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Proximality in Pisot tiling spaces

by

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Abstract. A substitution φ is strong Pisot if its abelianization matrix is nonsingular and all eigenvalues except the Perron–Frobenius eigenvalue have modulus less than one. For strong Pisot φ that satisfies a no cycle condition and for which the translation flow on the tiling space $\mathcal{T}_\varphi$ has pure discrete spectrum, we describe the collection $\mathcal{T}_\varphi^{\mathrm{P}}$ of pairs of proximal tilings in $\mathcal{T}_\varphi$ in a natural way as a substitution tiling space. We show that if ψ is another such substitution, then $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are homeomorphic if and only if $\mathcal{T}_\varphi^{\mathrm{P}}$ and $\mathcal{T}_\psi^{\mathrm{P}}$ are homeomorphic. We make use of this invariant to distinguish tiling spaces for which other known invariants are ineffective. In addition, we show that for strong Pisot substitutions, pure discrete spectrum of the flow on the associated tiling space is equivalent to proximality being a closed relation on the tiling space.

1. Introduction. Substitutions, and the tiling spaces associated with them, are of fundamental interest in recent investigations in number theory (numeration systems in various bases, diophantine approximation), physics (modeling quasicrystals), and dynamical systems (coding hyperbolic attractors, generating Markov partitions) (see [BFMS] for an excellent introduction to these topics).

The subtle recurrence properties of words generated by a substitution are expressed in the intricate details of the topology of the associated tiling space. Of particular interest are those substitutions whose abelianizations have a dominant eigenvalue that is a Pisot number, largely because of the connection with pure discrete spectrum of the tiling flow and its consequences. There are several algorithms that verify pure discrete spectrum, some of which apply generally ([Sie], [BK], [SS]), and others in special cases ([HS], [ARS], [Sid], [BD2]). For strong Pisot substitutions (see §2), in all known examples, the translation flow on the associated tiling space has

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pure discrete spectrum. The conjecture that strong Pisot substitutions always produce tiling flows with pure discrete spectrum is variously known as: the Pure Discrete Spectrum Conjecture, the Geometric Coincidence Conjecture, the Super Coincidence Conjecture, and, simply, the Pisot Conjecture. (See [BeS] for a survey of the progress on this conjecture, and §4 for a statement of the Geometric Coincidence Condition.)

Given a strong Pisot substitution φ whose tiling flow has pure discrete spectrum, the tiling dynamics on $\mathcal{T}_\varphi$ is measure theoretically and almost topologically conjugate with Anosov dynamics on a torus or solenoid of the appropriate dimension. This almost conjugacy is simply the quotient map that glues together proximal points in the tiling space $\mathcal{T}_\varphi$. That is, for such substitutions, the gross topology of the tiling space is that of a torus or solenoid; the intricate details lie in the structure of the collection of proximal points.

In this paper we isolate this proximal structure, symbolically encode it, and demonstrate its use as a distinguishing invariant for the topological type of Pisot tiling spaces. We also prove that, for strong Pisot substitutions, pure discrete spectrum for the tiling flow is equivalent to proximality being a closed relation.

Given a primitive and aperiodic substitution φ with associated tiling space $\mathcal{T}_\varphi$, tilings $T, T' \in \mathcal{T}_\varphi$ are *asymptotic* provided $\operatorname{dist}(T-t, T'-t) \to 0$ as $t \to \infty$ or $t \to -\infty$. In any such tiling space, there is a finite positive number of arc components (i.e., composants), all of whose tilings are asymptotic to the tilings in some other arc component. Such arc components are called the *asymptotic composants* of the tiling space, and any homeomorphism from one tiling space to another must take asymptotic composants to asymptotic composants. In [BD1], we exploited this fact to develop a complete topological invariant for 1-dimensional substitution tiling spaces. Unfortunately, this invariant is difficult to use.

More computable (but far from complete) invariants have recently emerged ([CE], [BSm], [BSw]). These are all cohomological in nature and depend, in one way or another, on the interaction between some relative cocycles associated with asymptotic composants and the cocycles of the space itself. The approach of this paper is to consider the less restrictive notion of proximal composants (tilings $T, T' \in \mathcal{T}_\varphi$ are *proximal* if $\inf_t \operatorname{dist}(T-t, T'-t) = 0$). This typically provides a much richer collection of composants to consider.

For general tiling spaces, it is not clear how to formulate proximality as a purely topological concept (that is, not tied to a particular flow). For example, in the case that φ is primitive, for any two tilings $T, T' \in \mathcal{T}_\varphi$, one can always find reparameterizations α, α' of $\mathbb{R}$ so that $\inf_t \operatorname{dist}(T-\alpha(t), T'-\alpha'(t)) = 0$. However, in the case that tiling spaces $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are strong

Pisot (with pure discrete spectrum tiling flows and a no cycle condition on periodic words), we prove that proximality is topological in the following sense: If h is a homeomorphism of $\mathcal{T}_\varphi$ with $\mathcal{T}_\psi$, and $\mathcal{C}, \mathcal{C}'$ are composants of $\mathcal{T}_\varphi$ containing proximal tilings, then $h(\mathcal{C})$ and $h(\mathcal{C}')$ are composants of $\mathcal{T}_\psi$ that contain proximal tilings (Theorem 4.16).

Our approach is to link proximality under the tiling flow to a symbolic analog that holds not only for tilings expressed in the language of the given substitution φ but also in the language associated with any substitution obtained from φ by certain cuttings and rewritings. Once this is accomplished, proximality becomes topological, since any homeomorphism between $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ can be isotoped to a homeomorphism that has a symbolic interpretation in terms of substitutions $\widetilde{\varphi}$ and $\widetilde{\psi}$ obtained from φ and ψ by suitable cutting and rewriting.

To achieve this symbolic linking, it is necessary to begin with a substitution φ that is strong Pisot and has a tiling flow with pure discrete spectrum. For this linkage to persist through the cutting and rewriting that yields $\widetilde{\varphi}$, we will need to know that $\widetilde{\varphi}$ is still (weakly) Pisot—the no cycle condition on φ, the subject of §3, guarantees this.

Moreover, we are able to precisely identify the proximal tilings in $\mathcal{T}_\varphi$ by showing that

$$\mathcal{T}_\varphi^{\mathrm{P}} = \{(T, T') : T, T' \text{ are proximal in } \mathcal{T}_\varphi\}$$

is itself a substitution tiling space, with an algorithmically identifiable underlying substitution φ_{EBP}. We obtain:

Theorem 4.15. *Suppose that φ and ψ are strong Pisot, satisfy the no cycle condition on periodic words, and have tiling flows with pure discrete spectrum. Then $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are homeomorphic if and only if $\mathcal{T}_{\varphi_{\mathrm{EBP}}}$ and $\mathcal{T}_{\psi_{\mathrm{EBP}}}$ are homeomorphic.*

In §4, we give examples of substitutions φ and ψ for which the additional structure in the proximal tiling spaces allows us to deduce that $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are not homeomorphic. (In these examples, the known cohomological invariants do not allow one to distinguish the two tiling spaces.) Section 2 contains definitions and background material; §3 is devoted to the no cycle condition and is technical in nature. The main results appear in §§4 and 5.

2. Notation and terminology. We introduce some of the notation and terminology necessary for the paper.

Let $\mathcal{A} = \{1, \ldots, \mathrm{card}(\mathcal{A})\}$ and $\mathcal{B} = \{1, \ldots, \mathrm{card}(\mathcal{B})\}$ be finite alphabets; $\mathcal{A}^*$ will denote the collection of finite nonempty words with letters in $\mathcal{A}$. Given a map $\tau : \mathcal{A} \to \mathcal{B}^*$, there is an associated transition matrix $A_\tau = (a_{ij})_{i\in\mathcal{B}, j\in\mathcal{A}}$ in which a_{ij} is the number of occurrences of i in the word $\tau(j)$;

A_τ is called the *abelianization* of the map τ. A map $\tau : \mathcal{A} \to \mathcal{B}^*$ extends naturally to $\tau : \mathcal{A}^* \to \mathcal{B}^*$.

A *substitution* is a map $\varphi : \mathcal{A} \to \mathcal{A}^*$; φ is *primitive* if $\varphi^n(i)$ contains j for all $i, j \in \mathcal{A}$ and sufficiently large n. Equivalently, φ is primitive if and only if the matrix A_φ is aperiodic, in which case A_φ has a simple eigenvalue λ_φ larger in modulus than its remaining eigenvalues, called the *Perron–Frobenius eigenvalue* of A_φ (and φ).

A word w is *allowed for* φ if for each finite subword (i.e., factor) w' of w, there are $i \in \mathcal{A}$ and $n \in \mathbb{N}$ such that w' is a subword of $\varphi^n(i)$; the *language of* φ, $\mathcal{L}_\varphi$, is the set of finite allowed words for φ. Let W_φ denote the set of allowed bi-infinite words for φ. We identify the 0th coordinate in a bi-infinite word w either by an indexing, as in $w = \dots w_{-1}w_0w_1\dots$, or by use of a decimal point (or both). Let $\sigma : W_\varphi \to W_\varphi$ denote the *shift map*:

$$\sigma(\dots w_{-1}.w_0w_1\dots) := \dots w_{-1}w_0.w_1\dots.$$

For $w \in W_\varphi$, the *shift class of* w is the equivalence class of bi-infinite words:

$$[w] := \{w' \in W_\varphi : w' \text{ is in the shift orbit of } w\}.$$

The substitution $\varphi : \mathcal{A} \to \mathcal{A}^*$ extends to $\varphi : W_\varphi \to W_\varphi$ where

$$\varphi(\dots w_{-1}w_0w_1\dots) := \dots\varphi(w_{-1}).\varphi(w_0)\varphi(w_1)\dots,$$

as well as to a map on equivalence classes:

$$\varphi([w]) := [\varphi(w)].$$

The word w is *periodic* for φ, or φ-*periodic*, if for some $m \in \mathbb{N}$,

$$\varphi^m(w) = \dots\varphi^m(w_{-1}).\varphi^m(w_0)\varphi^m(w_1)\dots = \dots w_{-1}.w_0w_1\dots.$$

Each primitive substitution φ has at least one allowed φ-periodic bi-infinite word which is necessarily uniformly recurrent under the shift. A substitution φ with precisely one periodic, hence fixed, bi-infinite word is called *proper*; φ is proper if and only if there are $b, e \in \mathcal{A}$ such that for all sufficiently large k and all $i \in \mathcal{A}$, $\varphi^k(i) = b\dots e$.

A primitive substitution φ is *aperiodic* if at least one (equivalently, each) φ-periodic bi-infinite word is not periodic under the natural shift map, in which case (W_φ, σ) is an infinite minimal dynamical system. If φ is aperiodic, then the map $\varphi : W_\varphi \to W_\varphi$ is one-to-one ([Mo]). If φ is *periodic* (that is, primitive and not aperiodic), then W_φ is finite.

The substitution φ is *weak Pisot* if φ is primitive, aperiodic, and all eigenvalues of $A = A_\varphi$ other than the Perron–Frobenius eigenvalue have modulus strictly less than 1; φ is *strong Pisot* if any nondominant eigenvalue for φ has modulus strictly between 0 and 1. If φ is strong Pisot, then φ is necessarily primitive and aperiodic ([BFMS] and [HZ]). If φ is weak Pisot, then the (hyperbolic) linear map on $\mathbb{R}^d$ defined by the matrix A has stable space E^{s} of dimension $d-1$ and unstable space E^{u} of dimension 1 spanned by the positive right Perron–Frobenius eigenvector ω_{R} for A. Also, if φ is strong Pisot, neither E^{s} nor E^{u} contain elements of the integer lattice other than the origin. (See Chapter 1 of [BFMS], for instance.)

Given a primitive substitution $\varphi : \mathcal{A} \to \mathcal{A}^*$ with $\operatorname{card}(\mathcal{A}) = d \geq 2$, let $\omega_{\mathrm{L},\varphi} := \omega_{\mathrm{L}} = (\omega_1, \dots, \omega_d)$ and $\omega_{\mathrm{R},\varphi} := \omega_{\mathrm{R}}$ be positive left and right eigenvectors (respectively) for the Perron–Frobenius eigenvalue, $\lambda = \lambda_\varphi$, of A. The intervals $P_i = [0, \omega_i]$, $i = 1, \dots, d$, are called *prototiles for* φ (consider P_i to be distinct from P_j for $i \neq j$ even if $\omega_i = \omega_j$). A *tiling* T of $\mathbb{R}$ by the prototiles for φ is a collection $T = \{T_i\}_{i=-\infty}^{\infty}$ of tiles T_i for which $\bigcup_{i=-\infty}^{\infty} T_i = \mathbb{R}$, each T_i is a translate of some P_j (in which case we say T_i is *of type* j), and $T_i \cap T_{i+1}$ is a singleton for each i. Generally we assume that the indexing is such that $0 \in T_0 \setminus T_1$.

There are occasions in this paper when we wish to define tilings by prototiles for φ but φ is not primitive. In each such case, the matrix for φ will have a unique Perron–Frobenius eigenvector, so prototiles and tilings will be well-defined.

If $\varphi(i) = i_1 \dots i_{k(i)}$, then $\lambda\omega_i = \sum_{j=1}^{k(i)} \omega_{i_j}$. Thus $|\lambda P_i| = \sum_{j=1}^{k(i)} |P_{i_j}|$, and λP_i is tiled by $\{T_j\}_{j=1}^{k(i)}$, where $T_j = P_{i_j} + \sum_{k=1}^{j-1} \omega_{i_k}$. This process is called *inflation and substitution* and extends to a map Φ taking a tiling $T = \{T_i\}_{i=-\infty}^{\infty}$ of $\mathbb{R}$ by prototiles to a new tiling, $\Phi(T)$, of $\mathbb{R}$ by prototiles defined by inflating, substituting, and suitably translating each T_i. More precisely, for $w = w_1 \dots w_n \in \mathcal{A}^*$, define

$$\mathcal{P}_w + t = \Big\{P_{w_1} + t, P_{w_2} + t + |P_{w_1}|, \dots, P_{w_n} + t + \sum_{i<n} |P_{w_i}|\Big\}.$$

Then $\Phi(P_i + t) = \mathcal{P}_{\varphi(i)} + \lambda t$ and $\Phi(\{P_{k_i} + t_i\}_{i\in\mathbb{Z}}) = \bigcup_{i\in\mathbb{Z}} (\mathcal{P}_{\varphi(k_i)} + \lambda t_i)$.

There is a natural topology on the collection Σ_φ of all tilings of $\mathbb{R}$ by prototiles ($\{T_i\}_{i=-\infty}^{\infty}$ and $\{T_i'\}_{i=-\infty}^{\infty}$ are "close" if there is an ϵ near 0 so that $\{T_i\}_{i=-\infty}^{\infty}$ and $\{T_i' + \varepsilon\}_{i=-\infty}^{\infty}$ are identical in a large neighborhood of 0; see [AP] for details). The space Σ_φ is compact and metrizable with this topology and $\Phi : \Sigma_\varphi \to \Sigma_\varphi$ is continuous. Given $T = \{T_i\}_{i=-\infty}^{\infty} \in \Sigma_\varphi$, let $\underline{w}(T) = \dots w_{-1} w_0 w_1 \dots$ denote the bi-infinite word with $w_i = j$ if and only if T_i is of type j. The *tiling space associated with* φ, $\mathcal{T}_\varphi$, is defined as

$$\mathcal{T}_\varphi = \{T : \underline{w}(T) \text{ is allowed for } \varphi\}.$$

There is a natural flow (translation) on Σ_φ defined by $(\{T_i\}_{i=-\infty}^{\infty}, t) \mapsto \{T_i - t\}_{i=-\infty}^{\infty}$. If φ is primitive and aperiodic, $\Phi : \mathcal{T}_\varphi \to \mathcal{T}_\varphi$ is a homeomorphism (this relies on the notion of *recognizability* or invertibility for such substitutions—see [Mo] and [So]). Each $T \in \mathcal{T}_\varphi$ is uniformly recurrent under the flow and has dense orbit (i.e., the flow is minimal on $\mathcal{T}_\varphi$). It follows that $\mathcal{T}_\varphi$ is a continuum.

Recall that a *composant* of a point x in a topological space X is the union of the proper compact connected subsets of X containing x. If φ is a primitive substitution, composants and arc components in $\mathcal{T}_\varphi$ are identical; in this case we use the terms interchangeably. For any substitution φ, the arc components of the tiling space $\mathcal{T}_\varphi$ coincide with the orbits of the natural flow (translation) on $\mathcal{T}_\varphi$. In particular, if $\mathcal{C}$ is an arc component of $\mathcal{T}_\varphi$, then

$$\{\underline{w}(T) : T \in \mathcal{C}\} = [\underline{w}(T)],$$

where $[\underline{w}(T)]$ is the shift class of $\underline{w}(T)$. We also call $[\underline{w}(T)]$ the *pattern of the arc component* (or *composant*) *of* T.

Tilings $T, T' \in \mathcal{T}_\varphi$ are *forward asymptotic* if $\lim_{t\to\infty} \operatorname{dist}(T-t, T'-t) = 0$. Equivalently, $T = \{T_i\}_{i=-\infty}^{\infty}$, $T' = \{T'_i\}_{i=-\infty}^{\infty}$ are *forward asymptotic* if there are $N, M \in \mathbb{Z}$ so that $T_{N+k} = T'_{M+k}$ for all $k \geq 0$. Composants are *forward asymptotic* if they contain forward asymptotic tilings. Backward asymptotic tilings and composants are defined similarly.

Given a primitive substitution $\varphi : \mathcal{A} \to \mathcal{A}^*$ with $\operatorname{card}(\mathcal{A}) = d \geq 2$, and left Perron–Frobenius eigenvector $\omega_{\mathrm{L}} = (\omega_1, \ldots, \omega_d)$, define $R_\varphi = \bigvee_{i=1}^{d} S_i$ as a wedge of d oriented circles $S_1, \ldots, S_d$ with the circumference of $S_i = \omega_i$, and let $f_\varphi : R_\varphi \to R_\varphi$ be the "linear" map, with expansion constant λ, that follows the pattern φ. That is, if $\varphi(i) = i_1 \ldots i_{k(i)}$, then f_φ maps the circle S_i around the circles $S_{i_1}, \ldots, S_{i_{k(i)}}$, in that order, preserving orientation and stretching distances locally by a factor of λ. We call f_φ *the map of the rose associated with* φ.

As with the case for tiling spaces, there are situations when we wish to define R_φ but φ is not primitive. In these situations, if A_φ does not have a unique left eigenvector, any positive left eigenvector will suffice to define the circumferences of the circles.

If $f : X \to X$ is a map of a compact connected metric space X, then the *inverse limit space* with single bonding map f is the space

$$\varprojlim f = \{(x_0, x_1, \ldots) : f(x_i) = x_{i-1} \text{ for } i = 1, 2, \ldots\}$$

with metric

$$\underline{d}(\underline{x}, \underline{y}) = \sum_{i \geq 0} \frac{d(x_i, y_i)}{2^i};$$

$\widehat{f} : \varprojlim f \to \varprojlim f$ will denote the natural (shift) homeomorphism

$$\widehat{f}(x_0, x_1, \ldots) = (f(x_0), x_0, x_1, \ldots).$$

We now define a second model of the tiling space called *strand space*. A *strand*, γ, in $\mathbb{R}^d$ is a collection of *segments* (sometimes called *edges*), $\gamma = \{S_n\}_{n=N}^{M}$, with each segment S_n a translate of a unit interval parallel to a coordinate axis: if $\{e_1, \ldots, e_d\}$ is the standard basis for $\mathbb{R}^d$, and $I_i := \{te_i : 0 \leq t \leq 1\}$, then $S_n = I_{i_n} + v_n$ for some $i_n \in \{1, \ldots, d\}$ and $v_n \in \mathbb{R}^d$.

Moreover, we require that γ be "connected" in the sense that $v_{n+1} = v_n + e_{i_n}$, $i = N, \dots, M-1$. Two strands are equal if they are identical as collections of segments. In particular, if $S_n = S'_{n+l}$ for $n = N, \dots, M$, then $\{S_n\}_{n=N}^{M} = \gamma = \gamma' = \{S'_n\}_{n=N+l}^{M+l}$. A strand $\gamma = \{S_n\}_{n=-\infty}^{\infty}$ is said to be *bi-infinite*. Let

$$\mathcal{F}^d = \{\gamma : \gamma \text{ is a bi-infinite strand in } \mathbb{R}^d\}.$$

The endpoints of segments in a strand are called the *vertices*, and if $S_n = I_{i_n} + v_n$, then $\min S_n := v_n$, $\max S_n := v_n + e_{i_n}$. If $\gamma = \{S_n\}_{n=N}^{M}$ and $\gamma' = \{S'_n\}_{n=N'}^{M'}$ are two strands with $M, N' < \infty$ and $\max S_M = \min S'_{N'}$, we can concatenate γ and γ' (that is, union and reindex) to obtain a single longer strand $\gamma \cup \gamma'$.

Given a substitution φ on the alphabet $\mathcal{A} = \{1, \dots, d\}$, define $\Phi(\{I_i\})$ to be the strand $\{S_{n,i}\}_{n=1}^{k(i)}$ with $S_{n,i} = I_{i_n} + (\sum_{j=1}^{n-1} e_{i_j})$, where $\varphi(i) = i_1 \dots i_{k(i)}$. That is, Φ applied to the singleton strand I_i is the strand with the origin as initial vertex that "follows the pattern" of the word $\varphi(i)$. Now extend Φ to arbitrary singleton strands by

$$\Phi(\{I_i + v\}) := \Big\{ I_{i_n} + \Big(\sum_{j=1}^{n-1} e_{i_j} \Big) + Av \Big\}_{n=1}^{k(i)}$$

and to arbitrary strands by concatenation:

$$\Phi(\{S_n\}_{n=N}^{M}) := \bigcup_{n=N}^{M} \Phi(\{S_n\}).$$

For $R \in \mathbb{R}$, let $\mathcal{F}_R^d$ denote the subspace of $\mathcal{F}^d$ consisting of those strands that lie in a cylinder of radius R centered on E^{u}. If φ is weak Pisot, then for sufficiently large R, $\mathcal{F}_R^d$ is mapped into itself by Φ; choose R_0 so that $\Phi(\mathcal{F}_{R_0}^d) \subset \mathcal{F}_{R_0}^d$. Define

$$\mathcal{F}_\varphi^d = \{\gamma = \{S_n\}_{n=-\infty}^{\infty} \in \mathcal{F}_{R_0}^d : \text{if } S_k \cap E^{\mathrm{s}} \neq \emptyset, \text{ then } i_{k-1} i_k i_{k+1} \in \mathcal{L}_\varphi\}.$$

The *strand space of* φ is

$$\mathcal{T}_\varphi^{\mathrm{S}} := \bigcap_{n \geq 0} \Phi^n(\mathcal{F}_\varphi^d).$$

A metric can be defined on $\mathcal{T}_\varphi^{\mathrm{S}}$ that has the property: the distance between $\gamma = \{S_n\}_{n=-\infty}^{\infty}$ and $\gamma' = \{S'_n\}_{n=-\infty}^{\infty}$ is small if there is $v \in \mathbb{R}^d$, $|v|$ small, and $N \in \mathbb{N}$, N large, so that $S_n = S'_n + v$ for $n = -N, \dots, N$ (where the indexing is such that $S_0 \cap E^{\mathrm{s}} \neq \emptyset$).

