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The classification of circle-like continua that admit expansive homeomorphisms

by

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Abstract. A homeomorphism $h : X \to X$ of a compactum X is *expansive* provided that for some fixed $c > 0$ and every $x, y \in X$ $(x \neq y)$ there exists an integer n, dependent only on x and y, such that $\mathrm{d}(h^n(x), h^n(y)) > c$. It is shown that if X is a solenoid that admits an expansive homeomorphism, then X is homeomorphic to a regular solenoid. It can then be concluded that *a circle-like continuum admits an expansive homeomorphism if and only if it is homeomorphic to a regular solenoid.*

1. Introduction. In 1955, R. F. Williams constructed an expansive homeomorphism on the dyadic solenoid [14]; this was the first example of an expansive homeomorphism on a continuum. In this paper we will show that the only circle-like continua that admit expansive homeomorphisms are regular solenoids like the dyadic solenoid. A homeomorphism $h : X \to X$ is called *expansive* provided that there exists a constant $c > 0$ such that for any distinct $x, y \in X$ there exists an integer n such that $\mathrm{d}(h^n(x), h^n(y)) > c$. Here, c is called the *expansive constant.* Expansive homeomorphisms exhibit sensitive dependence on initial conditions in the strongest sense in that no matter how close any two points are, either their images or preimages will at some point be at least a certain distance apart.

A continuum X is *circle-like* if it is the inverse limit of simple closed curves. Equivalently, a continuum is circle-like if for every $\epsilon > 0$ there exists a circle-chain cover $\mathcal{U}$ of X with $\mathrm{mesh}(\mathcal{U}) < \epsilon$. Let the simple closed curve S be defined by $\mathbb{R}/\mathbb{Z}$ with metric

$$\mathrm{d}_S(x, y) = \min\{|x - y|, 1 - |x - y|\}.$$

If $n \in \mathbb{N}$, let $(n) : S \to S$ be defined by

$$(n)x = (n)(x) = nx \bmod 1.$$

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A continuum is a *solenoid* if it is homeomorphic to $\varprojlim (S, (n(i)))_{i=1}^{\infty}$ where $\{n(i)\}_{i=1}^{\infty}$ is some sequence of natural numbers. A solenoid is a simple closed curve if and only if all but a finite number of elements of $\{n(i)\}_{i=1}^{\infty}$ are 1. Since it is well known that simple closed curves do not admit expansive homeomorphisms [12], we will only consider solenoids that are defined by bonding maps $\{(n(i))\}_{i=1}^{\infty}$ with an infinite number of elements greater than 1. It is well known that the *shift homeomorphism* of $\varprojlim (S, (n))_{i=1}^{\infty}$ is expansive when $n \geq 2$. Here, $\varprojlim (S, (n))_{i=1}^{\infty}$ is called a *regular solenoid.* Floris Takens gives a survey on the solenoid in [11] and shows that if M_n is a homeomorphism that is multiplication of the solenoid $\varprojlim (S, (n(i)))_{i=1}^{\infty}$ by n then

- each prime factor of n is a factor of each element of some infinite subsequence of $\{n(i)\}_{i=1}^{\infty}$,
- if $(\varprojlim (S, (n(i)))_{i=1}^{\infty}, M_n)$ is conjugated to a hyperbolic attractor, then $\varprojlim (S, (n(i)))_{i=1}^{\infty}$ is isomorphic to $\varprojlim (S, (n))_{i=1}^{\infty}$ as a topological group.

Alex Clark showed in [1] that a solenoid must be composite to admit an expansive homeomorphism. A solenoid is *composite* if there exists a prime number p that divides an infinite number of $\{n(i)\}_{i=1}^{\infty}$. Also, it is known that if X is tree-like [7] or separates the plane into two complementary domains [8], then X does not admit an expansive homeomorphism. The following related result has recently been shown:

Theorem 1 ([6]). *A circle-like continuum admits an expansive homeomorphism if and only if it is a solenoid.*

The goal of this paper is to prove the following theorem:

Theorem 2. *A solenoid Σ admits an expansive homeomorphism if and only if the solenoid is homeomorphic to $\varprojlim (S, (n))_{i=1}^{\infty}$ for some $n \geq 2$.*

Then combining this with Theorem 1 we get a complete classification of circle-like continua that admit expansive homeomorphisms:

Corollary 3. *A circle-like continuum admits an expansive homeomorphism if and only if it is homeomorphic to $\varprojlim (S, (n))_{i=1}^{\infty}$ for some $n \geq 2$.*

To do this we first need some basic results about expansive homeomorphisms. The following theorem is Corollary 5.22.1 in Walters [Wa]:

Theorem 4. *Let X be a compact metric space and $h : X \to X$ be an expansive homeomorphism. Then*

(1) *Expansiveness is independent of metric as long as the metric gives the topology of X.*
(2) *Let $\phi : X \to Y$ be a homeomorphism of compact spaces X and Y. Then $g = \phi \circ h \circ \phi^{-1}$ is an expansive homeomorphism.*
(3) *$h^n : X \to X$ is an expansive homeomorphism for each $n \in \mathbb{Z}^+$.*

Notice that part (2) implies that admitting an expansive homeomorphism is a topological invariant.

In Section 4, the motivation and an outline of the proof of the main theorem is given.

2. Inverse limits. In order to prove the main theorem, an understanding of inverse limits and the construction of homeomorphisms on inverse limits is necessary. Let $\{(X_i, \mathrm{d}_{X_i})\}_{i=1}^{\infty}$ be a collection of metric spaces such that $\{\mathrm{d}_{X_i}\}_{i=1}^{\infty}$ is uniformly bounded and for each i let $b_i : X_{i+1} \to X_i$ be a continuous function called a *bonding map*. The collection $(X_i, b_i)_{i=1}^{\infty}$ is called an *inverse system*. Each X_i is called a *factor space* of the inverse system. If each bonding map is the same map $b : Y \to Y$, then the inverse system can be written as $(Y, b)_{i=1}^{\infty}$. Every inverse system $(X_i, b_i)_{i=1}^{\infty}$ determines a topological space X called the *inverse limit* of the system and written $X = \varprojlim\,(X_i, b_i)_{i=1}^{\infty}$. The space X is the subspace of the Cartesian product $\prod_{i=1}^{\infty} X_i$ given by

$$X = \varprojlim\,(X_i, b_i)_{i=1}^{\infty} = \Big\{ \langle x_i \rangle_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i \,\Big|\, b_i(x_{i+1}) = x_i \Big\}.$$

X has the subspace topology induced on it by $\prod_{i=1}^{\infty} X_i$. If $\mathbf{x} = \langle x_i \rangle_{i=1}^{\infty}$ and $\mathbf{y} = \langle y_i \rangle_{i=1}^{\infty}$ are two points of the inverse limit, we define their distance to be

$$\mathrm{d}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} \frac{\mathrm{d}_{X_i}(x_i, y_i)}{2^i}.$$

If $b_i : X_{i+1} \to X_i$ for $n \le i < k$ then define $b_n^n : X_n \to X_n$ by $b_n^n = \mathrm{id}_{X_n}$ and $b_n^k : X_k \to X_n$ by $b_n^k = b_n \circ b_{n+1} \circ \cdots \circ b_{k-1}$. Next define the ith projection $\pi_i : X \to X_i$ by $\pi_i(\langle x_i \rangle_{i=1}^{\infty}) = x_i$. For more on inverse limits see [2], [3], [5] or [13].

