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Level by level equivalence and the number of normal measures over $P_\kappa(\lambda)$

by

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Abstract. We construct two models for the level by level equivalence between strong compactness and supercompactness in which if κ is λ supercompact and $\lambda \ge \kappa$ is regular, we are able to determine exactly the number of normal measures $P_\kappa(\lambda)$ carries. In the first of these models, $P_\kappa(\lambda)$ carries $2^{2^{[\lambda]^{<\kappa}}}$ many normal measures, the maximal number. In the second of these models, $P_\kappa(\lambda)$ carries $2^{2^{[\lambda]^{<\kappa}}}$ many normal measures, except if κ is a measurable cardinal which is not a limit of measurable cardinals. In this case, κ (and hence also $P_\kappa(\kappa)$) carries only κ^+ many normal measures. In both of these models, there are no restrictions on the structure of the class of supercompact cardinals.

1. Introduction and preliminaries. One of the advantages of an inner model for a particular type of measurable cardinal κ is that it provides canonical structure for the universe in which κ resides. In particular, in the usual sorts of inner models for measurability (see, e.g., the models constructed and analyzed in [12], [15], and [6]), if κ is a measurable cardinal, it is possible to determine exactly the number of normal measures κ carries.

Because of the limited inner model theory currently available for supercompactness, analogous results for κ-additive, fine, normal measures over $P_\kappa(\lambda)$ when $\lambda \ge \kappa$ is regular have been relatively few. Aside from the classical result (see [11]) that if κ is $2^{[\lambda]^{<\kappa}}$ supercompact, then $P_\kappa(\lambda)$ carries exactly $2^{2^{[\lambda]^{<\kappa}}}$ many κ-additive, fine, normal measures (the maximal number), and the more recent results of [4] that when $\lambda \ge \kappa$ is regular, it is consistent relative to the appropriate assumptions for κ to be λ supercompact and for $P_\kappa(\lambda)$ to carry fewer than the maximal number of κ-additive, fine, normal measures, not much has been known concerning models for supercompactness and the number of normal measures $P_\kappa(\lambda)$ can carry.

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The purpose of this paper is to rectify this situation by studying the number of normal measures $P_\kappa(\lambda)$ can carry when $\lambda \geq \kappa$ is regular in the context of the "inner model like" property of level by level equivalence between strong compactness and supercompactness. Specifically, we prove the following two theorems.

THEOREM 1. *Suppose $V \vDash$ "ZFC + GCH + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals + Level by level equivalence between strong compactness and supercompactness holds". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \vDash$ "ZFC + GCH + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals + Level by level equivalence between strong compactness and supercompactness holds". In $V^{\mathbb{P}}$, if κ is λ supercompact and $\lambda \geq \kappa$ is regular, then $P_\kappa(\lambda)$ carries exactly $2^{2^{[\lambda]^{<\kappa}}} = 2^{2^\lambda} = \lambda^{++}$ many κ-additive, fine, normal measures.*

THEOREM 2. *Suppose $V \vDash$ "ZFC + GCH + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals + Level by level equivalence between strong compactness and supercompactness holds". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \vDash$ "ZFC + GCH + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals + Level by level equivalence between strong compactness and supercompactness holds". In $V^{\mathbb{P}}$, if κ is a measurable cardinal which is not a limit of measurable cardinals, then κ carries exactly κ^+ many normal measures (and hence $P_\kappa(\kappa)$ carries exactly κ^+ many κ-additive, fine, normal measures). If this is not the case, i.e., if in $V^{\mathbb{P}}$, κ is a measurable cardinal which is a limit of measurable cardinals, then for any regular $\lambda \geq \kappa$ such that κ is λ supercompact, $P_\kappa(\lambda)$ carries exactly $2^{2^{[\lambda]^{<\kappa}}} = 2^{2^\lambda} = \lambda^{++}$ many κ-additive, fine, normal measures.*

In other words, there is a model for level by level equivalence between strong compactness and supercompactness (the one provided by Theorem 1) in which if κ is λ supercompact (and not necessarily more) and $\lambda \geq \kappa$ is regular, then $P_\kappa(\lambda)$ always carries the maximal number of κ-additive, fine, normal measures. On the other hand, there is also a model for level by level equivalence between strong compactness and supercompactness (the one provided by Theorem 2) in which under most circumstances, if κ is λ supercompact and $\lambda \geq \kappa$ is regular, then $P_\kappa(\lambda)$ carries the maximal number of κ-additive, fine, normal measures. However, in this model, this is not always the case. In particular, if κ is a measurable cardinal which is not a limit of measurable cardinals, then both κ and $P_\kappa(\kappa)$ carry fewer than the maximal number of normal measures.

We now very briefly give some preliminary information concerning notation and terminology. For anything left unexplained, readers are urged to consult [4]. When forcing, $q \geq p$ means that q is stronger than p. For κ a regular cardinal and $\lambda \geq \kappa$ any cardinal, $\mathrm{Add}(\kappa, 1)$ is the standard partial ordering for adding a single Cohen subset of κ, and $\mathrm{Coll}(\kappa,\lambda)$ is the standard collapse partial ordering (originally used by Cohen) for collapsing λ to κ. For κ a cardinal, the partial ordering $\mathbb{P}$ is *κ-directed closed* if every directed set of conditions of size less than κ has an upper bound. If G is V-generic over $\mathbb{P}$, we will abuse notation slightly and use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$. We will, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$ or $\check{x}$, especially if x is in the ground model V, or x is a variant of the generic set G.