It is proved in [BK] that if φ is a strong Pisot substitution, then $\Phi : \mathcal{T}_\varphi^{\mathrm{S}} \to \mathcal{T}_\varphi^{\mathrm{S}}$ is a homeomorphism (referred to as the $\mathbb{Z}$-*action*), $(\gamma, t) \mapsto \gamma + t\omega_{\mathrm{R}}$ defines a flow on $\mathcal{T}_\varphi^{\mathrm{S}}$ (referred to as the $\mathbb{R}$-*action*), and there is a homeomorphism

$h : \mathcal{T}_\varphi^{\mathrm{S}} \to \mathcal{T}_\varphi$ that conjugates to $\mathbb{Z}$- and $\mathbb{R}$-actions on these spaces (h is just the projection of a strand onto $E^{\mathrm{u}} \simeq \mathbb{R}$ along E^{s}).

In what follows, we refer to *rewriting* a substitution, a notion developed in [Dur]. We discuss rewriting with both starting and stopping rules in some detail in [BD1] (see Example 3.2 and the preceding discussion in [BD1]). We also include an example in §4 of this paper.

Finally, $X \simeq Y$ will mean that X and Y are homeomorphic.

3. The no cycle condition. The main result of this section is Corollary 3.4, from which we eventually deduce that if φ is strong Pisot, and $\widetilde{\varphi}$ is a *one-cut rewriting of* φ (see §4), then $\widetilde{\varphi}$ is weak Pisot. A reader willing to accept Corollary 3.4 can safely proceed to §4.

Given a substitution φ, let φ^+, $\varphi^- : \mathcal{A} \to \mathcal{A}$ be defined by

$$\varphi^+(a) = b \quad \text{if } \varphi(a) = b\ldots, \qquad \varphi^-(a) = c \quad \text{if } \varphi(a) = \ldots c.$$

Let

$$S^+ = \bigcap_{n\geq 0} (\varphi^+)^n(\mathcal{A}) \quad \text{and} \quad S^- = \bigcap_{n\geq 0} (\varphi^-)^n(\mathcal{A})$$

be the *eventual ranges* of φ^+ and φ^-, and let

$$\mathcal{P}_\varphi = \{(a,b) \in S^- \times S^+ : ab \in \mathcal{L}_\varphi\}.$$

Define an equivalence relation $\sim$ on $\mathcal{P}_\varphi$ by $(a,b) \sim (c,d)$ if $a = c$ or $b = d$ and extending by transitivity. A *cycle* in $\mathcal{P}_\varphi$ consists of a string of equivalences $(a_1,a_2) \sim (a_3,a_2) \sim (a_3,a_4) \sim \cdots \sim (a_1,a_{2n})$ with $a_1 \neq a_3 \neq \cdots \neq a_{2n-1}$, $a_2 \neq a_4 \neq \cdots \neq a_{2n}$, and $n \geq 2$.

No Cycle Condition. The substitution φ has *no cycles of periodic words* if $\mathcal{P}_\varphi$ has no cycles.

Note that if $(a,b),(c,b) \in \mathcal{P}_\varphi$, then the bi-infinite words obtained by iterating on $a.b$ and $c.b$ ($\lim_{n\to\infty} \varphi^n(a).\varphi^n(b)$ and $\lim_{n\to\infty} \varphi^n(c).\varphi^n(b)$ respectively) are periodic under φ and represent asymptotic composants. That is, a cycle in $\mathcal{P}_\varphi$ represents a cycle of asymptotic composants $\{\mathcal{C}_1,\ldots,\mathcal{C}_n\}$ ($\mathcal{C}_1,\mathcal{C}_2$ are forward asymptotic, $\mathcal{C}_2,\mathcal{C}_3$ are backward asymptotic, etc.) of a particular sort.

The existence of a cycle of asymptotic composants is a topological property of a space. However, whether a cycle of asymptotic composants in $\mathcal{T}_\varphi$ is associated with a cycle in $\mathcal{P}_\varphi$ is combinatorial and an artifact of the symbolic presentation of the tiling space. For instance, the Fibonacci substitution ($\psi(1) = 12, \psi(2) = 1$) has a cycle of two composants asymptotic in both directions but $\mathcal{P}_\psi$ has no cycles. On the other hand, the Morse–Thue substitution φ defined by $\varphi(1) = 12, \varphi(2) = 21$ has a cycle of four asymptotic composants, all associated with the four bi-infinite words periodic under φ:

$$\ldots 1.1\ldots,\quad \ldots 1.2\ldots,\quad \ldots 2.1\ldots,\quad \ldots 2.2\ldots,$$

which determine a cycle in $\mathcal{P}_\varphi$. There are several ways to rewrite φ to obtain a proper substitution φ' for which the tiling spaces $\mathcal{T}_\varphi$ and $\mathcal{T}_{\varphi'}$ are homeomorphic; since φ' has a single bi-infinite periodic word, its cycle of asymptotic composants is no longer associated with a cycle in $\mathcal{P}_{\varphi'}$.

The cycle of asymptotic composants in the tiling space $\mathcal{T}_\varphi$ (where φ is the Morse–Thue substitution) is associated with a nontrivial element of the cohomology of $\mathcal{T}_\varphi$ that is periodic under the action induced by inflation and substitution. That is, inflation and substitution has a root of unity eigenvalue on the level of cohomology. Thus any proper rewriting of φ has an abelianization that has a root of unity eigenvalue and hence is non-Pisot. The non-Pisot nature of the rewriting destroys the connection between the symbolic and the geometrical aspects of proximality that we exploit to obtain the results of this paper. In the case of the Fibonacci substitution, there is also a cohomology element associated with the cycle of asymptotic composants, but it is trivial, and does not correspond to a root of unity eigenvalue—in this case, our program can proceed.

For these reasons, in this paper we require that the substitutions we consider have no cycles of periodic words.

Let $\mathcal{P}_\varphi = \mathcal{P}$ be defined as above.

LEMMA 3.1. *If φ is weak Pisot, then $\mathcal{P}$ consists of a single equivalence class.*

Proof. Suppose that there are k equivalence classes in $\mathcal{P}$, where $k \geq 2$. Using the elements of $\mathcal{P}$ to define starting and stopping rules, we obtain a rewriting $\widetilde{\varphi}$ of φ with alphabet $\widetilde{\mathcal{A}}$ consisting of certain elements of $\mathcal{L}(\varphi)$. Let $v_1, \ldots, v_k$ denote the equivalence classes of $\sim$ in $\mathcal{P}$. Define a graph $G_{\widetilde{\varphi}} = G$ with vertices $v_1, \ldots, v_k$ and edges labeled by elements of $\widetilde{\mathcal{A}}$: the edge labeled $b \ldots a \in \widetilde{\mathcal{A}}$ starts at vertex $v_i = [(\ \ ,b)]$ and ends at vertex $v_j = [(a,\ \ )]$. Let $f_{\widetilde{\varphi}} : G \to G$ follow pattern $\widetilde{\varphi}$, R_φ be the rose with vertex v and edges labeled by $\mathcal{A}$, and $f_\varphi : R_\varphi \to R_\varphi$ follow pattern φ. Let $\varrho : \widetilde{\mathcal{A}} \to \mathcal{A}^*$ be the natural morphism, and $f_\varrho : G_{\widetilde{\varphi}} \to R_\varphi$ be the map following pattern ϱ. The maps f_φ, $f_{\widetilde{\varphi}}$ and f_ϱ satisfy $f_\varrho \circ f_{\widetilde{\varphi}} = f_\varphi \circ f_\varrho$. There is then an induced surjection $\widehat{f}_\varrho : \varprojlim f_{\widetilde{\varphi}} \to \varprojlim f_\varphi$. It is easily checked that $\widehat{f}_\varrho$ is one-to-one everywhere except at a single point: if $\underline{v}^i := (v_i, v_i, \ldots)$ for $i = 1, \ldots, k$ and $\underline{v} := (v, v, \ldots)$, then $(\widehat{f}_\varrho)^{-1}(\underline{v}) = \{\underline{v}^1, \ldots, \underline{v}^k\} := \mathcal{V}$. Thus

$$\varprojlim f_\varphi \simeq (\varprojlim f_{\widetilde{\varphi}})/\mathcal{V}.$$

We have the commuting diagram derived from the long exact sequences of pairs (the zeroth level is reduced and the coefficients are $\mathbb{Q}$):

$$\begin{tikzcd}[column sep=small]
 & & 0 \arrow[r] & 0 \arrow[r] \arrow[dd, no head] & \check{H}^1(\varprojlim f_\varphi, \{\underline{v}\}) \arrow[r] \arrow[dl, "\widehat{f}^*_\varphi"'] \arrow[dd, "f^*_\varrho"] & \check{H}^1(\varprojlim f_\varphi) \arrow[r] \arrow[dl, "\widehat{f}^*_\varphi"'] \arrow[dd, "f^*_\varrho"] & 0 \\
 & 0 \arrow[r] & 0 \arrow[r] & \check{H}^1(\varprojlim f_\varphi, \{\underline{v}\}) \arrow[r] \arrow[dd] & \check{H}^1(\varprojlim f_\varphi) \arrow[r] \arrow[dd] & 0 & \\
 & & 0 \arrow[r] & \check{H}^0(\mathcal{V}) \arrow[r] \arrow[dl, "\widehat{f}^*_{\widetilde{\varphi}}"'] & \check{H}^1(\varprojlim f_{\widetilde{\varphi}}, \mathcal{V}) \arrow[r] \arrow[dl, "\widehat{f}^*_{\widetilde{\varphi}}"] & \check{H}^1(\varprojlim f_{\widetilde{\varphi}}) \arrow[r] \arrow[dl, "\widehat{f}^*_{\widetilde{\varphi}}"] & 0 \\
 & 0 \arrow[r] & \check{H}^0(\mathcal{V}) \arrow[r] & \check{H}^1(\varprojlim f_{\widetilde{\varphi}}, \mathcal{V}) \arrow[r] & \check{H}^1(\varprojlim f_{\widetilde{\varphi}}) \arrow[r] & 0 &
\end{tikzcd}$$

The bottom rows split since all the homomorphisms are linear maps of finite-dimensional vector spaces. Let $\widehat{f_{\widetilde{\varphi}}}|_{\mathcal{V}} := \widehat{f}_{\widetilde{\varphi},1}$. Then $\widehat{f}^n_{\widetilde{\varphi},1} = \mathrm{id}$ for $n = k!$. We extract the commuting square

$$\begin{tikzcd}
\check{H}^1(\varprojlim f_\varphi, \{\underline{v}\}) \arrow[r, "\widehat{f}^*_\varphi"] \arrow[d, "h"'] & \check{H}^1(\varprojlim f_\varphi, \{\underline{v}\}) \arrow[d, "h"'] \\
\check{H}^0(\mathcal{V}) \oplus \check{H}^1(\varprojlim f_{\widetilde{\varphi}}) \arrow[r, "\widehat{f}^*_{\widetilde{\varphi},1} \oplus \widehat{f}^*_{\widetilde{\varphi},2}"] & \check{H}^0(\mathcal{V}) \oplus \check{H}^1(\varprojlim f_{\widetilde{\varphi}})
\end{tikzcd}$$

of vector space isomorphisms. Since $\dim \check{H}^0(\mathcal{V}) = k - 1 \geq 1$ and all eigenvalues of $\widehat{f}^*_{\varphi,1}$ are root of unity, $\widehat{f}^*_\varphi$ has root of unity eigenvalues.

Finally, by continuity of the Čech theory,

$$\check{H}^1(\varprojlim f_\varphi, \{\underline{v}\}) \simeq \check{H}^1(\varprojlim f_\varphi) \simeq \varinjlim(A^{\mathrm{tr}}_\varphi : \mathbb{Q}^d \to \mathbb{Q}^d) \simeq \mathrm{ER}_\varphi,$$

where ER_φ is the eventual range of A^{tr}_φ, and $\widehat{f}^*_\varphi$ is conjugated to $A^{\mathrm{tr}}_\varphi : \mathrm{ER}_\varphi \to \mathrm{ER}_\varphi$ by this isomorphism. Thus A_φ also has root of unity eigenvalues, contradicting the assumption that φ is weak Pisot. ∎

Remark. The proof of the above lemma only requires that φ is hyperbolic.

Suppose that $\psi_t : X \to X$ is a flow on the compact metric space X, and that for each $j = 1, \ldots, m$, $X_{1,j}, \ldots, X_{n_j,j}$ are asymptotic (forward, say) under ψ_t: that is, $d(\psi_t(X_{i,j}), \psi_t(X_{l,j})) \to 0$ as $t \to \infty$ for $i, l \in \{1, \ldots, n_j\}$ and $j \in \{i, \ldots, m\}$. Define $\overline{X}$ to be the quotient $\overline{X} = X/\!\sim$, where $\psi_t(X_{i,j}) \sim \psi_t(X_{l,j})$ for $t \geq 0$, $i, l \in \{1, \ldots, n_j\}$ and $j \in \{i, \ldots, m\}$.

Lemma 3.2. *The quotient map $p : X \to \overline{X}$ induces an isomorphism $p^* : \check{H}^k(\overline{X}) \to \check{H}^k(X)$ for all k.*

Proof. To avoid excessive notation, we assume $m = 1$. Let $\overline{\psi}_t$ be the semi-flow on $\overline{X}$ defined by $\overline{\psi}_t(p(x)) = p(\psi_t(x))$ for $t \geq 0$. Define

$$\varprojlim \overline{\psi}_t := \{\gamma : \mathbb{R} \to \overline{X} : \overline{\psi}_t(\gamma(s)) = \gamma(t+s) \text{ for all } t \geq 0,\ s \in \mathbb{R}\}$$

with the compact-open topology. Because the glued arcs are asymptotic, $\overline{X}$ is compact and Hausdorff, hence $\varprojlim \overline{\psi}_t$ is compact and Hausdorff. Let $\widehat{p} : X \to \varprojlim \overline{\psi}_t$ be defined by $\widehat{p}(x) = p(\psi_{\_}(x)) : \mathbb{R} \to X$. Then $\widehat{p}$ is clearly continuous.

To see that $\widehat{p}$ is a surjection, consider $\psi_t(p^{-1}(\gamma(-t)))$ for $\gamma \in \varprojlim \overline{\psi}_t$. Either this is a singleton or $\gamma(-t) \in \alpha := \{p(\psi_s(x_i)) : s \geq 0\}$. In the latter case, $p^{-1}(\gamma(-t)) = \{\psi_s(x_i) : i = 1, \ldots, n\}$ for some $s \geq 0$. Then

$$\psi_t(p^{-1}(\gamma(-t))) = \{\psi_{t+s}(x_i) : i = 1, \ldots, n\}$$

for some $s \geq 0$, and the diameter of this set goes to 0 as $t \to \infty$, uniformly in $s \geq 0$. In any event, $\operatorname{diam}(\psi_t(p^{-1}(\gamma(-t)))) \to 0$ as $t \to \infty$. Note also that

$$p(\psi_t(p^{-1}(\gamma(-t)))) = \overline{\psi}_t(\gamma(-t)) = \gamma(0),$$

so $x := \lim_{t\to\infty} \psi_t(p^{-1}(\gamma(-t)))$ is well-defined. Moreover, $\widehat{p}(x) = p(\psi_{\_}(x))$ and

$$\begin{aligned} p(\psi_t(x)) &= p\psi_t(\lim_{s\to\infty} \psi_s(p^{-1}(\gamma(-s)))) = \lim_{s\to\infty} p(\psi_{t+s}(p^{-1}(\gamma(-s)))) \\ &= \lim_{s\to\infty} \overline{\psi}_{t+s}(\gamma(-s)) = \lim_{s\to\infty} \gamma(t) = \gamma(t). \end{aligned}$$

Thus $\widehat{p}$ is surjective.

If $\widehat{p}(x) = \widehat{p}(y) = \gamma$, then $p(\psi_t(x)) = p(\psi_t(y)) = \gamma(t)$ for all t. Then $\psi_t(x), \psi_t(y) \in p^{-1}(\gamma(t))$ and $x, y \in \psi_{-t}(p^{-1}(\gamma(t)))$ for all t. But since

$$\operatorname{diam}(\psi_{-t}(p^{-1}(\gamma(t)))) \to 0 \quad \text{as } t \to -\infty,$$

we get $x = y$. Since X is compact and $\varprojlim \overline{\psi}_t$ is Hausdorff, $\widehat{p}$ is a homeomorphism.

Now let $\overline{\psi}_1 : \overline{X} \to \overline{X}$ be the time-one map of $\overline{\psi}$. If $h : \varprojlim \overline{\psi}_1 \to \varprojlim \overline{\psi}_t$ is defined by $h(\overline{x}_0, \overline{x}_1, \ldots) = \gamma$, where $\gamma(t) = \overline{\psi}_{n+t}(\overline{x}_n)$ for $n \geq -t$, then h is a homeomorphism. By continuity of the Čech theory,

$$\check{H}^k(X) \overset{\widehat{p}^*}{\simeq} \check{H}^k(\varprojlim \overline{\psi}_t) \overset{h^*}{\simeq} \check{H}^k(\varprojlim \overline{\psi}_1) \simeq \varinjlim \overline{\psi}_1^* : \check{H}_1(\overline{X}) \to \check{H}_1(\overline{X}).$$

Finally, since $\overline{\psi}_1$ is homotopic to $\overline{\psi}_0 = \mathrm{id}$ (where $\overline{\psi}_t$, $0 \leq t \leq 1$, provides the homotopy), $\overline{\psi}_1^* = \mathrm{id}$ and $\varinjlim \overline{\psi}_1^* \simeq \check{H}_1(\overline{X})$ (by projection onto the first coordinate). The composition of these isomorphisms is p^*. ∎

Recall that $\Phi : \mathcal{T}_\varphi \to \mathcal{T}_\varphi$ is the inflation and substitution map.

Proposition 3.3. *Suppose that φ is weak Pisot and has no cycles of periodic words. Then the linear map on $\varinjlim(A_\varphi^{\mathrm{tr}} : \mathbb{Q}^d \to \mathbb{Q}^d)$ induced by A_φ^{tr} is conjugate to the isomorphism $\Phi^* : \check{H}^1(\mathcal{T}_\varphi) \to \check{H}^1(\mathcal{T}_\varphi)$.*

Proof. By Lemma 3.1, $\mathcal{P}$ consists of a single equivalence class. For each $(a, b) \in \mathcal{P}$, let $T_{(a,b)}$ be the corresponding (periodic under Φ) tiling in $\mathcal{T}_\varphi$.

Let

$$\begin{aligned}\mathcal{P}^+ &:= \{b \in \mathcal{A} : \text{there are } a \neq c \text{ with } (a,b),(c,b) \in \mathcal{P}\} \\ &= \{b_1, \dots, b_k\}.\end{aligned}$$

For each $b_j \in \mathcal{P}^+$, let $\{a_{i,j}\}_{i=1}^{l_j}$ be a list of the letters for which $(a_{i,j}, b_j) \in \mathcal{P}$. Let $\overline{X}^+$ be the quotient space obtained from $\mathcal{T}_\varphi$ by the identifications $T_{(a_{i,j},b_j)} - t \sim T_{(a_{m,j},b_j)} - t$ for $t \geq 1$, $1 \leq i, m \leq l_j$, and $j = 1, \dots, k$. By Lemma 3.2, the quotient map $p_+ : \mathcal{T}_\varphi \to \overline{X}^+$ induces a vector space isomorphism $p_+^* : \check{H}^1(\overline{X}^+) \to \check{H}^1(\mathcal{T}_\varphi)$ (here and in what follows, coefficients are $\mathbb{Q}$). Furthermore, the inflation and substitution map, Φ, induces a map $F_+ : \overline{X}^+ \to \overline{X}^+$ so that

$$\begin{tikzcd} \check{H}^1(\overline{X}^+) \arrow[r, "F_+^*"] \arrow[d, "p_+^*"'] & \check{H}^1(\overline{X}^+) \arrow[d, "p_+^*"'] \\ \check{H}^1(\mathcal{T}_\varphi) \arrow[r, "\Phi^*"] & \check{H}^1(\mathcal{T}_\varphi) \end{tikzcd}$$

is a commuting diagram of vector space isomorphisms.

Let $S : \overline{X}^+ \to [0, \infty)$ be a continuous map with

$$S^{-1}(0) = \{[T_{a_{i,j},b_j} - 1] : j = i, \dots, k\}.$$

That is, S vanishes exactly at the branch points of $\overline{X}^+$. Moreover, using the local product structure of $\mathcal{T}_\varphi$, we may choose S so that if $(a,b),(a,c) \in \mathcal{P}$, then

$$S(p_+(T_{(a,b)} + t)) = S(p_+(T_{(a,c)} + t)) \quad \text{for all } t \geq 0.$$

Let $\tau : \mathcal{T}_\varphi \times \mathbb{R} \to \mathbb{R}$ be the solution to

$$\frac{d\tau}{dt}(T,t) = S \circ p_+(T - \tau(T,t)), \qquad \tau(T,0) = 0.$$

Then $(T,t) \mapsto T - \tau(T,t)$ is a flow on $\mathcal{T}_\varphi$ that descends to a flow ψ_t on $\overline{X}^+$ with rest points at exactly the branch points $\{[T_{a_{i,j},b_j}] : j = i, \dots, k\}$.

Now let

$$\begin{aligned}\mathcal{P}^- &:= \{a \in \mathcal{A} : \text{there are } b \neq c \in \mathcal{A} \text{ with } (a,b),(a,c) \in \mathcal{P}\} \\ &= \{a_1, \dots, a_l\}.\end{aligned}$$

For each $a_i \in \mathcal{P}^-$, let $\{b_{i,j}\}_{j=1}^{k_i}$ be a list of the letters for which $(a_i, b_{i,j}) \in \mathcal{P}$. Let $\overline{X}$ be the quotient space obtained from $\overline{X}^+$ by the identifications $\psi_{-t}(p_+(T_{(a_i,b_{i,j})})) \sim \psi_{-t}(p_+(T_{(a_i,b_{i,m})}))$ for $t \geq 1$, $1 \leq j, m \leq k_i$, and $i = 1, \dots, l$, with quotient map $p_- : \overline{X}^+ \to \overline{X}$. Then there is an induced map

$F : \overline{X} \to \overline{X}$ with $p_- \circ F_+ = F \circ p_-$ and, from Lemma 3.2, we have a commuting diagram of vector space isomorphisms:

$$\begin{tikzcd} \check{H}^1(\overline{X}) \arrow[r, "F^*"] \arrow[d, "p_-^*"'] & \check{H}^1(\overline{X}) \arrow[d, "p_-^*"'] \\ \check{H}^1(\overline{X}^+) \arrow[r, "F_+^*"] & \check{H}^1(\overline{X}^+) \end{tikzcd}$$

Letting $p = p_- \circ p_+$, and combining the above diagrams, we have

$$\begin{tikzcd} \check{H}^1(\overline{X}) \arrow[r, "F^*"] \arrow[d, "p^*"'] & \check{H}^1(\overline{X}) \arrow[d, "p^*"'] \\ \check{H}^1(\mathcal{T}_\varphi) \arrow[r, "\Phi^*"] & \check{H}^1(\mathcal{T}_\varphi) \end{tikzcd} \tag{1}$$

Let $q : \mathcal{T}_\varphi \to R_\varphi$ denote the map that takes a tiling to the location of its origin (that is, if the ith petal P_i of the rose R_φ is identified with the ith prototile $[0, \lambda_i]$, and $[0, \lambda_i] - t$ is the tile of T containing 0, then $q(T) = t \in P_i$). Let $f_\varphi : R_\varphi \to R_\varphi$ be the rose map and $\widehat{q} : \mathcal{T}_\varphi \to \varprojlim f_\varphi$ the continuous surjection defined by

$$\widehat{q}(T) = (q(T), q(f_\varphi^{-1}(T)), q(f_\varphi^{-2}(T)), \ldots).$$

There is then $\overline{q} : \overline{X} \to \varprojlim f_\varphi$ so that

$$\overline{q} \circ p = \widehat{q} \quad \text{and} \quad \overline{q} \circ F = \widehat{f_\varphi} \circ \overline{q}.$$

Let

$$\begin{aligned} B = \{&T_{(a_{i,j}, b_j)} - t : 0 \le t \le 1,\ i = 1, \ldots, l_j,\ j = 1, \ldots, k\} \\ &\cup \{T_{(a_i, b_{i,j})} + t : 0 \le t \le 1,\ j = 1, \ldots, k_i,\ i = 1, \ldots, l\} \end{aligned}$$

(where S must be adjusted so that $S \circ p_+(T_{(a_i, b_{i,j})} + t) := 1$ for $0 \le t \le 1,\ j = 1, \ldots, k_i,\ i = 1, \ldots, l$), and let

$$\overline{B} = p(B)$$

and

$$\widehat{B} = \overline{q}(\overline{B}) = \widehat{q}(B).$$

The fact that $\mathcal{P}$ is one equivalence class means that $\overline{B}$ is connected, and the assumption that $\mathcal{P}$ has no cycles means that $\overline{B}$ is acyclic; that is, $\overline{B}$ is a finite tree and is contractible.