The next theorem by Mioduszewski [5] states how to construct homeomorphisms between inverse limit spaces:

THEOREM 5. *Let* $X = \varprojlim\,(X_i, b_i)_{i=1}^{\infty}$, $Y = \varprojlim\,(Y_i, \beta_i)_{i=1}^{\infty}$. *Then a map* $h : X \to Y$ *is a homeomorphism if and only if there exist*

- *a decreasing sequence* $\{\epsilon_i\}_{i=1}^{\infty}$ *of positive numbers such that* $\epsilon_i \to 0$,
- *increasing sequences* $\{n_i\}_{i=1}^{\infty}$ *and* $\{m_i\}_{i=1}^{\infty}$ *of positive integers*,
- *maps* $f_i : X_{n_{2i-1}} \to Y_{m_{2i-1}}$ *and* $g_i : Y_{m_{2i}} \to X_{n_{2i}}$,

such that

- *the following diagrams are* $\epsilon_{m_{2i-1}}$*-commutative for all* $j \le m_{2i-1}$ *and* $k \ge i$:

$$\begin{array}{ccccc} & & X_{n_{2i-1}} & \xleftarrow{b_{n_{2i-1}}^{n_{2k}}} & X_{n_{2k}} \\ & & f_i \downarrow & & \uparrow g_k \\ Y_j & \xleftarrow{\beta_j^{m_{2i-1}}} & Y_{m_{2i-1}} & \xleftarrow{\beta_{m_{2i-1}}^{m_{2k}}} & Y_{m_{2k}} \end{array} \qquad \begin{array}{ccccc} & & X_{n_{2i-1}} & \xleftarrow{b_{n_{2i-1}}^{n_{2k-1}}} & X_{n_{2k-1}} \\ & & f_i \downarrow & & \downarrow f_k \\ Y_j & \xleftarrow{\beta_j^{m_{2i-1}}} & Y_{m_{2i-1}} & \xleftarrow{\beta_{m_{2i-1}}^{m_{2k-1}}} & Y_{m_{2k-1}} \end{array}$$

- *the following diagrams are $\epsilon_{n_{2i}}$-commutative for all $j \leq n_{2i}$ and $k \geq i$:*

$$\begin{array}{ccccc} X_j & \xleftarrow{b_j^{n_{2i}}} & X_{n_{2i}} & \xleftarrow{b_{n_{2i}}^{n_{2k}}} & X_{n_{2k}} \\ & & g_i \uparrow & & \uparrow g_k \\ & & Y_{m_{2i}} & \xleftarrow{\beta_{m_{2i}}^{m_{2k}}} & Y_{m_{2k}} \end{array} \qquad \begin{array}{ccccc} X_j & \xleftarrow{b_j^{n_{2i}}} & X_{n_{2i}} & \xleftarrow{b_{n_{2i}}^{n_{2k+1}}} & X_{n_{2k+1}} \\ & & g_i \uparrow & & \downarrow f_{k+1} \\ & & Y_{m_{2i}} & \xleftarrow{\beta_{m_{2i}}^{m_{2k+1}}} & Y_{m_{2k+1}} \end{array}$$

Furthermore,

$$h(\langle x_i \rangle) = \bigcap_{j=1}^{\infty} \overline{\bigcup_{i=j}^{\infty} \pi_{m_{2i-1}}^{-1}(f_i(x_{n_{2i-1}}))}, \quad h^{-1}(\langle x_i \rangle) = \bigcap_{j=1}^{\infty} \overline{\bigcup_{i=j}^{\infty} \pi_{n_{2i}}^{-1}(g_i(x_{m_{2i}}))}.$$

The next two corollaries show that we can compose or decompose the bonding maps without changing the topology of the inverse limit space, and "small" changes to $\{f_i\}_{i=1}^{\infty}$ do not change the homeomorphism.

COROLLARY 6. *Let $X = \varprojlim (X_i, b_i)_{i=1}^{\infty}$ and $\widehat{X} = \varprojlim (X_{n_i}, b_{n_i}^{n_{i+1}})_{i=1}^{\infty}$ where $\{n_i\}_{i=1}^{\infty}$ is an increasing sequence. Then $\widehat{X}$ and X are homeomorphic.*

COROLLARY 7. *Let $X = \varprojlim (X_i, b_i)_{i=1}^{\infty}$, $\{n_i\}_{i=1}^{\infty}$ be an increasing sequence, $f_i, \widehat{f}_i : X_{n_{i+1}} \to X_{n_i}$ and $h, \widehat{h} : X \to X$ be such that $h(\langle x_i \rangle_{i=1}^{\infty}) = \bigcap_{j=1}^{\infty} \overline{\bigcup_{i=j}^{\infty} \pi_{n_i}^{-1}(f_i(x_{n_{i+1}}))}$ and $\widehat{h}(\langle x_i \rangle_{i=1}^{\infty}) = \bigcap_{j=1}^{\infty} \overline{\bigcup_{i=j}^{\infty} \pi_{n_i}^{-1}(\widehat{f}_i(x_{n_{i+1}}))}$. If for every $\epsilon > 0$ there exists N_ϵ such that $\mathrm{d}(b_k^{n_i} \circ f_i(x_{n_{i+1}}), b_k^{n_i} \circ \widehat{f}_i(x_{n_{i+1}})) < \epsilon$ for every $i > N_\epsilon$ and $k \leq n_i$, then $h = \widehat{h}$.*

Let $\{f_\alpha\}_{\alpha \in \Omega}$ be a collection of maps $f_\alpha : X \to X$ such that $f_\alpha \circ f_\beta = f_\beta \circ f_\alpha$ for all $\alpha, \beta \in \Omega$. Let $\widehat{X} = \varprojlim (X, g_i)_{i=1}^{\infty}$ where $g_i \in \{f_\alpha\}_{\alpha \in \Omega}$ for each i. Then define $\mathcal{O}(f_\alpha, \widehat{X}) = |\{i : g_i = f_\alpha\}|$ and $\mathcal{O}_j(f_\alpha, \widehat{X}) = |\{i : g_i = f_\alpha \text{ and } i \geq j\}|$. Additionally suppose $p = (f_{\alpha_1})^{n_1} \circ \cdots \circ (f_{\alpha_k})^{n_k}$ where $\alpha_i \in \Omega$ and $\alpha_i = \alpha_j$ if and only if $i = j$. Then define $\mathcal{O}(f_\alpha, p) = n_i$ if $\alpha = \alpha_i$ and $\mathcal{O}(f_\alpha, p) = 0$ if $\alpha \notin \{\alpha_1, \ldots, \alpha_k\}$. This counts the number of times a particular function appears in the composition of p.

Let $\mathcal{A}$ be a finite multi-set, denoted by $[a_1, \ldots, a_n]$, of commuting self-maps on X. Then define $\mathcal{A}^* : X \to X$ by taking the composition of the elements of $\mathcal{A}$. Since the maps commute, the order of the composition does not matter. Then if $p = \mathcal{A}^*$ define $\mathcal{C}(p) = \mathcal{A}$. This allows us to compose multisets of functions into a function or decompose a function into a multiset of its component functions. For example, if $g_i^j = g_i \circ \cdots \circ g_{j-1}$, then $\mathcal{C}(g_i^j) = [g_i, \ldots, g_{j-1}]$ and $\mathcal{C}(g_i^j)^* = g_i^j$.

The next theorem shows that the order of the bonding maps does not matter if the maps are all pairwise commutative:

THEOREM 8. *Suppose that* $\widehat{X} = \varprojlim (X, g_i)_{i=1}^{\infty}$ *and* $\widehat{Y} = \varprojlim (X, \widetilde{g}_i)_{i=1}^{\infty}$ *where* $g_i, \widetilde{g}_i \in \{f_\alpha\}_{\alpha\in\Omega}$ *and* $\mathcal{O}(f_\alpha, \widehat{X}) = \mathcal{O}(f_\alpha, \widehat{Y})$ *for all* $\alpha \in \Omega$. *Then* $\widehat{X}$ *is homeomorphic to* $\widehat{Y}$.

Proof. For notational convenience we will assume $g_i : X_{i+1} \to X_i$, $\widetilde{g}_i : X'_{i+1} \to X'_i$, $p_i^j : X'_j \to X_i$ and $q_i^j : X_j \to X'_i$ where $X'_i = X = X_i$.

Let $p_1^1 = \mathrm{id}_X$ and $q_1^2 = g_1$. Then $g_1 = p_1^1 \circ q_1^2$. Also notice that

$$\mathcal{O}(f_\alpha, \widehat{Y}) = \mathcal{O}_2(f_\alpha, \widehat{X}) + \mathcal{O}(f_\alpha, q_1^2).$$

Thus there exists $n_1 > 2$ such that $\mathcal{C}(q_1^2) \subset \mathcal{C}(\widetilde{g}_1^{n_1})$. Let $p_2^{n_1} = (\mathcal{C}(\widetilde{g}_1^{n_1}) - \mathcal{C}(q_1^2))^*$. Then $\widetilde{g}_1^{n_1} = q_1^2 \circ p_2^{n_1}$.