Suppose V is a model of ZFC in which for all regular cardinals $\kappa < \lambda$, κ is λ strongly compact iff κ is λ supercompact, except possibly if κ is a measurable limit of cardinals δ which are λ supercompact. Such a model will be said to witness *level by level equivalence between strong compactness and supercompactness*. We will also say that κ is a witness to level by level equivalence between strong compactness and supercompactness iff for every regular cardinal $\lambda > \kappa$, κ is λ strongly compact iff κ is λ supercompact. Note that the exception is provided by a theorem of Menas [14], who showed that if κ is a measurable limit of cardinals δ which are λ strongly compact, then κ is λ strongly compact but need not be λ supercompact. Models in which level by level equivalence between strong compactness and supercompactness holds nontrivially were first constructed in [5]. Note that this property is considered to be "inner model like" in the sense that, like GCH and a combinatorial property such as $\diamondsuit$, it is the sort of regularity phenomenon one might expect in a "nice inner model" for supercompactness.

We assume familiarity with the large cardinal notions of measurability, strong compactness, and supercompactness. Readers are urged to consult [11] for further details. We do wish to mention, however, that we will use "supercompact ultrafilter over $P_\kappa(\lambda)$" and "κ-additive, fine, normal measure over $P_\kappa(\lambda)$" synonymously. In addition, we state explicitly that κ is κ supercompact iff κ is κ strongly compact iff κ is measurable, and if κ is measurable, there is a canonical correspondence between normal measures over κ and κ-additive, fine, normal measures over $P_\kappa(\kappa)$. This has as an immediate consequence that if κ is measurable, the number of normal measures over κ and the number of κ-additive, fine, normal measures over $P_\kappa(\kappa)$ is the same.

We conclude Section 1 by mentioning that there is one result critical to the proofs of Theorems 1 and 2 which will be taken as a "black box". For the convenience of readers, we provide a brief discussion of this fact here. The result is a corollary of Theorems 3 and 31 and Corollary 14 of Hamkins' paper [8]. This theorem is a generalization of Hamkins' Gap Forcing Theorem and Corollary 16 of [9] and [10] (and we refer readers to [9], [10], and [8] for further details). We therefore state the theorem we will be using now, along with some associated terminology. Suppose $\mathbb{P}$ is a partial ordering which can be written as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| \le \delta$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ^+-directed closed". In Hamkins' terminology of [8], $\mathbb{P}$ *admits a closure point at* δ. In Hamkins' terminology of [9] and [10], $\mathbb{P}$ is *mild with respect to a cardinal* κ iff every set of ordinals x in $V^{\mathbb{P}}$ of size below κ has a "nice" name τ in V of size below κ, i.e., there is a set y in V, $|y| < \kappa$, such that any ordinal forced by a condition in $\mathbb{P}$ to be in τ is an element of y. Also, as in the terminology of [9], [10], [8], and elsewhere, an embedding $j : \overline{V} \to \overline{M}$ is *amenable to* $\overline{V}$ when $j \restriction A \in \overline{V}$ for any $A \in \overline{V}$. The specific corollary of Theorems 3 and 31 and Corollary 14 of [8] we will be using is then the following.

THEOREM 3 (Hamkins). *Suppose that $V[G]$ is a forcing extension obtained by forcing that admits a closure point at some regular $\delta < \kappa$. Suppose further that $j : V[G] \to M[j(G)]$ is an embedding with critical point κ for which $M[j(G)] \subseteq V[G]$ and $M[j(G)]^{\delta} \subseteq M[j(G)]$ in $V[G]$. Then $M = V \cap M[j(G)]$ (so $M \subseteq V$). If the full embedding j is amenable to $V[G]$, then the restricted embedding $j \restriction V : V \to M$ is amenable to V. If j is definable from parameters (such as a measure or extender) in $V[G]$, then the restricted embedding $j \restriction V$ is definable from the names of those parameters in V. Finally, if $\mathbb{P}$ is mild with respect to κ and κ is λ strongly compact in $V[G]$ for any $\lambda \ge \kappa$, then κ is λ strongly compact in V.*

It immediately follows from Theorem 3 that any cardinal κ measurable in a generic extension obtained by forcing that admits a closure point below κ must also be measurable in the ground model V. In addition, Theorem 3 implies that if $V^{\mathbb{P}} \models$ "κ is λ strongly compact", $\mathbb{P}$ is mild with respect to κ, and $\mathbb{P}$ admits a closure point below κ, then $V \models$ "κ is γ strongly compact" for any ordinal γ such that $V^{\mathbb{P}} \models$ "$|\gamma| = \lambda$". Similarly, Theorem 3 implies that if $V^{\mathbb{P}} \models$ "κ is λ supercompact" and $\mathbb{P}$ admits a closure point below κ, then $V \models$ "κ is γ supercompact" for any ordinal γ such that $V^{\mathbb{P}} \models$ "$|\gamma| = \lambda$".

2. The proofs of Theorems 1 and 2

Proof of Theorem 1. Suppose $V \models$ "ZFC + GCH + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals + Level by level equivalence between strong compactness and supercompactness holds". The partial ordering $\mathbb{P}$ which will be used to prove Theorem 1 is the same one used in the proof of Theorem 3 of [1]. More specifically, $\mathbb{P}$ is the proper class reverse Easton iteration which does nontrivial forcing only at those stages δ which are V-regular cardinals. At such a stage, we force with $\mathrm{Add}(\delta, 1)$. By Theorem 3 of [1], $V^{\mathbb{P}} \models$ "ZFC + GCH + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals + Level by level equivalence between strong compactness and supercompactness holds". We must therefore show that $V^{\mathbb{P}} \models$ "If κ is λ supercompact and $\lambda \ge \kappa$ is regular, then $P_\kappa(\lambda)$ carries exactly $2^{2^{[\lambda]^{<\kappa}}} = 2^{2^\lambda} = \lambda^{++}$ many κ-additive, fine, normal measures".

