (where the aforementioned symmetries have been discarded). Therein, "$\boxed{6_2}$ 1 3" means the diagram obtained from that of 6_2 (given by its Dowker notation specified in §2) by switching crossings so that all crossings are positive except 1 and 3. It turns out that in all cases of $\sigma(D_0) = 0$ the diagram D_0 is unknotted. Then we start applying $\bar{t}_2'$ moves at (combinations of) positive crossings of D_0, noticing that either none of these moves changes σ, or some $\bar{t}_2'$ move gives a

Table 1. Proof of Proposition 6.1: the series of 6_2 and 7_6

$\boxed{6_2}$	1 3	$2 \to 0_1$		$\boxed{7_6}$	1 3	$2 \to 0_1$	$2\ 4 \to 3_1$	
		$4 \to 3_1$	$2*5* \to 0_1$			$4 \to 0_1$	$2\ 5 \to 0_1$	$2*5* \to 0_1$
		$5 \to 0_1$				$5 \to 0_1$	$4\ 5 \to 0_1$	$4*5* \to 0_1$
$2 \sim 5$		$6 \to 3_1$		$2 \underset{*}{\sim} 4$				
$3 \underset{*}{\sim} 4 \underset{*}{\sim} 6$				$3 \sim 6$	2 3	$1 \to 0_1$	$1\ 4 \to 3_1$	
	2 3	$1 \to 0_1$	$4\ 6 \to 3_1$	$5 \sim 7$		$4 \to 0_1$	$1\ 5 \to 0_1$	$1*5* \to 0_1$
		$4 \to 0_1$				$5 \to 0_1$	$4\ 5 \to 0_1$	$4*5* \to 0_1$
		$6 \to 0_1$	$1*4* \to 0_1$					
			$1*6* \to 0_1$		2 4	$1 \to 6_2$		
						$3 \to 0_1$	$3* \to 0_1$	
	3 4	$1 \to 3_1$				$5 \to 3_1$		
		$2 \to 0_1$	$2* \to 0_1$					
		$6 \to 0_1$	$6* \to 0_1$		2 5	$1 \to 0_1$		
			$2\ 6 \to 3_1$			$3 \to 0_1$	$1*3* \to 0_1$	
						$4 \to 3_1$		

knot diagram with $\sigma > 0$. In the latter case we exclude any further $\bar{t}'_2$ moves at that crossing. In the former case it turns out that we always obtain the unknot. (That arbitrarily many twists at some specific knot diagram give the unknot can be seen in each situation directly, but it also follows from checking the first two diagrams in the sequence because of the result of [ST].) The twisting procedure is denoted, exemplarily, in the following way:

$$\begin{array}{ccr} \boxed{6_2} & 1\ 3 & 4 \quad \to 3_1 \\ & & 2*5* \to 0_1 \end{array}$$

The notation means: the diagram $\boxed{6_2}$ 1 3, described above, with one twist at the crossing numbered 4 gives the trefoil (with $\sigma = 2$, so we cannot have twists at crossing 4), and arbitrarily many twists at crossings 2 and 5 give the unknot. Here a ',' (comma) on the left of a term '$x \to y$' means 'or', while 'and' is written as a space. Thus '4 4, 1 5' means double twist at crossing 4 *or* twists at crossings 1 *and* 5. The $\sim$- and $\underset{*}{\sim}$-equivalences for each generator are denoted below it to justify why certain crossings are not considered for symmetry reasons. ∎

Remark 6.1. Looking more carefully at our arguments, we see that we only needed the knot to be slice or achiral to ensure that the diagram has genus at most 2; then we only used the fact that the signature is zero. We could therefore hope to eliminate completely the condition of achirality or

Table 2. Proof of Proposition 6.1: the series of 6_3 and 7_7

					$\boxed{7_7}$	$1\ 3$	$2\to 0_1$	$2*\to 0_1$
							$4\to 3_1$	
$\boxed{6_3}$	$1\ 3$	$2\to 0_1$	$2\ 2\to 0_1$				$6\to 7_6$	
		$4\to 0_1$	$2\ 4\to 3_1$	$2*5*\to 0_1$	$2\sim 5$			
		$5\to 0_1$	$4\ 4\to 0_1$	$4*\to 0_1$	$4\sim 7$	$1\ 4$	$2\to 0_1$	$2*6*\to 0_1$
		$6\to 3_1$	$2\ 5\to 0_1$				$3\to 3_1$	
$2 \underset{*}{\sim} 4$							$6\to 0_1$	
$3 \underset{*}{\sim} 6$	$2\ 3$	$1,\ 4\to 0_1$	$1\ 4,\ 4\ 6,\ 5\ 6\to 3_1$	$4*5*\to 0_1$				
		$5,\ 6\to 0_1$	$1\ 5,\ 1\ 6,\ 4\ 5\to 0_1$	$1*5*\to 0_1$		$2\ 3$	$1\to 0_1$	
				$1*6*\to 0_1$			$4\to 0_1$	$1*4*\to 0_1$
							$6\to 3_1$	
	$2\ 4$	$1\to 6_2$						
		$3\to 0_1$	$3\ 6\to 3_1$	$3*\to 0_1$		$2\ 4$	$1\to 0_1$	
		$5\to 3_1$		$6*\to 0_1$			$3\to 0_1$	$1*3*6*\to 0_1$
		$6\to 0_1$					$6\to 0_1$	
	$2\ 5$	$1\to 0_1$				$2\ 6$	$1\to 0_1$	
		$3\to 0_1$	$3\ 6\to 3_1$	$1*3*\to 0_1$			$3\to 3_1$	$1*4*\to 0_1$
		$4\to 3_1$		$1*6*\to 0_1$			$4\to 0_1$	
		$6\to 0_1$						
						$3\ 4$	$1\to 3_1$	
	$3\ 6$	$1\to 3_1$					$2\to 0_1$	$2*6*\to 0_1$
		$2\to 0_1$	$2\ 2\to 0_1$	$4*\to 0_1$			$6\to 0_1$	
		$4\to 0_1$	$2\ 4\to 3_1$	$2*\to 0_1$				
		$5\to 6_2$	$4\ 4\to 0_1$			$3\ 6$	$1\to 7_6$	
							$2\to 3_1$	$4*\to 0_1$
							$4\to 0_1$	

sliceness by the condition of zero signature. (This would reprove the result of Przytycki and Taniyama [PT] that the only 2-almost positive zero signature knots are twist and additionally show that they have only the two obvious 2-almost positive diagrams.) For this we would basically need a version of the "slice Bennequin inequality" of [Ru] with signature replacing the slice genus. But the inequality $\sigma(D) \geq |w(D)| - n(D) + 1$ is *not* true for arbitrary diagrams. Lee Rudolph disappointed my hopes in this regard, quoting the braid representation of the untwisted Whitehead double of the trefoil in Bennequin's paper [Be, Fig., p. 121 bottom]. It is a 7-string braid (so $n(D) = 7$) consisting of eight positive bands (so $w(D) = 8$), but clearly $\sigma = 0$.

Table 3. Proof of Proposition 6.1: the series of 7_5 and the 8-to-10-crossing generators

$\boxed{7_5}$	2 3	$1 \to 0_1$	$1\ 1 \to 0_1$	
		$4 \to 0_1$	$1\ 4 \to 0_1$	$1*4* \to 0_1$
$1 \sim 6,\ 2 \underset{*}{\sim} 5,\ 3 \underset{*}{\sim} 4 \underset{*}{\sim} 7$		$5 \to 3_1$	$4\ 4 \to 0_1$	
$\boxed{8_{12}}$	1 3	$7 \to 5_2$	$4 \to 3_1$	$2*6* \to 0_1$
	2 3	$7 \to 3_1$	$5 \to 3_1$	$1*6* \to 0_1$
$1 \sim 4,\ 2 \sim 5,\ 3 \sim 7,\ 6 \sim 8$	2 6	$5 \to 5_2$	$8 \to 3_1$	$1*3* \to 0_1$
$\boxed{8_{14}}$	1 4	$7 \to 5_2$	$3 \to 3_1$	$2*6* \to 0_1$
	2 4	$5,\ 7 \to 3_1$	$1,3,6 \to 0_1$	$1*3*6* \to 0_1$
$2 \sim 5,\ 4 \underset{*}{\sim} 7,\ 6 \sim 8$	3 4	$1,7 \to 3_1$	$2,\ 6 \to 0_1$	$2*6* \to 0_1$
$\boxed{8_{15}}$	2 5	$4 \to 3_1$	$8 \to 3_1$	$1*3*7* \to 0_1$
$2 \underset{*}{\sim} 4,\ 3 \sim 6,\ 5 \underset{*}{\sim} 8$				
$\boxed{9_{23}}$	2 4	$5 \to 3_1$	$8 \to 3_1$	$1*3*7* \to 0_1$
$1 \sim 6,\ 2 \underset{*}{\sim} 5,\ 4 \underset{*}{\sim} 8,\ 7 \sim 9$				
$\boxed{9_{25}}$	2 5	$4 \to 3_1$	$8 \to 5_2$	$1*3*7* \to 0_1$
$2 \underset{*}{\sim} 4,\ 3 \sim 6,\ 5 \sim 8,\ 7 \sim 9$				
$\boxed{10_{58}}$	2 6	$5 \to 5_2$	$9 \to 5_2$	$1*3*8* \to 0_1$
$1 \sim 4,\ 2 \sim 5,\ 3 \sim 7,\ 6 \sim 9,\ 8 \sim 10$				

7. Almost positive knots. Almost positive knots, although very intuitively defined, are rather exotic—the simplest example $!10_{145}$ has 10 crossings. Therefore, not surprisingly, several properties of such knots have been proved. For example, they have positive σ, v_2 and v_3 (see [PT] and [St3]), so they are chiral and non-slice, and are non-alternating [St5]. Here we add the following property:

Theorem 7.1. *There is no almost positive knot of genus* 1.

Proof. Assume K is an almost positive knot of genus 1. By the Bennequin–Vogel inequality (or "slice Bennequin inequality" of [Ru]) an almost positive diagram D of K has genus at most 2. The description of genus 1 diagrams relatively easily excludes the cases where $\tilde{g}(D) = 1$ or D is composite. Thus again we need to consider the 24 series.

To have an almost positive diagram of an almost positive knot we need to switch (exactly) one crossing in the generator diagram to the negative, all others to the positive, and possibly apply $\bar{t}'_2$ moves at the positive crossings.

First note that the negative crossing must have no $\sim$-equivalent or $\underset{*}{\sim}$-equivalent crossing. Otherwise, after possible flypes, the negative crossing can be canceled by a Reidemeister II move or an easy tangle isotopy, giving a positive diagram.

Then note that the $\bar{t}_2'$ move at a positive crossing p in an almost positive diagram D changes ∇ (the Conway polynomial) by a multiple of ∇_L, where L is the link resulting by smoothing out the crossing p in D. Now, by [Cr1, Corollary 2.2, p. 539], ∇_L has only non-negative coefficients, hence such a $\bar{t}_2'$ move never reduces a coefficient in ∇, in particular not $[\nabla]_{z^4}$. Hence if at some point $[\nabla]_{z^4} > 0$, no further $\bar{t}_2'$ moves can produce a genus 1 knot.

In many cases $[\nabla]_{z^4} > 0$ already after the crossing switch (without $\bar{t}_2'$ moves) and we can exclude such cases *a priori*.

Finally, note that D must have at least 11 crossings, as the only almost positive knot of at most 10 crossings is $!10_{145}$, which has genus 2.

These arguments exclude after some check all but seven of the series. We discuss these cases in more detail.

The argument we apply for these cases basically repeats itself seven times and consists mainly in drawing and looking more carefully at the corresponding pictures to see how to eliminate the negative crossing by Reidemeister moves in most of the cases, and to check that in the remaining cases $\max\deg\Delta = 2$. I list the cases, leaving drawing the pictures to the reader.

6_2: We have $3 \underset{*}{\sim} 4 \underset{*}{\sim} 6$ and $2 \sim 5$. The negative crossing may be chosen to be 1. Then the diagram simplifies to a positive diagram, unless it is twisted at 3. However, if we apply no twists at one of 3, 4 and 6, then by flypes this crossing can be made to be 3, hence at all these three crossings there must be $\bar{t}_2'$ moves. The resulting 12-crossing diagram has $\max\deg\Delta = 2$.

6_3: $2 \underset{*}{\sim} 4$ and $3 \underset{*}{\sim} 6$. Then modulo flypes and inversion the negative crossing can only be 1 or 3. In case it is crossing 1, the resulting diagram can be transformed into a positive one, unless at both crossings 2 and 4, $\bar{t}_2'$ moves are applied, in which case $\max\deg\Delta = 2$. In case crossing 3 is changed to the negative, the transformation into a positive diagram is always possible.

7_6: $3 \sim 6$, $5 \sim 7$, $2 \underset{*}{\sim} 4$. This reduces to checking that the negative crossing is 1. Then the diagram can be transformed into a positive one, unless it is twisted at both 2 and 4, in which case $\max\deg\Delta = 2$.

7_7: $2 \sim 5$, $4 \sim 7$ and inversion symmetry leave us with the negative crossing being 1 or 3. The former case simplifies to a positive diagram unless it is twisted at crossings 3 and 6, and so does the latter case, unless it is twisted at crossings 1 and 6. In both remaining situations $\max\deg\Delta = 2$.