Continuing inductively suppose that $p_2^{n_1}, \ldots, p_{n_{2k-2}}^{n_{2k-1}}$ and $q_{n_1}^{n_2}, \ldots, q_{n_{2k-1}}^{n_{2k}}$ have been found. Then

$$\mathcal{O}_{n_{2k-1}}(f_\alpha, \widehat{Y}) = \mathcal{O}_{n_{2k}}(f_\alpha, \widehat{X}) + \mathcal{O}(f_\alpha, q_{n_{2k-1}}^{n_{2k}}).$$

Thus there exists $n_{2k+1} > n_{2k}$ such that $\mathcal{C}(q_{n_{2k-1}}^{n_{2k}}) \subset \mathcal{C}(\widetilde{g}_{n_{2k-1}}^{n_{2k+1}})$. Let $p_{n_{2k}}^{n_{2k+1}} = (\mathcal{C}(\widetilde{g}_{n_{2k-1}}^{n_{2k+1}}) - \mathcal{C}(q_{n_{2k-1}}^{n_{2k}}))^*$. Then $\widetilde{g}_{n_{2k-1}}^{n_{2k+1}} = q_{n_{2k-1}}^{n_{2k}} \circ p_{n_{2k}}^{n_{2k+1}}$.

Also,

$$\mathcal{O}_{n_{2k}}(f_\alpha, \widehat{X}) = \mathcal{O}_{n_{2k+1}}(f_\alpha, \widehat{Y}) + \mathcal{O}(f_\alpha, p_{n_{2k}}^{n_{2k+1}}).$$

Thus there exists $n_{2k+2} > n_{2k+1}$ such that $\mathcal{C}(p_{n_{2k}}^{n_{2k+1}}) \subset \mathcal{C}(g_{n_{2k}}^{n_{2k+2}})$. Let $q_{n_{2k+1}}^{n_{2k+2}} = (\mathcal{C}(g_{n_{2k}}^{n_{2k+2}}) - \mathcal{C}(q_{n_{2k}}^{n_{2k+1}}))^*$.

Hence $\widehat{X}$ and $\widehat{Y}$ are homeomorphic by Theorem 5. ∎

If $f : S \to S$ is a map, then define $\deg(f) = n$ if f is homotopic to (n). It quickly follows that $\deg(f \circ g) = \deg(f)\deg(g)$ (see [9]).

Let $f : S \to S$ be a map such that $\deg(f) = a$. Then define $F : \mathbb{R} \to \mathbb{R}$ to be the *lift* of f given by $F(x) = a\lfloor x\rfloor + f(x \bmod 1)$. Lower case letters will always define functions on S whereas the capitalization of that letter will represent the lift of that function—except in the case of multiplication by an integer a. In this case we define $(a) : S \to S$ and $a : \mathbb{R} \to \mathbb{R}$. Notice that $f(x) = F(x) \bmod 1$ and $(a)x = ax \bmod 1$. Normally a and b will denote positive integers.

Let $f, g : S \to S$. Then define $\mathrm{d}_S(f, g) = \sup_{x\in S} \mathrm{d}_S(f(x), g(x))$. Likewise define $\mathrm{d}_{\mathbb{R}}(F, G) = \sup_{x\in\mathbb{R}} \mathrm{d}_{\mathbb{R}}(F(x), G(x))$ where F and G are lifts of f and g respectively onto $\mathbb{R}$. Notice that

$$\mathrm{d}_S(f, g) = \min\{\mathrm{d}_{\mathbb{R}}(F, G) \bmod 1, 1 - (\mathrm{d}_{\mathbb{R}}(F, G) \bmod 1)\}.$$

From here on if $\Sigma = \varprojlim (S, (b_i))_{i=1}^{\infty}$, then define $S_i = \pi_i(\Sigma)$.

Suppose that $F:\mathbb{R}\to\mathbb{R}$ is a map. Then $[a,b]$ is an *up-horseshoe* of F if

- $F(a)=F(b)$,
- $F(x)\geq F(a)$ for all $x\in[a,b]$.

The interval $[a,b]$ is a *maximal up-horseshoe* of F if $[a,b]$ is an up-horseshoe and no interval $[c,d]$ that properly contains $[a,b]$ is an up-horseshoe.

PROPOSITION 9. *Suppose that $F:\mathbb{R}\to\mathbb{R}$ is a map and $k\in\mathbb{N}$ is such that $\mathrm{d}_{\mathbb{R}}(F,k)<\epsilon$. If $[a,b]$ is an up-horseshoe of F, then* $\mathrm{diam}(F([a,b]))<2\epsilon$.

Proof. Suppose on the contrary that $\mathrm{diam}(F([a,b]))\geq 2\epsilon$. Then there exists $x_m\in(a,b)$ such that $F(x_m)-F(b)=2\epsilon$. Since $|F(x_m)-kx_m|<\epsilon$, we have $kx_m-F(b)>\epsilon$. However, since k is increasing, $kb>kx_m$. Therefore $|kb-F(b)|=kb-F(b)>\epsilon$, which is a contradiction. ∎

PROPOSITION 10. *Suppose that $F:\mathbb{R}\to\mathbb{R}$ and $G:\mathbb{R}\to\mathbb{R}$ are onto maps and $k\in\mathbb{N}$ is such that $\mathrm{d}_{\mathbb{R}}(F\circ G,k)<\epsilon$. If $[a,b]$ is an up-horseshoe of F, then* $\mathrm{diam}(F([a,b]))<2\epsilon$.

Proof. Suppose that $[a,b]$ is an up-horseshoe of F. There exists an interval $[a',b']$ such that $G([a',b'])=[a,b]$, and either $G(a')=a$ and $G(b')=b$, or $G(a')=b$ and $G(b')=a$. Thus $[a',b']$ is an up-horseshoe of $F\circ G$. Therefore, by Proposition 9,

$$\mathrm{diam}(F([a,b]))=\mathrm{diam}(F\circ G([a',b']))<2\epsilon. \ ∎$$

PROPOSITION 11. *Suppose that $F:\mathbb{R}\to\mathbb{R}$ is a map such that one of the following is true:*

- $\lim_{x\to-\infty}F(x)=-\infty$ *and* $\lim_{x\to\infty}F(x)=\infty$,
- $\lim_{x\to-\infty}F(x)=\infty$ *and* $\lim_{x\to\infty}F(x)=-\infty$.

If $[a,b]$ and $[c,d]$ are maximal up-horseshoes of F, then $[a,b]=[c,d]$ or $[a,b]\cap[c,d]=\emptyset$.

Proof. Suppose on the contrary that $[a,b]$ and $[c,d]$ are maximal up-horseshoes such that $[a,b]\neq[c,d]$ and $[a,b]\cap[c,d]\neq\emptyset$. We may assume that $a<c<b<d$. Then since $c\in[a,b]$ and $b\in[c,d]$, we have $F(c)\geq F(b)$ and $F(b)\geq F(c)$. Therefore $F(a)=F(b)=F(c)=F(d)$ and $F(x)\geq F(a)$ for all $x\in[a,d]$. Thus $[a,d]$ is an up-horseshoe that properly contains $[a,b]$, which is a contradiction. ∎

PROPOSITION 12. *Suppose that $F:\mathbb{R}\to\mathbb{R}$ and $G:\mathbb{R}\to\mathbb{R}$ are onto maps and $k\in\mathbb{N}$ is such that $\mathrm{d}_{\mathbb{R}}(F\circ G,k)<\epsilon$. Then there exists a monotone map $\widehat{F}:\mathbb{R}\to\mathbb{R}$ such that $\mathrm{d}_{\mathbb{R}}(F,\widehat{F})<2\epsilon$.*