Since $F(\overline{B}) \supset \overline{B}$, we may homotope F to G with $G(\overline{B}) = \overline{B}$. Similarly, we may homotope $\widehat{f_\varphi}$ to g so that $g(\widehat{B}) = \widehat{B}$ and $\overline{q} \circ G = g \circ \overline{q}$. One can further check that $\overline{q}$ is one-to-one except on $\overline{B}$. We have the commuting diagram

$$\begin{tikzcd}
\check{H}^1(\varprojlim f_\varphi, \widehat{B}) \arrow[r, "g^*"] \arrow[d, "\overline{q}^*"'] & \check{H}^1(\varprojlim f_\varphi, \widehat{B}) \arrow[d, "\overline{q}^*"'] \\
\check{H}^1(\overline{X}, \overline{B}) \arrow[r, "G^*"] & \check{H}^1(\overline{X}, \overline{B})
\end{tikzcd}$$

Since $\overline{q} : \overline{X}/\overline{B} \to \varprojlim f_\varphi/\widehat{B}$ is a homeomorphism, $\overline{q}^*$ is an isomorphism. Thus

$$\begin{tikzcd}
\check{H}^1(\varprojlim f_\varphi) \arrow[r, "g^*"] \arrow[d, "\overline{q}^*"'] & \check{H}^1(\varprojlim f_\varphi) \arrow[d, "\overline{q}^*"'] \\
\check{H}^1(\overline{X}) \arrow[r, "G^*"] & \check{H}^1(\overline{X})
\end{tikzcd}$$

is a commuting diagram of isomorphisms. Replacing g^* by $\widehat{f_\varphi^*}$ and G^* by F^* (to which they are equal, respectively), and combining this last diagram with (1), we have the commuting diagram of vector space isomorphisms

$$\begin{tikzcd}
\check{H}^1(\varprojlim f_\varphi) \arrow[r, "\widehat{f_\varphi^*}"] \arrow[d, "q^*"'] & \check{H}^1(\varprojlim f_\varphi) \arrow[d, "q^*"'] \\
\check{H}^1(\mathcal{T}_\varphi) \arrow[r, "\Phi^*"] & \check{H}^1(\mathcal{T}_\varphi)
\end{tikzcd}$$

By continuity of the Čech theory, there exists an isomorphism between $\check{H}^1(\varprojlim f_\varphi)$ and $\varinjlim f_\varphi^*$ (where $f_\varphi^* : H^1(R_\varphi) \to H^1(R_\varphi)$) that conjugates $\widehat{f_\varphi^*}$ with $\widehat{f_\varphi^*} : \varinjlim f_\varphi^* \to \varinjlim f_\varphi^*$ defined by

$$\widehat{f_\varphi^*}([(\gamma, n)]) = [(f_\varphi^*(\gamma), n)].$$

Finally, identifying $H^1(R_\varphi)$ with the dual of $H_1(R_\varphi)$ and choosing as ordered basis for $H_1(R_\varphi)$ the oriented petals of R_φ, the matrix for $(f_\varphi)_*$ is A_φ, and A_φ^{tr} represents f_φ^* (where A^{tr} is the transpose of A). Thus $\widehat{f_\varphi^*}$ is conjugate to $A_\varphi^{\mathrm{tr}} : \varinjlim(A_\varphi^{\mathrm{tr}} : \mathbb{Q}^d \to \mathbb{Q}^d) \to \varinjlim(A_\varphi^{\mathrm{tr}} : \mathbb{Q}^d \to \mathbb{Q}^d)$. ∎

Corollary 3.4. *If φ is weak Pisot and has no cycles of periodic words, and ψ is a proper substitution with the property that Φ and Ψ are conjugate, then ψ is weak Pisot.*

Proof. Since ψ is proper, there is a homeomorphism of $\mathcal{T}_\psi$ with $\varprojlim f_\psi$ that conjugates Ψ_ψ with $\widehat{f}_\psi$. We have a conjugacy between A_φ^{tr} on $\varinjlim(A_\varphi^{\mathrm{tr}} : \mathbb{Q}^d \to \mathbb{Q}^d)$ and A_ψ^{tr} on $\varinjlim(A_\psi^{\mathrm{tr}} : \mathbb{Q}^d \to \mathbb{Q}^d)$. But $\varinjlim(A_\varphi^{\mathrm{tr}} : \mathbb{Q}^d \to \mathbb{Q}^d)$ and $\varinjlim(A_\psi^{\mathrm{tr}} : \mathbb{Q}^d \to \mathbb{Q}^d)$ are naturally isomorphic to the eventual ranges of $A_\varphi^{\mathrm{tr}} : \mathbb{Q}^d \to \mathbb{Q}^d$ and $A_\psi^{\mathrm{tr}} : \mathbb{Q}^{d'} \to \mathbb{Q}^{d'}$ respectively, hence the nonzero spectra of A_φ^{tr} and A_ψ^{tr} are the same. ■

4. Balanced pairs and proximality. The main result of this section is Theorem 4.16, in which we show that if φ and ψ are strong Pisot substitutions that satisfy both GCC and the no cycle condition on periodic words, and $h : \mathcal{T}_\varphi \to \mathcal{T}_\psi$ is a homeomorphism, then h carries composants containing proximal tilings in $\mathcal{T}_\varphi$ to composants containing proximal tilings in $\mathcal{T}_\psi$.

Suppose that φ is weak Pisot. Recall that $\mathcal{A} = \{1, \dots, d\}$ is the alphabet, $A = A_\varphi$ the incidence matrix, and $\mathcal{L}_\varphi = \mathcal{L} \subseteq \mathcal{A}^*$ the language of φ. Let $\omega_{\mathrm{L}} = (\omega_1, \dots, \omega_d)$ and ω_{R} be positive left and right Perron–Frobenius eigenvectors (respectively) for A. For $i \in \mathcal{A}$, let $|i|_g := \omega_i$, and for $w = w_1 \dots w_n \in \mathcal{A}^*$, let $|w|_g := \sum_{i=1}^n |w_i|_g$, $|w| := n$ and $l(w) := (a_1, \dots, a_d)^{\mathrm{tr}}$, where a_i is the number of occurrences of the letter i in w and v^{tr} is the transpose of v. A *balanced pair* for φ is a pair $\binom{u}{v}$ with $u, v \in \mathcal{L}$ and $l(u) = l(v)$. A *geometrically balanced pair* for φ is a pair $\binom{u}{v}$ with $u, v \in \mathcal{L}$ and $|u|_g = |v|_g$. Trivially, any balanced pair is geometrically balanced. If φ is strong Pisot, then the entries of ω_{L} are independent over $\mathbb{Q}$, so that if $|u|_g = |v|_g$, then $l(u) = l(v)$, and any geometrically balanced pair for φ is balanced. As the Morse–Thue substitution ($1 \to 12$, $2 \to 21$) shows, a geometrically balanced pair need not be balanced in general.

Define

$$\mathrm{BP}(\varphi) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : \begin{pmatrix} u \\ v \end{pmatrix} \text{ is a balanced pair for } \varphi \right\},$$

$$\mathrm{GBP}(\varphi) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : \begin{pmatrix} u \\ v \end{pmatrix} \text{ is a geometrically balanced pair for } \varphi \right\}.$$

If $x = \binom{u}{v} \in \mathrm{BP}(\varphi)$, the *dual* of x, written $\overline{x}$, denotes the balanced pair $\binom{v}{u}$. If $\binom{u}{v} \in \mathrm{BP}(\varphi)$ and there are $\binom{u_1}{v_1}, \binom{u_2}{v_2} \in \mathrm{BP}(\varphi)$ so that $u = u_1 u_2$, $v = v_1 v_2$, then $\binom{u}{v}$ is *reducible* and we write $\binom{u}{v} = \binom{u_1}{v_1}\binom{u_2}{v_2}$. Otherwise, $\binom{u}{v}$ is *irreducible*. We make similar definitions for elements of $\mathrm{GBP}(\varphi)$. Note that any balanced pair (geometrically balanced pair, respectively) factors uniquely as a finite product of irreducible balanced pairs (geometrically balanced pairs, respectively). Let $\mathcal{A}_{\mathrm{BP}}$ be the (possibly infinite) alphabet of irreducible balanced pairs, and let the substitution $\varphi_{\mathrm{BP}} : \mathcal{A}_{\mathrm{BP}} \to (\mathcal{A}_{\mathrm{BP}})^*$

be given by

$$\varphi_{\mathrm{BP}}\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} \varphi(u) \\ \varphi(v) \end{pmatrix},$$

factored as a word in $(\mathcal{A}_{\mathrm{BP}})^*$. One can define $\mathcal{A}_{\mathrm{GBP}}$ and φ_{GBP} similarly for irreducible geometrically balanced pairs. An irreducible balanced pair $\binom{u^0}{v^0}$ is *essential* if, for each $n \in \mathbb{N}$, there is $\binom{u^{-n}}{v^{-n}} \in \mathcal{A}_{\mathrm{BP}}$ so that $\binom{u^0}{v^0}$ is a factor of $\varphi^n_{\mathrm{BP}}\big(\binom{u^{-n}}{v^{-n}}\big)$. An *essential geometrically balanced pair* is defined similarly. Since φ is primitive, the trivial pair $\binom{i}{i}$ is essential for any $i \in A$.

Let $\mathcal{A}_{\mathrm{EBP}}$ be the alphabet consisting of essential balanced pairs for φ. Note that if $\binom{u}{v}$ is an essential balanced pair, then $\varphi_{\mathrm{BP}}\left(\binom{u}{v}\right)$ is a product of essential factors. That is, φ_{BP} restricted to $\mathcal{A}_{\mathrm{EBP}}$ determines a substitution $\varphi_{\mathrm{EBP}} : \mathcal{A}_{\mathrm{EBP}} \to (\mathcal{A}_{\mathrm{EBP}})^*$.

Example 4.1 (Essential balanced pairs). Define the substitution φ as follows:

$$\varphi(1) = 11122, \qquad \varphi(2) = 12.$$

The balanced pairs $\binom{12}{21}$, $\binom{112}{211}$, and $\binom{1122}{2121}$ (and their duals) are all essential:

$$\varphi\left(\begin{pmatrix}12\\21\end{pmatrix}\right) = \begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}112\\211\end{pmatrix}\begin{pmatrix}21\\12\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix},$$

$$\varphi\left(\begin{pmatrix}112\\211\end{pmatrix}\right) = \begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}112\\211\end{pmatrix}\begin{pmatrix}21\\12\end{pmatrix}\begin{pmatrix}112\\211\end{pmatrix}\begin{pmatrix}21\\12\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix},$$

$$\varphi\left(\begin{pmatrix}1122\\2121\end{pmatrix}\right) = \begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}112\\211\end{pmatrix}\begin{pmatrix}21\\12\end{pmatrix}\begin{pmatrix}1122\\2121\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}21\\12\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}.$$

Let a, b, c (and $\overline{a}, \overline{b}, \overline{c}$) denote the nontrivial essential balanced pairs above in the order given, and denote the trivial balanced pairs by the associated letter of $\mathcal{A}_\varphi$. Then φ_{EBP} must include at least the information:

$$\varphi_{\mathrm{EBP}}(a) = 1b\overline{a}2, \qquad \varphi_{\mathrm{EBP}}(b) = 1b\overline{a}b\overline{a}2, \qquad \varphi_{\mathrm{EBP}}(c) = 1b\overline{a}c1\overline{a}2,$$

along with the definitions $\varphi_{\mathrm{EBP}}(\overline{a}) = 1\overline{b}a2$, etc.

In Example 4.2, we provide, for a rewriting of φ, an essential geometrically balanced pair that is not balanced. ∎

Suppose that $T \in \mathcal{T}_\varphi$ is fixed by Φ and the tile T_0 of T containing 0 is a translate of the prototile $P_a = [0, \omega_a)$, say $T_0 = P_a - t_0$. We will use the location of 0 in T_0 to define a *one-cut rewriting of φ in the letter a* according to the two cases below:

Case 1: $t_0 > 0$ (0 is in the interior of T_0). In this case, $\varphi(a) = pas$ for some (unique) nonempty p, s in $\mathcal{A}^*$ with $|p|_g < \lambda t_0$ and $|s|_g < \lambda(\omega_a - t_0)$. We modify the alphabet $\mathcal{A}$ by "splitting" the letter a into two letters a_1 and a_2. Let $\mathcal{A}' = \mathcal{A} \cup \{a_1, a_2\} \setminus \{a\}$, and let $\alpha : \mathcal{A} \to \mathcal{A}'$ be the morphism that takes a to a_1a_2 and b to b if $b \neq a$. Define the substitution $\varphi' : \mathcal{A}' \to (\mathcal{A}')^*$ as follows:

$$\varphi'(a_1) = \alpha(p)a_1, \quad \varphi'(a_2) = a_2\alpha(s), \quad \varphi'(b) = \alpha(\varphi(b)) \quad \text{if } b \neq a_1, a_2.$$

Finally, let $\widetilde{\varphi}$ be the substitution obtained from φ' by rewriting, using starting rule $\{a_2\}$ and stopping rule $\{a_1\}$.

CASE 2: $t_0 = 0$. Let the tile type of T_{-1} be b. In this case, there are $s, p \in \mathcal{A}^*$ such that $\varphi(a) = as$, $\varphi(b) = pb$. Let $\widetilde{\varphi}$ denote the substitution obtained from φ by rewriting, using starting rule $\{a\}$ and stopping rule $\{b\}$.

EXAMPLE 4.2 (One-cut rewriting). Again, let φ be given by

$$\varphi(1) = 11122, \quad \varphi(2) = 12.$$

The bi-infinite word $\ldots 11122\,1\underline{1}122\,11122\,12\,12 \ldots$ is associated with a tiling T where 0 occurs in the interior of the tile associated with the underlined 1 and T is fixed under Φ.

We use the location of 0 to split 1 into 1_11_2. Then $\mathcal{A}' = \{1_1, 1_2, 2\}$, and φ' is defined by

$$\varphi'(1_1) = 1_11_21_1, \quad \varphi'(1_2) = 1_21_11_222, \quad \varphi'(2) = 1_11_22.$$

In rewriting using the starting rule 1_2 and the stopping rule 1_1, one obtains the three words/letters

$$a := 1_21_1, \quad b := 1_221_1, \quad c := 1_2221_1.$$

The one-cut rewriting $\widetilde{\varphi}$ is defined by

$$\widetilde{\varphi}(a) := \varphi'(1_21_1) = 1_21_11_2221_11_21_1 := aca, \quad \widetilde{\varphi}(b) = acba, \quad \widetilde{\varphi}(c) = acbba.$$

The geometrically balanced pair $\binom{ac}{bb}$ for $\widetilde{\varphi}$ is not balanced. The substitution φ is strong Pisot, with eigenvalues of $2+\sqrt{3}$ and $2-\sqrt{3}$, while $\widetilde{\varphi}$ is weak Pisot, with eigenvalues of $2+\sqrt{3}$, $2-\sqrt{3}$ and 0. ∎

There is a natural map that takes an arbitrary tiling $T \in \mathcal{T}_\varphi$ to a tiling $\widetilde{T} \in \mathcal{T}_{\widetilde{\varphi}}$, where $\widetilde{T}$ is obtained by marking T as follows. In Case 1, T is marked at the cut point in each tile of type a. In Case 2, T is marked at the beginning of each tile of type a that is preceded by a tile of type b. Either marking can be associated naturally with a tiling in $\mathcal{T}_{\varphi'}$ (Case 1) or $\mathcal{T}_\varphi$ (Case 2). These tiles are then amalgamated and relabeled, according to the rewriting (as seen in Example 4.2). The correspondence $T \mapsto \widetilde{T}$ is a homeomorphism that commutes with inflation and substitution as well as the flow on both spaces. That is, $\widetilde{\Phi(T)} = \widetilde{\Phi}(\widetilde{T})$ (where $\widetilde{\Phi}$ denotes inflation and substitution in $\mathcal{T}_{\widetilde{\varphi}}$) and $\widetilde{T-t} = \widetilde{T} - t$.

LEMMA 4.3. *If $\widetilde{\varphi}$ is obtained from φ by a one-cut rewriting, then $\widetilde{\varphi}$ is proper. If, in addition, φ is weak Pisot and satisfies the no cycle condition, then $\widetilde{\varphi}$ is weak Pisot.*

Proof. The first statement follows directly from the definition of one-cut rewriting. Since there is a homeomorphism of $\mathcal{T}_\varphi$ and $\mathcal{T}_{\widetilde{\varphi}}$ that conjugates Φ and $\widetilde{\Phi}$, the second statement then follows from Corollary 3.4. ∎

Furthermore, if $\binom{\widetilde{u}}{\widetilde{v}}$ is a geometrically balanced pair for $\widetilde{\varphi}$, there is an associated pair $\binom{u}{v}$ with $u, v \in \mathcal{L}_\varphi$. Specifically, if $\widetilde{\varphi}$ is determined by Case 2, then u and v begin with a and end with b, and $\widetilde{u}, \widetilde{v}$ result from rewriting u, v. If $\widetilde{\varphi}$ is determined by Case 1, then $\widetilde{u}$ and $\widetilde{v}$ are rewritings of $u' = a_2 x' a_1$ and $v' = a_2 y' a_1$ with $x' = \alpha(x)$, $y' = \alpha(y)$ for some $x, y \in \mathcal{L}_\varphi$; let $u = ax$, $v = ay$.

LEMMA 4.4. *Let φ be strong Pisot. If $\widetilde{\varphi}$ is a one-cut rewriting of φ, and $\binom{\widetilde{u}}{\widetilde{v}} \in \mathrm{GBP}(\widetilde{\varphi})$, then the associated pair $\binom{u}{v}$ is a balanced pair for φ (not necessarily irreducible).*

Proof. Since $|u|_g = |v|_g$, and φ is strong Pisot, $l(u) = l(v)$. ∎

In the following, if $T \in \mathcal{T}_\varphi$, $\gamma(T)$ will denote the corresponding strand in $\mathcal{T}_\varphi^{\mathrm{S}}$, and Φ will denote the inflation and substitution map in both $\mathcal{T}_\varphi$ and $\mathcal{T}_\varphi^{\mathrm{S}}$. The flow in $\mathcal{T}_\varphi$ will be denoted by $(T, t) \mapsto T - t$, and we assume that ω_{R} has been normalized so that $\gamma(T - t) = \gamma(T) - t\omega_{\mathrm{R}}$.

By a *state* I, we mean a line segment of length one, parallel to a coordinate axis and meeting E^{s}. We take I to be half-open, including its initial vertex but not its terminal vertex. If I is a state, $\widehat{\Phi}(I)$ will denote the unique state contained in $\Phi(I)$. Thus $\widehat{\Phi}$ maps states to states. If $\gamma \in \mathcal{T}_\varphi^{\mathrm{S}}$ or γ is a finite strand that intersects E^{s}, the state determined by γ is the segment (that is, edge) of γ that meets E^{s}.

Given tilings T and T' in $\mathcal{T}_\varphi$ with strands $\gamma = \gamma(T)$ and $\gamma' = \gamma(T')$, we say that γ and γ' are *coincident at zero* if the states determined by γ and γ' are identical; γ and γ' are *coincident* if there is $t \in \mathbb{R}$ so that $\gamma - t\omega_{\mathrm{R}}$ and $\gamma' - t\omega_{\mathrm{R}}$ are coincident at zero.

GEOMETRIC COINCIDENCE CONDITION (GCC). If I, J are states whose vertices are equivalent mod $\mathbb{Z}^d$, then for every $\varepsilon > 0$, there is $t \in \mathbb{R}$ such that $|t| < \varepsilon$, $I + t\omega_{\mathrm{R}}$ and $J + t\omega_{\mathrm{R}}$ are states, and for some $n \ge 0$, $\widehat{\Phi}^n(I + t\omega_{\mathrm{R}}) = \widehat{\Phi}^n(J + t\omega_{\mathrm{R}})$ (equivalently, for some $n \ge 0$, $\Phi^n(I + t\omega_{\mathrm{R}})$ and $\Phi^n(J + t\omega_{\mathrm{R}})$ are coincident at 0).

If φ is strong Pisot, and $\gamma \in \mathcal{T}_\varphi^{\mathrm{S}}$, let $v(\gamma)$ be any vertex of γ. Define $g : \mathcal{T}_\varphi^{\mathrm{S}} \to \mathbb{T}^d$ by $g(\gamma) = v(\gamma) \pmod{\mathbb{Z}^d}$. Then g is a continuous surjection and $g(\gamma - t\omega_{\mathrm{R}}) = g(\gamma) - t\omega_{\mathrm{R}} \pmod{\mathbb{Z}^d}$ for all $t \in \mathbb{R}$. Also, if $F_A : \mathbb{T}^d \to \mathbb{T}^d$ is defined by $F_A(p) = Ap \pmod{\mathbb{Z}^d}$, where A is the transition matrix for φ, then $g \circ \Phi = F_A \circ g$. If φ is unimodular, the map g is called *geometric realization*.

If φ is not unimodular, define $\widehat{g} : \mathcal{T}^{\mathrm{S}}_\varphi \to \varprojlim F_A$ by

$$\widehat{g}(\gamma) = (g(\gamma), g(\Phi^{-1}(\gamma)), g(\Phi^{-2}(\gamma)), \ldots);$$

in this case, $\widehat{g}$ is geometric realization. Let $(\underline{z})_t$ denote the flow on $\varprojlim F_A$ defined by

$$(z_0, z_1, \ldots)_t = (z_0 - t\omega_{\mathrm{R}}, z_1 - (t/\lambda)\omega_{\mathrm{R}}, z_2 - (t/\lambda^2)\omega_{\mathrm{R}}, \ldots),$$

where all coordinates are taken mod $\mathbb{Z}^d$, and let $\widehat{f}_A$ be the shift homeomorphism on $\varprojlim F_A$ given by

$$\widehat{F}_A(z_0, z_1, \ldots) = (F_A(z_0), z_0, z_1, \ldots).$$

Then $\widehat{g} \circ \Phi = \widehat{F}_A \circ \widehat{g}$, and $\widehat{g}(\gamma - t\omega_{\mathrm{R}}) = (\widehat{g}(\gamma))_t$ for all $t \in \mathbb{R}$.