Towards this end, assume that $V^{\mathbb{P}} \vDash$ "κ is λ supercompact and $\lambda \geq \kappa$ is regular". Clearly, $V \vDash$ "λ is regular". Since if λ is regular and we write $\mathbb{P} = \mathbb{P}_{\lambda+1} * \dot{\mathbb{P}}^{\lambda+1}$, then $\Vdash_{\mathbb{P}_{\lambda+1}}$ "$\dot{\mathbb{P}}^{\lambda+1}$ is λ^+-directed closed", we know that $V^{\mathbb{P}} \vDash$ "κ is λ supercompact" iff $V^{\mathbb{P}_{\lambda+1}} \vDash$ "κ is λ supercompact". Further, any normal measure over $P_\kappa(\lambda)$ in $V^{\mathbb{P}_{\lambda+1}}$ is also a normal measure over $P_\kappa(\lambda)$ in $V^{\mathbb{P}}$. It thus suffices to show that $V^{\mathbb{P}_{\lambda+1}} \vDash$ "If κ is λ supercompact and $\lambda \geq \kappa$ is regular, then $P_\kappa(\lambda)$ carries exactly $2^{2^{[\lambda]^{<\kappa}}} = 2^{2^\lambda} = \lambda^{++}$ many κ-additive, fine, normal measures".

To show this last fact, we begin by noting that as in Theorem 3 of [1], by Theorem 3 of this paper, it must be the case that $V \vDash$ "κ is λ supercompact". We now use a standard analysis found, e.g., in Lemma 1.1 of [3] or Lemma 6 of [7] in tandem with the usual argument (originally due to Silver) for lifting a supercompactness embedding after a reverse Easton iteration. Specifically, let $j : V \to M$ be an elementary embedding witnessing the λ supercompactness of κ generated by a supercompact ultrafilter over $P_\kappa(\lambda)$. Write $\mathbb{P}_{\lambda+1} = \mathbb{P}_\kappa * \dot{\mathbb{Q}}$, where $\dot{\mathbb{Q}}$ is a term for the partial ordering which adds Cohen subsets of each regular cardinal in the closed interval $[\kappa, \lambda]$. Then $j(\mathbb{P}_{\lambda+1}) = \mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, where $\dot{\mathbb{R}}$ is a term for the partial ordering which adds Cohen subsets of each regular cardinal in M in the open interval $(\lambda, j(\kappa))$. Let G be V-generic over $\mathbb{P}_\kappa$, and let H be $V[G]$-generic over $\mathbb{Q}$. Silver's standard arguments, as given, e.g., in the proof of Lemma 1.2 of [1], show that j lifts in $V[G][H]$ to $j^* : V[G][H] \to M[G][H][H'][H'']$, where H' and H'' are built in $V[G][H]$, H' is $M[G][H]$-generic over $\mathbb{R}$, and H'' is $M[G][H][H']$-generic over $j^*(\mathbb{Q})$ and contains a master condition for $j^{*\prime\prime}H$.

It is, though, the construction of H' here which is critical for our purposes. We therefore examine this more carefully. Since $\mathbb{R}$ has cardinality $j(\kappa)$ in $M[G][H]$, by GCH in both V and M, there are $2^{j(\kappa)} = j(\kappa^+)$ many dense open subsets of $\mathbb{R}$ present in $M[G][H]$. Again by GCH in both V and M, $V[G][H] \vDash$ "$|j(\kappa^+)| \leq |\{f \mid f : P_\kappa(\lambda) \to \kappa^+\}| = |\{f \mid f : \lambda \to \kappa^+\}| = 2^\lambda = \lambda^+$". This means, since $M[G][H]$ remains λ closed with respect to $V[G][H]$, that H' is constructed in $V[G][H]$ by meeting each member of an enumeration $\langle D_\alpha \mid \alpha < \lambda^+ \rangle \in V[G][H]$ of the dense open subsets of $\mathbb{R}$ present in $M[G][H]$. However, it is possible to build a tree $\mathcal{T}$ of height λ^+ which gives $2^{2^{[\lambda]^{<\kappa}}} = 2^{2^\lambda} = \lambda^{++}$ many distinct possible values for H'. More explicitly, the root of $\mathcal{T}$ is the empty condition. If p is an element at level $\alpha < \lambda^+$ of $\mathcal{T}$, then the successors of p at level $\alpha + 1$ are a maximal incompatible subset of D_α extending p. Note that by the definition of $\mathbb{P}$, there will be at least two successors of p at level $\alpha + 1$. Finally, if $\delta < \lambda^+$ is a limit ordinal, then the elements of $\mathcal{T}$ at height δ are upper bounds to any path through $\mathcal{T}$ of height δ.

Since $M[G][H]$ remains λ closed with respect to $V[G][H]$ and $M[G][H] \vDash$ "$\mathbb{R}$ is λ^+-directed closed", $\mathcal{T}$ is well-defined. In addition, by the fact that each node of $\mathcal{T}$ splits, there are $2^{\lambda^+} = \lambda^{++} = 2^{2^\lambda} = 2^{2^{[\lambda]^{<\kappa}}}$ many distinct paths through $\mathcal{T}$. As each path through $\mathcal{T}$ generates an $M[G][H]$-generic object over $\mathbb{R}$, there are $2^{2^{[\lambda]^{<\kappa}}} = 2^{2^\lambda} = \lambda^{++}$ many distinct possible values for H'. Hence, since for any lift of $j : V \to M$ to $j^* : V[G][H] \to M[G][H][H'][H'']$, $j^*(G * H) = G * H * H' * H''$, each distinct value of H' generates a different value of j^* and consequently, a distinct normal ultrafilter $\mathcal{U}$ over $P_\kappa(\lambda)$ given by $X \in \mathcal{U}$ iff $\langle j(\alpha) \mid \alpha < \lambda\rangle \in j^*(X)$. Thus, there are $2^{2^{[\lambda]^{<\kappa}}} = 2^{2^\lambda} = \lambda^{++}$ many different normal ultrafilters over $P_\kappa(\lambda)$ in $V[G][H]$. This completes the proof of Theorem 1. ∎

Proof of Theorem 2. Suppose $V \vDash$ "ZFC + GCH + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals + Level by level equivalence between strong compactness and supercompactness holds". Without loss of generality, by Theorem 1, we assume in addition that $V \vDash$ "If κ is λ supercompact and $\lambda \geq \kappa$ is regular, then $P_\kappa(\lambda)$ carries exactly $2^{2^{[\lambda]^{<\kappa}}} = 2^{2^\lambda} = \lambda^{++}$ many κ-additive, fine, normal measures".