Proof. By Proposition 11 the set of maximal up-horseshoes is a pairwise disjoint collection of intervals $\{[a_i, b_i]\}_{i\in\Omega}$. Define $\widehat{F} : \mathbb{R} \to \mathbb{R}$ by

$$\widehat{F}(x) = \begin{cases} F(x) & \text{if } x \in \mathbb{R} - \bigcup_{i\in\Omega}[a_i, b_i], \\ F(a_i) & \text{if } x \in [a_i, b_i]. \end{cases}$$

Then $\widehat{F}$ is monotone. Since $\operatorname{diam}(F([a_i, b_i])) < 2\epsilon$ by Proposition 10, we have $\mathrm{d}(\widehat{F}(x), F(x)) = \mathrm{d}(F(a_i), F(x)) < 2\epsilon$ if $x \in [a_i, b_i]$. Thus $\mathrm{d}_{\mathbb{R}}(F, \widehat{F}) < 2\epsilon$. ∎

THEOREM 13. *Suppose the following:*

- $\Sigma = \varprojlim (S, (b_i))_{i=1}^{\infty}$,
- $\{n_i\}_{i=1}^{\infty}$ *is a non-decreasing sequence of positive integers,*
- $f_i : S_{n_{i+1}} \to S_{n_i}$,
- $h : \Sigma \to \Sigma$ *is a homeomorphism such that*

$$h(\langle x_i\rangle_{i=1}^{\infty}) = \bigcap_{j=1}^{\infty} \overline{\bigcup_{i=j}^{\infty} \pi_{n_i}^{-1}(f_i(x_{n_{i+1}}))}.$$

Then there exist maps $\widehat{f_i} : S_{n_{i+1}} \to S_{n_i}$ whose lifts $\widehat{F_i}$ are monotone such that $h(\langle x_i\rangle_{i=1}^{\infty}) = \bigcap_{j=1}^{\infty} \overline{\bigcup_{i=j}^{\infty} \pi_{n_i}^{-1}(\widehat{f_i}(x_{n_{i+1}}))}$.

Proof. Let F_i be the lift of f_i. Then by Theorem 5 there exists a sequence $\{\epsilon_i\}_{i=1}^{\infty}$ of positive numbers, a sequence $\{k_i\}_{i=1}^{\infty}$ of positive integers and maps $G_i : \mathbb{R} \to \mathbb{R}$ such that

- $\epsilon_i \to 0$ as $i \to \infty$,
- $\mathrm{d}_{\mathbb{R}}(F_i \circ G_i, k_i) < \epsilon_i / b_1^{n_i}$.

Then it follows from Proposition 12 that there exists a monotone map $\widehat{F_i}$ such that $\mathrm{d}_{\mathbb{R}}(F_i, \widehat{F_i}) < 2\epsilon_i / b_1^{n_i}$. Let $\widehat{f_i} = \widehat{F_i} \bmod 1$; then the theorem follows from Corollary 7. ∎

Since h is an expansive homeomorphism if and only if h^2 is an expansive homeomorphism by Theorem 4, we may take each $\widehat{F_i}$ in Theorem 13 to be monotone increasing.

3. Expansive homeomorphisms on regular solenoids. In this section we will examine sufficient conditions for a solenoid to admit an expansive homeomorphism. This was first shown by Williams [14]. For completeness, the proof will be given.

Let $b \in \mathbb{Z}^+$ and define $(b) : S \to S$ by $(b)x = bx \bmod 1$. A solenoid Σ_b is *regular* if there exists a positive integer b such that $\Sigma_b = \varprojlim (S, (b))$.

Define the *shift homeomorphism* $h_b : \Sigma_b \to \Sigma_b$ by

$$h_b(\mathbf{x}) = h_b(\langle x_1, x_2, x_3, \ldots\rangle) = \langle (b)x_1, (b)x_2, (b)x_3, \ldots\rangle = \langle (b)x_1, x_1, x_2, \ldots\rangle.$$

Also, notice that $h_b^{-1}(\langle x_1, x_2, x_3, \ldots\rangle) = \langle x_2, x_3, x_4, \ldots\rangle$.

THEOREM 14. *Let Σ_b be a regular solenoid such that $b > 1$. Then Σ_b admits an expansive homeomorphism.*

Proof. We will show that the shift homeomorphism h_b is expansive.

Let the expansive constant be $1/(2b)$. Notice that if $x, y \in S$ and $\mathrm{d}_S(x, y) < 1/(2b)$ then $\mathrm{d}_S((b)x, (b)y) = b\mathrm{d}_S(x, y)$. Let $\mathbf{x}, \mathbf{y}$ be distinct elements in Σ_b. Then there exists i such that $x_i \neq y_i$. Furthermore, there exists a non-negative natural number n such that $1/(2b) < b^n \mathrm{d}_S(x_i, y_i) \le 1/2$. Hence,

$$\begin{aligned}\mathrm{d}(h_b^{n-i+1}(\mathbf{x}), h_b^{n-i+1}(\mathbf{y})) &= \mathrm{d}(h_b^n(h_b^{-i+1}(\mathbf{x})), h_b^n(h_b^{-i+1}(\mathbf{y}))) \\ &= \mathrm{d}(h_b^n(\langle x_i, x_{i+1}, \ldots\rangle), h_b(\langle y_i, y_{i+1}, \ldots\rangle)) \\ &> b^n \mathrm{d}_S(x_i, y_i) > \frac{1}{2b}.\end{aligned}$$

Hence h_b is expansive. ∎

Now in order to prove the main theorem we must show that if Σ is a solenoid that admits an expansive homeomorphism, then Σ must be regular.

4. A critical example and motivation of the proof of the main result. In this section we give an example of a homeomorphism of a solenoid that is continuum-wise expansive but not expansive. This example is the motivation for the main result. A homeomorphism is *continuum-wise expansive* if there exists a number $c > 0$ (called the *continuum-wise expansive constant*) such that for every non-degenerate subcontinuum A, there exists an integer n such that $\mathrm{diam}(h^n(A)) \ge c$. A homeomorphism is *positively continuum-wise expansive* if there exists an integer $n \ge 0$ such that $\mathrm{diam}(h^n(A)) \ge c$. All expansive homeomorphisms are continuum-wise expansive, but the converse is not true.

Let $\{p_i\}_{i=1}^{\infty}$ be an increasing sequence of primes and define $\{\widehat{b}_i\}_{i=1}^{\infty}$ by

$$\widehat{b}_i = \begin{cases} 2 & \text{if } i \text{ is odd}, \\ p_{i/2} & \text{if } i \text{ is even}. \end{cases}$$

Then define $\widehat{\Sigma}_2 = \varprojlim (S, (\widehat{b}_i))_{i=1}^{\infty}$.

Let $h : \widehat{\Sigma}_2 \to \widehat{\Sigma}_2$ be defined by $h(\langle x_i\rangle_{i=1}^{\infty}) = \langle (2)x_i\rangle_{i=1}^{\infty}$. It can be shown by using Theorem 5 that h is a homeomorphism.

In order to prove that neither the current example nor non-regular solenoids in general admit expansive homeomorphisms, we must find subsets that satisfy the assumption of the following lemma.

LEMMA 15. *Suppose that $h : X \to X$ is an expansive homeomorphism and $Y \subset X$ is such that $h(Y) = Y$ and $|Y| \ge 2$. Then the expansive constant for h cannot be greater than* $\mathrm{diam}(Y)$.

Proof. Let x and y be distinct elements of Y. Then $h^n(x), h^n(y) \in Y$ for all $n \in \mathbb{Z}$. Therefore, $\mathrm{d}(h^n(x), h^n(y)) \le \mathrm{diam}(Y)$ for all n. ∎

THEOREM 16. *The homeomorphism $h : \widehat{\Sigma}_2 \to \widehat{\Sigma}_2$ is continuum-wise expansive. However, h is not expansive.*

Proof. The continuum-wise expansive constant for h will be $1/4$. Let A be a proper subcontinuum of $\widehat{\Sigma}_2$. Then A is an arc with endpoints say $\mathbf{x} = \langle x_i\rangle_{i=1}^{\infty}$ and $\mathbf{y} = \langle y_i\rangle_{i=1}^{\infty}$. If $\pi_1(A) = S$, then $\operatorname{diam}(A) \geq (1/2)\mathrm{d}_S(0, 1/2) = 1/4$. So suppose that $\pi_1(A)$ is an arc $[x_1, y_1]$. Then there exists a positive integer n such that $|2^n x_1 - 2^n y_1| > 1$. Thus $(2)^n([x_1, y_1]) = S$. Therefore $\operatorname{diam}(h^n(A)) \geq 1/4$ and h is continuum-wise expansive.