8_{14}: $2 \sim 5$, $4 \underset{*}{\sim} 7$ and $6 \sim 8$ leave us with crossing 1 or 3. 1 simplifies unless 3 is twisted, in which case $\max\deg \Delta = 2$; 3 simplifies unless 1 is twisted, in which case again $\max\deg \Delta = 2$.

9_{39}: $1 \sim 4$, $2 \sim 6$, $3 \sim 8$ leave us with crossings 5, 7 and 9 possibly negative. When 5 is negative, then already $\max\deg \Delta = 2$. When one of 7 and 9 is negative, the diagram simplifies unless at the other one there is a $\bar{t}'_2$ move, in which case $\max\deg \Delta = 2$.

Finally, we have

9_{41}: $2 \sim 6$, $3 \sim 8$, $5 \sim 9$ leave 1, 4 and 7 to be negative. However, the diagram has (modulo S^2 moves) a $\mathbb{Z}_3$-symmetry (rotation through $2\pi/3$), hence we just need to deal with crossing 1 switched to the negative. This simplifies to a positive diagram unless it is twisted at both 4 and 7, in which case $\max\deg \Delta = 2$. ∎

Similar properties to the one I proved remain still open.

QUESTION 7.1. Is there an almost positive knot of 4-ball genus 1 or unknotting number 1?

The expected answer to both is negative. (Note that in this case the answer to the second part of the question is a consequence of the answer to the first part.) To give a negative answer, one could try to apply the argument excluding 10_{145}—namely that it has an almost positive genus 3 diagram—to the other knots occurring in our proof whose diagrams are not straightforwardly transformable into positive ones (instead of showing $\max\deg \Delta = 2$ for them), but this appears to require hard labor.

8. Unique and minimal positive diagrams. One of the achievements of the revolution initiated by the Jones polynomial was the proof of the fact that an alternating knot has an alternating diagram of minimal crossing number [Ka4, Mu, Th1]. Unfortunately, such a sharp tool is yet missing to answer the problem in the positive case. Hence the question whether there is a positive knot with no positive minimal diagram is unanswered. In [St5] I managed to give the negative answer to this question in case the positive knot is alternating, and subsequently I received a paper [N], where this result was proved independently. Moreover, it follows from [St4] that the answer is the same for (positive) knots of genus 1 (in fact, a positive genus 1 knot is an alternating pretzel knot). Here we extend this result to genus 2.

THEOREM 8.1. *Any positive genus* 2 *knot has a positive minimal diagram.*

With this theorem we also finish the proof of Corollary 2.2. The main tool we use to prove it is the Q polynomial of Brandt–Lickorish–Millett

[BLM] and Ho [Ho] (sometimes also called absolute polynomial) and some results about its maximal degree obtained by Kidwell [Ki]. (They were later extended by Thistlethwaite to the Kauffman polynomial.)

Recall that the *Q polynomial* is a Laurent polynomial in one variable z for links without orientation, defined by being 1 on the unknot and the relation

$$A_1 + A_{-1} = z(A_0 + A_\infty), \tag{2}$$

where A_i are the Q polynomials of links K_i, and K_i (with $i \in \mathbb{Z} \cup \{\infty\}$) possess diagrams equal except in one spot, where an i-tangle (in the Conway sense) is inserted (see Figure 8; orientation of any of the link components is unimportant for this polynomial).

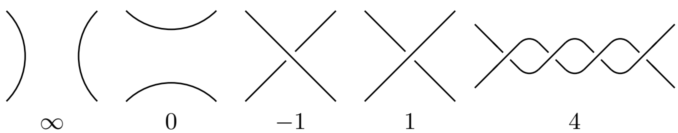


Fig. 8. The Conway tangles

The following result on $\max\deg Q$ will be applied.

THEOREM 8.2 (Kidwell [Ki]). *Let K be a knot. Then* $\max\deg Q(K) \leq c(K) - 1$ *with equality if and only if K is prime alternating.*

COROLLARY 8.1. *Let D be a positive diagram with* $\max\deg Q(D) = c(D) - 2$. *Then the knot K represented by D has a positive minimal diagram.*

Proof. By Theorem 8.2, either $c(K) = c(D)$, in which case the claim is trivial, or $c(K) = c(D) - 1$ and K is alternating, in which case the claim follows from the above mentioned result of [St5]. ■

LEMMA 8.1. *With the above notation in* (2), *the Q polynomial has the following property*:

$$A_n = (z^2 - 1)(A_{n-2} - A_{n-4}) + A_{n-6}. \tag{3}$$

Proof. From (2) we have

$$A_n + A_{n-2} = z(A_{n-1} + A_\infty). \tag{4}$$

Now, adding two copies of (4) for n and $n-2$ we obtain

$$A_n + 2A_{n-2} + A_{n-4} = 2zA_\infty + z(A_{n-1} + A_{n-3}) = (z^2 + 2z)A_\infty + z^2 A_{n-2}. \tag{5}$$

So

$$A_n = (z^2 + 2z)A_\infty + (z^2 - 2)A_{n-2} - A_{n-4}.$$

Therefore

$$A_n - (z^2-2)A_{n-2} + A_{n-4} = A_{n-2} - (z^2-2)A_{n-4} + A_{n-6},$$

which is equivalent to the assertion. ∎

Proof of Theorem 8.1. Take a positive diagram D of a positive genus 2 knot K. If D is composite, the genus 1 case shows that K is the connected sum of two alternating pretzel knots, hence K is alternating. Thus consider the prime case. The series of 12_{1097} and 13_{4233} and their progeny in Figure 6 contain only positive diagrams which are alternating, so these cases are trivial. Considering 11_{148}, the diagram is made positive by switching the negative clasp. But all diagrams arising by $\bar{t}'_2$ moves from this diagram can be simplified near the switched (and possibly twisted) clasp by one crossing, so as to become alternating. The same argument excludes (the series of) 10_{97}, 9_{25}, 8_{14}, 7_6 and 6_2. The case of 8_{12} is trivial, because it contains only rational knots, which are alternating. For 10_{58} and 7_7 apply the clasp argument separately to the two negative clasps.

This leaves us with 12_{1202}, 9_{41}, 9_{39} and 6_3. By Corollary 8.1 it suffices to check that for any positive diagram D in their series $\max\deg Q(D) = c(D)-2$. By Lemma 8.1 and Theorem 8.2 this reduces to calculating Q for at most one $\bar{t}'_2$ move applied near a crossing and a (reverse) clasp being positive or resolved. However, when the clasp is resolved, the diagram reduces to one in the series of some specialization, for which $\max\deg Q(D) = c(D) - 2$ or $\max\deg Q(D) = c(D)-3$ by the above discussion. The formula in Lemma 8.1 then shows that we need to consider just positive clasps without $\bar{t}'_2$ moves. This leaves a small number of diagrams. E.g., the diagram of 12_{1202} consists only of clasps, hence only one diagram has to be checked. Switching all crossings in the diagram of 12_{1202} to the positive, we obtain a diagram of the knot 12_{2169}, for which $\max\deg Q = 10$ is directly verified. 9_{39} and 9_{41} have three non-clasp crossings, hence there are eight diagrams to check, and for 6_3 we have 64 diagrams. Using various symmetries one can further reduce the work, but even that far I had no serious difficulty checking the $8+8+64 = 80$ relevant diagrams by computer. ∎

Our proof actually also shows the following:

COROLLARY 8.2. *Any positive* (*reduced*) *diagram of a positive genus* 2 *knot K has at most $c(K)+3$ crossings.* ∎

This is, in this special case, a much better estimate than the general bound $c(K)^2/2$ known from [St2].

The method of the proof can also be used further. In [St4] I exhibited the (p,q,r)-pretzel knots with $p,q,r > 1$ odd as positive knots with a unique positive diagram (up to inversion and moves in S^2) and asked whether these are the only examples. The reason behind this question was that (as I already expected at that point) the number of series generators grows rapidly with the genus and hence so does the number of diagram candidates for a positive knot of that genus. Here we observe that at least for genus 2 the variety on generators is not sufficiently large, so that such examples still exist. We take one of our generators.

EXAMPLE 8.1. The knot $!10_{120}$ has a unique positive diagram. To see this, first exclude the series of the knots up to 9_{25}, and 10_{58}. Positive diagrams in these series are, or simplify to, arborescent alternating diagrams, and $!10_{120}$ has no such diagram. The series of 9_{38}, 10_{101}, 11_{123}, 11_{329}, 12_{1097} and 13_{4233} are excluded because all positive diagrams in these series are alternating (and the only $\bar{t}'_2$ irreducible diagrams they contain are the generators themselves, and $\bar{t}'_2$ (ir)reducibility is preserved by flypes). 10_{97} is excluded, because by the above discussion the maximal degree of Q on positive diagrams in its series is equal to the crossing number minus 2, and hence all maximal degrees are even (whereas clearly $\max\deg Q(10_{120}) = 9$). The same argument excludes 12_{1202} and reduces checking positive diagrams in the series of 11_{148} only to the one with no $\bar{t}'_2$ moves applied, which belongs to 10_{101}, and the diagrams of 9_{39} and 9_{41} made positive by crossing changes and with exactly one $\bar{t}'_2$ move applied. In all the latter cases $\max\deg V = 11$, whereas $\max\deg V(!10_{120}) = 12$. To finish the argument, it remains to notice that the alternating diagram of 10_{120} does not admit a flype itself and because of the crossing number, no $\bar{t}'_2$ twisted diagram in its series can belong to it.

9. Some evaluations of the Jones and HOMFLY polynomials

9.1. *Roots of unity.* The first obstruction to particular values of $\tilde{g}$ is an inequality of Morton [Mo]: $\max\deg_m P/2 \le \tilde{g}$, which shows that $\tilde{g} > g$ for the untwisted Whitehead double of the trefoil [Mo, Remark 2] and also for one of the two 11-crossing knots with trivial Alexander polynomial, which according to [Ga1, Fig. 5] has genus 2 (I cannot identify which one). In this section we will discuss an alternative approach to such an obstruction, and apply it to exhibit the first examples of knots on which the weak genus inequality of Morton is not sharp.

THEOREM 9.1. *There exist knots K with $\tilde{g}(K) > 2 = \max\deg_m P(K)/2$.*

The present diagram description opens the search for alternative criteria which can be applied to exclude a knot from belonging to a given $\bar{t}'_2$ twist sequence. (We noted that some of the $\bar{t}'_2$ twist sequences contain others, so we need to consider only main $\bar{t}'_2$ twist sequences.) Such a criterion is the following fact, which is a direct consequence of the skein relations for the Jones [J1] and HOMFLY [F&] polynomials and has been probably first noted by Przytycki [Pr].

THEOREM 9.2 (Przytycki). *Let $a^{2k} = 1$, $a \neq \pm 1$. Then $V(a) \in \mathbb{C}$ and $P(ia, m) \in \mathbb{C}[m^2]$ (for $i = \sqrt{-1}$) are $\bar{t}_{2k}$(-move) invariant.*

COROLLARY 9.1. *The sets*

$$\mathcal{P}_{k,g} := \left\{ P_K \bmod \frac{(il)^{2k} - 1}{-l^2 - 1} \,\middle|\, \widetilde{g}(K) = g \right\}$$

and

$$\mathcal{V}_{k,g} := \left\{ V_K \bmod \frac{t^{2k} - 1}{t^2 - 1} \,\middle|\, \widetilde{g}(K) = g \right\}$$

are finite for any k and $g \in \mathbb{N}$.

Proof. From the theorem it is obvious that for every generator K, the sets of residues

$$V_{k,K'} := V_{K'} \bmod \frac{t^{2k} - 1}{t^2 - 1} \quad \text{and} \quad P_{k,K'} := P_{K'} \bmod \frac{(il)^{2k} - 1}{-l^2 - 1}$$

are finite on the series of K. $\mathcal{P}_{k,g}$ and $\mathcal{V}_{k,g}$ are finite unions of such sets. ∎

Proof of Theorem 9.1. We will explain how the knots have been found. The obvious idea is to compute the sets in Corollary 9.1 for some appropriate k in all 24 series and to hope not to find the value of any knot therein for which $\max\deg_m P \leq 4$. Note that the polynomials are preserved by mutations, so we need to consider just one diagram for any generating knot.

Table 4. The number of evaluations of V and P in the eighth and tenth roots of unity on each series, and in total. (The number of evaluations for V and P coincide for tenth roots of unity.)

series	5_1	6_2	6_3	7_5	7_6	7_7	8_{12}	8_{14}	8_{15}	9_{23}	9_{25}	9_{38}	9_{39}	9_{41}
$\# V_{4,K}$	47	121	202	226	136	119	52	302	702	418	479	1195	413	268
$\# P_{4,K}$	47	121	202	226	136	119	52	302	710	418	487	1231	413	268
$\# V_{5,K} = \# P_{5,K}$	112	408	919	988	538	456	146	1610	4281	2634	2554	8588	2271	1270

series	10_{58}	10_{97}	10_{101}	10_{120}	11_{123}	11_{148}	11_{329}	12_{1097}	12_{1202}	13_{4233}	total
$\# V_{4,K}$	157	980	2380	2587	2284	1041	2858	5791	197	5604	6645
$\# P_{4,K}$	163	1020	2429	2673	2349	1073	2970	6084	209	5915	6974
$\# V_{5,K} = \# P_{5,K}$	624	8161	23,714	27,510	22,817	8489	34,905	104,620	938	102,940	128,898

The cases $k = 2$ and $k = 3$ did not suggest themselves as particularly interesting at least for V, because the corresponding evaluations can be well controlled [LM2, Li]. Thus I started with $k = 4$. In the case of V, this is mainly the information given by its evaluations at $e^{\pi i/4}$ and $e^{3\pi i/4}$ (modulo conjugation and the value at i, which is equivalent to the Arf invariant [J2, §12] and hence not very informative). Table 4 summarizes the number of evaluations of each series.