Note that if φ is unimodular, then $F_A : \mathbb{T}^d \to \mathbb{T}^d$ is a homeomorphism, so that projection onto the first coordinate, π_0, yields a homeomorphism of $\varprojlim F_A$ with $\mathbb{T}^d$ that conjugates the $\mathbb{R}$- and $\mathbb{Z}$-actions on $\varprojlim F_A$ to those on $\mathbb{T}^d$ and produces the commuting diagram

$$\begin{tikzcd} \mathcal{T}^{\mathrm{S}}_\varphi \arrow[rr, "\widehat{g}"] \arrow[dr, "g"'] & & \varprojlim F_A \arrow[dl, "\pi_0"] \\ & \mathbb{T}^d & \end{tikzcd}$$

For efficiency in the following, we will denote geometric realization in the unimodular case by $\widehat{g}$ as well.

For a strong Pisot substitution φ, the *coincidence rank of* φ is defined as

$$\begin{aligned} \mathrm{cr}_\varphi = \max\{n :\ & \text{there are } \gamma_1, \ldots, \gamma_n \in \mathcal{T}^{\mathrm{S}}_\varphi \text{ and } \underline{x} \in \varprojlim F_A \text{ such that} \\ & \gamma_i \in \widehat{g}^{-1}(\underline{x}) \text{ for } 1 \le i \le n \text{ and } \gamma_i, \gamma_j \text{ are not coincident if } i \ne j\}. \end{aligned}$$

A balanced pair B for φ *terminates with coincidence* provided there is a finite collection $\{B_1, \ldots, B_k\}$ of irreducible balanced pairs so that (1) for each $n \in \mathbb{N} \cup \{0\}$, the pair $\varphi^n_{\mathrm{BP}}(B)$ factors as a product of the elements of $\{B_1, \ldots, B_k\}$, and (2) for each $i \in \{1, \ldots, k\}$, there is $n \in \mathbb{N}$ so that $\varphi^n_{\mathrm{BP}}(B_i)$ has a trivial balanced pair factor.

Proofs of the following results in the case φ is unimodular can be found in [BK]; for the nonunimodular case, see [BBK].

THEOREM 4.5. *Suppose that φ is strong Pisot. Then $\widehat{g}$ is bounded-to-one and $\#\widehat{g}^{-1}(\underline{x}) = \mathrm{cr}_\varphi$ for (Haar) almost every $\underline{x} \in \varprojlim F_A$. Moreover, the following are equivalent*:

(i) *φ satisfies GCC*,
(ii) $\mathrm{cr}_\varphi = 1$,

(iii) $\#\widehat{g}^{-1}(\underline{x}) = 1$ *for* (*Haar*) *almost every* $\underline{x} \in \varprojlim F_A$,
(iv) *every balanced pair* B *for* φ *terminates with coincidence*,
(v) *there are* $a, b \in \mathcal{A}$ *with* $a \neq b$ *so that the balanced pair* $\binom{a\,b}{b\,a}$ *terminates with coincidence*,
(vi) *the tiling flow*, $(T, t) \mapsto T - t$, *has pure discrete spectrum.*

If any of (i)–(vi) *holds*, *then* $\{\gamma \in \mathcal{T}_\varphi^{\mathrm{S}} : \#\widehat{g}^{-1}(g(\gamma)) = 1\}$ *is a set of full measure* (*with respect to the unique ergodic flow-invariant measure on* $\mathcal{T}_\varphi^{\mathrm{S}}$) *that contains a dense* G_δ.

Tilings $T, T' \in \mathcal{T}_\varphi$ are said to be *forward proximal* if there is a sequence $\{t_k\}$ of real numbers so that $t_k \to \infty$ and $\lim_{k\to\infty} d(T - t_k, T' - t_k) = 0$. If there is a sequence $\{t_k\}$ so that $t_k \to -\infty$ and $\lim_{k\to\infty} d(T - t_k, T' - t_k) = 0$, then T and T' are said to be *backward proximal.* If T and T' are either forward or backward proximal, they are *proximal.* Note that if T and T' are proximal, so are: $T - t$ and $T' - t$ for each $t \in \mathbb{R}$; $\Phi(T)$ and $\Phi(T')$; and $\Phi^{-1}(T)$ and $\Phi^{-1}(T')$ (since $\Phi^{-1}(T - t) = \Phi^{-1}(T) - t/\lambda$).

EXAMPLE 4.6 (Proximality). Let φ be defined by $\varphi(1) = 11122$, $\varphi(2) = 12$. The following three bi-infinite words (given in pairs) represent tilings that are fixed under Φ. In one case, the origin of the tiling is at an endpoint of a tile and the corresponding word is fixed under φ. The origin of each of the remaining two tilings is in the interior of the tile, and the associated words can be obtained by iterating around a fixed point associated with the underlined symbol. All three are (pairwise) proximal in both directions. Spacing is used to indicate the balanced pair structure for each pair of words.

$$\ldots 1\ 112\ 21\ 1\underline{1}2\ 21\ 2\ 1\ 211 \ldots$$
$$\ldots 1\ 211\ 12\ 21\underline{1}\ 12\ 2\ 1\ 112 \ldots$$

$$\ldots 1\ 112\ 21\ 1\underline{1}22\ 1\ 21\ 2\ 1\ 1 \ldots$$
$$\ldots 1\ 211\ 12\ 212.1\ 1\ 12\ 2\ 1\ 1 \ldots$$

$$\ldots 1\ 211\ 12\ 2\ 1\ \underline{1}12\ 21\ 112 \ldots$$
$$\ldots 1\ 211\ 12\ 2\ 1\ 2.11\ 12\ 211 \ldots$$

Note that each pair could also be generated by iterating under φ_{BP} (or φ_{EBP}) the balanced pair in which the fixed points of the bi-infinite words appear. (See Example 4.1 for φ_{EBP}.) ∎

PROPOSITION 4.7. *Suppose that* φ *is strong Pisot*, *and let* $T, T' \in \mathcal{T}_\varphi$. *If* T *and* T' *are either forward or backward proximal*, *then* $\gamma(T)$ *and* $\gamma(T')$ *have the same geometric realization. If* φ *satisfies GCC and* $\gamma(T)$ *and* $\gamma(T')$ *have the same geometric realization*, *then* T *and* T' *are proximal in both directions.*

Proof. Suppose that T and T' are either forward or backward proximal. Then so are the images of $\gamma(T)$ and $\gamma(T')$ under the irrational flow on $\mathbb{T}^d$ (or $\varprojlim F_A$, in the nonunimodular case). Since irrational flow is an isometry, this implies that $g(\gamma(T)) = g(\gamma(T'))$ (or $\widehat{g}(\gamma(T)) = \widehat{g}(\gamma(T'))$, respectively).

Suppose that φ satisfies GCC, and that $\gamma = \gamma(T)$ and $\gamma' = \gamma(T')$ have the same geometric realization. Choose $n_k \to \infty$ so that $\Phi^{-n_k}(\gamma)$ and $\Phi^{-n_k}(\gamma')$ converge, say to η and η' respectively. Then η and η' have the same geometric realization and, using GCC, there are $t > 0$ and $n_0 \in \mathbb{N}$ so that if I and I' are the states determined by $\eta + t\omega_{\mathrm{R}}$ and $\eta' + t\omega_{\mathrm{R}}$, then $\widehat{\Phi}^{n_0}(I) = \widehat{\Phi}^{n_0}(I')$. Moreover, we may choose t so that neither I nor I' has a vertex on E^{s}. Then, for sufficiently large k, the states I_{n_k} and I'_{n_k} determined by $\Phi^{-n_k}(\gamma) + t\omega_{\mathrm{R}}$ and $\Phi^{-n_k}(\gamma') + t\omega_{\mathrm{R}}$ satisfy $I = I_{n_k} + u_k$ and $I' = I'_{n_k} + u_k$ with $u_k \in \mathbb{R}^d$ and $u_k \to 0$ as $k \to \infty$. Thus

$$\widehat{\Phi}^{n_0}(I_{n_k} - u_k) = \widehat{\Phi}^{n_0}(I'_{n_k} - u_k) \quad \text{for sufficiently large } k.$$

That is, $\Phi^{n_0}(\Phi^{-n_k}(\gamma))$ and $\Phi^{n_0}(\Phi^{-n_k}(\gamma'))$ share an edge, call it L_k, whose initial vertex, $\min L_k$, satisfies $\langle \min L_k, \omega_{\mathrm{R}}\rangle < 0$, provided k is large enough so that $\langle t\omega_{\mathrm{R}} + u_k, \omega_{\mathrm{R}}\rangle > 0$ (here $\langle\,,\rangle$ is the usual Euclidean inner product). It follows that γ and γ' coincide along $\Phi^{n_k - n_0}(L_k)$. Since $\langle A^{n_k - n_0}(\min L_k), \omega_{\mathrm{R}}\rangle$ goes to $-\infty$ as $k \to \infty$ and the length of $\Phi^{n_k - n_0}(L_k)$ goes to ∞ as $k \to \infty$, T and T' are backward proximal. Choosing t as above but with $t < 0$ shows that T and T' are forward proximal. ∎

COROLLARY 4.8. *Suppose that φ is a strong Pisot substitution that satisfies GCC. Then proximality is an equivalence relation, and if T and T' are either forward or backward proximal, they are proximal in both directions. The proximality equivalence class of T is exactly the collection of T' for which $\gamma(T)$ and $\gamma(T')$ have the same geometric realization.*

Suppose that T and T' in $\mathcal{T}_\varphi$ are proximal, and that φ is a strong Pisot substitution satisfying GCC. If $\gamma = \gamma(T)$ and $\gamma' = \gamma(T')$, there are times t_k (for $k \in \mathbb{Z}$) such that $\lim_{k\to-\infty} t_k = -\infty$, $\lim_{k\to\infty} t_k = \infty$, and $\gamma - t_k\omega_{\mathrm{R}}$ and $\gamma' - t_k\omega_{\mathrm{R}}$ are coincident at zero for all k. Let

$$\mathrm{NC} = \{t : \gamma - t\omega_{\mathrm{R}} \text{ and } \gamma' - t\omega_{\mathrm{R}} \text{ are } \textit{not} \text{ coincident at zero}\}.$$

If D is a component of NC, then D is a bounded interval of the form $D = [a, b)$ (recall that we take the segments defining states to be closed on the left and open on the right). Let

$$\gamma_{[a,b)} = \gamma \cap \bigcup_{t\in[a,b)} (E^{\mathrm{s}} + t\omega_{\mathrm{R}}),$$

and let u be the word in $\mathcal{L}_\varphi$ determined by $\gamma_{[a,b)}$. Similarly, let v be the word determined by $\gamma'_{[a,b)}$. Then $\binom{u}{v}$ is a geometrically balanced, hence balanced pair for φ. If $\gamma_{[a,b)}$ and $\gamma'_{[a,b)}$ do not intersect in their interiors, $\binom{u}{v}$ is irreducible. In this case we say that $\binom{u}{v}$ is *obtained from a bubble in a proximal pair.*

Proposition 4.9. *Suppose that φ is strong Pisot and satisfies GCC. Let $\binom{u}{v}$ be an irreducible balanced pair for φ. Then $\binom{u}{v}$ is an essential balanced pair for φ if and only if $\binom{u}{v}$ is obtained from a bubble in a proximal pair or is a trivial pair $\binom{i}{i}$.*

Proof. Suppose that $\binom{u}{v}$ is obtained from a bubble in the proximal pair T, T'. For each $n \in \mathbb{N}$, $\Phi^{-n}(T)$ and $\Phi^{-n}(T')$ are also proximal. There is then a "geometrical bubble" in $\gamma(\Phi^{-n}(T))$ and $\gamma(\Phi^{-n}(T'))$ that maps, under Φ^n, over the geometrical bubble that determines $\binom{u}{v}$. Thus there is an irreducible balanced pair $\binom{u^{-n}}{v^{-n}}$ (the linguistic equivalent of the geometrical bubble in $\gamma(\Phi^{-n}(T))$ and $\gamma(\Phi^{-n}(T')))$ that maps, under φ_{BP}^n, into a word having $\binom{u}{v}$ as a factor, thus $\binom{u}{v}$ is essential.

Conversely, if $\binom{u}{v}$ is essential, for each $n \in \mathbb{N}$, let $\binom{u^{-n}}{v^{-n}}$ be an irreducible balanced pair with $\binom{u}{v}$ a factor of $\varphi_{\mathrm{BP}}^n\big(\binom{u^{-n}}{v^{-n}}\big)$. For each n, let $\gamma_{-n}, \gamma'_{-n}$ be a pair of finite strands that realize the patterns u^{-n} and v^{-n} and have the same initial and terminal points (so that $\gamma_{-n} \cup \gamma'_{-n}$ is a geometrical bubble) located a bounded (minimum) distance from the origin in $\mathbb{R}^d$. By adjusting by translation in the E^{u} direction, we can ensure that $\Phi^n(\gamma_{-n}) \cup \Phi^n(\gamma'_{-n})$ contains a bubble linguistically equivalent to $\binom{u}{v}$ that meets E^{s}. There is a subsequence $\{n_i\}$ so that the increasingly long strands $\Phi^{n_i}(\gamma_{-n_i})$ converge to a strand $\gamma \in \mathcal{T}_\varphi^{\mathrm{S}}$. There is a further subsequence so that the strands $\Phi^{n_{i_j}}(\gamma'_{-n_{i_j}})$ converge to a strand $\gamma' \in \mathcal{T}_\varphi^{\mathrm{S}}$. It is clear that γ and γ' have the same geometric realization, so that if T and T' satisfy $\gamma = \gamma(T)$ and $\gamma' = \gamma'(T')$, then T and T' are proximal. Finally, $\binom{u}{v}$ is obtained from a bubble in the proximal pair T, T'. ∎

Recall that $\mathcal{A}_{\mathrm{EBP}}$ is the alphabet consisting of essential balanced pairs for φ. According to the following result, if φ is strong Pisot and satisfies GCC, the substitution $\varphi_{\mathrm{EBP}} : \mathcal{A}_{\mathrm{EBP}} \to (\mathcal{A}_{\mathrm{EBP}})^*$ is on a finite alphabet.

Lemma 4.10. *If φ is strong Pisot and satisfies GCC, then $\mathcal{A}_{\mathrm{EBP}}$ is finite.*

Proof. If $\mathcal{A}_{\mathrm{EBP}}$ is infinite, then according to Proposition 4.9, there are $T^n, S^n \in \mathcal{T}_\varphi$ so that T^n and S^n are proximal and $\gamma(T^n), \gamma(S^n)$ determine an irreducible geometrical bubble of length at least $2n|\omega_{\mathrm{R}}|$. By translating, we may assume that these geometric bubbles extend at least from $E^{\mathrm{s}} - n\omega_{\mathrm{R}}$ to $E^{\mathrm{s}} + n\omega_{\mathrm{R}}$. Choose a subsequence $\{n_i\}$ so that $T^{n_i} \to T \in \mathcal{T}_\varphi$ and $S^{n_i} \to S \in \mathcal{T}_\varphi$. Since T^{n_i} has the same geometric realization as S^{n_i} for each i, T and S have the same geometric realization, hence T and S must be proximal. But $\gamma(T)$ and $\gamma(S)$ do not intersect, so T and S cannot be proximal. ∎

Suppose that $\widetilde{\mathcal{A}} = \{e_1, \dots, e_m\}$ is a collection of balanced pairs for φ that contains the trivial balanced pairs $e_i = \binom{i}{i}$, $i = 1, \dots, d$. Suppose also that $\widetilde{\varphi}(e_j) := \varphi_{\mathrm{BP}}(e_j)$ factors as a product of elements of $\widetilde{\mathcal{A}}$ for each $j = 1, \dots, m$. The matrix for $\widetilde{\varphi}$ then has block triangular form

$$A_{\widetilde{\varphi}} = \begin{pmatrix} A_\varphi & B \\ 0 & C \end{pmatrix}.$$

(In particular, $\widetilde{\varphi}$ is not primitive if $m > d$.)

For the tiling space $\mathcal{T}_{\widetilde{\varphi}}$, let the length of the prototile corresponding to a balanced pair $e_i = \binom{u_i}{v_i}$ be given by $|u_i|_g = \omega_i$, let

$$|e_{i_1} \dots e_{i_k}|_g = \sum_{s=1}^{t} \omega_{i_s}$$

be the geometrical length of a word in $\widetilde{\mathcal{A}}^*$, and let $\widetilde{\Phi}$ denote the inflation and substitution homeomorphism on $\mathcal{T}_{\widetilde{\varphi}}$. (Ordinarily, the invertibility of a substitution φ or associated map Φ on strand space is *recognizability*, which depends on primitivity. But in this case, Φ is invertible, so $\widetilde{\Phi}$ is invertible.) Then $|\widetilde{\Phi}(e_i)|_g = \lambda |e_i|_g$ for $i = 1, \dots, m$, so that $\omega_{\mathrm{L},\widetilde{\varphi}} := (\omega_1, \dots, \omega_m)$ is a positive left eigenvector for $A_{\widetilde{\varphi}}$ with eigenvalue λ.

From the primitivity of φ, it follows that the positive right Perron–Frobenius eigenvector $\omega_{\mathrm{R}} = (f_1 \dots, f_d)^{\mathrm{tr}}$ of A_φ, normalized so that $|\omega_{\mathrm{R}}|_1 = \sum_{i=1}^d |f_i| = 1$, has entries f_i equal to the frequency of occurrence of the tiles of type i in any tiling $T = \{T_n\}_{n=-\infty}^{\infty} \in \mathcal{T}_\varphi$: for $i = 1, \dots, d$,

$$f_i = \lim_{N\to\infty} \frac{1}{2N+1} \#\{n : -N \le n \le N \text{ and } T_n \text{ has type } i\}.$$

We extend this result to $\mathcal{T}_{\widetilde{\varphi}}$.

LEMMA 4.11. *Assume that φ is strong Pisot and let $\widetilde{\mathcal{A}} = \{e_1, \dots, e_m\}$ be any finite collection of balanced pairs for φ with $e_i = \binom{i}{i}$, $i = 1, \dots, d$, the trivial balanced pairs. Suppose that $\widetilde{\mathcal{A}}$ satisfies*: (i) *$\varphi_{\mathrm{BP}}(e_i)$ factors as a product of elements of $\widetilde{\mathcal{A}}$ for each $i \in \{1, \dots, m\}$, and* (ii) *each $e_i \in \widetilde{\mathcal{A}}$ terminates with coincidence. Let $\widetilde{\varphi}$ be the restriction of φ_{BP} to $\widetilde{\mathcal{A}}$. Let f_i be as above for $i = 1, \dots, d$, and for $i = d+1, \dots, m$, let $f_i = 0$. Then, for any tiling $T = \{T_n\}_{n=-\infty}^{\infty} \in \mathcal{T}_{\widetilde{\varphi}}$,*

$$f_i = \lim_{N\to\infty} \frac{1}{2N+1} \#\{n : -N \le n \le N \text{ and } T_n \text{ has type } e_i\}$$

for all $i = 1, \dots, m$, and the limit is uniform in T.

Proof. Since φ is primitive and the elements of $\widetilde{\mathcal{A}}$ terminate with coincidence, $(A_{\widetilde{\varphi}})^n$ has the form $\left(\begin{smallmatrix} (A_\varphi)^n & B_n \\ 0 & C^n \end{smallmatrix}\right)$ with B_n strictly positive for large enough n, say $n \geq M$. Then

$$A_{\widetilde{\varphi}}^{lM} = \begin{pmatrix} A_{\varphi}^{lM} & [\sum_{i=1}^{l} A_{\varphi}^{(l-i)M} B_M C^{(i-1)M}] \\ 0 & C^{lM} \end{pmatrix}$$

for $l = 1, 2, \ldots$. Let $\omega_{\mathrm{L}} = (\omega_1, \ldots, \omega_d)$ and $v = (\omega_{d+1}, \ldots, \omega_m)$ so that $(\omega_{\mathrm{L}}, v) A_{\widetilde{\varphi}} = (\lambda \omega_{\mathrm{L}}, \lambda v)$. Then

$$\omega_{\mathrm{L}} \Big(\sum_{i=1}^{l} A_{\varphi}^{(l-i)M} B_M C^{(i-1)M} \Big) + v C^{lM} = \lambda^{lm} v.$$

That is,

$$(*) \qquad \Big(\sum_{i=1}^{l} \lambda^{(l-i)M} \omega_{\mathrm{L}} B_M C^{(i-1)M} \Big) + v C^{lM} = \lambda^{lm} v,$$

with $\omega_{\mathrm{L}} B_M$ and v strictly positive.

Let β be the spectral radius of C. Since $C \geq 0$, β is a real eigenvalue of C and since $\omega_{\mathrm{L}} B_M, v$ are strictly positive, there is a constant $K > 0$ so that $|\omega_{\mathrm{L}} B_M C^{(i-1)M}|_1 \geq K \beta^{(i-1)M}$ and $|v C^{lM}|_1 \geq K \beta^{lM}$ for all $l \in \mathbb{N}$. Thus, from $(*)$,

$$\lambda^{lm} |v|_1 \geq \Big(\beta^{lm} + \sum_{i=1}^{l} \lambda^{(l-i)M} \beta^{(i-1)M} \Big) K$$

for all $l \in \mathbb{N}$. This implies that $\beta < \lambda$. It follows (again using the fact that $B_n > 0$ for $n \geq M$) that, up to scale, $\omega_{\mathrm{R}, \widetilde{\varphi}} := (\omega_{\mathrm{R}}, 0) = (f_1, \ldots, f_d, 0, \ldots, 0)$ is the unique eigenvector for $A_{\widetilde{\varphi}}$ with all entries nonnegative and that

$$\frac{A_{\widetilde{\varphi}}^n w}{|A_{\widetilde{\varphi}}^n w|} \to \omega_{\mathrm{R}, \widetilde{\varphi}}$$

for any nonnegative $w \in \mathbb{R}^m$ with at least one nonzero entry. In particular, for any $e_i \in \mathcal{A}_{\widetilde{\varphi}}$ and $k \in \{1, \ldots, m\}$,

$$\frac{(l(\widetilde{\varphi}^n(e_i)))_k}{|l(\widetilde{\varphi}^n(e_i))|} \to f_k,$$

where the numerator represents the kth component of the abelianization vector l of the word $\widetilde{\varphi}^n(e_i)$. Thus, for each $k \in \{1, \ldots, m\}$ and $j \in \mathbb{N}$, there is $\varepsilon_j > 0$ so that for all $i \in \{1, \ldots, m\}$,

$$\left| \frac{(l(\widetilde{\varphi}^j(e_i)))_k}{|l(\widetilde{\varphi}^j(e_i))|} - f_k \right| < \varepsilon_j$$

and $\varepsilon_j \to 0$ as $j \to \infty$.

Now for $N \in \mathbb{N}$, let $w = w_{-N} \ldots w_N$ be the word in $\mathcal{A}_{\widetilde{\varphi}}^*$ corresponding to the central portion of a tiling $T = \{T_n\}_{n=-\infty}^{\infty}$ (that is, T_n has type w_n).