On the other hand, let c be the expansive constant for h. Pick $i > 2$ such that

$$\frac{1}{2}\sum_{j=2i}^{\infty}\frac{1}{2^j} < c$$

and consider the set $\{j/p_i\}_{j=1}^{p_i-1}$. Then since 2 and p_i are relatively prime, $(2)(\{j/p_i\}_{j=1}^{p_i-1}) = \{j/p_i\}_{j=1}^{p_i-1}$. Thus

$$h\left(\pi_{2i+1}^{-1}\left(\left\{\frac{j}{p_i}\right\}_{j=1}^{p_i-1}\right)\right) \subset \pi_{2i+1}^{-1}\left((2)\left(\left\{\frac{j}{p_i}\right\}_{j=1}^{p_i-1}\right)\right) = \pi_{2i+1}^{-1}\left(\left\{\frac{j}{p_i}\right\}_{j=1}^{p_i-1}\right).$$

Let $Y = \bigcap_{n=0}^{\infty} h^n(\pi_{2i+1}^{-1}(\{j/p_i\}_{j=1}^{p_i-1}))$. Then $h(Y) = Y$. Furthermore, since $\pi_{2i+1}(Y) = \{j/p_i\}_{j=1}^{p_i-1}$, we have $|Y| \geq 2$. Finally, as $(b_{2i})(j/p_i) = (p_i)(j/p_i) = 0$, it follows that

$$\operatorname{diam}(Y) \leq \operatorname{diam}\left(\pi_{2i+1}^{-1}\left(\left\{\frac{j}{p_i}\right\}_{j=1}^{p_i-1}\right)\right) \leq \operatorname{diam}(\pi_{2i}^{-1}(0)) < \frac{1}{2}\sum_{j=2i}^{\infty}\frac{1}{2^j} < c.$$

Thus by Lemma 15, c is not an expansive constant for h. Consequently, h is not an expansive homeomorphism. ∎

In order to prove the main result, it will also be necessary to find arbitrarily small invariant sets under h. However, since we are only guaranteed ϵ-commuting diagrams and not commuting diagrams to describe an arbitrary homeomorphism on an inverse limit space, this process becomes more difficult and technical. An outline of the proof of the main theorem is the following:

- Without loss of generality, we can assume that $h : \Sigma \to \Sigma$ is a positively continuum-wise expansive homeomorphism and an expansive homeomorphism with expansive constant $c > 0$ of a solenoid Σ.
- From Section 2, we can assume that $\Sigma = \varprojlim (S, (b_i))_{i=1}^{\infty}$ where $\{b_i\}_{i=1}^{\infty}$ is a sequence of primes.
- In Section 5, it is shown that Σ is homeomorphic to one of five types of solenoids (see Section 5 for precise definitions):

(a) Types 1A, 2A and 3A have the following properties:

(i) there exist at least one prime number p that occurs infinitely often in $\{b_i\}_{i=1}^{\infty}$,
(ii) there are an infinite number of distinct primes in $\{b_i\}_{i=1}^{\infty}$,
(iii) there are strings of increasing length of the same primes that follow a certain pattern.

(b) Type 4 solenoids have the following properties:

(i) no prime number occurs infinitely often in $\{b_i\}_{i=1}^{\infty}$,
(ii) there are an infinite number of distinct primes in $\{b_i\}_{i=1}^{\infty}$.

(c) Solenoids that are type 5A have the property that there are only a finite number of distinct prime numbers in $\{b_i\}_{i=1}^{\infty}$, each of which occurs infinitely often. These solenoids are regular and admit expansive homeomorphisms as shown in Section 3.

- It is shown in Theorems 13 and 19 and Proposition 18 that there exist a non-decreasing sequence $\{n_i\}_{i=1}^{\infty}$ and maps $f_i : S_{n_{2i}} \to S_{n_{2i-1}}$ such that for each i:

(a) the lift F_i of f_i is monotone increasing,
(b) there exist relatively prime positive integers a and b such that $\deg(f_i)/b_{n_{2i-1}}^{n_{2i}} = a/b$,
(c) $h(\langle x_i\rangle_{i=1}^{\infty}) = \bigcap_{j=1}^{\infty} \overline{\bigcup_{i=j}^{\infty} \pi_{n_{2i-1}}^{-1}(f_i(x_{n_{2i}}))}$.

- The *multiplying factor of* h is defined to be $M(h) = a/b$. The multiplying factor is essentially the factor to which the diameters of small subarcs of Σ expand under h.
- It is shown in Corollary 34 that if h is positively continuum-wise expansive, then $M(h) > 1$.
- It follows from Corollary 20 that if Σ is a type 4 solenoid, then $M(h)=1$ and hence h cannot be continuum-wise expansive. It follows that h is not expansive.
- Next, under the assumption that Σ is a type 1A, 2A, or 3A solenoid it can be shown by using Theorem 22 and Corollary 23 that there exist an increasing sequence $\{s_i\}_{i=1}^{\infty}$ of positive integers and an integer β such that $\{n_i\}_{i=1}^{\infty}$ and $\{f_i\}_{i=1}^{\infty}$ can be chosen to have the following additional properties:

(a) $\deg(f_i) = a$,
(b) $n_{2_i} = n_{2i-1} + \beta = \alpha_i' + \beta$,
(c) $b_{n_{2i-1}}^{n_{2i}} = b_{\alpha_i'}^{\alpha_i'+\beta} = b$,
(d) $s_i < \alpha_i'$,

(e) b_{s_i} is relatively prime to both a and b,

(f) f_i becomes arbitrarily close to (a) with great control. (Note: finding this control on the closeness is a major part of the technical discussion in the following.)

- As $\{s_i\}_{i=1}^{\infty}$ is increasing, there exists an N_1 such that $\mathrm{diam}(\pi_{s_i}^{-1}(0)) < c/6$ for all $i > N_1$. Then since $(b_{s_i})(j/b_{s_i}) = 0$ for $j \in \mathbb{Z}$, it follows that
$$\mathrm{diam}\left(\pi_{s_i+1}^{-1}\left(\left\{\frac{j}{b_{s_i}}\right\}_{j=1}^{b_{s_i}-1}\right)\right) < c/6.$$
(Note: b_{s_i} is also represented by ρ_i and ρ in what follows.)
- Also, since b_{s_i} is relatively prime to a and b, it follows that
$$(a)\left(\left\{\frac{j}{b_{s_i}}\right\}_{j=1}^{b_{s_i}-1}\right) = (b)\left(\left\{\frac{j}{b_{s_i}}\right\}_{j=1}^{b_{s_i}-1}\right) = \left\{\frac{j}{b_{s_i}}\right\}_{j=1}^{b_{s_i}-1}.$$
- Since $a > b$ for an appropriate choice of $\epsilon > 0$, there is an $r_\epsilon > 1$ such that
$$\bigcup_{j=1}^{b_{s_i}-1}\left[\frac{j}{b_{s_i}} - r_\epsilon b\epsilon, \frac{j}{b_{s_i}} + r_\epsilon b\epsilon\right] \subset \bigcup_{j=1}^{b_{s_i}-1}(a)\left(\left[\frac{j}{b_{s_i}} - \epsilon, \frac{j}{b_{s_i}} + \epsilon\right]\right).$$
- Since f_i can be taken arbitrarily close to (a) with some control, and by Theorem 41 and 42 and Corollaries 43 and 44, it follows that for some choice of ϵ we have
$$\bigcup_{j=1}^{b_{s_i}-1}\left[\frac{j}{b_{s_i}} - b\epsilon, \frac{j}{b_{s_i}} + b\epsilon\right] \subset \bigcup_{j=1}^{b_{s_i}-1} f_i\left(\left[\frac{j}{b_{s_i}} - \epsilon, \frac{j}{b_{s_i}} + \epsilon\right]\right).$$
This choice of ϵ also requires quite a bit of technical details in the following.
- Thus, if
$$Y = \pi_{\alpha_i'}^{-1}\left(\bigcup_{j=1}^{b_{s_i}-1}\left[\frac{j}{b_{s_i}} - b\epsilon, \frac{j}{b_{s_i}} + b\epsilon\right]\right) = \pi_{\alpha_i'+\beta}^{-1}\left(\bigcup_{j=1}^{b_{s_i}-1}\left[\frac{j}{b_{s_i}} - \epsilon, \frac{j}{b_{s_i}} + \epsilon\right]\right),$$
then $Y \subset h(Y)$. So there a non-degenerate set $Y' \subset \Sigma$ such that $h(Y') = Y'$ and $\mathrm{d}_H(Y', \pi_{s_i+1}^{-1}(\{j/b_{s_i}\}_{j=1}^{b_{s_i}-1})) < c/6$ by Theorem 51.
- It follows from Lemma 15 that c is not an expansive constant for h.