As noted in Remark 2.2 (and established also in [Ga1]), $\widetilde{g}=g=\max\deg\Delta$ for $\leq$10-crossing knots, so we need to look at more complicated examples. Examining Thistlethwaite's tables, I found 2010 non-alternating 11-to-15-crossing knots for which $\max\deg_m P \leq 4$. (Among these 2010 knots only the expected 12 pretzel knots had $\max\deg_m P \leq 2$.) The unity root test for V and $k = 4$ does not exclude any of these 2010 knots from having $\widetilde{g} = 2$. The test for P with $k = 4$ produced the same disappointing result. (The above table shows that it does not bring much improvement compared to V.)

However, examining V with $k = 5$ exhibited four 15-crossing knots of the type sought. These examples are shown in Figure 9. One explanation of this outcome may be that for $k = 5$ all four relevant evaluations (at $e^{k\pi i/5}$ for $k = 1, 2, 3, 4$) admit very little control. The only known result about them is Jones's norm bound for $k = 1$ in terms of the braid index and bridge number ([J2, Propositions 15.3 and 15.6]) and the fact that this evaluation is finite on closed 3-braids (see [J2, (12.8)]). Experiments with P and $k = 5$, however, turned out to be significantly more time and memory consuming, and all the values on all of the 24 series reported by my C++ program repeated those of V, so considering P appears little rewarding. ■

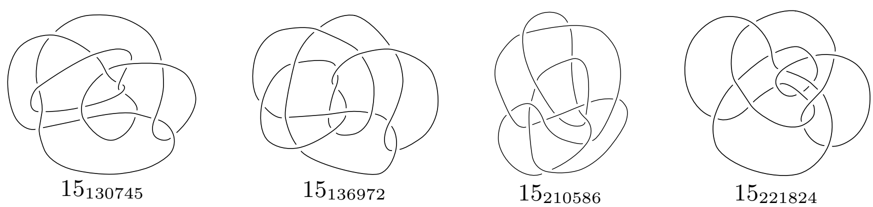
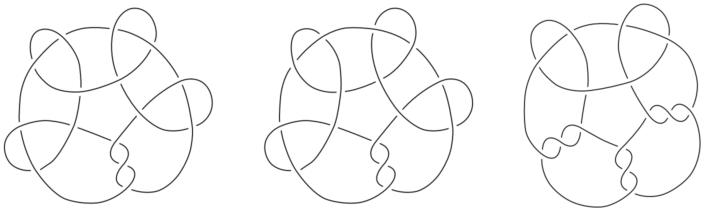


Fig. 9. The simplest examples of knots for which $\widetilde{g} > 2$ can be proved using Jones polynomial unity root evaluations, but not using Morton's inequality

REMARK 9.1. M. Hirasawa was able to find a genus 2 Seifert surface for the last example in Figure 9, 15_{221824}, so that the ordinary genus is not an applicable obstruction to weak genus 2 either in this case. Later Jake Rasmussen showed that the other five knots from Figures 9 and 11 all have genus 2 as well, though his argument, based on direct calculation of the knot Floer homology, is not constructive.

It would clearly be helpful to find some nice properties of the sets occurring in Corollary 9.1, but such seem unlikely to exist or at least are obscured by the electronic way of obtaining them.

Here is a more special example.



15_{184486} 15_{184487} 15_{197572}

Fig. 10. The three 15-crossing pretzel knots (two of them mutants) for which we can show at least that they have no (reduced) diagram of even crossing number of genus 2, but which have $\max\deg_m P = 4$

EXAMPLE 9.1. Consider the knots 15_{184486} and 15_{184487} of Figure 10. These knots are slice (generalized) pretzel knots, which are mutants. Their (common) Jones polynomial is

$$\begin{aligned} V(15_{184486}) &= V(15_{184487}) \\ &= (-1\ 3\ -6\ 13\ -16\ 17\ -19\ 12\ -7\ [4]\ 4\ -5\ 4\ -3\ 1). \end{aligned}$$

A check of the evaluation of V mod $\frac{t^{10}-1}{t^2-1}$ shows that the polynomial modulus is not realized in any main series of even crossing number. Thus these knots do not have a (reduced) genus 2 diagram of even crossing number (although they clearly have some in the series of 5_1). A similar situation occurs for 15_{197572}.

9.2. *The Jones polynomial on the unit circle.* While the unity root values of V have been useful for practical purposes, we can continue the discussion of the polynomial evaluations in a more theoretical direction.

More generally, it is possible to say something about the evaluations of the polynomials on the unit circle. Here are two slightly weaker but hopefully also useful modifications of Corollary 9.1. They are also possible for P, but I restrict myself to V for simplicity.

PROPOSITION 9.1. *Let $z \in \mathbb{C}$ with $|z| = 1$ and $z \neq -1$. Then the set $\{V_K(z) : \widetilde{g}(K) = g\} \subset \mathbb{C}$ is bounded for any $g \in \mathbb{N}$.*

Proof. We use the Jones skein relation to expand the Jones polynomial of a knot in the $\bar{t}'_2$ twist sequence of a diagram in terms of the Jones polynomials of the diagram and all its crossing-changed versions. We obtain a complex expression of partial sums of the Neumann series for z^2 and z^{-2}. Then we use the boundedness of these partial sums if $|z| = 1$ and $z \neq -1$. (The value $V(1) \equiv 1$ is of little interest.) ∎

PROPOSITION 9.2. *Let $z \in \mathbb{C}$ with $|z| \leq 1$ and $z \neq -1$. Then the set $\{V_K(z) : K$ is positive and $g(K) = g\} \subset \mathbb{C}$ is bounded for any $g \in \mathbb{N}$.*

Proof. In the case of positive $\bar{t}'_2$ twists only, the Neumann series for z^{-2} do not occur, and we are done as before. ∎

This result seems similar to the boundedness of some other sets of evaluations of V on closed braids of given strand number considered by Jones [J2, §14]. The nature of our sets is quite different, though. Note, for example, that their closures are countable (so in particular their sets of norms have empty interior) for $|z|<1$, while Jones showed that for the evaluations he considered, the closure is an interval.

THEOREM 9.3. *The map $f = f_g : S^1 \to \mathbb{R}$ defined for $g \in \mathbb{N}_+$ by*

$$f_g(q) := \sup\{|V_K(q)| : \widetilde{g}(K) = g\}$$

has the following properties:

(i) $f(\overline{q}) = f(q)$, *where the bar denotes complex conjugation*;
(ii) $f(1) = 1$, $f(-1) = \infty$;
(iii) f *is upper semicontinuous on* $S^1 \setminus \{-1\}$, *that is*, *for* $q \in S^1$ *and* $q \neq -1$ *we have*

$$\limsup_{q_n \neq q,\, q_n \to q} f_g(q_n) \geq f_g(q).$$

(iv) f_g *satisfies the bound*

$$f_g(q) \leq \max_L |V_L(q)| \cdot \left(\frac{2}{|1+q|} + 1\right)^{d_g},$$

where the maximum is taken over L being a(*n alternating*) *link diagram obtained by smoothing out some sets of crossings in an alternating $\bar{t}'_2$ irreducible diagram of genus g. In particular*, *the order of the singularity of f_g at -1 is at most d_g.*

The same properties hold if we modify the definition of f_g by taking the supremum only over positive or alternating knots.

Proof. The explicit estimate follows from the same argument as in the proof of Proposition 9.1. If V_n denote the Jones polynomials of L_n, where L_n are links with diagrams equal except in one room, where n antiparallel half-twist crossings are inserted, then from the skein relation for the Jones polynomial we have

$$V_{2n+1}(q) = q^{2n} V_1(q) + \frac{q^{2n}-1}{q^2-1}\,(q^{1/2} - q^{-1/2}) V_\infty(q),$$

with V_∞ denoting the Jones polynomial of L_∞, and L_∞ being the link obtained by smoothing out a(ny) crossing in the room.

Expand this relation with respect to any of the d_k crossings at which $\bar{t}'_2$ moves can be applied, obtaining 2^{d_k} terms on the right, and take the norm, applying the triangle inequality and using $|q| = 1$.

The upper semicontinuity of f_g is straightforward from its definition and the continuity of V. Thus the only fact remaining to prove is $f_g(-1) = \infty$. For this, one first easily observes that the determinant (even the whole Alexander polynomial) depends linearly on the number of $\bar{t}'_2$ twists. Thus we could achieve arbitrarily high and low determinants in the $\bar{t}'_2$ twist sequence (and at least one of both types in alternating or positive diagrams), unless all linear coefficients in this dependence are zero. But the fact that the determinant never changes sign under a $\bar{t}'_2$ twist implies that all knots in the series have the same signature, and as any diagram can be unknotted by crossing changes, it must be 0. This is clearly not the case, and so we have a contradiction. ∎

9.3. *Jones's denseness result for knots.* This subsection is unrelated to our discussion as far as weak genus 2 knots are considered. However, it is interesting in connection with (or rather in contrast to) the properties of their Jones polynomial unity root evaluations.

In [J2, Proposition 14.6], Jones proved the denseness of the norms of $V(e^{2\pi i/k})$ on closed 3-braids in $[0, 4\cos^2\pi/k]$ if $k \in \mathbb{N} \setminus \{1,2,3,4,6,10\}$.

Here we modify this result restricting our attention to *knots*, which are closed 3-braids.

PROPOSITION 9.3. *If* $k \in \mathbb{N} \setminus \{1,2,3,4,6,10\}$, *then*

$$\left.\begin{array}{lr} [0, 4\cos^2\pi/5 - 1], & k = 5 \\ [4\cos^2\pi/k - 3, 4\cos^2\pi/k - 1], & k \geq 7 \end{array}\right\} \tag{6}$$
$$\subset \overline{\{|V_K(e^{2\pi i/k})| : K \textit{ is a closed } 3\textit{-braid knot}\}} \subset [0, 4\cos^2\pi/k].$$

Proof. We closely follow Jones's proof. The second inclusion is due to him. The essential point is the first inclusion.

In the following we denote by ψ the (reduced) Burau representation. If β is a braid, then $\psi_\beta = \psi(\beta)$ is its Burau matrix. We also write ψ_n for the n-strand Burau representation, when dealing with different strand numbers. (Since numbers and braids are disjoint, the subscripts of ψ cannot be interpreted ambiguously.)

By Jones's proof, for $\beta \in B_3$ with even exponent sum $[\beta]$ (in particular when β's closure $\hat{\beta}$ is a knot) we have

$$\frac{1}{4\cos^2\pi/k} V_{\hat{\beta}}(e^{2\pi i/k}) = f(\mathrm{tr}(\psi_\beta)) := 1 - \frac{1}{2\cos^2\pi/k} + \frac{1}{4\cos^2\pi/k}\,\mathrm{tr}(\psi_\beta), \tag{7}$$

with ψ being the reduced Burau representation of B_3.

Now by [Sq], up to conjugation (not affecting the trace), $\psi(\beta) \in U(2)$, and hence, if additionally k divides $[\beta]$, then

$$\psi_\beta(e^{2\pi i/k})\big|_{\{\beta\in B_3 : k\,|\,[\beta]\}} \subset SU(2),$$

in particular $\mathrm{tr}(\psi(\beta))$ is real.

Now

$$\Gamma' := \{\beta \in B_3 : \widehat{\beta} \text{ is a knot and } k \,|\, [\beta]\}$$

is a coset in B_3/Γ, where Γ is the kernel of $\beta \mapsto (\sigma(\beta), e^{2\pi i[\beta]/k}) \in S_3 \times \mathbb{Z}_k$. (Here σ is *not* the signature, but the induced permutation homomorphism $B_3 \to S_3$.) Again $\Gamma \subset B_3$ is normal and of finite index, hence the closure of $\psi(\Gamma) \subset SU(2)$ has non-trivial connected sets. In particular the connected component of 1 contains an $S^1 \ni -1$. Therefore, $\overline{\psi(\Gamma')}$ with each ψ' also contains a coset of S^1 we call $G_{\psi'}$ (not necessarily a subgroup), with $G_{\psi'} \ni -\psi'$.

If now for some $\psi' \in \psi(\Gamma')$ we had $\mathrm{tr}(\psi') = \tau$ (where $\tau \in \mathbb{R}$), then $|f|\,\big|_{G_{\psi'}}$ would be a continuous function on $G_{\psi'}$ admitting the values $f(-\tau)$ and $f(\tau)$, and for $\tau \neq 0$ we would apply Jones's argument.