For δ any ordinal, define $\gamma_\delta = \omega$ if δ is less than or equal to the least measurable cardinal, and γ_δ as the least inaccessible cardinal above the supremum of all of the measurable cardinals below δ otherwise. The partial ordering $\mathbb{P}$ used in the proof of Theorem 2 will be defined as $\mathbb{P}^0 * \dot{\mathbb{P}}^1$, where $\mathbb{P}^0$ and $\mathbb{P}^1$ are two reverse Easton iterations. $\mathbb{P}^0$ does nontrivial forcing only at those stages δ which are V-measurable cardinals which are not in V limits of measurable cardinals. At such a stage, we force with $\mathrm{Add}(\gamma_\delta, 1) * \dot{\mathrm{Coll}}(\delta^+, \delta^{++})$. If there are only set many measurable cardinals in V, we let Ω be their supremum, and conclude the definition of $\mathbb{P}^0$ by forcing with $\mathrm{Add}(\gamma_\Omega, 1)$ (if there are any inaccessibles above Ω). Terminology we will use later at a nontrivial stage of forcing δ is that $\mathbb{P}^0$ (or some portion thereof) acts nontrivially on the ordinals γ_δ, δ^+, and δ^{++}. At all other ordinals, $\mathbb{P}^0$ acts trivially.

We observe that $\mathbb{P}^0$ is either a set or a proper class, depending upon whether the collection of measurable cardinals in V is a set or a proper class. Regardless if $\mathbb{P}^0$ is a set or a proper class, routine arguments show that $V^{\mathbb{P}^0} \vDash$ "ZFC + GCH" and that the only cardinals collapsed in $V^{\mathbb{P}^0}$ have the form $(\delta^{++})^V$, where δ is in V a measurable cardinal which is not a limit of measurable cardinals.

LEMMA 2.1. *If $V^{\mathbb{P}^0} \vDash$ "κ is a measurable cardinal which is not a limit of measurable cardinals", then $V \vDash$ "κ is a measurable cardinal which is not a limit of measurable cardinals".*

Proof. Suppose $V^{\mathbb{P}^0} \models$ "κ is a measurable cardinal which is not a limit of measurable cardinals". Note that it is possible to write $\mathbb{P}^0$ as $\mathbb{Q}*\dot{\mathbb{R}}$, where $|\mathbb{Q}| = \omega$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is $\aleph_1$-directed closed". Hence, as we observed at the end of Section 1, since any cardinal which is measurable in $V^{\mathbb{P}^0}$ had to have been measurable in V, $V \models$ "κ is measurable". Thus, to prove Lemma 2.1, it suffices to show that $V \models$ "κ is not a limit of measurable cardinals".

To do this, assume to the contrary that $V \models$ "κ is a limit of measurable cardinals". In particular, $V \models$ "κ is a limit of measurable cardinals δ which themselves are not limits of measurable cardinals". For any such measurable cardinal δ, write $\mathbb{P}^0 = \mathbb{P}^0_\delta * \dot{\mathrm{A}}\mathrm{dd}(\gamma_\delta, 1) * \dot{\mathrm{C}}\mathrm{oll}(\delta^+, \delta^{++}) * \dot{\mathbb{Q}}'$. Since $|\mathbb{P}^0_\delta * \dot{\mathrm{A}}\mathrm{dd}(\gamma_\delta, 1)| < \delta$, by the Lévy-Solovay results [13], $V^{\mathbb{P}^0_\delta * \dot{\mathrm{A}}\mathrm{dd}(\gamma_\delta,1)} \models$ "δ is measurable". Since $\Vdash_{\mathbb{P}^0_\delta * \dot{\mathrm{A}}\mathrm{dd}(\gamma_\delta,1)}$ "$\dot{\mathrm{C}}\mathrm{oll}(\delta^+, \delta^{++}) * \dot{\mathbb{Q}}'$ is δ^+-directed closed", $V^{\mathbb{P}^0_\delta * \dot{\mathrm{A}}\mathrm{dd}(\gamma_\delta,1) * \dot{\mathrm{C}}\mathrm{oll}(\delta^+,\delta^{++}) * \dot{\mathbb{Q}}'} = V^{\mathbb{P}^0} \models$ "δ is measurable". Thus, $V^{\mathbb{P}^0} \models$ "κ is a measurable cardinal which is a limit of measurable cardinals". This contradiction completes the proof of Lemma 2.1. ∎

LEMMA 2.2. $V^{\mathbb{P}^0} \models$ *"If κ is a measurable cardinal which is not a limit of measurable cardinals, then κ carries exactly κ^+ many normal measures".*