5. Homeomorphisms on solenoids. In this section we reduce the class of solenoids to types of solenoids whose bonding maps occur in a manageable order. Also we show that a homeomorphism h on a solenoid has a multiplication factor $M(h)$ associated with it. This multiplication factor helps to determine the dynamics of the homeomorphism.

Proposition 17. *Suppose that* $f, g : S \to S$ *are such that* $\mathrm{d}_S(f, g) < 1/16$. *Then* $\deg(f) = \deg(g)$.

Proof. Suppose on the contrary that $\deg(f) > \deg(g)$ and let F and G be the respective lifts of f and g. Then $F(1) - F(0) \geq G(1) - G(0) + 1$. Define $D(x) = F(x) - G(x) - (F(0) - G(0))$. Then $D(0) = 0$ and $D(1) \geq 1$. Therefore by the Intermediate Value Theorem, there exists $y \in (0, 1)$ such that $D(y) = 1/4$. But since $|f(0) - g(0)| < 1/16$, it follows that $|F(0) - G(0)| < 1/16$ or $15/16 < |F(0) - G(0)| < 1$. Thus $3/16 < |F(y) - G(y)| < 5/16$, $11/16 < |F(y) - G(y)| < 3/4$ or $19/16 < |F(y) - G(y)| < 5/4$. Therefore $\mathrm{d}_S(f, g) > 1/16$, which is a contradiction. ∎

Proposition 18. *Suppose that* $f, g, \phi, \psi : S \to S$ *are such that*

$$d_S(\psi \circ f(x), g \circ \phi(x)) < 1/16 \quad \text{for each } x \in S.$$

Then $\deg(f)\deg(\psi) = \deg(g)\deg(\phi)$.

Proof. This follows from the fact that

$$\deg(f)\deg(\psi) = \deg(\psi \circ f), \qquad \deg(g)\deg(\phi) = \deg(g \circ \phi)$$

and Proposition 17. ∎

Theorem 19. *Suppose the following:*

(1) $\{n_i\}_{i=1}^{\infty}$ *is a non-decreasing sequence of positive integers,*
(2) $f_i : S_{n_{2i}} \to S_{n_{2i-1}}$ *and* $g_i : S_{n_{2i+1}} \to S_{n_{2i}}$,
(3) $\deg(f_i) b_{n_{2i}}^{n_{2j}} = \deg(f_j) b_{n_{2i-1}}^{n_{2j-1}}$ *for* $i < j$,
(4) $\deg(f_i)\deg(g_i) = b_{n_{2i-1}}^{n_{2i+1}}$.

Then there exist relatively prime positive integers a *and* b *such that*

- $\deg(f_i)/b_{n_{2i-1}}^{n_{2i}} = a/b$ *and* $\deg(g_i)/b_{n_{2i}}^{n_{2i+1}} = b/a$,
- ab *divides* $b_{n_{2i-1}}^{n_{2i+1}}$,

for every $i \in \mathbb{Z}^+$.

Proof. It follows from hypothesis (3) that

$$\deg(f_i) b_{n_{2j-1}}^{n_{2j}} b_{n_{2i}}^{n_{2j-1}} = \deg(f_j) b_{n_{2i}}^{n_{2j-1}} b_{n_{2i-1}}^{n_{2i}}.$$

Thus

$$\frac{\deg(f_i)}{b_{n_{2i-1}}^{n_{2i}}} = \frac{\deg(f_j)}{b_{n_{2j-1}}^{n_{2j}}}$$

for all $i < j$. Therefore there exist relatively prime positive integers a and b such that $\deg(f_i)/b_{n_{2i-1}}^{n_{2i}} = a/b$. Then $\deg(g_i)/b_{n_{2i}}^{n_{2i+1}} = b/a$ follows from hypothesis (4). Also since $a = b\deg(f_i)/b_{n_{2i-1}}^{n_{2i}}$, a divides $\deg(f_i)$ and similarly b divides $\deg(g_i)$. Hence ab divides $b_{n_{2i-1}}^{n_{2i+1}}$ by hypothesis (4). ∎

Let $\Sigma = \varprojlim (S, (b_i))_{i=1}^{\infty}$ and $h : \Sigma \to \Sigma$ be a homeomorphism. Then by Theorem 5, there exists a non-decreasing sequence $\{n_i\}_{i=1}^{\infty}$ and maps $g_i : S_{n_{2i+1}} \to S_{n_{2i}}$ and $f_i : S_{n_{2i}} \to S_{n_{2i-1}}$ such that

- $h(\langle x_i\rangle_{i=1}^{\infty}) = \bigcap_{j=1}^{\infty} \overline{\bigcup_{i=j}^{\infty} \pi_{n_{2i-1}}^{-1}(f_i(x_{n_{2i}}))}$,
- $h^{-1}(\langle x_i\rangle_{i=1}^{\infty}) = \bigcap_{j=1}^{\infty} \overline{\bigcup_{i=j}^{\infty} \pi_{n_{2i}}^{-1}(g_i(x_{n_{2i+1}}))}$,
- $\mathrm{d}(f_i \circ (b_{n_{2i}}^{n_{2j}}), (b_{n_{2i-1}}^{n_{2j-1}}) \circ f_j) < 1/16$ for all $i < j$.

Then Theorem 19 and Proposition 18 imply that there exist relatively prime numbers a and b such that $\deg(f_i)/b_{n_{2i-1}}^{n_{2i}} = a/b$ and $\deg(g_i)/b_{n_{2i}}^{n_{2i+1}} = b/a$. So define the *multiplying factor* of h to be $M(h) = a/b$. Notice that $M(h^{-1}) = (M(h))^{-1} = b/a$.

The next corollary now follows from the previous definition and Theorem 19:

Corollary 20. *Suppose that $h : \Sigma \to \Sigma$ is a homeomorphism where $\Sigma = \varprojlim (S, (b_i))_{i=1}^{\infty}$. If $M(h) = a/b$ where a and b are relatively prime and k is a prime number that divides ab, then there exists an increasing sequence $\{m_i\}_{i=1}^{\infty}$ such that k divides each b_{m_i}.*

Let

$$P(\{b_i\}_{i=1}^{\infty}) = \{p \mid p \text{ is prime and } |\{i : b_i = p\}| = \infty\},$$
$$Q(\{b_i\}_{i=1}^{\infty}) = \{q \mid q \text{ is prime and } 0 \le |\{i : b_i = q\}| < \infty\}.$$

That is, $P(\{b_i\}_{i=1}^{\infty})$ is the set of primes that repeat an infinite number of times in $\{b_i\}_{i=1}^{\infty}$, and $Q(\{b_i\}_{i=1}^{\infty})$ is the set of primes that repeat a finite number of times in $\{b_i\}_{i=1}^{\infty}$. Also define $W(\{b_i\}_{i=1}^{\infty}) = \{w_i\}_{i=1}^{\infty}$ in the following way:

$$\begin{aligned} w_i &= b_1 && \text{if } i = 1 \bmod 2,\\ w_i &= b_2 && \text{if } i = 2 \bmod 4,\\ & && \vdots\\ w_i &= b_k && \text{if } i = 2^{k-1} \bmod 2^k. \end{aligned}$$

For example, let $\{p_i\}_{i=1}^{\infty}$ be the sequence of all primes. Then $W(\{p_i\}_{i=1}^{\infty})$ begins with the following pattern:

$$2,3,2,5,2,3,2,7,2,3,2,5,2,3,2,11,2,3,2,5,2,3,2,7,2,3,2,5,2,3,2,13,$$
$$2,3,2,5,2,3,2,7,2,3,2,5,2,3,2,11,2,3,2,5,2,3,2,7,2,3,2,5,2,3,2,17,\ldots$$

Next, we will partition the collection of infinite sequences of primes into five types:

A sequence $\{b_i\}_{i=1}^\infty$ is *type 1* if each b_i is a prime, $1 \le |P(\{b_i\}_{i=1}^\infty)| < \infty$ and $|Q(\{b_i\}_{i=1}^\infty)| = \infty$.