For any $j \in \mathbb{N}$, w can be factored as $w = u\widetilde{\varphi}^j(e_{i_1})\dots\widetilde{\varphi}^j(e_{i_t})v$ with $|u|,|v|$ bounded independently of T and N. It follows that

$$\begin{aligned}
&\frac{-\varepsilon_j - \dfrac{|u|+|v|}{\sum_{s=1}^t |l(\widetilde{\varphi}^j(e_{i_s}))|} f_k}{\dfrac{|u|+|v|}{\sum_{s=1}^t |l(\widetilde{\varphi}^j(e_{i_s}))|}+1} \\
&\quad\le \frac{\dfrac{\sum_{s=1}^t (l(\widetilde{\varphi}^j(e_{i_s})))_k}{\sum_{s=1}^t |l(\widetilde{\varphi}^j(e_{i_s}))|} - f_k - \dfrac{|u|+|v|}{\sum_{s=1}^t |l(\widetilde{\varphi}^j(e_{i_s}))|} f_k}{\dfrac{|u|+|v|}{\sum_{s=1}^t |l(\widetilde{\varphi}^j(e_{i_s}))|}+1} \\
&\quad\le \frac{\sum_{s=1}^t (l(\widetilde{\varphi}^j(e_{i_s})))_k}{|u|+|v|+\sum_{s=1}^t |l(\widetilde{\varphi}^j(e_{i_s}))|} - f_k \\
&\quad\le \frac{1}{2N+1}\,\#\{n : -N \le n \le N \text{ and } T_n \text{ has type } e_k\} - f_k \\
&\quad\le \frac{|u|+|v|+\sum_{s=1}^t (l(\widetilde{\varphi}^j(e_{i_s})))_k}{\sum_{s=1}^t |l(\widetilde{\varphi}^j(e_{i_s}))|} - f_k \le \frac{|u|+|v|}{\sum_{s=1}^t |l(\widetilde{\varphi}^j(e_{i_s}))|} + \varepsilon_j.
\end{aligned}$$

Since $\sum_{s=1}^t |l(\widetilde{\varphi}^j(e_{i_s}))| \to \infty$ as $N \to \infty$, we have

$$\limsup_{N\to\infty}\left|\frac{1}{2N+1}\,\#\{n : -N \le n \le N \text{ and } T_n \text{ has type } e_k\} - f_k\right| \le \varepsilon_j.$$

Since $\varepsilon_j \to 0$ as $j \to \infty$, we have the desired result. ∎

We have just proved that if φ satisfies GCC, then in the tiling space $\mathcal{T}_{\varphi_{\mathrm{EBP}}}$, the tile types corresponding to nontrivial essential balanced pairs occur with zero frequency in any tiling. Equivalently, if T, T' is a proximal pair in $\mathcal{T}_\varphi$, then

$$\mu\{t : t_0 \le t \le t_0 + \tau,\ t \in \mathrm{NC}\}/\tau \to 0,$$

uniformly in t_0, as $\tau \to \infty$ (where μ is Lebesgue measure, and NC is the previously defined set of noncoincident times).

LEMMA 4.12. *Suppose that φ is strong Pisot and satisfies both GCC and the no cycle condition. If $\widetilde{\varphi}$ is derived from φ by a one-cut rewriting, then*

$$\mathcal{A}_{\widetilde{\varphi},\mathrm{EGBP}} := \left\{ \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} : \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} \text{ is an essential geometrically balanced pair for } \widetilde{\varphi} \right\}$$

is finite.

Proof. Suppose that $\binom{\widetilde{u}^{-n}}{\widetilde{v}^{-n}}$ is a sequence of irreducible geometrically balanced pairs for $\widetilde{\varphi}$ with $\binom{\widetilde{u}}{\widetilde{v}}$ a factor of $\widetilde{\varphi}^n_{\mathrm{GBP}}\big(\binom{\widetilde{u}^{-n}}{\widetilde{v}^{-n}}\big)$ for $n \in \mathbb{N}$. Then the corresponding balanced pairs $\binom{u}{v}$ and $\binom{u^{-n}}{v^{-n}}$ for φ have the property that $\binom{u}{v}$ is a factor of $\varphi^n_{\mathrm{BP}}\big(\binom{u^{-n}}{v^{-n}}\big)$ for $n \in \mathbb{N}$. It follows that the irreducible factors of $\binom{u}{v}$ are all essential. Now, the cut letter a (or the word ba, in the case $\widetilde{\varphi}$ is obtained from φ as in Case 2 of the definition of $\widetilde{\varphi}$) occurs with bounded gap. That is, there is an $N \in \mathbb{N}$ so that if $W \in \mathcal{L}_\varphi$ has length at least N, then W contains a (ba, respectively) as a factor. It follows from Lemma 4.11 that there is a $K \in \mathbb{N}$ so that if $W \in \mathcal{L}_{\varphi_{\mathrm{EBP}}}$ with $|W| \geq K$, then W contains a factor that is itself a product of N trivial balanced pairs. Thus, if $\binom{\widetilde{u}}{\widetilde{v}}$ is an essential geometrically balanced pair for $\widetilde{\varphi}$ that is long enough so that the corresponding $\binom{u}{v}$ factors into a product of at least K essential balanced pairs for φ, then

$$\begin{pmatrix} u \\ v \end{pmatrix} = \dots \begin{pmatrix} i_1 \\ i_1 \end{pmatrix} \dots \begin{pmatrix} i_N \\ i_N \end{pmatrix} \dots$$

with $i_j \in \mathcal{A}$ for $j = 1, \dots, N$. There must be j with $i_j = a$ (or $i_j i_{j+1} = ba$, respectively), hence $\binom{\widetilde{u}}{\widetilde{v}}$ is not irreducible and thus not essential. It follows that there are only finitely many essential geometrically balanced pairs for $\widetilde{\varphi}$. ■

Suppose, for the remainder of this section, that φ *is strong Pisot and satisfies GCC and the no cycle condition on periodic words.* In addition, $\widetilde{\varphi}$ *is obtained from* φ *by a one-cut rewriting.* Let

$$\mathcal{T}^{\mathrm{P}}_\varphi = \left\{ \begin{pmatrix} T \\ T' \end{pmatrix} : T, T' \in \mathcal{T}_\varphi \text{ and } T, T' \text{ are proximal} \right\}$$

have the natural (product) topology. It follows from Proposition 4.9 and Lemma 4.10 that $\mathcal{T}^{\mathrm{P}}_\varphi \simeq \mathcal{T}_{\varphi_{\mathrm{EBP}}}$. Let $\widetilde{\varphi}_{\mathrm{EGBP}} : \mathcal{A}_{\widetilde{\varphi},\mathrm{EGBP}} \to (\mathcal{A}_{\widetilde{\varphi},\mathrm{EGBP}})^*$ be the substitution (on a finite alphabet, by Lemma 4.12) given by

$$\widetilde{\varphi}_{\mathrm{EGBP}} \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} = \begin{pmatrix} \widetilde{\varphi}(\widetilde{u}) \\ \widetilde{\varphi}(\widetilde{v}) \end{pmatrix},$$

factored as a product of essential geometrically balanced pairs. Let $\mathcal{A}_{\widetilde{\varphi}} = \{1, \dots, \widetilde{d}\}$. (Note that the symbol $i \in \mathcal{A}_\varphi$ is not equal to the symbol $i \in \mathcal{A}_{\widetilde{\varphi}}$, since $\widetilde{\varphi}$ is a one-cut rewriting of φ and not an extension of φ.) If we order the elements of $\mathcal{A}_{\widetilde{\varphi},\mathrm{EGBP}}$ as $\{\widetilde{e}_1, \dots, \widetilde{e}_n\}$, where for $i = 1, \dots, \widetilde{d}$, $\widetilde{e}_i$ denotes the trivial geometrically balanced pair $\binom{i}{i}$ for $\widetilde{\varphi}$, then the matrix for $\widetilde{\varphi}_{\mathrm{EGBP}}$ has the form $\left(\begin{smallmatrix} \widetilde{A} & \widetilde{B} \\ 0 & \widetilde{C} \end{smallmatrix}\right)$ where $\widetilde{A} = A_{\widetilde{\varphi}}$. Again, if

$$\begin{pmatrix} \widetilde{A} & \widetilde{B} \\ 0 & \widetilde{C} \end{pmatrix}^n = \begin{pmatrix} \widetilde{A}^n & \widetilde{B}_n \\ 0 & \widetilde{C}^n \end{pmatrix},$$

then $\widetilde{B}_n$ is strictly positive for sufficiently large n. Thus (see Lemma 4.11) if $\widetilde{S} = \{\widetilde{S}_k\}_{k=-\infty}^{\infty} \in \mathcal{T}_{\widetilde{\varphi}_{\mathrm{EGBP}}}$, the tiles $\widetilde{S}_k$ are predominantly of types $\{\widetilde{e}_1, \dots, \widetilde{e}_{\widetilde{d}}\}$.

In fact, there must be $\{k_m\}_{m=-\infty}^{\infty}$ with $\lim_{m\to-\infty} k_m = -\infty$, $\lim_{m\to\infty} k_m = \infty$, and $\widetilde{S}_{k_m}, \widetilde{S}_{k_m+1}, \dots, \widetilde{S}_{k_m+|m|}$ all of trivial type for each m.

We can interpret $\widetilde{S}$ as a pair $\binom{\widetilde{T}}{\widetilde{T}'}$ of tilings in $\mathcal{T}_{\widetilde{\varphi}}$: if $\widetilde{S}_k$ is a tile of type $\widetilde{e}_i = \binom{\widetilde{u}}{\widetilde{v}}$, tiling the interval $[a,b)$, then in $\widetilde{T}$ and $\widetilde{T}'$, the interval $[a,b)$ is tiled following the patterns $\widetilde{u}$ and $\widetilde{v}$, respectively. Under this interpretation, $\widetilde{T}$ and $\widetilde{T}'$ are proximal. Conversely, if $\widetilde{T}$ and $\widetilde{T}'$ are proximal in $\mathcal{T}_{\widetilde{\varphi}}$, then the corresponding T, T' in $\mathcal{T}_{\varphi}$ are proximal and the pair $\binom{T}{T'}$ determines $S \in \mathcal{T}_{\varphi_{\mathrm{EBP}}}$. The tiling S in turn uniquely determines $\widetilde{S} \in \mathcal{T}_{\widetilde{\varphi}_{\mathrm{EGBP}}}$ [for instance, any two consecutive occurrences of $\binom{a}{a}$ in S (or $\binom{b}{b}\binom{a}{a}$ in the case $\widetilde{\varphi}$ is constructed as in Case 2) uniquely determine the decomposition of the associated section of $\binom{\widetilde{T}}{\widetilde{T}'}$ into essential geometrically balanced pairs].

Thus $\mathcal{T}_{\widetilde{\varphi}}^{\mathrm{P}} \simeq \mathcal{T}_{\widetilde{\varphi}_{\mathrm{EGBP}}}$ and, since the homeomorphism $T \to \widetilde{T}$ takes proximal pairs to proximal pairs, we have

Proposition 4.13. *Suppose that φ is strong Pisot and satisfies GCC and the no cycle condition. Then*

$$\mathcal{T}_{\varphi_{\mathrm{EBP}}} \simeq \mathcal{T}_{\varphi}^{\mathrm{P}} \simeq \mathcal{T}_{\widetilde{\varphi}}^{\mathrm{P}} \simeq \mathcal{T}_{\widetilde{\varphi}_{\mathrm{EGBP}}}.$$

The definition of *weak equivalence* for substitutions appears in [BD1]. For substitutions $\widetilde{\varphi}$ and $\widetilde{\psi}$, this reads as follows: $\widetilde{\varphi}$ and $\widetilde{\psi}$ are *weak equivalent*, denoted by $\widetilde{\varphi} \sim_{\mathrm{w}} \widetilde{\psi}$, if for $i \in \mathbb{N}$, there are $n_i, m_i \in \mathbb{N}$ and morphisms $\tau_i : \mathcal{A}_{\widetilde{\varphi}} \to (\mathcal{A}_{\widetilde{\psi}})^*$, $\sigma_i : \mathcal{A}_{\widetilde{\psi}} \to (\mathcal{A}_{\widetilde{\varphi}})^*$ so that $\widetilde{\varphi}^{m_i} = \sigma_i\tau_i$ and $\widetilde{\psi}^{n_i} = \tau_i\sigma_{i+1}$:

(2)
$$\begin{tikzcd}
{} & & {} \arrow[ll, "(\widetilde{\varphi})^{m_1}"'] \arrow[dl, "\tau_1"] & & {} \arrow[ll, "(\widetilde{\varphi})^{m_2}"'] \arrow[dl, "\tau_2"] & & \cdots \\
& {} \arrow[ul, "\sigma_1"] & & {} \arrow[ll, "(\widetilde{\psi})^{n_1}"] \arrow[ul, "\sigma_2"] & & {} \arrow[ll, "(\widetilde{\psi})^{n_2}"] \arrow[ul, "\sigma_3"] &
\end{tikzcd}$$

The next result follows immediately from known results. However, we make use of the details of the argument in the proof of Theorem 4.15 and include them for completeness.

Lemma 4.14. *Suppose that φ and ψ are strong Pisot and satisfy both GCC and the no cycle condition. Let $h : \mathcal{T}_{\varphi} \to \mathcal{T}_{\psi}$ be a homeomorphism. There are one-cut rewritings $\widetilde{\varphi}$ of φ and $\widetilde{\psi}$ of ψ so that $\widetilde{\varphi} \sim_{\mathrm{w}} \widetilde{\psi}$ and*

$$\varprojlim f_{\widetilde{\varphi}} \simeq \mathcal{T}_{\widetilde{\varphi}} \simeq \mathcal{T}_{\varphi} \simeq \mathcal{T}_{\psi} \simeq \mathcal{T}_{\widetilde{\psi}} \simeq \varprojlim f_{\widetilde{\psi}}.$$

Proof. Let $h : \mathcal{T}_{\varphi} \to \mathcal{T}_{\psi}$ be a homeomorphism. Assume that h is orientation preserving; otherwise replace h by its *reverse* (see [BD1]). Also, we may assume that φ and ψ are such that all asymptotic composants in $\mathcal{T}_{\varphi}$ and $\mathcal{T}_{\psi}$ are fixed by inflation and substitution ([BD1]). Since h takes asymptotic composants to asymptotic composants, and any composant fixed under inflation and substitution contains a tiling fixed under inflation and substitution, we may modify h by an isotopy to a homeomorphism h' so that (1) for $T \in \mathcal{T}_\varphi$, $h'(T) = h(T) + t$, where $t = t(T)$, and (2) for some $T \in \mathcal{T}_\varphi$, $\Phi(T) = T$ and $\Psi(h'(T)) = h'(T)$. To simplify notation, we assume that h itself has this property.

We use these fixed tilings to determine the one-cut rewritings $\widetilde{\varphi}$ of φ and $\widetilde{\psi}$ of ψ. The roses $R_{\widetilde{\varphi}}$ and $R_{\widetilde{\psi}}$ are formed by taking the disjoint unions of the collections of prototiles and identifying all of the endpoints to a single branch point b. The prototiles P_a, for $a \in \mathcal{A}_{\widetilde{\varphi}}$ or $a \in \mathcal{A}_{\widetilde{\psi}}$, then become the petals of the roses. The rose maps $f_{\widetilde{\varphi}} : R_{\widetilde{\varphi}} \to R_{\widetilde{\varphi}}$ and $f_{\widetilde{\psi}} : R_{\widetilde{\psi}} \to R_{\widetilde{\psi}}$ fix b and map the petals following the patterns described by $\widetilde{\varphi}$ and $\widetilde{\psi}$, locally stretching arc length by a factor of $\lambda = \lambda_{\widetilde{\varphi}}$ or $\lambda = \lambda_{\widetilde{\psi}}$. Since $\widetilde{\varphi}$ and $\widetilde{\psi}$ are proper, it follows from Theorem 4.3 of [AP] that

$$\varprojlim f_{\widetilde{\varphi}} \simeq \mathcal{T}_{\widetilde{\varphi}} \simeq \mathcal{T}_\varphi \simeq \mathcal{T}_\psi \simeq \mathcal{T}_{\widetilde{\psi}} \simeq \varprojlim f_{\widetilde{\psi}}.$$

The homeomorphism $\widehat{p}_{\widetilde{\varphi}} : \mathcal{T}_{\widetilde{\varphi}} \to \varprojlim f_{\widetilde{\varphi}}$ is defined by

$$\widehat{p}_{\widetilde{\varphi}}(T) = (p_{\widetilde{\varphi}}(T), p_{\widetilde{\varphi}}(\widetilde{\Phi}^{-1}(T)), p_{\widetilde{\varphi}}(\widetilde{\Phi}^{-2}(T)), \ldots),$$

where $p_{\widetilde{\varphi}}(T) = s \in P_a \subseteq R_{\widetilde{\varphi}}$ provided T_0, the tile in T containing 0, is the translated prototile P_a with $T_0 = P_a - s$. It follows that the homeomorphism of $\varprojlim f_{\widetilde{\varphi}}$ with $\varprojlim f_{\widetilde{\psi}}$ takes $(b, b, \ldots)$ to $(b, b, \ldots)$, where b denotes the branch point in both $R_{\widetilde{\varphi}}$ and $R_{\widetilde{\psi}}$. According to the proof of Theorem 1.16 of [BJV], $\widetilde{\varphi}$ is weakly equivalent to $\widetilde{\psi}$. ∎

THEOREM 4.15. *Suppose that φ and ψ are strong Pisot and satisfy both GCC and the no cycle condition. Then $\mathcal{T}_\varphi \simeq \mathcal{T}_\psi$ if and only if $\mathcal{T}_{\varphi_{\rm EBP}} \simeq \mathcal{T}_{\psi_{\rm EBP}}$.*

Proof. Suppose that $\mathcal{T}_{\varphi_{\rm EBP}} \simeq \mathcal{T}_{\psi_{\rm EBP}}$. The original tiling spaces $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ sit in $\mathcal{T}_{\varphi_{\rm EBP}}$ and $\mathcal{T}_{\psi_{\rm EBP}}$ as distinguished subspaces; for example, they are the unique subspaces irreducible with respect to the property of being indecomposable. It follows that $\mathcal{T}_\varphi \simeq \mathcal{T}_\psi$.

Now suppose that $\mathcal{T}_\varphi \simeq \mathcal{T}_\psi$. According to Proposition 4.13, in order to show that $\mathcal{T}_{\varphi_{\rm EBP}} \simeq \mathcal{T}_{\psi_{\rm EBP}}$, it is enough to show that $\mathcal{T}_{\widetilde{\varphi}_{\rm EGBP}} \simeq \mathcal{T}_{\widetilde{\psi}_{\rm EGBP}}$ for an appropriate choice of $\widetilde{\varphi}, \widetilde{\psi}$. We take $\widetilde{\varphi}$ and $\widetilde{\psi}$ to be as in the proof of Lemma 4.14, so that $\widetilde{\varphi}$ is weakly equivalent to $\widetilde{\psi}$.

Let s_i and t_i be the matrices (abelianizations) of the morphisms σ_i and τ_i (see (2)), so that $(A_{\widetilde{\varphi}})^{m_i} = s_i t_i$ and $(A_{\widetilde{\psi}})^{n_i} = t_i s_{i+1}$.

Suppose that $\binom{\widetilde{u}}{\widetilde{v}}$ is a geometrically balanced pair for $\widetilde{\varphi}$, and let $\omega_{{\rm L},\widetilde{\varphi}}$ denote a left Perron–Frobenius eigenvector for $A_{\widetilde{\varphi}}$. Then $|\widetilde{u}|_g = |\widetilde{v}|_g$, so $\sum |\widetilde{u_i}|_g = \sum |\widetilde{v_i}|_g$ and $\langle l(\widetilde{u}), \omega_{{\rm L},\widetilde{\varphi}}\rangle = \langle l(\widetilde{v}), \omega_{{\rm L},\widetilde{\varphi}}\rangle$ (where $\langle\, ,\rangle$ denotes the usual Euclidean inner product). That is, $\langle l(\widetilde{u}) - l(\widetilde{v}), \omega_{{\rm L},\widetilde{\varphi}}\rangle = 0$, which implies that

$l(\widetilde{u})-l(\widetilde{v}) \in E^{\mathrm{s}}_{\widetilde{\varphi}}$ and $(A_{\widetilde{\varphi}})^n(l(\widetilde{u})-l(\widetilde{v})) \to 0$ as $n \to \infty$. But $l(\widetilde{u})-l(\widetilde{v}) \in \mathbb{Z}^{\widetilde{d}}$, so $(A_{\widetilde{\varphi}})^n(l(\widetilde{u})-l(\widetilde{v})) = 0$ for some n, and $l(\widetilde{u})-l(\widetilde{v}) \in N_{\widetilde{\varphi}}$, the generalized null space of $A_{\widetilde{\varphi}}$.

Choose j large enough so that $m := \sum_{i=2}^{j} m_i \geq \widetilde{d}$, and define $n = \sum_{i=1}^{j-1} n_i$. The fact that $t_1(A_{\widetilde{\varphi}})^m = (A_{\widetilde{\psi}})^n t_j$ implies that $t_j(N_{\widetilde{\varphi}}) \subseteq N_{\widetilde{\psi}}$. Therefore $t_j(l(\widetilde{u})) - t_j(l(\widetilde{v})) \in N_{\widetilde{\psi}}$ for sufficiently large j. That is, $\langle t_j(l(\widetilde{u})) - t_j(l(\widetilde{v})), \omega_{\mathrm{L},\widetilde{\psi}}\rangle = 0$, so that $|\tau_j(\widetilde{u})|_g = |\tau_j(\widetilde{v})|_g$. In other words, for sufficiently large j, τ_j takes geometrically balanced pairs for $\widetilde{\varphi}$ to geometrically balanced pairs for $\widetilde{\psi}$.

Similarly, for large enough j, $s_j(N_{\widetilde{\psi}}) \subseteq N_{\widetilde{\varphi}}$, and σ_j takes geometrically balanced pairs for $\widetilde{\psi}$ to geometrically balanced pairs for $\widetilde{\varphi}$. Thus we have a weak equivalence between $\widetilde{\varphi}_{\mathrm{GBP}}$ and $\widetilde{\psi}_{\mathrm{GBP}}$. It is clear that this restricts to a weak equivalence between $\widetilde{\varphi}_{\mathrm{EGBP}}$ and $\widetilde{\psi}_{\mathrm{EGBP}}$. A weak equivalence between substitutions induces a homeomorphism between their tiling spaces ([BD1]), so that $\mathcal{T}_{\widetilde{\varphi}_{\mathrm{EGBP}}} \simeq \mathcal{T}_{\widetilde{\psi}_{\mathrm{EGBP}}}$. ■

THEOREM 4.16. *Suppose that φ and ψ are strong Pisot substitutions that satisfy both GCC and the no cycle condition, and let $h : \mathcal{T}_\varphi \to \mathcal{T}_\psi$ be a homeomorphism. If T, T' are proximal in $\mathcal{T}_\varphi$, there is t_0 so that $h(T)$ and $h(T') + t_0$ are proximal in $\mathcal{T}_\psi$.*

Proof. Suppose that $h : \mathcal{T}_\varphi \to \mathcal{T}_\psi$ is a homeomorphism, and let $\widetilde{\varphi}$ and $\widetilde{\psi}$ be one-cut rewritings of φ and ψ associated with h defined as in the proof of Lemma 4.14. We also use $h : \mathcal{T}_{\widetilde{\varphi}} \to \mathcal{T}_{\widetilde{\psi}}$ to denote the homeomorphism between $\mathcal{T}_{\widetilde{\varphi}}$ and $\mathcal{T}_{\widetilde{\psi}}$ induced by the homeomorphisms $T \to \widetilde{T}$ associated with each of φ and ψ. Other notation in the following argument will serve to avoid confusion.