Therefore, we are interested in some ψ' where $|\tau|$ is maximal. Now if $\xi_{1,2}$ are the eigenvalues of ψ' (with $|\xi_{1,2}| = 1$), then because of $\Gamma'^k := \{\gamma^k : \gamma \in \Gamma'\} \subset \Gamma'$ for any $3 \nmid k$, we consider the maximal trace of ψ'^k with $3 \nmid k$, which is

$$\mu(\xi) := \sup_{3 \nmid k} |1 + \xi^k|$$

with $\xi := \xi_1/\xi_2$. One sees that μ is minimized by $\xi = e^{\pm 2\pi i/3}$, where it is 1. Therefore, f ranges at least between $f(-1)$ and $f(1)$ on one of the $G_{\psi'^k}$, which implies the assertion. ∎

While this is likely not the maximum we can get in our restricted situation for 3-braids, Jones's corollary holds in full strength restricting the evaluations to knots.

Corollary 9.2. *If $k \in \mathbb{N} \setminus \{1, 2, 3, 4, 6, 10\}$, then*

$$\overline{\{|V_K(e^{2\pi i/k})| : K \textit{ is a knot}\}} = [0, \infty).$$

Proof. Use the fact that 1 is always in the interior of the interval on the left of (6) and apply connected sums. ∎

Now we attempt to generalize Corollary 9.2 to the case $k = 10$. According to Jones [J3, p. 263 top], by the work of Coxeter and Moser [CMo], the image of B_3 in the Hecke algebra is finite, so we need to start with 4-braids, which makes the situation somewhat more subtle.

Proposition 9.4. $\overline{\{|V_K(e^{\pi i/5})| : K \textit{ is a knot}\}} = [0, \infty)$.

Proof. First we show that $\overline{\{|V_K(e^{\pi i/5})| : K \text{ is a 4-braid knot}\}}$ contains an interval. This argument starts along similar lines as the proof of Proposition 9.3.

Consider $\varGamma \subset B_4$, which is the kernel of

$$B_4 \ni \beta \mapsto ([\beta] \bmod 10, \sigma(\beta), \psi_3(\bar{\beta})) \in \mathbb{Z}_{10} \times S_4 \times H(e^{\pi i/5}, 3),$$

where $H(e^{\pi i/5}, 3)$ denotes the 3-strand Hecke algebra of parameter $e^{\pi i/5}$, $\bar{\cdot}$ is the homomorphism $B_4 \to B_3$ with $\bar{\sigma}_{1,3} = \sigma_1, \bar{\sigma}_2 = \sigma_2$, and all other notations are as before. ($\psi_3 = \psi$ is the reduced 3-strand Burau representation.) Again $\varGamma \subset B_3$ is normal and of finite index, hence the closure of $\psi_4(\varGamma) \subset SU(3)$ is non-discrete.

All subgroups S^1 of $SU(3)$ are conjugate to subgroups of the standard maximal toral subgroup, which are of the form

$$u \in [0,1] \mapsto \begin{pmatrix} e^{2k\pi iu} & 0 & 0 \\ 0 & e^{2l\pi iu} & 0 \\ 0 & 0 & e^{-2(k+l)\pi iu} \end{pmatrix}$$

for some $k, l \in \mathbb{Z}$ with $(k,l) = 1$. We will refer to these S^1's as *standard* S^1's and denote them by $S^1_{k,l}$. (The case of $(k,l) > 1$ gives no new subgroups, at least as *subsets* of $SU(3)$.) Therefore, $\overline{\psi_4(\varGamma)}$ contains some $AS^1_{k,l}A^{-1}$ for some $A \in SU(3)$.

Now, consider some $\beta \in B_4$ with $\sigma(\beta)$ a 4-cycle, and write down the weighted trace sum for 4-braids. The result is

$$(8) \qquad V_{\widehat{\beta}}(e^{\pi i/5}) = \pi_0(\beta) := 8c^3 - 6c + \frac{1}{2c} + \frac{1}{c}\operatorname{tr}(\psi_3(\bar{\beta})) + \left(6c - \frac{3}{2c}\right)\operatorname{tr}(\psi_4(\beta))$$

with $c := \cos\pi/10$. (Keep in mind that $\psi_{3,4}$ denote Burau representations of different braid groups.)

If $|\pi_0|\,\big|_{\beta\varGamma}$ is not constant, we obtain the desired interval. Therefore, assume that in particular $|\pi_0|\,\big|_{\varPhi}$ is constant for the set

$$\varPhi = \beta \cdot [\psi_4^{-1}(AS^1_{k,l}A^{-1}) \cap \varGamma].$$

Now, $\psi_3(\bar{\beta})$ is constant on any coset of $B_4/\varGamma$, and $S^1_{k,l}$ acts on $A^{-1}\psi_4(\beta)A$ by multiplying by unit norm complex numbers its columns, so in particular the diagonal entries ξ_i $(i = 1,2,3)$. Therefore, for these ξ_i,

$$f(u) = f_{\xi_1,\xi_2,\xi_3}(u) = e^{2\pi iku}\xi_1 + e^{2\pi ilu}\xi_2 + e^{-2\pi i(k+l)u}\xi_3$$

must lie for all $u \in [0,1]$ in a sphere (boundary of some ball) in $\mathbb{C}$, which is specified from (8).

That this happens only in exceptional cases follows by holomorphy arguments. Namely, with (8),

$$\gamma = -\left(8c^3 - 6c + \frac{1}{2c} + \frac{1}{c}\operatorname{tr}(\psi_3(\bar{\beta}))\right)\bigg/\left(6c - \frac{3}{2c}\right)$$

must be the center of this sphere, i.e.

$$u \mapsto f_{\xi_1,\xi_2,\xi_3}(u) - \gamma$$

must be of constant norm on $[0,1]$. Then so is

$$|f(u)-\gamma|^2 = (f(u)-\gamma)(\overline{f(u)}-\overline{\gamma}),$$

which is holomorphic, since $\overline{f(u)} = f_{\bar\xi_1,\bar\xi_2,\bar\xi_3}(-u)$. Thus $|f(u)-\gamma|^2$ is constant for any $u \in \mathbb{C}$.

Assume now that $\mathrm{tr}(\psi_4(\beta)) = \xi_1+\xi_2+\xi_3 \neq 0$. We claim that $\xi_i\lambda_i = 0$ for $i=1,2,3$, with $\lambda_1 := k$, $\lambda_2 := l$, $\lambda_3 := -(k+l)$. In particular, since (at least) two of the λ_i's are non-zero, (at least) two of the ξ_i's are zero.

Assume the contrary, that is, some $\xi_i\lambda_i \neq 0$. Then, since $(\lambda_1,\lambda_2,\lambda_3)$ is completely characterized by being a triple of relatively prime integers summing up to 0, we can by symmetry assume that $\xi_1 \neq 0 \neq k$. Since any $\alpha \in \mathbb{C}\setminus\{0\}$ is of the form $e^{2\pi i u}$ for $u \in \mathbb{C}$, we find that

$$P(\alpha) = \alpha^k\xi_1 + \alpha^l\xi_2 + \alpha^{-k-l}\xi_3 - \gamma$$

has constant norm for any $\alpha \in \mathbb{C}\setminus\{0\}$. Letting $\alpha \to 0$ or $\alpha \to \infty$, we see that this is possible only if $P(\alpha) \equiv C \in \mathbb{C}[\alpha,\alpha^{-1}]$ is a constant as Laurent polynomial in α. This in turn is possible (up to interchanging $\lambda_{2,3}$ and $\xi_{2,3}$) only if (i) $\xi_1 = \xi_2 = 0$ and $k = -l = 1$ or (ii) $\xi_1 = -\xi_2$, $\xi_3 = 0$ and $k = l = \pm 1$. These cases contradict the assumptions $\xi_1 \neq 0$ or $\mathrm{tr}(\psi_4(\beta)) \neq 0$ resp.

Thus we have shown that if $|\pi_0|\,\big|_{\Phi}$ is constant, then $A^{-1}\psi_4(\beta)A \in \mathcal{M}$, where $\mathcal{M}$ is the (closed) subset of $U(3)$ consisting of the matrices with zero trace or at least two zero diagonal entries.

But if $\sigma(\beta)$ is a 4-cycle, so is $\sigma(\beta^{2k+1})$ for any $k \in \mathbb{Z}$, so that in particular by the same argument any odd power of $A^{-1}\psi_4(\beta)A$ must lie in $\mathcal{M}$. Taking $\beta = \sigma_1\sigma_2\sigma_3^{-1}$ and setting $U := e^{-\pi i/5}A^{-1}\psi_4(\beta)A$ we obtain an element of infinite order in $SU(3)$, with all odd powers in $\mathcal{M}$. But now, $\overline{U^{\mathbb{Z}}} \subset SU(3)$ is an Abelian closed non-discrete subgroup, and hence $\overline{U^{\mathbb{Z}}}$ contains some S^1. But $\overline{U^{\mathbb{Z}}}$ contains the dense subset $U^{2\mathbb{Z}+1}$, which is also a subset of $\mathcal{M}$, and hence $\overline{U^{\mathbb{Z}}}$ is contained itself in $\mathcal{M}$. Therefore, $\mathcal{M} \cap SU(3)$ contains an $S^1 = A' S^1_{m,n} A'^{-1}$.

To show that this is impossible, consider again the trace. If $\mathrm{tr} \neq 0$, we see, from the two zero entries and Cauchy–Schwarz for the third, that $|\mathrm{tr}| \leq 1$ on the whole $\mathcal{M}$. But integrating the (conjugacy invariant) squared trace norm on the standard S^1, and using the equality, for any $X \in \mathbb{Z}[t,t^{-1}]$,

$$[X(t)X(1/t)]_{t^0} = \int_0^1 |X(e^{2\pi i u})|^2\,du,$$

we obtain

$$\int_0^1 |e^{2\pi imu}+e^{2\pi inu}+e^{-2(m+n)\pi iu}|^2\,du=\begin{cases}3 & \text{for } |\{m,n,-m-n\}|=3,\\ 5 & \text{for } |\{m,n,-m-n\}|=2.\end{cases}$$

Thus we must have $|\mathrm{tr}|>1$ somewhere on the standard S^1, and hence on any other $S^1\subset SU(3)$, providing us with the desired contradiction.

In summary, we showed that $|V_K(e^{\pi i/5})|$ is dense in some interval when taking knots K ranging over closed 4-braids. From this the proposition follows by taking connected sums once we can show that there are knots $K_{1,2}$ with $|V_{K_1}(e^{\pi i/5})|>1$ and $0\neq|V_{K_2}(e^{\pi i/5})|<1$. Luckily, already $K_1=3_1$ (trefoil) and $K_2=5_1$ ($(2,5)$-torus knot) do the job, and we are done. ∎

REMARK 9.2. V. Jones pointed out that for $l=0,\dots,n-1$, $|V(e^{l\pi i/n})|$ is invariant under an n-move (adding or deleting subwords $\sigma_i^{\pm n}$). Thus for $4\nmid k$ our result follows directly from his, in particular for $k=10$. However, since no proof was given in this case in [J2], it is worth including one here anyway.

There is another way to prove the last two statements on norm denseness in $[0,\infty)$, avoiding any braid group theory, and just applying connected sums. It would go via showing for every k the existence of knots $K_{1,2}$ such that $\ln|V_{K_1}(e^{2\pi i/k})|/\ln|V_{K_2}(e^{2\pi i/k})|$ is irrational. It is unclear how to find such knots for general k, but for specific values this is a matter of some calculation. The following example deals with $k=10$, and thus indicates an alternative (but much less insightful) proof of Proposition 9.4.

EXAMPLE 9.2. Consider the knots 6_3, 9_{42}, 11_{391} and 15_{134298}. Writing

$$V_{[n]}(t):=\frac{t^{n+1/2}+t^{-n-1/2}}{t^{1/2}+t^{-1/2}},$$

note that $V(4_1)=V_{[2]}$ is (up to units) the minimal polynomial of $e^{\pi i/5}$. The polynomials of our four knots are given by:

$$V(9_{42})=V_{[3]},\qquad V(6_3)=-V_{[3]}+V_{[2]}+1,\qquad V(11_{391})=2-V_{[2]}^2,$$
$$V(15_{134298})=3-2V_{[2]}^2.$$

Their evaluations at $e^{\pi i/5}$ are $(1\pm\sqrt{5})/2$, 2 and 3 resp. Then we use the fact that the first two numbers are inverses of each other up to sign, and $\ln 3/\ln 2$ is irrational. (Except for 6_3, the knots are not amphicheiral, although they were chosen to have self-conjugate V to make its evaluation at $e^{\pi i/5}$ as simple as possible.)

To apply our results in this subsection to the weak genus, we obtain

Corollary 9.3. *For any even $k > 6$ and any $g \in \mathbb{N}_+$ there are infinitely many knots K with braid index*

$$b(K) \leq \begin{cases} 3, & k \neq 10, \\ 4, & k = 10, \end{cases}$$

which are not k-equivalent to a knot of canonical genus $\leq g$. ∎

Note that, when replacing k-equivalence just by isotopy, this is well-known, because of the result of Birman and Menasco [BM, Theorem 2] that there exist only finitely many knots of given (Seifert) genus and given braid index. We will consider the k-moves in more detail later.