Suppose that the sequence $\{b_i\}_{i=1}^\infty$ is type 1 and let $P(\{b_i\}_{i=1}^\infty) = \{p_i\}_{i=1}^k$ and $Q(\{b_i\}_{i=1}^\infty) = \{q_i\}_{i=1}^\infty$ where $\{q_i\}_{i=1}^\infty$ is increasing. Let γ_i be the number of times that each q_i occurs in $\{b_i\}_{i=1}^\infty$, and let $\{n_i\}_{i=0}^\infty$ be a sequence defined in the following way: $n_0 = 0$, $n_1 = 2kq_1$ and continuing inductively $n_{2i} = n_{2i-1} + \gamma_i$ and $n_{2i+1} = n_{2i} + 2kq_{i+1}$. Then $\{b_i\}_{i=1}^\infty$ is *type 1A* if it has the following additional properties:

- $b_{n_{2i}+j} = p_{j \bmod k}$ for $j \in \{1, \ldots, 2kq_{i+1}\}$ and $i \ge 0$,
- $b_{n_{2i+1}+j} = q_{i+1}$ for $j \in \{1, \ldots, \gamma_{i+1}\}$.

For example, if $P(\{b_i\}_{i=1}^\infty) = \{2, 11\}$ and $Q(\{b_i\}_{i=1}^\infty) = \{3, 5, 7, 13, 17, \ldots\}$ with $\gamma_i = i$ and $\{b_i\}_{i=1}^\infty$ is type 1A, then $\{b_i\}_{i=1}^\infty$ would begin in the following way (numbers in bold are elements of $Q(\{b_i\}_{i=1}^\infty)$):

$$2, 11, 2, 11, 2, 11, 2, 11, 2, 11, 2, 11, \mathbf{3}, 2, 11, 2, 11, 2, 11, 2, 11, 2, 11,$$
$$2, 11, 2, 11, 2, 11, 2, 11, 2, 11, \mathbf{5}, \mathbf{5}, 2, 11, 2, 11, 2, 11, \ldots$$

A sequence $\{b_i\}_{i=1}^\infty$ is *type 2* if each b_i is a prime, $|P(\{b_i\}_{i=1}^\infty)| = \infty$ and $|Q(\{b_i\}_{i=1}^\infty)| < \infty$.

A sequence $\{b_i\}_{i=1}^\infty$ is *type 2A* if there is an increasing sequence $\{p_i\}_{i=1}^\infty$ of primes such that $\{b_i\}_{i=1}^\infty = W(\{p_i\}_{i=1}^\infty)$. Notice that if a sequence is type 2A, then it is type 2 since $P(\{b_i\}_{i=1}^\infty) = \{p_i\}_{i=1}^\infty$ and $Q(\{b_i\}_{i=1}^\infty) = \emptyset$.

A sequence $\{b_i\}_{i=1}^\infty$ is *type 3* if each b_i is a prime, $|P(\{b_i\}_{i=1}^\infty)| = \infty$ and $|Q(\{b_i\}_{i=1}^\infty)| = \infty$.

A sequence $\{b_i\}_{i=1}^\infty$ is *type 3A* if there exist

- increasing sequences $\{p_i\}_{i=1}^\infty$ and $\{q_i\}_{i=1}^\infty$ of prime numbers such that $\{p_i\}_{i=1}^\infty \cap \{q_i\}_{i=1}^\infty = \emptyset$,
- an increasing sequence $\{n_i\}_{i=1}^\infty$ defined in the following way: $n_1 = 2q_1$, $n_2 = n_1 + \gamma_1 + 2q_2$ and continuing inductively $n_i = n_{i-1} + \gamma_{i-1} + 2q_i$ where γ_i is the number of times that q_i occurs in $\{b_i\}_{i=1}^\infty$.

Then $\{b_i\}_{i=1}^\infty$ is defined by

$$\begin{aligned}
b_i &= w_i && \text{if } i \le n_1,\\
b_i &= q_1 && \text{if } n_1 < i \le n_1 + \gamma_1,\\
b_i &= w_{i-n_1-\gamma_1} && \text{if } n_1 + \gamma_1 < i \le n_2,\\
& && \vdots\\
b_i &= q_k && \text{if } n_k < i \le n_k + \gamma_k,\\
b_i &= w_{i-n_k-\gamma_k} && \text{if } n_k + \gamma_k < i \le n_{k+1},
\end{aligned}$$

where $\{w_i\}_{i=1}^\infty = W(\{p_i\}_{i=1}^\infty)$. Notice that if $\{b_i\}_{i=1}^\infty$ is type 3A, then $\{b_i\}_{i=1}^\infty$ is type 3 since $P(\{b_i\}_{i=1}^\infty) = \{p_i\}_{i=1}^\infty$ and $Q(\{b_i\}_{i=1}^\infty) = \{q_i\}_{i=1}^\infty$.

For example, let $\{\widehat{p}_i\}_{i=1}^\infty$ be the sequence of primes, $p_i = \widehat{p}_{2i-1}$, $q_i = \widehat{p}_{2i}$ and $\gamma_i = i$. Then define $\{b_i\}_{i=1}^\infty$ from $\{p_i\}_{i=1}^\infty$ and $\{q_i\}_{i=1}^\infty$ so that $\{b_i\}_{i=1}^\infty$ has type 3A. Then $\{b_i\}_{i=1}^\infty$ begins in the following way (numbers in bold are elements of $Q(\{b_i\}_{i=1}^\infty)$):

$$2,5,2,11,2,5,\mathbf{3},2,5,2,11,2,5,2,17,2,5,2,11,2,5,\mathbf{7},\mathbf{7},2,5,2,11,2,5,2,17,\\ 2,5,2,11,2,5,2,23,2,5,2,11,2,5,2,17,2,5,\mathbf{13},\mathbf{13},\mathbf{13},2,5,2,\ldots$$

A sequence $\{b_i\}_{i=1}^\infty$ is *type 4* if each b_i is a prime, $|P(\{b_i\}_{i=1}^\infty)| = 0$ and $|Q(\{b_i\}_{i=1}^\infty)| = \infty$.

A sequence $\{b_i\}_{i=1}^\infty$ is *type 5* if each b_i is a prime, $0 < |P(\{b_i\}_{i=1}^\infty)| < \infty$ and $|Q(\{b_i\}_{i=1}^\infty)| < \infty$.

A sequence $\{b_i\}_{i=1}^\infty$ is *type 5A* or *regular* if $b_i = b$ for some $b \in \mathbb{Z}^+$ and all i.