Proof. Suppose $V^{\mathbb{P}^0} \models$ "κ is a measurable cardinal which is not a limit of measurable cardinals". By Lemma 2.1, $V \models$ "κ is a measurable cardinal which is not a limit of measurable cardinals". Therefore, in analogy to the proof of Lemma 2.1, write $\mathbb{P}^0 = \mathbb{P}^0_\kappa * \dot{\mathrm{A}}\mathrm{dd}(\gamma_\kappa, 1) * \dot{\mathrm{C}}\mathrm{oll}(\kappa^+, \kappa^{++}) * \dot{\mathbb{Q}}'$, where $|\mathbb{P}^0_\kappa| < \kappa$ and $\Vdash_{\mathbb{P}^0_\kappa * \dot{\mathrm{A}}\mathrm{dd}(\gamma_\kappa,1) * \dot{\mathrm{C}}\mathrm{oll}(\kappa^+,\kappa^{++})}$ "Forcing with $\dot{\mathbb{Q}}'$ adds no bounded subsets of the least inaccessible cardinal above κ". Thus, since by the results of [13], $\Vdash_{\mathbb{P}^0_\kappa}$ "κ is a measurable cardinal which is not a limit of measurable cardinals", the proof of Lemma 2.2 will be complete once we have shown that $\Vdash_{\mathbb{P}^0_\kappa * \dot{\mathrm{A}}\mathrm{dd}(\gamma_\kappa,1) * \dot{\mathrm{C}}\mathrm{oll}(\kappa^+,\kappa^{++})}$ "κ is a measurable cardinal which is not a limit of measurable cardinals carrying exactly κ^+ many normal measures".

To do this, we use an argument due to Cummings, which also appears in the proof of the Main Theorem of [4] and the proof of Lemma 2.1 of [2]. First, note that by our assumptions on V, $V \models$ "κ carries exactly $\kappa^{++} = 2^{2^\kappa}$ many normal measures". Let $\overline{V} = V^{\mathbb{P}^0_\kappa}$. By the results of [13], $\overline{V} \models$ "κ carries exactly κ^{++} many normal measures" as well. Suppose G_0 is $\overline{V}$-generic over $\mathrm{Add}(\gamma_\kappa, 1)$ and G_1 is $\overline{V}[G_0]$-generic over $\mathrm{Coll}(\kappa^+, \kappa^{++})$. Again by the results of [13], since $|\mathrm{Add}(\gamma_\kappa, 1)| < \kappa$, $\overline{V}[G_0] \models$ "κ is a measurable cardinal carrying exactly κ^{++} many normal measures". These remain normal measures over κ in $\overline{V}[G_0][G_1]$, since no additional subsets of κ are added by the collapse forcing. Thus, since $(\kappa^+)^V$ is preserved to $\overline{V}[G_0][G_1]$, there are at least κ^+ many normal measures over κ in $\overline{V}[G_0][G_1]$.

Conversely, suppose that $\mathcal{U}$ is a normal measure over κ in $\overline{V}[G_0][G_1]$, with the associated ultrapower embedding $j : \overline{V}[G_0][G_1] \to M[G_0][j(G_1)]$. In particular, $X \in \mathcal{U}$ iff $\kappa \in j(X)$ for all $X \subseteq \kappa$ in $\overline{V}[G_0][G_1]$. By Theorem 3, it follows that the restriction $j \restriction \overline{V} : \overline{V} \to M$ is a definable class in $\overline{V}$. Once more by the results of [13], since $|\mathrm{Add}(\gamma_\kappa, 1)| < \kappa$, $j \restriction \overline{V}$ lifts uniquely to $\overline{V}[G_0]$, and so $j \restriction \overline{V}[G_0] : \overline{V}[G_0] \to M[G_0]$ is a definable class in $\overline{V}[G_0]$. The key observation is now that because $\overline{V}[G_0]$ and $\overline{V}[G_0][G_1]$ have the same subsets of κ, one can reconstruct $\mathcal{U}$ inside $\overline{V}[G_0]$ by observing $X \in \mathcal{U}$ iff $\kappa \in j(X)$, using only $j \restriction \overline{V}[G_0]$. Thus, $\mathcal{U} \in \overline{V}[G_0]$. Consequently, every normal measure over κ in $\overline{V}[G_0][G_1]$ is actually in $\overline{V}[G_0]$. The number of such normal measures, therefore, is at most $(\kappa^{++})^{\overline{V}[G_0]}$, which is κ^+ in $\overline{V}[G_0][G_1]$, because $(\kappa^{++})^{\overline{V}[G_0]}$ was collapsed by G_1. Hence, in $\overline{V}[G_0][G_1]$, there are exactly κ^+ many normal measures over κ, as desired. Since forcing with $\mathrm{Add}(\gamma_\kappa, 1) * \dot{\mathrm{C}}\mathrm{oll}(\kappa^+, \kappa^{++})$ does not change the fact that κ is not a limit of measurable cardinals, this completes the proof of Lemma 2.2. ∎

For any (measurable) cardinal κ, define θ_κ as the least cardinal such that κ is not θ_κ strongly compact.

Lemma 2.3. *Suppose in V, κ is λ^+ strongly compact and $\theta_\kappa = \lambda^{++}$, where $\lambda > \kappa$ is a measurable cardinal which is not a limit of measurable cardinals. Then $V^{\mathbb{P}^0} \vDash$ "κ is not λ^+ strongly compact".*

Proof. By the definition of $\mathbb{P}^0$, we may write $\mathbb{P}^0 = \mathbb{P}^0_\kappa * \dot{\mathbb{Q}} * \dot{\mathrm{C}}\mathrm{oll}(\lambda^+, \lambda^{++}) * \dot{\mathbb{R}}$, where $\Vdash_{\mathbb{P}^0_\kappa * \dot{\mathbb{Q}} * \dot{\mathrm{C}}\mathrm{oll}(\lambda^+, \lambda^{++})}$ "Forcing with $\dot{\mathbb{R}}$ adds no bounded subsets of the least inaccessible cardinal above λ". It thus suffices to show that $V^{\mathbb{P}^0_\kappa * \dot{\mathbb{Q}} * \dot{\mathrm{C}}\mathrm{oll}(\lambda^+, \lambda^{++})} \vDash$ "κ is not λ^+ strongly compact".