Recall that the homeomorphism $\widehat{p}_{\widetilde{\varphi}} : \mathcal{T}_{\widetilde{\varphi}} \to \varprojlim f_{\widetilde{\varphi}}$ is defined by

$$\widehat{p}_{\widetilde{\varphi}}(T) = (p_{\widetilde{\varphi}}(T), p_{\widetilde{\varphi}}(\widetilde{\Phi}^{-1}(T)), p_{\widetilde{\varphi}}(\widetilde{\Phi}^{-2}(T)), \ldots),$$

where $p_{\widetilde{\varphi}}(T) = s \in P_a \subseteq R_{\widetilde{\varphi}}$ if T_0, the tile in T containing 0, is the translated prototile P_a with $T_0 = P_a - s$. Given $T = \{T_i\}_{i=-\infty}^{\infty}$, let $\underline{w}(T) = \ldots w_{-1}w_0w_1\ldots$ denote the bi-infinite word representing T (that is, $w_i = j$ if and only if T_i is of type j). Recall that $[\underline{w}(T)]$, the shift class of $\underline{w}(T)$, is the pattern of the composant of T. Define the kth projection map $\pi_k : \varprojlim f_{\widetilde{\varphi}} \to R_{\widetilde{\varphi}}$ by $\pi_k(x_0, x_1, \ldots) = x_k$. Then the path $t \mapsto \pi_k(\widehat{p}(T+t))$ winds around $R_{\widetilde{\varphi}}$ following the pattern of the composant of $\widetilde{\Phi}^{-k}(T)$.

The weak equivalence between $\widetilde{\varphi}$ and $\widetilde{\psi}$ induced by the homeomorphism $\widehat{h} := \widehat{p}_{\widetilde{\psi}} \circ h \circ \widehat{p}_{\widetilde{\varphi}}^{-1}$ arises as follows (see [BJV] for details). For $a \in \mathcal{A}_{\widetilde{\varphi}}$, let $\mathring{P}_a = P_a \setminus \{b\} \subseteq R_{\widetilde{\varphi}}$. For each $k \in \mathbb{N}$ and $a \in \mathcal{A}_{\widetilde{\varphi}}$, the set $\overline{\pi_k^{-1}(\mathring{P}_a)}$ is homeomorphic to the product of a Cantor set ($\pi_k^{-1}(\{x\})$, where $x \in \mathring{P}_a$) and an arc. The larger is k, the longer and skinnier is this product and the closer are all of its endpoints $\overline{\pi_k^{-1}(\mathring{P}_a)} \cap \pi_k^{-1}(\{b\})$ to the point $\underline{b} = (b, b, \ldots)$. A similar statement can be made for $\varprojlim f_{\widetilde{\psi}}$.

Thus, there is an l_1 large enough so that, for each $a \in \mathcal{A}_{\widetilde{\psi}}$, the points $\pi_0(\widehat{h}^{-1}(\overline{\pi_{l_1}^{-1}(\mathring{P}_a)} \cap \pi_{l_1}^{-1}(\{b\})))$ in $R_{\widetilde{\varphi}}$ are all close to b, and all the arcs of $\widehat{h}^{-1}(\overline{\pi_{l_1}^{-1}(\mathring{P}_a)})$ map under π_0 around $R_{\widetilde{\varphi}}$ in the same pattern, which we denote by $\sigma_1(a) \in \mathcal{A}^*_{\widetilde{\varphi}}$.

There is now an m_1 large enough so that for each $a \in \mathcal{A}_{\widetilde{\varphi}}$, the points $\pi_{l_1}(\widehat{h}(\overline{\pi_{m_1}^{-1}(\mathring{P}_a)} \cap \pi_{m_1}^{-1}(\{b\})))$ are all close to b in $R_{\widetilde{\psi}}$, and the arcs of $\widehat{h}(\overline{\pi_{m_1}^{-1}(\mathring{P}_a)})$ all map in the same well-defined pattern, which we label $\tau_1(a) \in \mathcal{A}^*_{\widetilde{\psi}}$, around $R_{\widetilde{\psi}}$, etc. We see from this description that if $[\underline{w}]$ is the pattern of the composant of $\widetilde{\Phi}^{-m_1}(T)$, then $[\tau_1(\underline{w})]$ is the pattern of the composant of $\widetilde{\Psi}^{-l_1}(h(T))$.

Suppose that T and T' are proximal in $\mathcal{T}_\varphi$. Then $\widetilde{T}$ and $\widetilde{T}'$ are proximal in $\mathcal{T}_{\widetilde{\varphi}}$, and hence so are $\widetilde{\Phi}^{-m_1}(\widetilde{T})$ and $\widetilde{\Phi}^{-m_1}(\widetilde{T}')$. Thus the pair of words $\binom{\widetilde{\Phi}^{-m_1}(\widetilde{T})}{\widetilde{\Phi}^{-m_1}(\widetilde{T}')}$ factors as a bi-infinite product of essential geometrically balanced pairs for $\widetilde{\varphi}$. Apply τ_1 to this product; the result factors as a bi-infinite product of essential geometrically balanced pairs for $\widetilde{\psi}$ (see the proof of Theorem 4.15). Thus the patterns $[\tau_1 \underline{w}(\widetilde{\Phi}^{-m_1}(\widetilde{T}))]$ and $[\tau_1 \underline{w}(\widetilde{\Phi}^{-m_1}(\widetilde{T}'))]$, appropriately shifted, balance geometrically. This means that $\widetilde{\Psi}^{-l_1}(h(T))$ and $\widetilde{\Psi}^{-l_1}(h(T')) + t_1$ are proximal in $\mathcal{T}_{\widetilde{\psi}}$ for some t_1, and hence so are $h(T)$ and $h(T') + t_0$, where $t_0 = (\lambda_{\widetilde{\psi}})^{l_1} t_1$. ∎

Example 4.17 (Using proximality to distinguish tiling spaces). Define φ and ψ as follows:

$$\varphi(1) = 11122111221112211212, \quad \varphi(2) = 1112212,$$
$$\psi(1) = 1112211121212121212, \quad \psi(2) = 1112212.$$

The substitution φ is the second iterate of that considered in Examples 4.1, 4.2 and 4.6, and the substitution ψ is obtained by modifying φ slightly.

The basic structure of the asymptotic composants is the same: A proper substitution on two symbols can have at most four asymptotic composants ([BDH]), and $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ both have a pair of backward asymptotic and a pair of forward asymptotic composants. For φ, these are represented by the bi-infinite words (spaced to indicate balanced pairs)

$$\ldots 1\ 2\ 1\ 1\ 1\ 2\ 2\ 1\ \underline{1}12\ 21\ 112\ 21\ 2 \ldots$$
$$\ldots 1\ 2\ 1\ 1\ 1\ 2\ 2\ 1\ 2.11\ 12\ 211\ 12\ 2 \ldots$$

and

$$\ldots 2\ 211\ 12\ 2\ 1\ 112\ 2\underline{1}\ 2\ 1\ 2\ 1\ 1\ 1\ \ldots$$
$$\ldots 1\ 112\ 21\ 2\ 1\ 211\ 1\underline{2}\ 2\ 1\ 2\ 1\ 1\ 1\ \ldots$$

and for ψ by

$$\ldots 1\ 1\ 1\ 2\ 2\ 1\ \underline{1}12\ 1\ 2\ 12\ 1\ 21\ 2 \ldots$$
$$\ldots 1\ 1\ 1\ 2\ 2\ 1\ 2.11\ 1\ 2\ 21\ 1\ 12\ 2 \ldots$$

and

$$\ldots 1\ 12\ 12\ 1\ 2\underline{1}\ 2\ 1\ 2\ 1\ 1\ 1\ \ldots$$
$$\ldots 1\ 21\ 21\ 1\ 1\underline{2}\ 2\ 1\ 2\ 1\ 1\ 1\ \ldots$$

It is easy to check that φ and ψ are strong Pisot. All strong Pisot substitutions on two symbols satisfy GCC (see [BD2], [HS], and [BK] for the unimodular case).

Tilings, asymptotic composants, and essential balanced pairs are identical for all iterates of a substitution, and we saw in Examples 4.1 and 4.6 that φ has at least three essential balanced pairs which generate the backward asymptotic (and forward proximal) tilings, T and T', and a third tiling T'' proximal in both directions with each of the first two. This can be seen from φ_{EBP}, which includes at least the information (recall that φ is the second iterate of the substitution in Example 4.1):

$$\begin{aligned}
\varphi_{\mathrm{EBP}}(a) &= 111221b\overline{a}b\overline{a}21\overline{b}a212,\\
\varphi_{\mathrm{EBP}}(b) &= 111221b\overline{a}b\overline{a}21\overline{b}a21b\overline{a}b\overline{a}21\overline{b}a212,\\
\varphi_{\mathrm{EBP}}(c) &= 111221b\overline{a}b\overline{a}21\overline{b}a21b\overline{a}c111221\overline{b}a212,
\end{aligned}$$

along with the definition of φ_{EBP} on duals. In particular, φ_{EBP} has two additional backward asymptotic composants indicating that T'' is proximal to T and T' in $\mathcal{T}_\varphi$:

$$\ldots 1b\overline{a}\underline{b}\overline{a}21 \ldots$$
$$\ldots 1b\overline{a}\underline{c}1\overline{a}2 \ldots$$

along with the three that capture the two original backward asymptotic composants:

$$\ldots 111221\underline{1}122 \ldots$$
$$\ldots 1112212.122 \ldots$$
$$\ldots 111221\underline{b}\overline{a}b\overline{a} \ldots.$$

Suppose that $\mathcal{T}_\varphi \simeq \mathcal{T}_\psi$. According to Theorem 4.15, $\mathcal{T}_{\varphi_{\mathrm{EBP}}} \simeq \mathcal{T}_{\psi_{\mathrm{EBP}}}$. As in the proof of Lemma 4.14, we assume the homeomorphism is orientation preserving. Since such a homeomorphism must take backward asymptotic composants to backward asymptotic composants, $\mathcal{T}_{\psi_{\mathrm{EBP}}}$ must have at least five backward asymptotic composants.

The proof of the next lemma appears in the Appendix.

LEMMA 4.18. *The balanced pairs* $\binom{1}{1}, \binom{2}{2}, \binom{21}{12}, \binom{211}{122}$, *and their duals are the only essential balanced pairs for* ψ.

It follows from Lemma 4.18 that the substitution ψ_{EBP} is entirely given by

$$\begin{aligned}\psi_{\mathrm{EBP}}(a) &= 111221b12a1\overline{a}\,\overline{a}1a212,\\ \psi_{\mathrm{EBP}}(b) &= 111221b12a1\overline{a}\,\overline{a}\,\overline{a}1a21b12a1\overline{a}\,\overline{a}\,\overline{a}1a212\end{aligned}$$

and the implied definition on trivial balanced pairs and the duals of a, b. The only backward asymptotic words for ψ_{EBP} are

$$\begin{gathered}\ldots 111221\underline{1}121\ldots\\ \ldots 111221\underline{b}12a\ldots\\ \ldots 111221\underline{2}111\ldots\end{gathered}$$

which code the original backward asymptotic composants. That is, $\mathcal{T}_{\psi_{\mathrm{EBP}}}$ has only three backward asymptotic composants, and $\mathcal{T}_\varphi \not\approx \mathcal{T}_\psi$. ∎

Suppose that φ and ψ are strong Pisot and satisfy both GCC and the no cycle condition. If $\mathcal{T}_\varphi \simeq \mathcal{T}_\psi$, then $\mathcal{T}_\varphi^{\mathrm{P}}$ is homeomorphic to $\mathcal{T}_\psi^{\mathrm{P}}$ under a homeomorphism that maps $\mathcal{T}_\varphi$ to $\mathcal{T}_\psi$, hence $\mathcal{T}_\varphi^{\mathrm{P}}/\mathcal{T}_\varphi \simeq \mathcal{T}_\psi^{\mathrm{P}}/\mathcal{T}_\psi$ (we are identifying $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ with the "diagonals" in $\mathcal{T}_\varphi^{\mathrm{P}}$ and $\mathcal{T}_\psi^{\mathrm{P}}$). The space $\mathcal{T}_\varphi^{\mathrm{P}}/\mathcal{T}_\varphi$ has a local product structure everywhere but at $[\mathcal{T}_\varphi]$. The element $[\mathcal{T}_\varphi]$ itself is the center of an m-od, where $m = 2(\#\text{ asymptotic pairs})$. Let $R_{\varphi_{\mathrm{EBP}}}$ be the rose associated with $\mathcal{T}_\varphi^{\mathrm{P}} \simeq \mathcal{T}_{\varphi_{\mathrm{EBP}}}$, and let $f_{\varphi_{\mathrm{EBP}}} : R_{\varphi_{\mathrm{EBP}}} \to R_{\varphi_{\mathrm{EBP}}}$ be the rose map. Collapsing R_φ (which is embedded in $R_{\varphi_{\mathrm{EBP}}}$) to the branch point induces a substitution φ_P on just the symbols in $\mathcal{A}_{\mathrm{EBP}}$ that correspond to nontrivial essential balanced pairs and the map $f_{\varphi_P} : R_{\varphi_P} := R_{\varphi_{\mathrm{EBP}}}/R_\varphi \to R_{\varphi_P}$. Furthermore, $\mathcal{T}_\varphi^{\mathrm{P}}/\mathcal{T}_\varphi \simeq \varprojlim f_{\varphi_P}$.

Assuming still that φ and ψ are strong Pisot and satisfy both GCC and the no cycle condition, suppose that $\mathcal{T}_\varphi \simeq \mathcal{T}_\psi$. Any homeomorphism of $\mathcal{T}_\varphi^{\mathrm{P}}$ with $\mathcal{T}_\psi^{\mathrm{P}}$ not only takes $\mathcal{T}_\varphi$ to $\mathcal{T}_\psi$ but must also take arc components of $\mathcal{T}_\varphi^{\mathrm{P}}$ that are asymptotic to asymptotic composants of $\mathcal{T}_\varphi$ to arc components of $\mathcal{T}_\psi^{\mathrm{P}}$ that are asymptotic to asymptotic composants of $\mathcal{T}_\psi$ (again, $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are identified with the "diagonals" in $\mathcal{T}_\varphi^{\mathrm{P}}$ and $\mathcal{T}_\psi^{\mathrm{P}}$). Let $\mathcal{T}_\varphi^{\mathrm{A}}$ be the minimal subcontinuum of $\mathcal{T}_\varphi^{\mathrm{P}}$ that contains all arc components of $\mathcal{T}_\varphi^{\mathrm{P}}$ that are asymptotic to asymptotic composants of $\mathcal{T}_\varphi$. Thus $\mathcal{T}_\varphi \simeq \mathcal{T}_\psi$ implies $\mathcal{T}_\varphi^{\mathrm{A}} \simeq \mathcal{T}_\psi^{\mathrm{A}}$.

Note that $\binom{T'}{T''} \in \mathcal{T}_\varphi^{\mathrm{P}}$ is asymptotic to $\binom{T}{T} \in \mathcal{T}_\varphi \subset \mathcal{T}_\varphi^{\mathrm{P}}$ if and only if T, T', T'' are all asymptotic (in the same direction). Thus the pairs $\binom{T'}{T''} \in \mathcal{T}_\varphi^{\mathrm{A}}$ are precisely those proximal pairs all of whose bubbles come from asymptotic pairs (see Proposition 4.9). More precisely, let

$$\mathrm{ABP}_\varphi = \left\{ \binom{u}{v} : \binom{u}{v} \text{ is a trivial balanced pair for } \varphi \right.$$
$$\left. \text{or } \binom{u}{v} \text{ is obtained from a bubble in an asymptotic pair for } \varphi \right\}.$$

As Φ takes asymptotic pairs to asymptotic pairs, for $\binom{u}{v} \in \mathrm{ABP}_\varphi$, $\binom{\varphi(u)}{\varphi(v)}$ can be factored as a product of elements of ABP_φ; this defines $\varphi_{\mathrm{ABP}} : \mathrm{ABP}_\varphi \to (\mathrm{ABP}_\varphi)^*$. We see that $\mathcal{T}_{\varphi_{\mathrm{ABP}}} \simeq \mathcal{T}_\varphi^{\mathrm{A}}$. Letting φ_{A} be the substitution on the nontrivial elements of ABP_φ that is the composition of φ_{ABP} with the morphism that forgets the trivial balanced pairs, and letting $f_{\varphi_{\mathrm{A}}} : R_{\varphi_{\mathrm{A}}} \to R_{\varphi_{\mathrm{A}}}$ be the associated rose map, we see (just as in the preceding paragraph) that $\mathcal{T}_\varphi^{\mathrm{A}}/\mathcal{T}_\varphi \simeq \varprojlim f_{\varphi_{\mathrm{A}}}$. We have almost completed the proof of

Proposition 4.19. *Suppose that φ and ψ are strong Pisot and satisfy both GCC and the no cycle condition. If $\mathcal{T}_\varphi \simeq \mathcal{T}_\psi$, then*

(i) $\varprojlim f_{\varphi_{\mathrm{P}}} \simeq \varprojlim f_{\psi_{\mathrm{P}}}$,
(ii) $\varprojlim f_{\varphi_{\mathrm{A}}} \simeq \varprojlim f_{\psi_{\mathrm{A}}}$.

Moreover, (i) *is equivalent to* $\varphi_{\mathrm{P}} \sim_{\mathrm{w}} \psi_{\mathrm{P}}$ *and* (ii) *is equivalent to* $\varphi_{\mathrm{A}} \sim_{\mathrm{w}} \psi_{\mathrm{A}}$. (*The definition of* $\sim_{\mathrm{w}}$ *precedes Lemma* 4.14.)

Proof. (i) and (ii) follow from the discussion preceding this proposition. The branch points of $\varprojlim f_{\varphi_{\mathrm{P}}}$, $\varprojlim f_{\psi_{\mathrm{P}}}$, $\varprojlim f_{\varphi_{\mathrm{A}}}$ and $\varprojlim f_{\psi_{\mathrm{A}}}$ are distinguished, so by [BJV], $\varprojlim f_{\varphi_{\mathrm{P}}} \simeq \varprojlim f_{\psi_{\mathrm{P}}}$ if and only if $\varphi_{\mathrm{P}} \sim_{\mathrm{w}} \psi_{\mathrm{P}}$, and $\varprojlim f_{\varphi_{\mathrm{A}}} \simeq \varprojlim f_{\psi_{\mathrm{A}}}$ if and only if $\varphi_{\mathrm{A}} \sim_{\mathrm{w}} \psi_{\mathrm{A}}$. ∎

Remark. It can be tedious to identify all essential balanced pairs, even for relatively simple substitutions. At the same time, the alphabet for the substitution φ_{A} on nontrivial balanced pairs arising from bubbles formed by asymptotic pairs is no more difficult to determine than the asymptotic pairs themselves. As a result, the use of (ii) of Proposition 4.19 to distinguish nonhomeomorphic tiling spaces $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ is more straightforward when it applies. We illustrate this with the next example.

Example 4.20 (Distinguishing tiling spaces using the reduced substitution on balanced pairs). As part of his program to classify hyperbolic one-dimensional attractors (up to topological conjugacy), R. F. Williams ([Wil]) sought to determine the shift equivalence classes of all (there are 46) substitutions on two letters that are proper and whose abelianizations have characteristic polynomial $x^2 - 3x - 2$. Shift equivalence of proper substitutions is equivalent to topological conjugacy of the corresponding inflation and substitution homeomorphisms of the tiling spaces. Williams reduced the problem to the consideration of four particular substitutions, two of which:

$$\varphi(1) = 11221, \qquad \varphi(2) = 1,$$

and

$$\psi(1) = 112222, \qquad \psi(2) = 12,$$

were finally shown in [DA] not to be shift equivalent. We show here that $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are not even homeomorphic. This completes the topological classification of the spaces arising from the characteristic polynomial $x^2 - 3x - 2$: two of these spaces are homeomorphic if and only if their inflation and substitution homeomorphisms are conjugate, and there are exactly three topological equivalence classes (see [BSw]).

Each of $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ has one pair of forward and one pair of backward asymptotic composants. Since a tiling space for a proper substitution on two letters has at most four asymptotic composants ([BDH]), there are no others.

A proper substitution has no cycles of periodic words. Also, since φ and ψ are strong Pisot substitutions on two letters, they satisfy GCC (see [BD2], [HS], and [BK] for the unimodular case—the nonunimodular case is a straightforward generalization). It is tedious to ensure that all essential balanced pairs have been identified for either of φ and ψ, so we make use of Proposition 4.19 and work only with essential balanced pairs associated with asymptotic composants.

For φ, the only essential balanced pair associated with asymptotic composants is $a := \binom{1122}{2211}$ and its dual $\overline{a}$. If $1 := \binom{1}{1}$ and $2 := \binom{2}{2}$, then $\varphi_{\mathrm{ABP}}(i) = \varphi(i)$ for $i = 1, 2$, and

$$\varphi_{\mathrm{ABP}}(a) = 11\overline{a}1\overline{a}1, \qquad \varphi_{\mathrm{ABP}}(\overline{a}) = 11a1a1.$$

The reduced substitution is

$$\varphi_{\mathrm{A}}(a) = \overline{a}\overline{a}, \qquad \varphi_{\mathrm{A}}(\overline{a}) = aa,$$

and the inverse limit of the rose map, $\varprojlim f_{\varphi_{\mathrm{A}}}$, is homeomorphic to a pair of dyadic solenoids joined at a point.

As for ψ, there are ten essential balanced pairs associated with asymptotic composants: $a := \binom{12}{21}$, $b := \binom{2221}{1222}$, $c := \binom{1211222}{2212121}$, $d := \binom{22121}{11222}$, $e := \binom{1122}{2211}$, and their duals. A computation yields

$$\psi_{\mathrm{A}}(a) = ab, \qquad \psi_{\mathrm{A}}(b) = \overline{a}c, \qquad \psi_{\mathrm{A}}(ac) = ada\overline{e}\overline{c},$$
$$\psi_{\mathrm{A}}(d) = \overline{a}ec, \qquad \psi_{\mathrm{A}}(e) = a\overline{e}d$$

(and the dual statements). A quick check shows that ψ_{A} is primitive and aperiodic, so $\varprojlim f_{\psi_{\mathrm{A}}}$ is an indecomposable continuum, thus $\varprojlim f_{\varphi_{\mathrm{A}}} \not\approx \varprojlim f_{\psi_{\mathrm{A}}}$.

(Alternatively, $\check{H}^1(\varprojlim f_{\varphi_{\mathrm{A}}}) \simeq \varinjlim\left(\begin{smallmatrix}0&2\\2&0\end{smallmatrix}\right) \simeq \mathbb{Z}\oplus\mathbb{Z}$, while $\check{H}^1(\varprojlim f_{\psi_{\mathrm{A}}})$ is homeomorphic to the direct sum of ten copies of $\mathbb{Z}$, as A is nonsingular.) Thus, by Proposition 4.19(ii), $\mathcal{T}_\varphi \not\approx \mathcal{T}_\psi$. ■

5. GCC if and only if proximality is closed. The main result of this section is Theorem 5.4, in which we show that a strong Pisot substitution φ satisfies GCC if and only if proximality is a closed relation on $\mathcal{T}_\varphi$.