10. k-moves and the Brandt–Lickorish–Millett–Ho polynomial

10.1. *The minimal coefficients of Q.* It becomes clear from the previous discussion that the Jones polynomial evaluations for themselves will unlikely give some significantly more powerful and applicable criteria for showing $\widetilde{g} > 2$ than Morton's inequality, so it is interesting to find additional methods that sometimes provide an efficient amplification. Here we study the Q polynomial in this regard. This is where the effort in examining the 8th roots of unity of V can be used in practice.

First, we have the following (not maximally sharp, but easy to apply) criterion on the low degree coefficients of Q.

Proposition 10.1. *Let k be a prime. Then $Q \bmod (k, z^k)$ is $\bar{t}_{4k}$ invariant.*

Proof. As in the proof of Lemma 8.1, adding two copies of (4) for n and $n-2$, we get (5). Now we iterate this procedure to obtain

$$\sum_{i=0}^{k} \binom{k}{i} A_{n-2i} = z^k A_{n-k} + z\,\frac{z^k - 2^k}{z-2}\,A_\infty. \tag{9}$$

Now, when we orient K_n, the twists are antiparallel, and thus K_{n-k} is a knot, even if k is odd (so that $\min\deg A_{n-k} = 0$). From the primality of k, so that modulo k the left hand side collapses to two terms, we get, modulo k and z^k,

$$A_n + A_{n-2k} = \left(z\,\frac{z^k - 2^k}{z-2}\right) \cdot A_\infty.$$

Subtracting two copies of this equality for n and $n-2k$ instead of n gives the assertion. ∎

Remark 10.1. The proof also shows that $Q \bmod (k, z^{k-1})$ is invariant under a t_{4k} move (because always $\min\deg A_{n-k} \geq -1$).

Since in the study of V we came to treat unity roots of order 8 and 10, it turns out useful to consider the criterion for $k = 5$. This criterion has some chance to give partial information as long as the number of cases left over by the unity root evaluations is sufficiently less than the total number of values of Q mod $(5, z^5)$, which is very likely $5^5 = 3125$.

Another criterion for the Kauffman polynomial $F(a, z + 1/z)$ follows again from Przytycki's work (see [Pr, Corollary 1.17, p. 629]). The Kauffman polynomial is a powerful invariant, but, especially when dealing with many and/or high crossing number diagrams, too complex for practical computations. Hence, to make this result more computationally manageable, we set again $a = 1$ and use the Q polynomial. Then from Corollary 1.17(b) of [Pr] it follows that $Q(z + 1/z)$ is invariant under a $\bar{t}_{2k}$ move for kth roots of unity z. However, we need to prove this condition in a slightly sharper form, replacing the order k by $2k$. Our proof is somewhat different from (and less technical than) Przytycki's, since it uses generating series.

Proposition 10.2. *$Q(z + 1/z)$ mod $\frac{z^{2k}-1}{z^{3+(-1)^k}-1}$ is $\bar{t}_{2k}$ invariant, and in particular $Q(z + 1/z)$ mod $\frac{z^{4k}-1}{z^4-1}$ is $\bar{t}_{4k}$ invariant.*

Proof. We use the formula in the proof of Theorem 3.2 of [St8]. We observed there that the formula (4) and Lemma 8.1 imply that the generating series

$$f(z, x) := \sum_{n=0}^{\infty} A_{2n}(z)x^n$$

is of the form

$$f = \frac{P(z, x)}{(1 - x)(1 + (2 - z^2)x + x^2)}$$

for some $P \in \mathbb{Z}[z, x]$. The invariance of $Q(z)$ under a $2k$-move is equivalent to the denominator dividing $x^k - 1$. Thus we need to choose z so that the zeros of $1+(2-z^2)x+x^2$ are distinct kth roots of unity, different from 1. Now if x_0 and x_1 are these zeros, then $x_0x_1 = 1$. Thus $x_{0,1} = e^{\pm 2l\pi i/k}$ for some $0 \le l \le k - 1$. We must assume that $l \ne k/2$ (for even k) and $l \ne 0$, since otherwise $x_0 = x_1$ is a double zero. Then $2 - z^2 = -x_0 - x_1 = 2\cos(2l\pi/k)$, hence

$$z^2 = 2 + 2\cos\left(\frac{2\pi}{k} \cdot l\right) = 4\cos^2\left(\frac{\pi}{k} \cdot l\right),$$

and $z = \pm 2\cos(\pi l/k)$. (Since l can be replaced by $k - l$, the sign freedom is fictitious.)

Thus $Q(z + 1/z)$ is invariant if $z + 1/z = 2\cos(\pi l/k)$, with $1 \le l \le k - 1$ and $l \ne k/2$ for even k, which means $z = e^{\pm l\pi i/k}$ for such l, and these are exactly the zeros of the modulo-polynomials stated above. ∎

In the following we decide to use the second property in Proposition 10.2 for $k = 5$. (One could also take $k = 10$ for the first property.)

Clearly, the (Przytycki type) criterion in Proposition 10.2 is more efficient than the one in Proposition 10.1, already because the number of values of the invariant is infinite. But our first criterion is easier to compute, and at least it is not a consequence of the second one, as is shown by the following

EXAMPLE 10.1. Consider $k = 5$. The knots 11_{367} and 9_1 have Q polynomials that leave the same rest modulo $(z^{20} - 1)/(z^4 - 1)$. But modulo 5 they differ in the z^4-term, so 11_{367} and 9_1 are not $\bar{t}_{20}$ equivalent.

In this example, the difference of $Q(z)$ mod (k, z^k) comes out in the highest coefficient covered (that of z^{k-1}). Surprisingly, this turns out to be the case for *all* other examples I found, that is, Proposition 10.2 was suggested to imply the weaker version of Proposition 10.1 for t_{4k} moves noted in Remark 10.1. Later I indeed deduced this implication rigorously, but the argument (using properties of Bernoulli polynomials) requires some space, and I would rather omit it here.

I tested all prime and composite knots of at most 16 crossings for $k = 3, 5, 7$; for $k = 3$ there were about a million coincidences of $Q(z + 1/z)$ mod $\frac{z^{4k}-1}{z^4-1}$ with different $Q(z)$ mod (k, z^k), for $k = 5$ they were about 3200, and for $k = 7$ only one, so in this range of knots for higher k there are too few coincidences of $Q(z + 1/z)$ mod $\frac{z^{4k}-1}{z^4-1}$ to have an interesting picture.

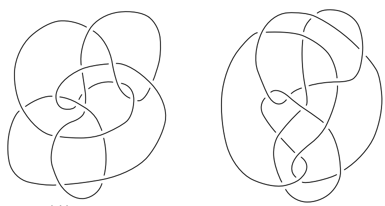
10.2. *Excluding weak genus 2 with the Q polynomial.* The original intention for the Q polynomial criteria was to exclude further knots in the set of 2010 from having $\tilde{g} = 2$. Then I was fairly surprised that the most promising candidates (that is, the knots whose V moduli appeared the least number of times in the series) showed up in (at least one of) the series of 12_{1097} and 13_{4233}. Thus in practice the above criteria have been useful to reduce the number of diagrams in the series to be considered to identify these knots. The identification was done using KnotScape.

First, I considered diagrams in the series of 13_{4233} and 12_{1097} obtained by switching crossings and performing at most one $\bar{t}'_2$ move at each crossing/clasp, that is, with ≤ 4 crossings in each $\sim$-equivalence class. (Resolving clasps gives diagrams in the subseries of 13_{4233} and 12_{1097} in Figure 6.) Then I added all the (other) diagrams in these series of at most 17, resp. 18, crossings. From the set of diagrams thus obtained, I selected diagram candidates for any knot with $\max\deg_m P \leq 4$ by calculating the Jones polynomial, and tracking down coincidences. Finally, on the diagrams with matching polynomials, Thistlethwaite's diagram transformation tool `knotfind` was applied to identify the knot. By this procedure I managed to identify all the $\leq$15-crossing knots with $\max\deg_m P \leq 4$ in genus 2 diagrams except six. We already know four of them—they were given in Figure 9, and the other two are shown in Figure 11.

Thus these two knots deserved closer consideration under the Q polynomial criteria. These criteria proved that the two knots share the status of those in Figure 9. We give some details just for the first knot, the other one is examined in the same way.

EXAMPLE 10.2. Consider 15_{216607} (Figure 11). We have

$$V(15_{216607}) \bmod \frac{t^{10}-1}{t^2-1} = ([-3]\ 0\ -2\ -3\ 1\ -5\ 2\ -4).$$



15_{216607} 15_{217802}

Fig. 11. The two prime knots of at most 15 crossings, for which one can use the Q polynomial to show that the lower bound 2 for $\widetilde{g}$, coming from Morton's inequality, is not sharp. In all remaining (including composite) cases it is sharp (if it is 2), except for the four knots in Figure 9.

It turns out that in the series of 13_{4233} the modulus of V for $k = 5$ appears 28 times. They can be encoded by the *twist vectors*:

```
{2,-1, 1, 1, 1,-2,-1, 0,-1},     { 1, 0,-1, 1, 1, 1, 1,-2,-2},
{1, 1, 1, 2,-1, 1,-1, 2,-1},     { 1,-1, 1, 1, 1,-1,-1, 1,-2},
{1, 1, 1, 0, 1, 1,-2,-2,-1},     { 1,-1, 1,-1, 1, 1,-1, 0,-1},
{1, 1, 1, 0, 1,-1,-2,-2, 1},     { 0, 1, 0, 1,-2, 1, 0,-2, 0},
{1, 1, 1, 0, 0, 0,-2,-2, 0},     { 0, 1,-2, 1, 0,-2, 0, 1, 0},
{1, 1, 1,-1,-1, 1,-1, 2, 2},     { 0, 0, 0, 1, 1, 1, 0,-2,-2},
{1, 1, 1,-1,-1, 1,-1,-2, 1},     { 0,-2, 0, 1, 1,-2, 0, 1, 0},
{1, 1,-1, 2,-2, 1, 1,-2,-1},     {-1, 2, 1, 1, 1, 1,-1, 2,-1},
{1, 1,-1,-1, 1, 1,-2, 1,-1},     {-1,-1, 1, 1, 1, 1, 2, 2,-1},
{1, 1,-1,-1,-2, 1, 1,-1, 1},     {-1,-1, 1, 1, 1, 1, 1,-2,-1},
{1, 1,-1,-1,-2, 1, 1,-2, 2},     {-2, 2, 1, 1, 1,-1,-1,-2, 1},
{1, 1,-2,-1, 2, 1,-1, 0,-1},     {-2, 1, 1, 1, 0, 0, 0,-2, 0},
{1, 1,-2,-2, 0, 0, 0, 1, 0},     {-2,-1, 1, 1, 1,-1, 2,-2, 1},
{1, 0, 1, 1, 1, 1,-1,-2,-2},     {-2,-1, 1, 1, 1,-1, 1,-1, 1}.
```

We explain this notation. First, the crossings are numbered as specified above in the order of the Dowker notation of 13_{4233} given by

$$6\ 12\ 22\ \underline{26}\ 16\ \underline{4}\ 20\ 24\ \underline{8}\ \underline{14}\ 2\ 10\ 18.$$

In this notation one skips an entry of a crossing appearing in a clasp with some crossing (entry) on its left. For example, crossings denoted by '6' and

'26' in the notation form a clasp, so the fourth entry '26' is skipped, and crossing number 4 in the list refers to the crossing represented by the fifth integer '16' in the above Dowker notation. To facilitate this renumbering, the integers of the crossings to be skipped are underlined. An entry x_i at position i $(1 \le i \le 9)$ in some list denotes the switching and number of $\bar{t}'_2$ moves applied to the crossing at number i. There are two possibilities.

If the crossing numbered i is a single element in its $\sim$-equivalence class, then $x_i = -1$ means a switched crossing in the alternating diagram, $x_i = 0$ the crossing in the alternating diagram as it is, and for $x_i \ge 1$ (resp. $x_i < -1$) the crossing in the alternating diagram (resp. the switched one) with x_i (resp. $-1 - x_i$) $\bar{t}'_2$ moves applied to it.

If the crossing i builds (up to flype) a reverse clasp with another crossing (that is, there are two elements in its $\sim$-equivalence class), '$x_i > 0$' means the clasp unswitched with $\bar{t}'_2$ moves applied $x_i - 1$ times, '$x_i = 0$' means the clasp resolved, and '$x_i < 0$' means the clasp switched with $-1 - x_i$ twists applied.

Note that all the values of x_i have to be considered, and hence are meant, only modulo 5 ([1]).

Similarly for the other main series (clearly only such have to be considered) the modulus of V appears 22 times for the series of 12_{1097} and once for 10_{97}.

Checking the 51 diagrams resulting from these vectors modulo 5, we obtain the following values for $Q(z + 1/z)$ mod $\frac{z^{10}-1}{z^2-1}$:

$$([-13]\ 0\ 0\ 30\ 28\ 42\ 28\ 30), \qquad ([-23]\ 0\ 0\ 40\ 38\ 62\ 38\ 40),$$
$$([-25]\ 0\ 0\ 54\ 54\ 84\ 54\ 54).$$

But

$$Q(15_{216607}) \bmod \frac{z^{10}-1}{z^2-1} = ([-17]\ 0\ 0\ 38\ 40\ 56\ 40\ 38)$$

does not occur among them. Thus the Q polynomial criterion in Proposition 10.2 excludes all remaining possibilities, and so $\tilde{g}(15_{216607}) > 2$. (In Remark 9.1 we mentioned that $g(15_{216607}) = 2$.)