To do this, note that $\mathbb{P}^0_\kappa * \dot{\mathbb{Q}} * \dot{\mathrm{C}}\mathrm{oll}(\lambda^+, \lambda^{++})$ may be written as $\mathbb{R}_0 * \dot{\mathbb{R}}_1$, where $|\mathbb{R}_0| = \omega$, $\mathbb{R}_0$ is nontrivial, and $\Vdash_{\mathbb{R}_0}$ "$\dot{\mathbb{R}}_1$ is $\aleph_1$-directed closed". In addition, by its definition, $\mathbb{P}^0_\kappa * \dot{\mathbb{Q}} * \dot{\mathrm{C}}\mathrm{oll}(\lambda^+, \lambda^{++})$ is easily seen to be mild with respect to κ. Since $V^{\mathbb{P}^0_\kappa * \dot{\mathbb{Q}} * \dot{\mathrm{C}}\mathrm{oll}(\lambda^+, \lambda^{++})} \vDash$ "$|(\lambda^{++})^V| = \lambda^+$" and $\theta_\kappa = (\lambda^{++})^V$, by Theorem 3, $V^{\mathbb{P}^0_\kappa * \dot{\mathbb{Q}} * \dot{\mathrm{C}}\mathrm{oll}(\lambda^+, \lambda^{++})} \vDash$ "κ is not λ^+ strongly compact". This is because otherwise, as we observed at the end of Section 1, κ would have had to have been λ^{++} strongly compact in V, which contradicts the fact that $\theta_\kappa = (\lambda^{++})^V$. This completes the proof of Lemma 2.3. ∎

We remark that the exact same proof as given in Lemma 2.3 (without the reference to mildness, which is unnecessary in the context of supercompactness) shows that if $V \vDash$ "κ is λ^+ supercompact but not λ^{++} supercompact and $\lambda > \kappa$ is a measurable cardinal which is not a limit of measurable cardinals", then $V^{\mathbb{P}^0} \vDash$ "κ is not λ^+ supercompact". This observation will be used later.

LEMMA 2.4. *Suppose in V, κ is λ supercompact for $\lambda \geq \kappa$ a regular cardinal and κ is a measurable cardinal which is a limit of measurable cardinals. Suppose further that for any cardinal $\gamma > \kappa$ which is a measurable cardinal which is not a limit of measurable cardinals, if $V \vDash$ "κ is γ^+ supercompact", then $V \vDash$ "κ is γ^{++} supercompact" as well. Then $V^{\mathbb{P}^0} \vDash$ "κ is λ supercompact".*

Proof. Write $\mathbb{P}^0 = \mathbb{P}^0_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{Q}}'$, where $\dot{\mathbb{Q}}$ is forced to act (either trivially or nontrivially) on ordinals in the closed interval $[\kappa, \lambda]$, and $\dot{\mathbb{Q}}'$ is a term for the rest of $\mathbb{P}^0$. If λ is a nontrivial stage of the forcing, i.e., if λ is a measurable cardinal which is not a limit of measurable cardinals, then $\Vdash_{\mathbb{P}^0_\kappa * \dot{\mathbb{Q}}}$ "Forcing with $\dot{\mathbb{Q}}'$ adds no bounded subsets of λ^+". If λ is a trivial stage of the forcing, then $\Vdash_{\mathbb{P}^0_\kappa * \dot{\mathbb{Q}}}$ "Forcing with $\dot{\mathbb{Q}}'$ adds no bounded subsets of the least inaccessible cardinal above λ". Thus, to show $V^{\mathbb{P}^0} \vDash$ "κ is λ supercompact", it suffices to show that $V^{\mathbb{P}^0_\kappa * \dot{\mathbb{Q}}} \vDash$ "κ is λ supercompact".

To do this, let $j : V \to M$ be an elementary embedding witnessing the λ supercompactness of κ generated by a supercompact ultrafilter over $P_\kappa(\lambda)$. Note that by hypothesis, if $\lambda = \gamma^+$ where γ is a measurable cardinal which is not a limit of measurable cardinals, then κ is actually $\lambda^+ = \gamma^{++}$ supercompact in V (so under these circumstances, M may be taken as being λ^+ closed). Consequently, regardless if this is the case, M has enough closure so that $j(\mathbb{P}^0_\kappa * \dot{\mathbb{Q}}) = \mathbb{P}^0_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, where the first ordinal at which $\dot{\mathbb{R}}$ is forced to act nontrivially is above λ. Silver's standard lifting arguments, as given, e.g., in the proof of Lemma 1.2 of [1] (and mentioned in the proof of Theorem 1), once again show that if G is V-generic over $\mathbb{P}^0_\kappa$ and H is $V[G]$-generic over $\mathbb{Q}$, then j lifts in $V[G][H]$ to $j^* : V[G][H] \to M[G][H][H'][H'']$ which witnesses the λ supercompactness of κ, where H' and H'' are the generic objects constructed in $V[G][H]$ for $\mathbb{R}$ and $j^*(\mathbb{Q})$, and H'' contains a master condition for $j^{*\prime\prime}H$. Hence, $V^{\mathbb{P}^0_\kappa * \dot{\mathbb{Q}}} \vDash$ "κ is λ supercompact". This completes the proof of Lemma 2.4. ∎

LEMMA 2.5. *$V^{\mathbb{P}^0} \vDash$ "$\mathcal{K}$ is the class of supercompact cardinals".*

Proof. As in the proof of Lemma 2.1, we may write $\mathbb{P}^0 = \mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| = \omega$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is $\aleph_1$-directed closed". Hence, by Theorem 3, any cardinal supercompact in $V^{\mathbb{P}^0}$ had to have been supercompact in V. However, by Lemma 2.4, any cardinal supercompact in V remains supercompact in $V^{\mathbb{P}^0}$. Thus, $V^{\mathbb{P}^0} \vDash$ "$\mathcal{K}$ is the class of supercompact cardinals". This completes the proof of Lemma 2.5. ∎