LEMMA 5.1. *Suppose that A is a nonsingular, hyperbolic, integer $d\times d$ matrix and $F_A : \mathcal{T}_d \to \mathcal{T}_d$ is the associated Anosov endomorphism. Suppose also that there is a compact metric space X with homeomorphism $F : X \to X$ and a covering map $c : X \to \varprojlim F_A$ that semi-conjugates F with the shift homeomorphism $\widehat{f}_A : \varprojlim F_A \to \varprojlim F_A$. Then there are $d\times d$ integer matrices B and R with $RB = AR$ and a homeomorphism $h : X \to \varprojlim F_B$ so that the diagram*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "c"'] & & \varprojlim F_B \arrow[dl, "\widehat{F}_R"] \\ & \varprojlim F_A & \end{tikzcd}$$

commutes, where $\widehat{F}_R$ is induced by $F_R : \mathbb{T}_d \to \mathbb{T}_d$.

Proof. Let c be an m-to-one covering map and let $r = \deg F_A = |\det A|$. Let $\pi_k : \varprojlim F_A \to \mathbb{T}_d$ be projection onto the kth coordinate. Given $\delta > 0$, let $\{U_i\}$ be a finite cover of $\mathbb{T}_d$ by open $\delta/4$-balls with the property that if $U_i \cap U_l = \emptyset$, then $\overline{U}_i \cap \overline{U}_l = \emptyset$. For each $k\in\mathbb{N}$, $F_A^{-k}(U_i) = U_i^1 \cup \cdots \cup U_i^{r^k}$ is a disjoint union of topological balls. For large enough k, $\operatorname{diam}(\pi_k^{-1}(U_i^j)) < \delta$ for all i,j. Then, for sufficiently small δ,

$$c^{-1}(\pi_k^{-1}(U_i^j)) = W_i^{j,1} \cup \cdots \cup W_i^{j,m}$$

is a disjoint union with $c|W_i^{j,s} : W_i^{j,s} \to \pi_k^{-1}(U_i^j)$ a homeomorphism for all i,j,s. Define the relation $\sim$ on X by $x \sim y$ if and only if $x, y \in W_i^{j,s}$ for some i,j,s and $\pi_k\circ c(x) = \pi_k \circ c(y)$. Note that if $x \in W_i^{j,s}\cap W_l^{t,q}$, $y \in W_i^{j,s}$ and $\pi_k\circ c(x) = \pi_k\circ c(y)$, then $y \in W_l^{t,q}$. It follows that $\sim$ is an equivalence relation. Let $X_1 = X/\!\sim$, and define $p_1 : X_1 \to \mathbb{T}^d$ by

$$p_1([x]) = \pi_k(c(x)).$$

Then p_1 is exactly m-to-one everywhere, and if $p_1([x]) = p_1([y])$ with $[x]\neq[y]$, then there are s,i,j,q,t,l with $x\in W_i^{j,s}$, $y \in W_l^{t,q}$, and $W_i^{j,s}\cap W_l^{t,q} = \emptyset$. Since $\overline{W}_i^{j,s} \cap \overline{W}_l^{t,q} = \emptyset$ by the assumption on U_i and U_l, the distance between $[x]$ and $[y]$ must be at least as large as the minimum of all the minimum distances between pairs of disjoint compact sets $\overline{W}_b^{c,a}$ and $\overline{W}_e^{f,g}$. That is, there is $\eta > 0$ so that if $p_1([x]) = p_1([y])$ and $[x] \neq [y]$, then $d([x],[y]) > \eta$. Thus p_1 is a covering map and since X_1 is compact, $X_1 \simeq \mathbb{T}^d$.

Now if $x \sim y$, say $x, y \in W_i^{j,s}$ with $\pi_k \circ c(x) = \pi_k \circ c(y)$, then

$$\begin{aligned}\pi_k \circ c(F(x)) &= \pi_k \circ \widehat{f}_A(c(x)) = F_A(\pi_k(c(x)))\\ &= F_A(\pi_k(c(y))) = \pi_k \circ \widehat{f}_A(c(y))\\ &= \pi_k \circ c(F(y)).\end{aligned}$$

We show that $F(x), F(y) \in W_l^{t,q}$ for some q, t, l, so that $F(x) \sim F(y)$. In order to guarantee this, we will adjust δ (and k correspondingly): Let $\varrho = \inf\{d(u,v) : u \neq v,\ c(u) = c(v)\} > 0$, and choose $\varepsilon > 0$ small enough so that if $d(x,y) < \varepsilon$, then $d(F(x),F(y)) < \varrho/2$. Now, for $\delta > 0$ sufficiently small, $\operatorname{diam}(W_i^{j,s}) < \varepsilon$ for all i, j, s. With this δ and k, let $\pi_k \circ c(F(x)) \in U_l^t$. Then $\pi_k \circ c(F(x)) = \pi_k \circ c(F(y)) \in U_l^t$, hence for some p, q, $F(x) \in W_l^{t,p}$ and $F(y) \in W_l^{t,q}$. Since $x, y \in W_i^{j,s}$, $d(x,y) < \varepsilon$. It follows that $d(F(x),F(y)) < \varrho/2$, hence $p = q$ and $F(x) \sim F(y)$.

Thus there is an induced map $F_1 : X_1 \to X_1$, and

$$\begin{aligned}p_1 \circ F_1([x]) &= p_1([F(x)]) = \pi_k \circ c(F(x))\\ &= \pi_k \circ \widehat{f}_A(c(x)) = F_A \circ \pi_k(c(x))\\ &= F_A \circ p_1([x]).\end{aligned}$$

That is, we have a commuting diagram of toral endomorphisms

$$\begin{tikzcd} X_1 \arrow[d, "p_1"'] & X_1 \arrow[l, "F_1"'] \arrow[d, "p_1"] \\ \mathbb{T}^d & \mathbb{T}^d \arrow[l, "F_A"'] \end{tikzcd}$$

It now follows that there are integer matrices R and B (with $|\det R| = m$) which satisfy $RB = AR$ and a homeomorphism $h_1 : X_1 \to \mathbb{T}^d$ so that $F_B = h_1 \circ F_1 \circ h_1^{-1}$, $F_R \circ h_1 = p_1$ and the diagram

$$\begin{tikzcd} \mathbb{T}^d \arrow[d, "F_R"'] & \mathbb{T}^d \arrow[l, "F_B"'] \arrow[d, "F_R"] \\ \mathbb{T}^d & \mathbb{T}^d \arrow[l, "F_A"'] \end{tikzcd}$$

commutes.

Claim. $\varprojlim F_1 \simeq X$.

The commuting diagram

$$\begin{tikzcd}
& X \arrow[dl, "\pi"'] \arrow[dd, "c"] & & X \arrow[ll, "F"'] \arrow[dl, "\pi"'] \arrow[dd, "c"] \\
X_1 \arrow[dd, "p_1"'] & & X_1 \arrow[ll, "F_1"'] \arrow[dd, "p_1"] & \\
& \varprojlim F_A \arrow[dl, "\pi_k"] & & \varprojlim F_A \arrow[ll, "\widehat{f}_A"'] \arrow[dl, "\pi_k"] \\
\mathbb{T}^d & & \mathbb{T}^d \arrow[ll, "F_A"] &
\end{tikzcd}$$

in which $\pi : X \to X/\!\sim\, = X_1$ is the quotient map, induces the commuting diagram

$$\begin{tikzcd}
& X \arrow[dl, "\widehat{\pi}"'] \arrow[dd, "\widehat{c}"] & & X \arrow[ll, "F"'] \arrow[dl, "\widehat{\pi}"'] \arrow[dd, "\widehat{c}"] \\
\varprojlim F_1 \arrow[dd, "\widehat{p}_1"'] & & \varprojlim F_1 \arrow[ll, "\widehat{f}_1"'] \arrow[dd, "\widehat{p}_1"] & \\
& \varprojlim F_A \arrow[dl, "\widehat{\pi}_k"] & & \varprojlim F_A \arrow[ll, "\widehat{f}_A"'] \arrow[dl, "\widehat{\pi}_k"] \\
\varprojlim F_A & & \varprojlim F_A \arrow[ll, "\widehat{f}_A"] &
\end{tikzcd}$$

Here $\widehat{\pi}_k = (\widehat{f_A})^{-k}$, $\widehat{c}$ is exactly m-to-one, and $\widehat{p}_1$ is exactly m-to-one. Thus $\widehat{\pi}$ is a homeomorphism. Moreover,

$$\begin{tikzcd}
X \arrow[rrr, "F"] \arrow[dr, "\widehat{\pi}"] \arrow[ddd, "c"'] & & & X \arrow[dl, "\widehat{\pi}"'] \arrow[ddd, "c"] \\
& \varprojlim F_1 \arrow[r, "\widehat{f}_1"] \arrow[d, "\widehat{h}_1"'] & \varprojlim F_1 \arrow[d, "\widehat{h}_1"] & \\
& \varprojlim F_B \arrow[r, "\widehat{f}_B"] \arrow[dl, "\widehat{F}_R"] & \varprojlim F_B \arrow[dr, "\widehat{F}_R"'] & \\
\varprojlim F_A \arrow[rrr, "\widehat{f}_A"'] & & & \varprojlim F_A
\end{tikzcd}$$

commutes. Letting $h = \widehat{h}_1 \circ \widehat{\pi}$, we have the desired result. ∎

Auslander proves under fairly general hypotheses that if $\sim_p$ is a closed relation, then it is an equivalence relation ([Aus]). We require this fact in Theorem 5.4 and include a brief proof in (3) of Lemma 5.2 for completeness.

LEMMA 5.2. *Suppose that φ is strong Pisot and that $\sim_p$ is a closed relation on $\mathcal{T}_\varphi \simeq \mathcal{T}_\varphi^{\mathrm{S}}$. Then*:

(1) *$\gamma \in \mathcal{T}_\varphi^{\mathrm{S}}$ is forward proximal to $\gamma' \in \mathcal{T}_\varphi^{\mathrm{S}}$ if and only if γ is backwards proximal to γ'.*
(2) *If $\{B_i\} = \mathcal{B}$ is the collection of all irreducible balanced pairs formed by proximal pairs in $\mathcal{T}_\varphi^{\mathrm{S}}$, then $\mathcal{B}$ is finite and each B_i terminates with coincidence.*
(3) *$\sim_p$ is an equivalence relation.*
(4) *If $\gamma \nsim_p \gamma'$ and $\widehat{g}(\gamma) = \widehat{g}(\gamma')$, then γ and γ' do not share an edge.*

Proof. (1) Suppose that γ and γ' are forward proximal, so that there is a sequence $\{t_n\}$ with $d(\gamma - t_n\omega_{\mathrm{R}}, \gamma' - t_n\omega_{\mathrm{R}}) \to 0$ as $t_n \to \infty$. If γ and γ' are not backward proximal, there are $t_0 \in \mathbb{R}$ and $\varepsilon > 0$ with $d(\gamma - t\omega_{\mathrm{R}}, \gamma' - t\omega_{\mathrm{R}}) \geq \varepsilon$ for $t \leq t_0$. Choose $\{s_n\}$ so that $s_n \to -\infty$ and $\gamma - s_n\omega_{\mathrm{R}} \to \eta$, $\gamma - s_n\omega_{\mathrm{R}} \to \eta'$. Since $\sim_p$ is closed, $\eta \sim_p \eta'$. There is $t \in \mathbb{R}$ so that $d(\eta - t\omega_{\mathrm{R}}, \eta' - t\omega_{\mathrm{R}}) < \varepsilon$. Then, for sufficiently large n, $d(\gamma - (s_n + t)\omega_{\mathrm{R}}, \gamma' - (s_n + t)\omega_{\mathrm{R}}) < \varepsilon$ and $s_n + t \leq t_0$, a contradiction. The converse can be proved by a symmetric argument.

(2) By (1), proximal pairs determine a bi-infinite product of irreducible balanced pairs. Suppose that there are arbitrarily long products of nontrivial balanced pairs that arise in such factorizations. Let $\gamma_i \sim_p \gamma_i'$ with γ_i and γ_i' forming the products W_i of nontrivial irreducible balanced pairs, where W_i is centered on E^{s} with $|W_i|_g \to \infty$. Without loss of generality, $\gamma_i \to \eta$ and $\gamma_i' \to \eta'$. The relation $\sim_p$ is closed, hence $\eta \sim_p \eta'$. But the definition of γ_i and γ_i' implies that $\eta \nsim_p \eta'$, a contradiction. Thus $\mathcal{B}$ is finite. Since $\gamma \sim_p \gamma'$ implies that $\Phi(\gamma) \sim_p \Phi(\gamma')$, each $B_i \in \mathcal{B}$ terminates with coincidence.

(3) Let $\widetilde{\varphi}$ be the restriction of the substitution on balanced pairs, $\varphi_{\mathrm{BP}} : \mathrm{BP}(\varphi) \to \mathrm{BP}(\varphi)$, to the set of irreducible (trivial and nontrivial) balanced pairs associated with proximal pairs. According to Lemma 4.11, tiles of type corresponding to trivial balanced pairs have frequency 1 in every element of $\mathcal{T}_{\widetilde{\varphi}}$. That is, coincidence in proximal pairs of $\mathcal{T}_\varphi^{\mathrm{S}}$ occurs with frequency 1, hence proximality is transitive.

(4) Suppose that $\widehat{g}(\gamma) = \widehat{g}(\gamma')$ and that γ and γ' coincide along some segment I. Let $m = \mathrm{cr}_\varphi$. By Theorem 4.5, for some $\underline{x} \in \varprojlim F_A$, $\widehat{g}^{-1}(\underline{x})$ consists of m strands, any two of which are nowhere coincident. By minimality of the flow on $\varprojlim F_A$, in every preimage $\widehat{g}^{-1}(\underline{y})$, there are at least m pairwise

nowhere coincident strands. It follows that

$$\mathcal{G} := \{\underline{y} : \widehat{g}^{-1}(\underline{y}) \text{ contains exactly } m \text{ pairwise nowhere coincident strands}\}$$

is a set of full measure, as is $\bigcap_{n\geq 0} \widehat{F}_A^{-n}(\mathcal{G})$, since $\widehat{F}_A$ is measure preserving. Pick $\underline{y} \in \bigcap_{n\geq 0} \widehat{F}_A^{-n}(\mathcal{G})$ and let $\widehat{g}^{-1}(\underline{y}) = \{\eta_1, \dots, \eta_m\}$. Then $\Phi^n(\eta_i)$ and $\Phi^n(\eta_j)$ are nowhere coincident for all $n \geq 0$ and $i \neq j$. Choose $\{t_k\}$ so that

$$\eta_1 - t_k\omega_{\mathrm{R}} \to \gamma := \gamma_1 \quad \text{and} \quad \eta_i - t_k\omega_{\mathrm{R}} \to \gamma_i \in \mathcal{T}_\varphi^{\mathrm{S}} \quad \text{for } i \in \{2, \dots, m\}.$$

Then $\gamma_1, \dots, \gamma_m$ are pairwise nowhere coincident and $\{\gamma_1, \dots, \gamma_m, \gamma\} \subset \widehat{g}^{-1}(\widehat{g}(\gamma))$. Using minimality again, choose $s_k \to \infty$ so that $\gamma_i - s_k\omega_{\mathrm{R}} \to \eta_{j(i)}$ and $\gamma' - s_k\omega_{\mathrm{R}} \to \eta'$ with $i \mapsto j(i)$ a bijection on $\{1, \dots, m\}$. Then $\eta' = \eta_{j(i)}$ for some i_0, so that γ' is proximal to γ_{i_0}. Either γ_{i_0} coincides with γ' along the segment I, in which case γ_{i_0} is coincident with γ_1 along I, so that $\gamma_{i_0} = \gamma_1 = \gamma$ and we have $\gamma' \sim_p \gamma$, or the segment I occurs in a bubble formed by γ_{i_0} and γ'. By (2), this bubble must terminate with coincidence. That is, there is $t \in \mathbb{R}$ so that $E^{\mathrm{s}} + t\omega_{\mathrm{R}}$ meets I in its interior, and there is $n \in \mathbb{N}$ so that $\Phi^n(\gamma_{i_0} - t\omega_{\mathrm{R}})$ and $\Phi^n(\gamma' - t\omega_{\mathrm{R}})$ are coincident along a common strand meeting E^{s}. But so are $\Phi^n(\gamma' - t\omega_{\mathrm{R}})$ and $\Phi^n(\gamma - t\omega_{\mathrm{R}})$. Thus $\Phi^n(\gamma_{i_0})$ and $\Phi^n(\gamma_1) = \Phi^n(\gamma)$ are coincident along a common strand J meeting $E^{\mathrm{s}} + \lambda^n t\omega_{\mathrm{R}}$. Now $\Phi^n(\eta_{i_0} - t_k\omega_{\mathrm{R}})$ and $\Phi^n(\eta_1 - t_k\omega_{\mathrm{R}})$ converge to $\Phi^n(\gamma_{i_0})$ and $\Phi^n(\gamma)$ respectively, so for large k, $\Phi^n(\eta_{i_0} - t_k\omega_{\mathrm{R}})$ and $\Phi^n(\eta_1 - t_k\omega_{\mathrm{R}})$ must coincide along J as well. Thus $\Phi^n(\eta_{i_0})$ and $\Phi^n(\eta_1)$ coincide along $J + \lambda^n t_k\omega_{\mathrm{R}}$, so that $i = 1$ and $\gamma \sim_p \gamma'$. ∎

The next result for the case in which φ is unimodular is Theorem 12.1 of [BK]. We need a generalization to the nonunimodular case. Because the proof is somewhat technical, we include it in an appendix.

THEOREM 5.3 (Asubharmonicity). *Suppose that φ is strong Pisot with abelianization A. If B and R are nonsingular integer matrices with $AR = RB$, p is a continuous surjection so that the diagram*

$$\begin{tikzcd} \mathcal{T}_\varphi^{\mathrm{S}} \arrow[rr, "p"] \arrow[dr, "\widehat{g}"'] & & \varprojlim F_B \arrow[dl, "\widehat{F}_R"] \\ & \varprojlim F_A & \end{tikzcd}$$

commutes, and p, $\widehat{F}_R$, and $\widehat{g}$ semi-conjugate the $\mathbb{R}$- and $\mathbb{Z}$-actions on the various spaces, then $\widehat{F}_R$ is a homeomorphism.

THEOREM 5.4. *Suppose that φ is strong Pisot. Then φ satisfies GCC if and only if proximality is a closed relation on $\mathcal{T}_\varphi$, in which case $\mathcal{T}_\varphi/\sim_p \simeq \varprojlim F_A$.*

Proof. If φ satisfies GCC, then $\sim_p$ is a closed equivalence relation whose equivalence classes are precisely the preimages of points under the geometric realization $\widehat{g}$ (Corollary 4.8), so that

$$\mathcal{T}_\varphi/\sim_p \simeq \varprojlim F_A.$$

Suppose that $\sim_p$ is a closed relation. According to Lemma 5.2, $\sim_p$ is an equivalence relation. Using the strand space model $\mathcal{T}_\varphi^{\mathrm{S}}$ of $\mathcal{T}_\varphi$, we have a commuting diagram of continuous surjections

$$\begin{tikzcd}
\mathcal{T}_\varphi \simeq \mathcal{T}_\varphi^{\mathrm{S}} \arrow[rr, "p"] \arrow[dr, "\widehat{g}"'] & & \mathcal{T}_\varphi^{\mathrm{S}}/\sim_p \arrow[dl, "c"] \\
& \varprojlim F_A &
\end{tikzcd}$$

in which p is the quotient map $\gamma \to [\gamma] := \{\gamma' : \gamma' \sim_p \gamma\}$, $\widehat{g}$ is geometric realization, and $c([\gamma]) := \widehat{g}(\gamma)$ is well-defined by Proposition 4.7. Note that the maps in the diagram also commute with the $\mathbb{Z}$- and $\mathbb{R}$-actions on the various spaces (which, on $\mathcal{T}_\varphi^{\mathrm{S}}/\sim_p$, are $[\gamma] \mapsto [\Phi(\gamma)] := \Phi[\gamma]$ and $([\gamma], t) \mapsto [\gamma]_t := [\gamma - t\omega_{\mathrm{R}}]$, respectively).

Claim. *c is a covering map.*

According to Theorem 4.5, $\widehat{g}$, and hence c, is bounded-to-one. Choose $\underline{x} \in \varprojlim F_A$ with $m = \#c^{-1}(\underline{x}) < \infty$ maximal. Let $c^{-1}(\underline{x}) = \{[\gamma_1], \dots, [\gamma_m]\}$ and choose $\underline{x}' \in \varprojlim F_A$. Since the flow on $\varprojlim F_A$ is minimal, there is $\{t_n\} \subset \mathbb{R}$ with $(\underline{x})_{t_n} \to \underline{x}'$. For some subsequence (which without loss of generality, and to simplify notation, we assume to be $\{t_n\}$ itself), $\{\gamma_i - t_n\omega_{\mathrm{R}}\}$ converges for each $i \in \{1, \dots, m\}$, say to $\gamma_i' \in \mathcal{T}_\varphi^{\mathrm{S}}$. Since γ_i and γ_j are not proximal for $i \neq j$, there is $\delta > 0$ so that

$$d(\gamma_i - t\omega_{\mathrm{R}}, \gamma_j - t\omega_{\mathrm{R}}) \geq \delta$$

for all $t \in \mathbb{R}$ and for all $i \neq j \in \{1, \dots, m\}$. It follows that

$$d(\gamma_i' - t\omega_{\mathrm{R}}, \gamma_j' - t\omega_{\mathrm{R}}) \geq \delta$$

for all $t \in \mathbb{R}$ and $i \neq j$, so that $[\gamma_i'] \neq [\gamma_j']$ for $i \neq j$. Thus $\#c^{-1}(\underline{x}') \geq m$. By maximality of m, $\#c^{-1}(\underline{x}') = m$ and c is m-to-one everywhere.

Suppose that for some $\gamma \in \mathcal{T}_\varphi^{\mathrm{S}}$, c is not one-to-one on any neighborhood of $[\gamma]$. There are then $\{[\gamma_n]\}, \{[\gamma_n']\}$ converging to $[\gamma]$ with $[\gamma_n] \neq [\gamma_n']$ and $c([\gamma_n]) = c([\gamma_n'])$ for all $n \in \mathbb{N}$. Without loss of generality, $\gamma_n \to \eta$ and $\gamma_n' \to \eta'$. Since $\sim_p$ is closed, $\eta, \eta' \in [\gamma]$, i.e., η, η' are proximal. According to (4) of Lemma 5.2, there is $\delta > 0$ so that $d(\gamma_n - t\omega_{\mathrm{R}}, \gamma_n' - t\omega_{\mathrm{R}}) \geq \delta$ for all $t \in \mathbb{R}$ and all $n \in \mathbb{N}$. On the other hand, if $t \in \mathbb{R}$ is chosen so that $d(\eta - t\omega_{\mathrm{R}}, \eta' - t\omega_{\mathrm{R}}) < \delta$, then for large enough n, $d(\gamma_n - t\omega_{\mathrm{R}}, \gamma_n' - t\omega_{\mathrm{R}}) < \delta$.

This contradiction proves that c is locally one-to-one and hence an m-to-one covering map.