REMARK 10.2. It is striking that if we take, as above, the rest $Q(z+1/z)$ mod $\frac{z^{10}-1}{z^2-1}$ to be an honest polynomial P in z of degree ≤ 7, then always $[P]_1 = [P]_2 = [P]_4 - [P]_6 = [P]_3 - [P]_7 = 0$ (with $[P]_i = [P]_{z^i}$). This is in fact true whatever $Q \in \mathbb{Z}[z]$ may be, because the subalgebra of $\mathbb{Z}[z, 1/z]/\frac{z^{10}-1}{z^2-1}$

([1]) To avoid confusion, let us remark that in a previous(ly cited) version of the paper a *different* convention for the twist vectors was used. There, for every crossing an entry $x_i = 0$ meant the crossing in the alternating diagram switched, $x_i = 1$ the crossing in the alternating diagram as it is, and $x_i \ge 2$ (resp. $x_i < 0$) the crossing in the alternating diagram (resp. the switched one) with $x_i - 1$ (resp. $-x_i$) $\bar{t}'_2$ moves applied to it. Thus if a crossing builds a (reversely oriented) clasp with another one, as before '1' means the clasp as it is, '0' means the clasp resolved, and '-1' the clasp switched.

generated by $z + 1/z = -z^3 - z^5 - z^7$ is a $\mathbb{Z}$-module with basis $1, z^5, z^4 + z^6$ and $z^3 + z^7$, and hence is a rank 4 subalgebra of an algebra of rank 8 over $\mathbb{Z}$. Therefore, Proposition 10.2 becomes less efficient whenever this subalgebra (considered also with 10 replaced by other values of n) is small.

For n divisible by 5 an additional restriction comes from the Jones–Rong result [J4, Rn], showing that (depending on the parity of $\dim_{\mathbb{Z}_5} H_1(D_K, \mathbb{Z}_5)$) $Q(z + 1/z) \bmod \frac{z^5-1}{z-1}$ is always of the form $\pm 5^k$ or $\pm 5^k(2z^3 + 2z^2 + 1)$ for some natural number k.

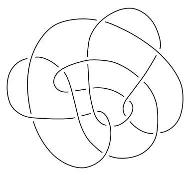
REMARK 10.3. For both knots in Figure 11 not only the modulus of the Jones polynomial, but the whole polynomial itself, and even the HOMFLY polynomial, are realized by weak genus 2 knots (14_{27627}, 14_{34335} and 15_{123857} for 15_{217802}, and 14_{35025} for 15_{216607}), so that the HOMFLY polynomial cannot give complete information on the weak genus 2 property.

We find in summary that the six knots in Figures 9 and 11 are indeed the only examples up to 15 crossings where Morton's weak genus estimate $\widetilde{g} \geq 2$ is not exact. This reveals Morton's inequality as extremely effective, at least for $\widetilde{g} = 2$, even at that "high" (in comparison to Rolfsen's classical tables) crossing numbers.

Besides the ones given above, this quest produced some further interesting examples with no minimal crossing number diagram of weak genus 2. In contrast, using a similar argument to that in the proof of Theorem 8.1 for the maximal degree of the Q polynomial on the non-alternating pretzel knots, one can show that for $\widetilde{g} = 1$ any (weak genus 1) knot has a genus 1 minimal diagram.

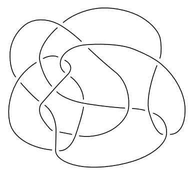
10.3. *16-crossing knots.* After the verification of 15-crossing knots, the 16-crossing knot tables were released by Thistlethwaite. A check therein shows that there are 2249 non-alternating 16-crossing prime knots with $\max\deg_m P \leq 4$. (There were no knots with $\max\deg_m P \leq 2$.) Most of these knots again have weak genus 2. There are 19 knots that can be excluded using $V \bmod \frac{t^{10}-1}{t^2-1}$, and three using additionally $Q(z+1/z) \bmod \frac{z^{10}-1}{z^2-1}$. As a counterpart to the knots in Figure 10, there is one knot, $16_{1265905}$, whose V modulus occurs only in (main) series of even crossing number, so this knot can have no reduced genus 2 diagram of odd crossing number.

As another novelty, there is one knot, 16_{686716}, which cannot be decided upon. It has the same V and Q moduli (in fact the same V and P, but not Q polynomial) as two weak genus 2 knots, 16_{619178} and 16_{733071}. Thus our criteria cannot exclude weak genus 2. (Apparently Przytycki's Kauffman polynomial criteria do not apply either.) But, after testing all (still potentially relevant) diagrams in the series of 12_{1097} and 13_{4233} of ≤ 49 crossings corresponding to twist vectors with all $|x_i| \leq 9$, I was unable to find a diagram of this knot.



$16_{1265905}$

Fig. 12. A weak genus 2 knot with no reduced weak genus 2 diagram of odd crossing number.



16_{686716}

Fig. 13. Does this knot have weak genus 2?

REMARK 10.4. It is interesting to remark that unusually many of the above examples are slice, *inter alia* 15_{184486}, 15_{184487} and 15_{221824} from Figures 9 and 10, and the knots in Figures 11, 12 and 13. It is quite unclear so far what relation (if any) exists between sliceness and exceptional behavior regarding Morton's inequality.

10.4. *Unknotting numbers and the 3-move conjecture.* Among the family of k-moves defined above, 3-moves are of particular interest because of their relation to unknotting numbers. An important conjecture of Nakanishi [Na] is

CONJECTURE 10.1 (Nakanishi's 3-move conjecture). *Any link is* 3-*unlinked*, *that is*, 3-*equivalent to some* (*unique*) *unlink.*

By trivial arguments, this conjecture is true for rational and arborescent links, and by non-trivial work of Coxeter it has been made checkable for closures of braids of at most five strands, as he showed in [Cx] that $B_n/\langle\sigma_i^3\rangle$ is finite for $n \le 5$, so proving the conjecture reduces to verifying (a representative of) a finite number of classes. Qi Chen in his thesis settled all of them except (the class of) the 5-braid $(\sigma_1\sigma_2\sigma_3\sigma_4)^{10}$.

As for our context, we get a finite case simplification for the conjecture for knots of any given weak genus. The weak genus 1 case is arborescent and hence trivial, and we can now do by hand the proof of the 3-move conjecture for weak genus 2 knots.

PROPOSITION 10.3. *Any weak genus* 2 *knot is* 3-*unlinked.*

Proof. Applying 3-moves near the $\bar{t}_2'$ twisted crossings in the 24 generators, we can simplify any genus 2 knot diagram to one of the generators with possibly a crossing eliminated or switched, and a clasp resolved or reduced to one crossing. We obtain this way a link diagram of at most nine crossings. These links are easy to check directly, but this has also been done previously by Qi Chen [Ch]. ∎

REMARK 10.5. A few years after our work was originally done, Dąbkowski and Przytycki disproved the 3-move conjecture [DP].

10.5. *On the 4-move conjecture.* Similar arguments to those for the 3-move conjecture allow us to give a proof of Przytycki's 4-move conjecture for weak genus 2 knots.

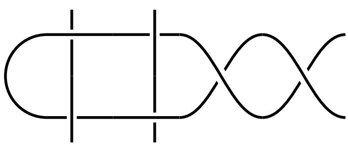
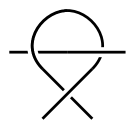
CONJECTURE 10.2 (Przytycki [Pr]). *Any knot is* 4*-equivalent to the unknot.*

Thus we have

PROPOSITION 10.4. *Any weak genus* 2 *knot is* 4*-equivalent to the unknot.*

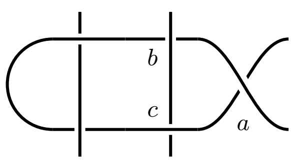
Proof. By 4-moves we can simplify any genus 2 knot diagram to one of the generators of the 24 series with possibly crossings switched. As the conjecture is verified by Nakanishi for knots of up to 10 crossings, we need to consider just the diagrams of the six last generators (with possibly crossings switched). In their diagrams we still have the freedom to change clasps.

The 11-crossing generators and 13_{4233} have one of the tangles

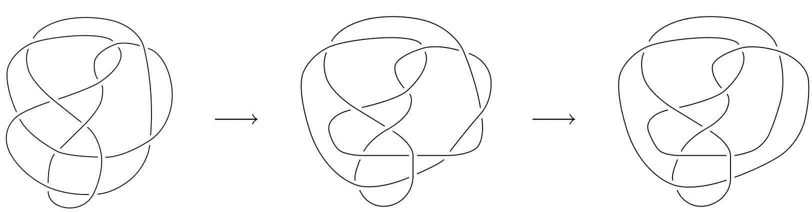
$T_1 =$  and $T_2 =$ 

It is easily observed that, whichever way the non-clasp crossings are changed, the clasps can be adjusted so as to simplify the diagram by eliminating one crossing (and then it still has genus ≤ 2). Then for the 11-crossing generators we are done, while for 13_{4233} we work inductively on the crossing number.

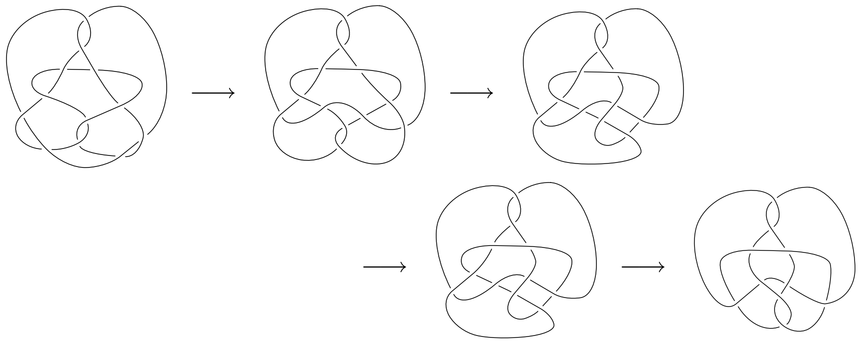
12_{1097} has the tangle



and the same argument as for T_1 applies, unless none (or all) of crossings a, b and c are switched. In this case, by switching the lower clasp in the diagram of 12_{1097}, one simplifies the diagram by two crossings independently of how the remaining crossings are switched:



Finally, the procedure for 12_{1202} (and it clasp-switched variants) is shown below:



■

11. An asymptotical estimate for the Seifert algorithm. The Seifert algorithm gives us the possibility to construct a lot of Seifert surfaces for a knot, and although there is not always a minimal one, we may hope that these cases are rather exceptional. Theorem 3.1 of [St4] together with a property of the Alexander polynomial gives us the tools to confirm this in a way we make precise below.

THEOREM 11.1. *Fix* $g \in \mathbb{N}_+$. *Then*

$$\frac{\#\{D : \max\deg \Delta(D) = g([D]) = g(D) = g,\, c(D) \le n\}}{\#\{D : g(D) = g,\, c(D) \le n\}} \xrightarrow[n\to\infty]{} 1, \tag{10}$$

where D *is a knot diagram*, $g(D)$ *denotes its genus*, *and* $[D]$ *the knot it represents.*

This theorem says that for an arbitrary genus g diagram with many crossings, the probability for the canonical Seifert surface to be of minimal genus is very high. For the proof we use the Alexander polynomial.

REMARK 11.1. There is a purely topological result due to Gabai, which can also be applied (see corollary 2.4 of [Ga2]), as a $\bar{t}_2'$ move corresponds to change of the Dehn filling of a torus in the complement of the Seifert surface. (Gabai needs the manifold obtained from the knot complement by cutting out this torus to be Haken, but this is true for any 3-manifold whose boundary is a collection of tori.) This leads to a slightly weaker version of the theorem, in which the property of the degree of the Alexander polynomial does not appear.

The proof of our theorem bases on the following lemma.

LEMMA 11.1. *Let* S *be a subset of* $\mathbb{Z}^n$ *with the following property*: *if* $(x_1, \dots, x_{k-1}, a, x_{k+1}, \dots, x_n) \in S$ *and* $(x_1, \dots, x_{k-1}, b, x_{k+1}, \dots, x_n) \in S$ *for some* $a \ne b$, *then* $(x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_n) \in S$ *for all* $x_k \in \mathbb{Z}$.