LEMMA 2.6. *$V^{\mathbb{P}^0} \vDash$ "Level by level equivalence between strong compactness and supercompactness holds".*

Proof. Suppose $V^{\mathbb{P}^0} \vDash$ "$\kappa < \lambda$ are such that κ is λ strongly compact and λ is a regular cardinal". By its definition, $\mathbb{P}^0$ is mild with respect to κ. Therefore, by the factorization of $\mathbb{P}^0$ given in Lemmas 2.1 and 2.5 and Theorem 3, it must be true that $V \vDash$ "κ is λ strongly compact". By level by level equivalence between strong compactness and supercompactness, $V \vDash$ "Either κ is λ supercompact, or κ is a measurable limit of cardinals δ which are λ supercompact". By Lemma 2.3 and the paragraph immediately following, it cannot be the case that $V \vDash$ "$\lambda = \gamma^+$, γ is a measurable cardinal which is not a limit of measurable cardinals, and either strong compactness or supercompactness first fails at γ^{++} for any cardinal δ which is either λ strongly compact or λ supercompact". Hence, by Lemma 2.4, $V^{\mathbb{P}^0} \vDash$ "Either κ is λ supercompact, or κ is a measurable limit of cardinals δ which are λ supercompact", i.e., $V^{\mathbb{P}^0} \vDash$ "Level by level equivalence between strong compactness and supercompactness holds". This completes the proof of Lemma 2.6. ∎

Because λ^{++} is collapsed if κ is exactly λ supercompact and λ is in V a measurable cardinal which is not a limit of measurable cardinals, we need to do an additional forcing to ensure that if κ is λ supercompact, $\lambda \geq \kappa$ is regular, and κ is a limit of measurable cardinals, then $P_\kappa(\lambda)$ carries exactly $2^{2^{[\lambda]^{<\kappa}}} = 2^{2^\lambda} = \lambda^{++}$ many κ-additive, fine, normal measures. To this end, let $\overline{V} = V^{\mathbb{P}^0}$. The partial ordering $\mathbb{P}^1 \in \overline{V}$ we use to complete the proof of Theorem 2 is the reverse Easton iteration which begins by adding a Cohen subset of ω and then does nontrivial forcing only at those stages δ which are $\overline{V}$-measurable cardinals which are not in $\overline{V}$ limits of measurable cardinals. At such a stage δ, we force with $\mathrm{Add}(\delta^*, 1)$, where δ^* is (in either $\overline{V}$ or $\overline{V}^{\mathbb{P}^1_\delta}$) the least inaccessible cardinal above δ. Regardless if $\mathbb{P}^1$ is a set or a proper class, routine arguments show that forcing with $\mathbb{P}^1$ preserves all cardinals and cofinalities and $\overline{V}^{\mathbb{P}^1} \vDash$ "ZFC + GCH".

Lemma 2.7. *If $\overline{V}^{\mathbb{P}^1} \vDash$ "κ is a measurable cardinal which is not a limit of measurable cardinals", then $\overline{V} \vDash$ "κ is a measurable cardinal which is not a limit of measurable cardinals".*

Proof. We mimic to a certain extent the proof of Lemma 2.1. The exact same arguments as in Lemma 2.1 show that $\overline{V} \vDash$ "κ is measurable". Thus, it once again suffices to show that $\overline{V} \vDash$ "κ is not a limit of measurable cardinals". As before, to do this, we assume to the contrary that $\overline{V} \vDash$ "κ is a limit of measurable cardinals", so that in particular, $\overline{V} \vDash$ "κ is a limit of measurable cardinals which themselves are not limits of measurable cardinals". For any such measurable cardinal δ, write $\mathbb{P}^1 = \mathbb{P}^1_\delta * \dot{\mathbb{Q}}$. Since $|\mathbb{P}^1_\delta| < \delta$, by the results of [13], $\overline{V}^{\mathbb{P}^1_\delta} \vDash$ "δ is measurable". Since $\Vdash_{\mathbb{P}^1_\delta}$ "$\dot{\mathbb{Q}}$ is δ^+-directed closed", $\overline{V}^{\mathbb{P}^1_\delta * \dot{\mathbb{Q}}} = \overline{V}^{\mathbb{P}^1} \vDash$ "δ is measurable". Hence, $\overline{V}^{\mathbb{P}^1} \vDash$ "κ is a measurable cardinal which is a limit of measurable cardinals", a contradiction which then completes the proof of Lemma 2.7. ∎

LEMMA 2.8. $\overline{V}^{\mathbb{P}^1} \models$ *"If κ is a measurable cardinal which is not a limit of measurable cardinals, then κ carries exactly κ^+ many normal measures".*

Proof. Suppose $\overline{V}^{\mathbb{P}^1} \models$ "κ is a measurable cardinal which is not a limit of measurable cardinals". By Lemma 2.7, $\overline{V} \models$ "κ is a measurable cardinal which is not a limit of measurable cardinals". Therefore, in analogy to the proof of Lemma 2.7, write $\mathbb{P}^1 = \mathbb{P}^1_\kappa * \dot{\mathbb{Q}}$. By Lemma 2.2, $\overline{V} \models$ "κ carries exactly κ^+ many normal measures". Hence, since $|\mathbb{P}^1_\kappa| < \kappa$, by the results of [13], $\overline{V}^{\mathbb{P}^1_\kappa} \models$ "κ is a measurable cardinal which is not a limit of measurable cardinals and κ carries exactly κ^+ many normal measures". Consequently, as $\Vdash_{\mathbb{P}^1_\kappa}$ "Forcing with $\dot{\mathbb{Q}}$ adds no bounded subsets of the least inaccessible cardinal above κ", $\overline{V}^{\mathbb{P}^1_\kappa * \dot{\mathbb{Q}}} = \overline{V}^{\mathbb{P}^1} \models$ "κ is a measurable cardinal which is not a limit of measurable cardinals and κ carries exactly κ^+ many normal measures". This completes the proof of Lemma 2.8. ∎