By Lemma 5.1, there are $d \times d$ nonsingular integer matrices B, R with $RB = AR$ and a homeomorphism h so that the diagram

$$\begin{tikzcd} \mathcal{T}_\varphi^{\mathrm{S}}/{\sim_p} \arrow[rr, "h"] \arrow[dr, "c"'] & & \varprojlim F_B \arrow[dl, "\widehat{F}_R"] \\ & \varprojlim F_A & \end{tikzcd}$$

commutes. As $B = R^{-1}AR$, B is also Pisot: Let $\omega_{\mathrm{R},B}$ be the right eigenvector of B with $R(\omega_{\mathrm{R},B}) = \omega_{\mathrm{R},A}$, the right Perron–Frobenius eigenvector of A. Note that h and $\widehat{F}_R$ semi-conjugate the flows on $\mathcal{T}_\varphi^{\mathrm{S}}/{\sim_p}$ and $\varprojlim F_A$ with $(\underline{y}, t) \mapsto (y_1 - t\omega_{\mathrm{R},B}, y_2 - (t/\lambda)\omega_{\mathrm{R},B}, \ldots) = (\underline{y})_t$ on $\varprojlim F_B$, and that h semi-conjugates the action $[\gamma] \mapsto [\Phi(\gamma)]$ on $\mathcal{T}_\varphi^{\mathrm{S}}/{\sim_p}$ with $\widehat{f_B}$ on $\varprojlim F_B$. Now we have the commuting diagram

$$\begin{tikzcd} \mathcal{T}_\varphi^{\mathrm{S}} \arrow[rr, "h\circ p"] \arrow[dr, "\widehat{g}"'] & & \varprojlim F_B \arrow[dl, "\widehat{F}_R"] \\ & \varprojlim F_A & \end{tikzcd}$$

of surjections that semi-conjugate all of the $\mathbb{Z}$- and $\mathbb{R}$- actions. By the "asubharmonicity theorem" (Theorem 6.1 of [BK] in the unimodular case, and Theorem 5.3 of this paper for the general case), $\widehat{F}_R$ is one-to-one. Thus c is a homeomorphism and $\mathcal{T}_\varphi/{\sim_p} \simeq \varprojlim F_A$.

If $\widehat{g}(\gamma) = \widehat{g}(\gamma')$, then $c(p(\gamma)) = c(p(\gamma'))$, so that $p(\gamma) = p(\gamma')$. That is, γ and γ' are proximal, hence γ and γ' share an edge. Then $\mathrm{cr}_\varphi = 1$, and, by Theorem 4.5, φ satisfies GCC. ∎

EXAMPLE 5.5 (Proximality not closed). As the Morse–Thue example ($\varphi(1) = 12$, $\varphi(2) = 21$) shows, proximality need not be a closed relation, even for a weak Pisot substitution. Also, the fixed words represent tilings that are proximal in one direction but not the other. ∎

6. Appendix.

We now complete the proofs of Lemma 4.18 and Theorem 5.3.

We first prove

LEMMA 4.18. *The balanced pairs* $\binom{1}{1}$, $\binom{2}{2}$, $\binom{21}{12}$, $\binom{211}{122}$, *and their duals are the only essential balanced pairs for* ψ.

To this end, we prove a slightly stronger statement:

LEMMA 4.18$'$. *Given any balanced pair* $\binom{u}{v}$ *for* ψ, $\binom{\psi(u)}{\psi(v)}$ *is the concatenation of the irreducible balanced pairs* $\binom{1}{1}$, $\binom{2}{2}$, $\binom{21}{12}$, $\binom{211}{122}$, $\binom{21211}{11212}$, *and their duals.*

Since $\binom{\psi(21)}{\psi(12)}$, $\binom{\psi(211)}{\psi(112)}$ and $\binom{\psi(21211)}{\psi(11212)}$ are concatenations of $\binom{1}{1}$, $\binom{2}{2}$, $\binom{21}{12}$, $\binom{211}{122}$, and their duals, it follows that for any balanced pair $\binom{u}{v}$, $\binom{\psi^2(u)}{\psi^2(v)}$ is also such a concatenation, and Lemma 4.18 is proved.

Proof of Lemma 4.18$'$. The following proof consists of demonstrating that for any irreducible balanced pair $\binom{u}{v} = \binom{u_1\ldots u_n}{v_1\ldots v_n}$, the balanced pair $\binom{\psi(u)}{\psi(v)}$ can be factored into a product of irreducible balanced pairs by first reducing $\binom{\psi(u_1)}{\psi(v_1)}$ (with remainder), then reducing $\binom{\psi(u_1u_2)}{\psi(v_1v_2)}$ (with remainder), etc.; each step in this sequence is represented by an edge (and the adjoining vertices) in a finite graph G. The complete factorization of $\binom{\psi(u)}{\psi(v)}$ is then represented by a path of length n in G with the ith edge of the path labeled $\binom{u_i}{v_i}$, and the adjoining vertices indicating, in part, remainders in the reductions of

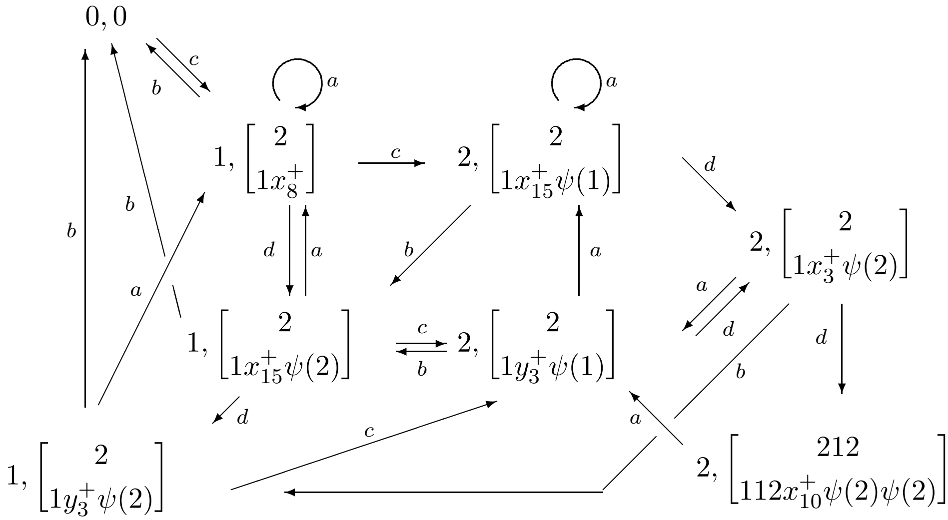
$$\begin{pmatrix}\psi(u_1\ldots u_i)\\ \psi(v_1\ldots v_i)\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\psi(u_1\ldots u_{i+1})\\ \psi(v_1\ldots v_{i+1})\end{pmatrix}.$$

The proof will be completed by noting that for any $\binom{u}{v}$, only the vertices and edges of the graph G will be visited, and the irreducible balanced pairs arising in each step of the factorization are included in those listed in Lemma 4.18$'$ above.

Given two words u, v of the same length allowed for the substitution ψ, we define the *discrepancy vector* of u and v to be the difference of the content vectors of u and v, $l(u) - l(v)$. The discrepancy vector has integer entries, and since ψ is on a two-letter alphabet, it is of the form $(m, -m)$, where $m \in \mathbb{Z}$. Let $\mathrm{dis}(u, v) = |m|$ be the *discrepancy number* of u and v. In the case of ψ, $|m| = 0, 1$ or 2. Note that $\mathrm{dis}(u, v) = 0$ if and only if $\binom{u}{v}$ is balanced.

Let $\binom{u}{v} = \binom{u_1\ldots u_n}{v_1\ldots v_n}$ be an irreducible balanced pair for ψ. Without loss of generality, $u_1 = 2$, $v_1 = 1$. Since $|\psi(2)| = 7$ while $|\psi(1)| = 19$, and $\binom{u}{v}$ is irreducible, $|\psi(u_1\ldots u_i)| < |\psi(v_1\ldots v_i)|$ for all $1 \leq i < n$, and $\mathrm{dis}(u_1\ldots u_i, v_1\ldots v_i)$ indicates how many more 2's appear in $u_1\ldots u_i$ than in $v_1\ldots v_i$. In the graph G below, each edge is labeled by a pair of symbols $\begin{bmatrix}a_1\\ a_2\end{bmatrix}$. Each vertex has two components: an integer representing a discrepancy number, and a pair of words $\begin{bmatrix}w_1\\ w_2\end{bmatrix}$ representing, under the right circumstances, the remainder in the reduction of a particular pair of words. As described above, for an irreducible balanced pair $\binom{2u_2\ldots u_n}{1v_2\ldots v_n}$, the reduction of $\binom{\psi(2u_2\ldots u_n)}{\psi(1v_2\ldots v_n)}$ is represented by a path of length n through G that begins and ends at $(0, 0)$; the ith vertex in the path indicates both $\mathrm{dis}(u_1\ldots u_i, v_1\ldots v_i)$ and the remainder upon writing $\binom{\psi(u_1\dots u_i)}{\psi(v_1\dots v_i)}$ in irreducible balanced pairs. The edge to the $(i+1)$st vertex is labeled with $\left[\begin{smallmatrix}a_1\\a_2\end{smallmatrix}\right] = \left[\begin{smallmatrix}u_{i+1}\\v_{i+1}\end{smallmatrix}\right]$, and the $(i+1)$st vertex consists of $\operatorname{dis}(u_1\dots u_{i+1}, v_1\dots v_{i+1})$ and the remainder in the reduction of $\binom{\psi(u_1\dots u_{i+1})}{\psi(v_1\dots v_{i+1})}$.

To simplify the writing of remainders, we use the notation x_i^+ (y_i^+, respectively) to denote the subword $x_i\dots x_{19}$ of $\psi(1)$ ($y_i\dots y_7$ of $\psi(2)$, respectively). In the following diagram, a, b, c and d denote $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$, and $\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]$, respectively:



Some transitions between vertices are not allowed. For instance, the edge $c = \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$ cannot leave a vertex with discrepancy 2, since this would result in a discrepancy of 3, not allowed for ψ. Also, the edges $b = \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ and $d = \left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]$ cannot leave the vertices with terminal factors of $\psi(2)\psi(2)$ or $y_3^+\psi(2)$ in their remainders, since the subword 222 is not allowed for ψ. Since every other edge appears in G, every irreducible balanced pair is represented by a path in G.

We illustrate the use of G with the balanced pair $\binom{211}{112}$. The path representing the reduction of $\binom{\psi(211)}{\psi(112)}$ will involve edges labeled by $c = \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$, $a = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, and $b = \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$, in that order, discrepancy numbers of 1, 1, and 0, and remainders of $\left[\begin{smallmatrix}2\\1x_8^+\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}2\\1x_8^+\end{smallmatrix}\right]$, and 0.

We invite the reader to verify some of the calculations. Note that movement from one vertex to another can be checked independently of the remainder of the graph.

We leave the reader to check that all reductions except that involving the single edge from 2, $\left[\begin{smallmatrix}2\\112v_{10}^+\psi(2)\psi(2)\end{smallmatrix}\right]$ involve only the irreducible balanced pairs described in the statement of Lemma 4.18; this exception also includes $\binom{21211}{11212}$ in its reduction. It is easy to verify that the reduction of $\binom{\psi(21211)}{\psi(11212)}$ also involves only the balanced pairs of Lemma 4.18. This proves Lemma 4.18′. ■

We now prove Theorem 5.3.

THEOREM 5.3 (Asubharmonicity). *Suppose that φ is strong Pisot with abelianization A. If B and R are nonsingular integer matrices with $AR = RB$, p is a continuous surjection so that the diagram*

$$\begin{tikzcd} \mathcal{T}_\varphi^{\mathrm{S}} \arrow[rr, "p"] \arrow[dr, "\widehat{g}"'] & & \varprojlim F_B \arrow[dl, "\widehat{F}_R"] \\ & \varprojlim F_A & \end{tikzcd}$$

commutes, and p, $\widehat{F}_R$, and $\widehat{g}$ semi-conjugate the $\mathbb{R}$- and $\mathbb{Z}$-actions on the various spaces, then $\widehat{F}_R$ is a homeomorphism.

Proof. Given a strand (finite or infinite) γ that meets E^{s}, let $\widehat{\gamma}$ denote the state determined by γ (i.e., the edge of γ, closed on the initial end and open on the terminal end, that meets E^{s}). We write $\gamma \sim_0 \eta$ if γ and η are strands that meet E^{s} for which $\widehat{\Phi^n(\gamma)} = \widehat{\Phi^n(\eta)}$ for some $n \in \mathbb{N}$. Note that $\widehat{\Phi(\widehat{\gamma})} = \widehat{\Phi(\gamma)}$. Given $\gamma \in \mathcal{T}_\varphi^{\mathrm{S}}$, let

$$\mathcal{F}_\gamma = \Big\{v \in \bigcup_{n \geq 0} A^{-n}\mathbb{Z}^d : \gamma \sim_0 \gamma + v\Big\}.$$

Recall that ω_{R} is the right eigenvector for A. Define

$$W^{\mathrm{s}}(\gamma) = \{\eta \in \mathcal{T}_\varphi^{\mathrm{S}} : d(\Phi^n(\gamma), \Phi^n(\eta)) \to 0 \text{ as } n \to \infty\}$$

and let pr_A denote the projection of $\mathbb{R}^d$ onto E_A^{u} along E_A^{s}. Letting

$$\mathrm{ret}(\gamma) = \{t \in \mathbb{R} : \gamma - t\omega_{\mathrm{R}} \in W^s(\gamma)\}$$

be the set of return times for γ, we see that, as long as $\Phi^n(\gamma)$ does not have a vertex on E^{s} for any $n \geq 0$, $v \in \mathcal{F}_\gamma$ if and only if $\mathrm{pr}_A v = t\omega_{\mathrm{R}}$ for some $t \in \mathrm{ret}(\gamma)$. Let

$$\mathcal{G} = \{\gamma \in \mathcal{T}_\varphi^{\mathrm{S}} : \Phi^n(\gamma) \text{ does not have a vertex on } E^{\mathrm{s}} \text{ for any } n \geq 0\}.$$

Then $\mathcal{G}$ has full measure in $\mathcal{T}_\varphi^{\mathrm{S}}$. Let $H = \langle \bigcup_{\gamma \in \mathcal{G}} \mathcal{F}_\gamma \rangle$ be the subgroup of $\bigcup_{n \geq 0} A^{-n}\mathbb{Z}^d$ generated by $\bigcup_{\gamma \in \mathcal{G}} \mathcal{F}_\gamma$.

Since $(\Phi^{n+1}(\Phi^{-1}(\gamma)+A^{-1}v))^\wedge = (\Phi^n(\gamma+v))^\wedge$, and $\gamma\in\mathcal{G}$ if and only if $\Phi^{-1}(\gamma)\in\mathcal{G}$, we can conclude that

$$A^{-1}(H)\subset H. \tag{A.1}$$

For $i\in\{1,\dots,d\}$, let

$$\begin{aligned}\Theta(i)=\{v\in\mathbb{Z}^d : {}&\text{for some (and hence any) } \gamma\in\mathcal{T}_\varphi^{\mathrm{S}},\ \text{there are edges}\\ &I,J \text{ of } \gamma,\ \text{both of type } i,\ \text{with } \min(J)-\min(I)=v\}\end{aligned}$$

be the set of return vectors for type i. Note that for $v\in\Theta(i)$ and $\gamma\in\mathcal{T}_\varphi^{\mathrm{S}}$, γ and $\gamma+v$ share an edge, say I. It follows that if $E^{\mathrm{s}}+t\omega_{\mathrm{R}}$ meets I in its interior, and $\gamma-t\omega_{\mathrm{R}}\in\mathcal{G}$ (as will be the case for most such t), then $(\gamma-t\omega_{\mathrm{R}})\sim_0(\gamma-t\omega_{\mathrm{R}})+v$, hence $v\in\mathcal{F}_{\gamma-t\omega_{\mathrm{R}}}$. Thus

$$\Theta(i)\subset H\quad \text{for all } i\in\{1,\dots,d\}. \tag{A.2}$$

Now fix $\gamma\in\mathcal{G}$, and define

$$[v+H]^+=\{i : i \text{ is the type of an edge } I \text{ of } \gamma \text{ with } \min I\in v+H\},$$
$$[v+H]^-=\{i : i \text{ is the type of an edge } I \text{ of } \gamma \text{ with } \max I\in v+H\}.$$

If $i\in[v+H]^+\cap[u+H]^+$, then $\min I+H=v+H$ and $\min J+H=u+H$ for some edges I,J of γ of type i. Then $u-v+H=\min J-\min I+H=H$, since $\min J-\min I\in\Theta(i)\subset H$, hence $u-v\in H$. A similar statement can be made if $i\in[v+H]^-\cap[u+H]^-$, so that

(A.3) If $[v+H]^+\cap[u+H]^+\neq\emptyset$ or $[v+H]^-\cap[u+H]^-\neq\emptyset$, then $u-v\in H$.

Recall that the flow on $\mathcal{T}_\varphi^{\mathrm{S}}$ is uniquely ergodic, and that if for each $i\in\{1,\dots,d\}$, f_i denotes the frequency of occurrence of tiles of type i in $\gamma\in\mathcal{T}_\varphi^{\mathrm{S}}$,

$$f_i:=\lim_{N\to\infty}\frac{1}{2N}\,\mu\{t\in[-N,N] : (\gamma-t\omega_{\mathrm{R}})^\wedge \text{ has type } i\},$$

then $(f_1,\dots,f_d)^{\mathrm{tr}}$ is a right Perron–Frobenius eigenvector for A. Since the characteristic polynomial of A is irreducible over $\mathbb{Q}$, the entries of $(f_1,\dots,f_d)$ are independent over $\mathbb{Z}$. It follows from (A.2) that an edge of type $[v+H]^-$ in γ must be followed by an edge of type $[v+H]^+$, and an edge of type $[v+H]^+$ must be preceded by an edge of type $[v+H]^-$. Thus occurrences in γ of edges of type $[v+H]^-$ are in one-to-one correspondence with occurrences of edges of type $[v+H]^+$. That is, the frequency of type $[v+H]^+$ equals the frequency of type $[v+H]^-$:

$$\sum_{i\in[v+H]^+} f_i=\sum_{i\in[v+H]^-} f_i.$$

As the f_i are linearly independent over $\mathbb{Z}$, it must be that

$$[v+H]^+=[v+H]^-. \tag{A.4}$$

Now fix $i \in \{1, \ldots, d\}$, and let $e_i = (0, \ldots, 1, \ldots, 0)$ be the unit vector in the i direction. Let I be an edge of γ of type i, $u = \min I$ and $v = \max I$. Then $i \in [u+H]^+ = [u+H]^-$ and $i \in [v+H]^-$. Thus $v - u \in H$ by (A.3) so that $e_i \in H$ and thus $\mathbb{Z}^d \subset H$. From (A.1) we now have $\bigcup_{n\geq 0} A^{-n}\mathbb{Z}^d \subset H$. Thus

$$H = \Big\langle \bigcup_{n\geq 0} A^{-n}\mathbb{Z}^d \Big\rangle. \tag{A.5}$$

Suppose that $t \in \mathrm{ret}(\gamma)$ (i.e., t is a return time for γ). Then t is also a return time for $p(\gamma) \in \varprojlim F_B$, since p semi-conjugates the $\mathbb{R}$- and $\mathbb{Z}$-actions on $\mathcal{T}_\varphi^{\mathrm{S}}$ with those on $\varprojlim F_B$. That is, if $t \in \mathrm{ret}(\gamma)$, then

$$t\omega_{\mathrm{R},B} \in \mathrm{pr}_B\Big(\bigcup_{n\geq 0} B^{-n}\mathbb{Z}^d\Big),$$

where pr_B is the projection of $\mathbb{R}^d$ onto E_B^{u} along E_B^{s} and $\omega_{\mathrm{R},B} := R^{-1}(\omega_{\mathrm{R},A})$ is a right eigenvector of B corresponding to the Perron–Frobenius eigenvalue λ of A.

Choose $\gamma \in \mathcal{G}$ and $v \in \mathcal{F}_\gamma$. Then for $t \in \mathrm{ret}(\gamma)$ and $u \in \bigcup_{n\geq 0} B^{-n}\mathbb{Z}^d$,

$$\mathrm{pr}_A(v) = t\omega_{\mathrm{R},A} = Rt\omega_{\mathrm{R},B} = R\,\mathrm{pr}_B(u).$$

But $R\,\mathrm{pr}_B(u) = \mathrm{pr}_A(Ru)$ since $R^{-1}AR = B$ and, by irreducibility of the characteristic polynomial of A, $v = Ru$. That is, $\mathcal{F}_\gamma \subset R(\bigcup_{n\geq 0} B^{-n}\mathbb{Z}^d)$. Thus

$$H = \Big\langle \bigcup_{\gamma\in\mathcal{G}} \mathcal{F}_\gamma \Big\rangle \subset R\Big(\bigcup_{n\geq 0} B^{-n}\mathbb{Z}^d\Big).$$

In addition, since $RB = AR$, it follows that $R(B^{-n}\mathbb{Z}^d) \subset A^{-n}\mathbb{Z}^d$, so that by (A.5),

$$\bigcup_{n\geq 0} A^{-n}\mathbb{Z}^d = R\Big(\bigcup_{n\geq 0} B^{-n}\mathbb{Z}^d\Big). \tag{A.6}$$

Since $\widehat{F}_R$ is a covering map and $\varprojlim F_B$ is compact, $\widehat{F}_R^{-1}(\underline{0} := (0, 0, \ldots))$ is finite. Let $\underline{x} \in \widehat{F}_R^{-1}(\underline{0})$. As $\widehat{F}_R$ semi-conjugates $\widehat{f}_B$ with $\widehat{f}_A$, $\underline{x}$ must be periodic under $\widehat{f}_B$; without loss of generality, assume $\underline{x} = (x, x, \ldots)$ is fixed, say $x = y + \mathbb{Z}^d \in \mathbb{T}^d$. Since $\widehat{F}_R(\underline{x}) = \underline{0}$, $Ry = v \in \mathbb{Z}^d$. Also, $\underline{x} \in \varprojlim F_B$, hence $B(y) = y + w$, where $w \in \mathbb{Z}^d$. We have

$$\begin{tikzcd} y+w \arrow[d, mapsto, "R"'] & y \arrow[l, mapsto, "B"'] \arrow[d, mapsto, "R"] \\ Av & v \arrow[l, mapsto, "A"'] \end{tikzcd}$$

That is, $Av = R(y+w)$. Since $R(y+w) \in \mathbb{Z}^d$,

$$v = A^{-1}(R(y+w)) \in \bigcup_{n\geq 0} A^{-n}\mathbb{Z}^d.$$

Thus, by (A.6), there are $n \in \mathbb{N}$ and $u \in \mathbb{Z}^d$ so that $RB^{-n}u = v$. Now $y = R^{-1}v = B^{-n}u$, so that

$$\begin{aligned} By &= B^{-n+1}u = y + w,\\ y &= B^{-n+1}u - w,\\ By &= B^{-n+2}u - Bw = y + w,\\ y &= B^{-n+2}u - Bw - w. \end{aligned}$$

Continuing, we obtain

$$y = u - B^{n-2}u - B^{n-3}u - \cdots - Bw - w \in \mathbb{Z}^d.$$

That is, $\underline{x} = \underline{0}$. Hence $\#\widehat{f}_R^{-1}(0) = 1$. Since $\widehat{F}_R$ is a covering map, and $\varprojlim F_B$ is connected, $\widehat{F}_R$ is a homeomorphism. ∎

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