Then

$$\forall n\ \exists \varepsilon_n, k_n\ \forall k \geq k_n : \quad \frac{|S \cap [-k,k]^n|}{(2k+1)^n} \geq \varepsilon_n \ \Rightarrow\ S \supset [-k,k]^n. \tag{11}$$

Proof. Fix some parameter $p \in \mathbb{N}$ and use induction on n. For $n = 1$ the claim is evident: set $\varepsilon_1 = 1/2p$ and $k_1 = p$. Assume now the assertion holds for $n-1$. Let $S \subset \mathbb{Z}^n$ and set

$$n_{i,k} := \#(S \cap ([-k,k]^{n-1} \times \{i\})), \qquad |i| \leq k.$$

Set $\varepsilon_n := 1 - (1-\varepsilon_{n-1})^2$. If there is k_0' such that for all $k \geq k_0'$ there is at most one i_0 such that

$$\frac{n_{i_0,k}}{(2k+1)^{n-1}} \geq \varepsilon_{n-1},$$

then for each such k,

$$\sum_{i=-k}^{k} \frac{n_{i,k}}{(2k+1)^n} < \frac{1}{2k+1} + \varepsilon_{n-1} \xrightarrow[k\to\infty]{} \varepsilon_{n-1} < \varepsilon_n.$$

Therefore, there exists k_0'' such that for all $k \geq k_0''$,

$$\frac{|S \cap [-k,k]^n|}{(2k+1)^n} < \varepsilon_n,$$

and, choosing k_n large enough, there is nothing to prove, as the premise of (11) does not hold. Therefore, assume that for every k_0' there exist $k \geq k_0'$ and $i_0 \neq i_1$ such that

$$\frac{n_{i_0,k}}{(2k+1)^{n-1}} \geq \varepsilon_{n-1}, \qquad \frac{n_{i_1,k}}{(2k+1)^{n-1}} \geq \varepsilon_{n-1}.$$

Set $k_n := k_{n-1}$. Then for $k \geq k_n$ there exists $k' \geq k$ for which $S \supset [-k',k']^{n-1} \times \{i_0, i_1\}$. Then $S \supset [-k',k']^n \supset [-k,k]^n$. ∎

Note that yet we have the freedom to vary the parameter p. We need this now.

LEMMA 11.2. *Lemma* 11.1 *can be modified by replacing "$\forall n\ \exists \varepsilon_n, k_n$" by "$\forall n\ \forall \varepsilon\ \exists k_{n,\varepsilon}$".*

Proof. Let $p \to \infty$ in the proof of Lemma 11.1. ∎

Proof of Theorem 11.1. Clearly (even taking care of possible flypes) it suffices to prove the assertion for the $\bar{t}_2'$ twist sequence of one fixed diagram D, which we parametrize using the twist vectors $(x_1,\dots,x_n)$ introduced in §10.2 by $\{D(x_1,\dots,x_n)\}_{x_i=-\infty}^{\infty}$, so that a positive parameter corresponds to a $\bar{t}_2'$ twisted positive crossing.

Then we apply the previous lemma to

$$S := \{(x_1,\dots,x_n) : \tilde{g}(D(x_1,\dots,x_n)) > \max\deg \Delta(D(x_1,\dots,x_n))\}.$$

The property needed for S in the preceding lemma is established by the simple fact that the Alexander polynomials of knots in a 1-parameter $\bar{t}_2'$ twist sequence form an arithmetic progression.

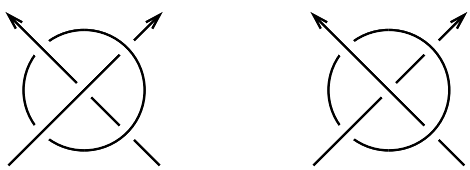
Denoting by $c_{g,n}$ the fraction on the left of (10), assume $\liminf_{n\to\infty} c_{g,n} < 1$. It is equivalent to use the k-ball around 0 in the $\|\cdot\|_1$ or $\|\cdot\|_\infty$ norm, so this means to assume

$$\exists \varepsilon > 0\ \forall k_0\ \exists k \geq k_0 : \quad |S \cap [-k,k]^n| > \varepsilon(2k+1)^n.$$

Then by Lemma 11.2 we have $S \supset [-k,k]^n$ for $k \geq k_{n,\varepsilon}$, hence $S = \mathbb{Z}^n$. But this is clearly impossible, since for example the property of positive diagrams to have minimal genus canonical Seifert surfaces implies that $S \cap \mathbb{N}_+^n = \emptyset$. Hence $\liminf_{n\to\infty} c_{g,n} = 1$. Therefore $\lim_{n\to\infty} c_{g,n}$ exists, and it is 1. ∎

12. Estimates and applications of the hyperbolic volume. We conclude the discussion of the weak genus in general, and weak genus 2 in particular, by some remarks concerning the hyperbolic volume. Surprisingly, it turned out that with regard to the hyperbolic volume, the setting of [St4] had been previously considered in a preprint of Brittenham [Br2], of which I learned only with great delay. Parts of the material in this section (for example, the reference to [Ad2]) have been completed using Brittenham's work.

Definition 12.1. For an alternating knot K define a link $\widetilde{K}$ by adding a circle with linking number $\mathrm{lk} = 0$ (i.e. disjoint from the canonical Seifert surface) around a crossing in each $\sim$-equivalence class of an alternating diagram of K.

(12) 

(The orientation of the circles is not important.)

In this language one can obtain all weak genus g knots by $1/n_i$-Dehn surgery along the unknotted components of $\widetilde{K}$ for the genus g generators K. (In fact, the main generators suffice, and the cases of composite generators can be discarded.)

In this situation we can apply a result of Thurston (see [NZ]). To state it, here and below $\mathrm{vol}(K)$ denotes the hyperbolic volume of (the complement of) K, or 0 if K is not hyperbolic. $K(n_1,\dots,n_l)$ denotes, as in [St6], the knot in the series of K with twist vector $(n_1,\dots,n_l)$, as explained in §10.2.

THEOREM 12.1 (Thurston). *If* $\operatorname{vol}(\widetilde{K})>0$, *then for all vectors* $(n_1,\dots,n_l) \in \mathbb{Z}^l$,

$$\operatorname{vol}(K(n_1,\dots,n_l)) < \operatorname{vol}(\widetilde{K}),$$

and

$$\operatorname{vol}(K(n_1,\dots,n_l)) \to \operatorname{vol}(\widetilde{K}) \qquad \text{as } \min_{i=1}^{l} |n_i| \to \infty.$$

As a consequence, we obtain the following theorem.

THEOREM 12.2. *Let*

$$S_g := \{\operatorname{vol}(\widetilde{K}) : K \text{ main generator of genus } g\}.$$

Then

$$\sup\{\operatorname{vol}(K) : \tilde{g}(K) = g\} = \max S_g.$$

Proof. The $\widetilde{K}$ are augmented alternating links in the sense of Adams [Ad2], and hence by his result are hyperbolic, if K is a prime alternating knot different from a torus knot. Applying Thurston's result, it remains to prove that the alternating torus knot is never a main generator. This is an easy exercise. ■

This theorem shows in particular that the hyperbolic volume of knots of bounded weak genus is bounded, with an explicitly computable exact upper estimate.

In particular, we obtain from Theorem 12.2 by explicit calculation:

COROLLARY 12.1.

$$\sup\{\operatorname{vol}(K) : \tilde{g}(K) = 1\} = \operatorname{vol}(\widetilde{3_1}) \approx 14.6554495068355,$$
$$\sup\{\operatorname{vol}(K) : \tilde{g}(K) = 2\} = \operatorname{vol}(\widetilde{13_{4233}}) \approx 58.6217980273420. \ ■$$

The (approximate) volumes of $\widetilde{K}$ for the main generating knots K of genus 2 are as follows:

K	$\operatorname{vol}(\widetilde{K})$
6_3	36.6386237671
9_{41}	38.7476335870
10_{97}	43.9663485205
11_{148}	43.9663485205
12_{1097}	58.6217980273
12_{1202}	38.7476335870
13_{4233}	58.6217980273

There is a further application of the hyperbolic volume.

PROPOSITION 12.1. *If* $\operatorname{vol}(\widetilde{K}) > \operatorname{vol}(\widetilde{K'})$ *for two generators* K *and* K', *then a generic alternating knot in the series of* K *has no diagram in the series of* K'. ■

To make precise what 'generic' means we make a definition:

DEFINITION 12.2. A subclass $\mathcal{B} \subset \mathcal{C}$ in a class $\mathcal{C}$ of links is called *asymptotically dense* or *generic* if

$$\lim_{n\to\infty} \frac{|\{K \in \mathcal{B} : c(K) = n\}|}{|\{K \in \mathcal{C} : c(K) = n\}|} = 1.$$

For example, in [Th2] Thistlethwaite showed that the non-alternating links are generic in the class of all links. Similarly, a result of [St9] is that any generic subclass of the class of alternating links contains mutants.

The proof of Proposition 12.1 is similar to the arguments in §11, but simpler, and is hence omitted. (Again avoiding K' to be a torus knot is easy.)

EXAMPLE 12.1. We have

$$\mathrm{vol}(\widetilde{9}_{38}) \approx 47.2069898171 > \mathrm{vol}(\widetilde{10}_{97}) \approx 43.9663485205,$$

so that a generic alternating knot in the series of 9_{38} will not have a diagram in the series of 10_{97}. (Note that both series have seven $\sim$-equivalence classes and thus the number of diagrams in them grows comparably.)

The fact that $\mathrm{vol}(\widetilde{13}_{4233})$ and $\mathrm{vol}(\widetilde{12}_{1097})$ are equal is unfortunate, as otherwise we would be able to conclude that a generic genus 2 alternating knot of one of the crossing number parities has no genus 2 diagrams of the other crossing number parity (as we did for specific examples before using the values of the Jones polynomial at roots of unity). Also, this value is much higher than the volume of any non-alternating $\leq$16-crossing knot. (The maximal volume of such a knot is about 32.9, and the maximal volume among those knots with $\max\deg_m P \leq 4$ is about 22.9.) Thus the volume does not seem to have much practical significance as an obstruction to $\widetilde{g} = 2$. On the other hand, we can use the fact that $\mathrm{vol}(\widetilde{13}_{4233}) = \mathrm{vol}(\widetilde{12}_{1097})$ is higher than $\mathrm{vol}(\widetilde{K})$ for the other main generators K. From this, and Proposition 12.1, we obtain

COROLLARY 12.2. *A generic alternating genus* 2 *knot has no non-special genus* 2 *diagrams* (*i.e. such diagrams with a separating Seifert circle*). ∎

This is not true for weak genus 1, because of the alternating knots of even crossing number. For odd crossing number genus 1 alternating knots it is, in contrast, trivial. However, being such a narrow class, genus 1 diagrams are not interesting anyway.

To estimate $\max S_g$, Brittenham uses a remark of W. Thurston that any link L satisfies $\mathrm{vol}(L) \leq 4V_0 c(L)$, with V_0 being the volume of the ideal tetrahedron. (In [GL, §1.5], Garoufalidis and Le quote private communication with I. Agol and D. Thurston, stating $\mathrm{vol}(L) \leq v_8(c(L)-2)$ for a knot L, where $v_8 \approx 3.66386$ is the volume of the ideal octahedron.) Then Brittenham studies

$$C_g := \{c(\widetilde{K}') : K \text{ main generator of genus } g\},$$

where $\widetilde{K}'$ is obtained from $\widetilde{K}$ by resolving in K clasps of $\sim$-equivalence classes with two crossings. (This move preserves the link complement.) Brittenham shows that $\max C_g \leq 30g - 3$.

We conclude this section by giving an estimate for $\max C_g$, best possible for $g \geq 6$.

PROPOSITION 12.2. $\max\, C_g \leq 30g - 15$, *and this inequality is sharp for* $g \geq 6$.

In particular, we have a slight improvement of Brittenham's volume estimate:

COROLLARY 12.3. $\sup\{\mathrm{vol}(K) : \widetilde{g}(K) = g\} \leq (120g - 60)V_0$. ∎

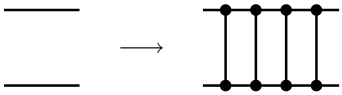
However, we also know now that a significant further improvement of Brittenham's volume estimate is possible only by studying the volume of the $\widetilde{K}$ directly, and not via their crossing number.

Proof of Proposition 12.2. We know from [STV] that $d_g \leq 6g-3$, and in each $\sim$-equivalence class we need four crossings for the trivial loop, and at most one crossing for the generating knot. (Recall that d_g are the numbers introduced at the end of §2.) If some $\sim$-equivalence class of the generating diagram has two $\sim$-equivalent crossings, their clasp can be resolved, since this preserves the link complement. Thus each $\sim$-equivalence class contributes at most five crossings to $c(\widetilde{K}')$, showing the estimate claimed.

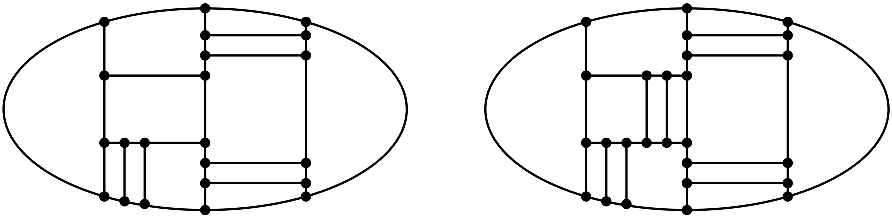
To show that the estimate is sharp, we need to construct a prime alternating knot $K = K_g$ of genus $g \geq 6$ with $6g - 3$ $\sim$-equivalence classes, all consisting of a single crossing.