LEMMA 2.9. *If $\overline{V} \models$ "κ is λ supercompact and $\lambda > \kappa$ is regular", then $\overline{V}^{\mathbb{P}^1} \models$ "κ is λ supercompact".*

Proof. We mimic to a certain extent the proof of Lemma 2.4. Write $\mathbb{P}^1 = \mathbb{P}^1_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{Q}}'$, where $\dot{\mathbb{Q}}$ is forced to act (either trivially or nontrivially) on ordinals in the closed interval $[\kappa, \lambda]$, and $\dot{\mathbb{Q}}'$ is a term for the rest of $\mathbb{P}^1$. Since $\Vdash_{\mathbb{P}^1_\kappa * \dot{\mathbb{Q}}}$ "$\dot{\mathbb{Q}}'$ is λ^+-directed closed", to show $\overline{V}^{\mathbb{P}^1} \models$ "κ is λ supercompact", it suffices to show that $\overline{V}^{\mathbb{P}^1_\kappa * \dot{\mathbb{Q}}} \models$ "κ is λ supercompact".

To do this, let $j : \overline{V} \to M$ be an elementary embedding witnessing the λ supercompactness of κ generated by a supercompact ultrafilter over $P_\kappa(\lambda)$. Then M has enough closure so that $j(\mathbb{P}^1_\kappa * \dot{\mathbb{Q}}) = \mathbb{P}^1_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, where the first ordinal at which $\dot{\mathbb{R}}$ is forced to act nontrivially is above λ. As before, Silver's standard lifting arguments, as given, e.g., in the proof of Lemma 1.2 of [1] (and mentioned in the proof of Theorem 1 and Lemma 2.4), once again show that if G is $\overline{V}$-generic over $\mathbb{P}^1_\kappa$ and H is $\overline{V}[G]$-generic over $\mathbb{Q}$, then j lifts in $\overline{V}[G][H]$ to $j^* : \overline{V}[G][H] \to M[G][H][H'][H'']$ which witnesses the λ supercompactness of κ, where H' and H'' are the generic objects constructed in $\overline{V}[G][H]$ for $\mathbb{R}$ and $j^*(\mathbb{Q})$, and H'' contains a master condition for $j^{*\prime\prime}H$. Hence, $\overline{V}^{\mathbb{P}^1_\kappa * \dot{\mathbb{Q}}} \models$ "κ is λ supercompact". This completes the proof of Lemma 2.9. ∎

LEMMA 2.10. $\overline{V}^{\mathbb{P}^1} \models$ *"Level by level equivalence between strong compactness and supercompactness holds".*

Proof. We mimic to a certain extent the proof of Lemma 2.6. Suppose $\overline{V}^{\mathbb{P}^1} \models$ "$\kappa < \lambda$ are such that κ is λ strongly compact and λ is a regular cardinal". By its definition, $\mathbb{P}^1$ is mild with respect to κ. In addition, it is possible to factor $\mathbb{P}^1$ as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| = \omega$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is $\aleph_1$-directed closed". Therefore, by Theorem 3, it must be the case that $\overline{V} \models$ "κ is λ strongly compact". By level by level equivalence between strong compactness and supercompactness in $\overline{V}$, $\overline{V} \models$ "Either κ is λ supercompact, or κ is a measurable limit of cardinals δ which are λ supercompact". Hence, by Lemma 2.9, $\overline{V}^{\mathbb{P}^1} \models$ "Either κ is λ supercompact, or κ is a measurable limit of cardinals δ which are λ supercompact", i.e., $\overline{V}^{\mathbb{P}^1} \models$ "Level by level equivalence between strong compactness and supercompactness holds". This completes the proof of Lemma 2.10. ∎

The proof of Theorem 1 now applies almost verbatim to show that $\overline{V}^{\mathbb{P}^1} \models$ "If κ is λ supercompact, $\lambda \geq \kappa$ is regular, and κ is a limit of measurable cardinals, then $P_\kappa(\lambda)$ carries exactly $2^{2^{[\lambda]^{<\kappa}}} = 2^{2^\lambda} = \lambda^{++}$ many κ-additive, fine, normal measures". The same proof as in Lemma 2.5 (replacing a reference to Lemma 2.4 with a reference to Lemma 2.9) shows that $\overline{V}^{\mathbb{P}^1} \models$ "$\mathcal{K}$ is the class of supercompact cardinals". Therefore, by letting $\mathbb{P} = \mathbb{P}^0 * \dot{\mathbb{P}}^1$, Lemmas 2.1–2.10 and the intervening remarks complete the proof of Theorem 2. ∎

As we remarked at the beginning of this paper, if κ exhibits enough supercompactness, it will be the case that $P_\kappa(\lambda)$ carries the maximal number of κ-additive, fine, normal measures. However, since this may not always be the case, we conclude by asking what the other possibilities are for the number of normal measures over $P_\kappa(\lambda)$ in a universe containing supercompact cardinals in which level by level equivalence between strong compactness and supercompactness holds. In particular, if κ is λ supercompact, $\lambda \geq \kappa$ is regular, and κ is not λ^+ supercompact, is it possible, in a model satisfying GCH and level by level equivalence between strong compactness and supercompactness, for $P_\kappa(\lambda)$ to carry exactly 1 normal measure? What about δ many normal measures, where δ is an arbitrary cardinal less than $2^{2^{[\lambda]^{<\kappa}}}$?

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