Once this is done, it is easy to show that $c(\widetilde{K}') = c(\widetilde{K}) = 30g - 15$. Let $L_1, \dots, L_n$ be the trivial components of $\widetilde{K}$. Then $K \sqcup L_i$ is non-split for any i, since $1/n_i$-surgery on L_i changes K, as it may give an alternating knot of higher crossing number. Also, as this knot is prime (by [Me] and the primality of the diagram), L_i cannot be enclosed in a sphere intersecting K in an unknotted arc (otherwise the result from K after $1/n_i$-surgery on L_i will always have K as prime factor). Thus L_i and K have at least four mixed crossings in any diagram of $\widetilde{K}$. Since K appears in a reduced alternating diagram in the diagram of $\widetilde{K}$ obtained by the replacements (12), it is also of minimal crossing number.

We give the K_g in terms of their Seifert graphs; since all K_g are special alternating, these graphs determine uniquely a special alternating diagram of K_g (see e.g. [Cr1]; these graphs are trivalent and bipartite). We include the graphs only for $g = 6$ and $g = 7$. (The genus can be determined easily, since the number of regions of the graph is $2g+1$.) Given a graph of a knot of genus g, one can obtain a graph of a knot of genus $g+2$ by the replacement



performed so that the number of edges in each face remains even.



■

REMARK 12.1. Brittenham uses his proof that weak genus bounds the volume to show in [Br1] that there are (hyperbolic) knots of genus 1 and arbitrarily large weak genus. Because of the use of Thurston's theorem, however, for a given lower weak genus bound, his construction cannot concretely identify the example. Such examples, although not hyperbolic, have been previously given in [Mr, St10].

REMARK 12.2. While K_1 is the trefoil, one can check that for $1 < g \leq 5$ the knots K_g do not exist. This follows for $g = 1$ from [St4], for $g = 2$ from our discussion, and for $g = 3$ from the calculation given later in §13.2. For $g = 4, 5$, one can establish this in the following way. It follows from the results of [MS] and [SV] that the Seifert graphs of the alternating diagrams of K_g are exactly the planar, 3-connected, bipartite, 3-valent graphs with $4g - 2$ vertices and an odd number of spanning trees. A list of candidates for such graphs was generated and then examined with MATHEMATICA. It showed that for $1 < g \leq 5$ no such graphs exist.

13. Genus 3

13.1. *The homogeneity of* 10_{151}*,* 10_{158} *and* 10_{160}. After having some success with $\widetilde{g} = 2$, I was encouraged to face the combinatorial explosion and to try to obtain at least some partial results about $\widetilde{g} = 3$. One motivation for this attempt were the three undecided genus 3 knots in [Cr1, appendix]. They can now be settled, and thus, together with Corollary 4.1, Cromwell's table completed.

PROPOSITION 13.1. *The knots* 10_{151}, 10_{158} *and* 10_{160} *are non-homogeneous.*

Proof. These knots all have monic Alexander polynomial, and hence a homogeneous diagram must be a genus 3 diagram of at most 12 crossings [Cr1, Corollary 5.1] with no $\bar{t}'_2$ move applied (see proof of [Cr1, Theorem 4]). As crossing changes commute with flypes, deciding about homogeneity reduces to looking for homogeneous diagrams obtained by flypes and crossing changes from a $\bar{t}'_2$ irreducible alternating diagram of between 10 and 12 crossings. We can exclude special alternating series generators, as homogeneous diagrams therein are alternating (and positive). Since the leading coefficient of Δ is multiplicative under Murasugi sum, and invariant up to sign under mirroring, the monicness of the Alexander polynomial is preserved by passing from the homogeneous to the alternating diagram. Therefore, it suffices to consider only (alternating) generating knots, whose Alexander polynomial is itself monic. There are 37 such knots.

Unfortunately, (non-)homogeneity of a diagram, unlike alternation and positivity, is a condition not necessarily preserved by flypes. Thus we must apply flypes on the 37 generators, obtaining 275 (alternating) generating diagrams.

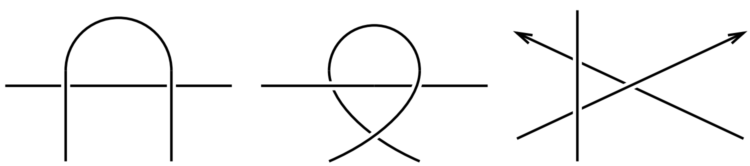


Fig. 14. Fragments to exclude, together with their obverses, in a homogeneity test. Unoriented lines may have both orientations. The first and third fragments above make the diagram non-homogeneous even after flypes. The second one may or may not do so (depending on the orientation) but if the diagram is homogeneous, then this property is not spoiled by reducing the fragment to a clasp (so there is a simpler homogeneous diagram).

We must now consider the diagrams obtained from these 275 by crossing changes, and then test homogeneity. However, it is useful to make a preselection. There are several simple fragments in a diagram, which either render it non-homogeneous, or reveal a simpler homogeneous diagram. See Figure 14. Thus it suffices to consider diagrams without such fragments. More generally than (excluding) the first fragment, if $p \sim q$ or $p \underset{*}{\sim} q$, then p and q must have equal sign. We also apply Kidwell's inequality (see Theorem 8.2) to discard diagrams with long bridges (we have $\max\deg Q = 8$ for all three knots). There remain 1430 diagrams to be considered.

Homogeneity test on these 1430 diagrams gives 430 homogeneous ones. It is easy to check that none of them matches the Alexander polynomial of any of the three knots we seek, and so we are done. ■

13.2. *The complete classification.* The above three knots were a motivation to find the $\bar{t}_2'$ irreducible alternating genus 3 knots at least up to 12 crossings. However, a complete classification of the $\bar{t}_2'$ irreducible genus 3 alternating knots is considerably more difficult.

Theorem 3.1 of [St4] shows that at least at $c_3 \leq 8c_2 + 6 = 110$ crossings the series will terminate. The situation becomes then more optimistic, though. If one repeats the discussion at the end of §2 for a $\bar{t}_2'$ irreducible alternating genus 3 diagram, this leads to expect c_3 to be around 23. Then we found in [STV] that it is indeed equal to 23. The method there used the list of maximal Wicks forms compiled as described in [BV]. This method becomes increasingly efficient when the crossing number grows beyond 15. After some optimization, I was able to process with it also the crossing numbers below 23, finally reaching 17 crossings. For fewer crossings, one can select generators directly from the alternating knot tables. (I also processed 16 crossings by both methods to check that the results are consistent.) The number of generating knots is shown in Table 5. (The list of knots is available electronically [St1].) In particular $d_3 = 15$. These data show that there are a huge number of generators, which render discussions by hand, or with moderately reasonable electronic calculation, as for $\widetilde{g} = 2$, practically impossible in most cases.

Table 5. The number of $\bar{t}_2'$ irreducible prime genus 3 alternating knots tabulated by crossing number c and number of $\sim$-equivalence classes ($\# \sim$).

$\#\sim$ \ c	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total
6						4												4
7	1	2	5	8	11			9										36
8		6	10	21	22	30	44			13								146
9			4	16	42	72	64	55	68			7						328
10				2	15	51	104	159	119	52	45			2				549
11					1	10	49	120	194	211	130	20	14					749
12						1	5	32	112	220	229	154	75	2	1			831
13							1	2	17	63	170	252	178	48	18			749
14									1	4	22	63	132	163	82			467
15											2	3	12	25	47	46	23	158
total	1	8	19	47	91	168	267	377	511	563	598	499	411	240	148	46	23	4017

Nonetheless, one can obtain some interesting information already from the data in the table, for example:

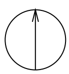
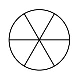
Proposition 13.2. *The number of alternating genus* 3 *knots of odd and even crossing number grows in the ratio* 42/37. ■

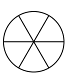
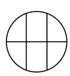
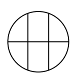
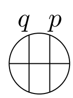
This is certainly not a fact one would expect from considering the genus 2 case.

13.3. *The achiral alternating knots.* Achirality is a relatively restrictive condition on a knot, and so I tried, just as for genus 2, to consider the achiral alternating knots of genus 3, hoping to reduce significantly the number of cases and to obtain an interesting collection of knots. As we saw, in order for a knot to generate a series with an achiral alternating knot, it must be in particular of even crossing number, zero signature and even number of $\sim$-equivalence classes of crossings. (In fact, among these classes there must be equally many of both signs for the same number 1 or 2 of elements.) From the generators compiled above, 68 passed these tests.

To deal with these 68 cases more conveniently, it is worth mentioning a further simple criterion which can often be useful. It uses Gauß sums (see for the definitions [F, FS, St2, PV]).

Proposition 13.3. *Let K be the alternating generator of a series containing an alternating achiral knot K'. Then the following Gauß sums vanish on a(ny) reduced alternating diagram of K*:

 w_p (*writhe*)  $\frac{w_p+w_q}{2}$  $w_pw_qw_r$

 $w_p+w_q+w_r$  $w_pw_qw_r$  $w_p+w_q+w_r$ 

$\frac{w_p+w_q}{2}$

Proof. The intersection graph of the Gauß diagram (IGGD) of K' has an automorphism taking each vertex to one with the opposite sign. But building K out of K' means reducing the number of elements in a $\sim$-equivalence class in the IGGD to 1 or 2 according to their parity, and hence the above automorphism carries over to (the IGGD of) K. But the above Gauß sums are clearly invariants of the intersection graph (and not only of the Gauß diagram). They change sign under mirroring the knot diagram, and hence the result follows. ■

The proof suggests that more is likely.

Conjecture 13.1. *If K is the alternating generator of a series containing an alternating achiral knot*, *then*

(i) *K is achiral*, *or*
(ii) *K is an iterated mutant of its obverse*, *or*
(iii) *K has self-conjugate HOMFLY and/or Kauffman polynomial.*

Clearly (i) and (ii) are stronger than our result. But note that (iii) is not. Remarkably some of the above simple Gauß sums can sometimes do better in distinguishing an alternating knot from its obverse than the HOMFLY and/or Kauffman polynomial, as one can see from the examples 10_{48} and 10_{71}.

It is a good exercise to apply the above criteria by hand in some simple examples. However, for many and/or more complicated diagrams it is easier and safer to use computer.

Applying Proposition 13.3 on the 68 knots, only the 30 achiral (*without* regarding orientation) knots remained. Up to 14 crossings the list is 8_9, 8_{17}, 8_{18}, 10_{43}, 10_{45}, 10_{81}, 10_{88}, 10_{115}, 12_{125}, 12_{273}, 12_{477}, 12_{510}, 12_{960}, 12_{1124}, 12_{1251}, 14_{1202}, 14_{5678}, 14_{15366}, 14_{16078}, 14_{16857} and 14_{17247}. There are six knots of 16 crossings, two of 18 and one of 20 crossings.

Again one can study their series in more detail, as we did for $\widetilde{g} = 2$. For example, we have

PROPOSITION 13.4. *The fibered achiral alternating genus* 3 *knots are*: 8_9, 8_{17}, 8_{18}, 10_{43}, 10_{45}, 10_{81}, 10_{88}, 10_{115}, 12_{125}, 12_{477} *and* 12_{1124}. ■

Since the maximal number of $\sim$-equivalence classes of these 30 knots is 12 (16_{277679}, 16_{309640} and the two 18 crossing knots have that many), we have

PROPOSITION 13.5. *The number of prime achiral alternating genus* 3 *knots of* n *crossings is* $O^{\asymp}(n^5)$. ■

14. Questions

QUESTION 14.1. Are there any composite (other than the obvious ones) or satellite knots of $\widetilde{g} = 2$?

The lack of "exotic" composite $\widetilde{g} = 2$ knots is suggested by a conjecture of Cromwell:

CONJECTURE 14.1 (Cromwell [Cr2]). *If* D *is a diagram of a composite knot* $K = K_1 \# K_2$ *and* $g(D) = \widetilde{g}(K)$, *then* D *is composite.*

The conjecture is true by Cromwell's work if D is a diagram of a closed positive braid and by Menasco's work [Me] if D is alternating. However, the conjecture in general turns out to be false, as is shown by the example of Figure 15, discovered in the course of the work previously described here.

We can pose, however, a different problem:

QUESTION 14.2. Does any knot have only finitely many reduced diagrams of minimal (weak) genus? (Of course, we exclude resolved clasps in a $\sim$-equivalence class.)

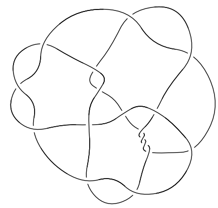
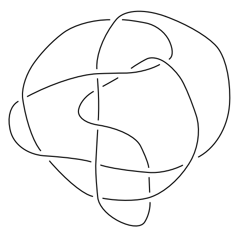


Fig. 15. A counterexample to a conjecture of Cromwell: a prime genus 2 diagram of the knot $5_2 \,\#\, !5_2$

It is an easy observation (similar to the proof of Proposition 10.4) that there are infinitely many slice knots of $\widetilde{g} = 2$. (See also Remark 10.4.) Take 13_{4233}. Then switching two of the clasps we obtain a knot bounding a ribbon disc with two singularities, and can change by twists the half-twist crossings:



Question 14.3. Can one decide more exactly which weak genus 2 knots are slice?

Finally, we point out two general problems:

Question 14.4. Is $\widetilde{g}$ always additive under connected sum?

In this case the combinatorial nature of $\widetilde{g}$ seems to make the problem much more involved than for g (for which there is an easy cut-and-paste argument, see [Ad1]). Note again that the answer would be positive if Cromwell's conjecture had been true.

Question 14.5. Is $\widetilde{g}$ invariant under mutation?

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