THEOREM 21. *Suppose that Σ is a solenoid that is not a simple closed curve. Then Σ is homeomorphic to $\varprojlim\,(S,(b_i))_{i=1}^\infty$ where $\{b_i\}_{i=1}^\infty$ is a type 1A, 2A, 3A, 4 or 5A sequence.*

Proof. By Corollary 6, the solenoid Σ is homeomorphic to $\varprojlim\,(S,(b_i'))_{i=1}^\infty$ where $\{b_i'\}_{i=1}^\infty$ is a sequence of primes. Thus, $\{b_i'\}_{i=1}^\infty$ is a type 1, 2, 3, 4 or 5 sequence. If $\{b_i'\}_{i=1}^\infty$ is a type 1, 2, 3, or 5 sequence, then by Theorem 8, Σ is homeomorphic to $\varprojlim\,(S,(b_i))_{i=1}^\infty$ where $\{b_i\}_{i=1}^\infty$ is a type 1A, 2A, 3A, or 5A sequence. ■

It has already been shown in Section 3 that if $\{b_i\}_{i=1}^\infty$ is a type 5A sequence, then $\varprojlim\,(S,(b_i))_{i=1}^\infty = \varprojlim\,(S,(b))_{i=1}^\infty$ admits an expansive homeomorphism. Therefore it *suffices to show that if $\{b_i\}_{i=1}^\infty$ is a type 1A, 2A, 3A or 4 sequence, then $\varprojlim\,(S,(b_i))_{i=1}^\infty$ does not admit an expansive homeomorphism.* The next theorem begins this process:

THEOREM 22. *Suppose that $\{b_i\}_{i=1}^\infty$ is type 1A, 2A or 3A. Let $\alpha,\beta \in \mathbb{Z}^+$ be such that $b_i \in P(\{b_i\}_{i=1}^\infty)$ for $i \in \{\alpha,\ldots,\alpha+\beta-1\}$ and let $b_\alpha^{\alpha+\beta} = \prod_{i=\alpha}^{\alpha+\beta-1} b_i$. Then there exist increasing sequences $\{s_i\}_{i=1}^\infty$, $\{\alpha_i\}_{i=1}^\infty$ and $\{\alpha_i'\}_{i=1}^\infty$ such that*

- $b_{\alpha_i}^{\alpha_i+\beta} = b_{\alpha_i'}^{\alpha_i'+\beta} = b_\alpha^{\alpha+\beta}$,
- $\alpha_i + \beta < s_i < \alpha_i'$,
- *b_{s_i} does not divide $b_\alpha^{\alpha+\beta}$,*
- $\lim_{i\to\infty}(\alpha_i' - s_i) = \infty$.

Proof. There are three cases to consider, each corresponding to the three types:

CASE 1: $\{b_i\}_{i=1}^\infty$ is a type 1A sequence. Let $k = |P(\{b_i\}_{i=1}^\infty)|$ and $\{n_i\}_{i=1}^\infty$ be as in the definition of type 1A. Then there exists a non-negative integer

M such that

$$n_{2M} < \alpha < \alpha + \beta < n_{2M+1}.$$

Let $r = \alpha - n_{2M}$ and $u = n_{2M+1} - \alpha$ and define $\alpha_i = n_{4i+2M} + r$, $\alpha'_i = n_{4i+2M+3} - u$ and $s_i = n_{4i+2M+2}$. Then for each $j \in \{0, \ldots, \beta - 1\}$, $b_{\alpha_i+j} = b_{\alpha'_i+j} = b_{\alpha+j}$. Thus $b_{\alpha_i}^{\alpha_i+\beta} = b_{\alpha'_i}^{\alpha'_i+\beta} = b_{\alpha}^{\alpha+\beta}$. Furthermore, $b_{s_i} \in Q(\{b_i\}_{i=1}^{\infty})$ and $b_j \in P(\{b_i\}_{i=1}^{\infty})$ for $\alpha < j \leq \alpha + \beta - 1$. So $b_{s_i} \notin \{b_\alpha, \ldots, b_{\alpha+\beta-1}\}$ and hence b_{s_i} divides $b_\alpha^{\alpha+\beta}$ for no i. Finally,

$$\lim_{i\to\infty} (\alpha'_i - s_i) = \lim_{i\to\infty} (n_{4i+2M+3} - u - n_{4i+2M+2}) = \lim_{i\to\infty} 2q_{2i+M+2} - u = \infty.$$

Case 2: $\{b_i\}_{i=1}^{\infty}$ is a type 2A sequence. Let $\{p_i\}_{i=1}^{\infty}$ be a sequence of primes such that $\{b_i\}_{i=1}^{\infty} = W(\{p_i\}_{i=1}^{\infty})$. There exists an integer M such that $\alpha + \beta \leq 2^M$. Then if $j \leq 2^M$ and $i \bmod 2^{M+1} = j$, it follows that $b_i = b_j$. Let $\alpha_i = \alpha + 2^{2i+M}$ and $\alpha'_i = \alpha + 2^{2i+M+1} - 2^{M+1}$. Then $b_{\alpha+j} = b_{\alpha_i+j} = b_{\alpha'_i+j}$ whenever $j \in \{0, \ldots, \beta - 1\}$. Thus, $b_{\alpha_i}^{\alpha_i+\beta} = b_{\alpha'_i}^{\alpha'_i+\beta} = b_\alpha^{\alpha+\beta}$. Let $s_i = 2^{2i+M} + 2^{2i+M-1}$. Then $b_{s_i} = p_{2i+M-1} \notin \{b_i\}_{i=1}^{2^M}$. Therefore b_{s_i} does not divide $b_\alpha^{\alpha+\beta}$. Finally,

$$\begin{aligned}\lim_{i\to\infty} (\alpha'_i - s_i) &= \lim_{i\to\infty} (\alpha + 2^{2i+M+1} - 2^{M+1} - 2^{2i+M} - 2^{2i+M-1}) \\ &= \lim_{i\to\infty} 2^{2i+M-1} - 2^{M+1} + \alpha = \infty.\end{aligned}$$

Case 3: $\{b_i\}_{i=1}^{\infty}$ is a type 3A sequence. Since $\{b_i\}_{i=n_k+\gamma_k+1}^{n_{k+1}} = \{w_i\}_{i=1}^{2q_{k+1}}$ where $\{w_i\}_{i=1}^{\infty} = W(\{p_i\}_{i=1}^{\infty})$ and $q_k \to \infty$, the proof of this case is similar to the proof of Case 2. ∎

Corollary 23. *Let $X = \varprojlim (S_i, (b_i))_{i=1}^{\infty}$ where $\{b_i\}_{i=1}^{\infty}$ is a type 1A, 2A or 3A sequence. Suppose that $\{k_i\}_{i=1}^{\infty}$ is an increasing sequence of positive integers, a and b are relatively prime positive integers and $\{f_i\}_{i=1}^{\infty}$ is a collection of maps such that $f_i : S_{k_{2i}} \to S_{k_{2i-1}}$ and $\deg(f_i)/b_{k_{2i-1}}^{k_{2i}} = a/b$ for each i. Then there exists a positive integer β, increasing sequences $\{m(i)\}_{i=1}^{\infty}$, $\{s_i\}_{i=1}^{\infty}$, $\{\alpha_i\}_{i=1}^{\infty}$ and $\{\alpha'_i\}_{i=1}^{\infty}$ and a collection $\{f'_i\}_{i=1}^{\infty}$ of maps such that*

(1) $\alpha_i < \alpha_i + \beta < s_i < \alpha'_i < \alpha'_i + \beta$,
(2) $b_{\alpha_i}^{\alpha_i+\beta} = b_{\alpha'_i}^{\alpha'_i+\beta} = b'$,
(3) $\{b_{s_i}\}_{i=1}^{\infty}$ *is an increasing sequence of primes,*
(4) b_{s_i} *divides* b_j *for no* $j \in \{\alpha_i, \ldots, \alpha_i + \beta - 1\}$,
(5) $\lim_{i\to\infty}(\alpha'_i - s_i) = \infty$,
(6) $f'_i : S_{\alpha_{m(i)}+\beta} \to S_{\alpha_{m(i)}}$,
(7) $f_i \circ (b_{k_{2i}}^{\alpha_{m(i)}+\beta}) = (b_{k_{2i-1}}^{\alpha_{m(i)}}) \circ f'_i$,
(8) $\deg(f'_i) = b'a/b = a'$.

Proof. Parts (1)–(5) can be found in Theorem 22. Let $\{m(i)\}_{i=1}^{\infty}$ be an increasing sequence such that $k_{2i} < \alpha_{m(i)}$ for each i and define

$$f_i'(x) = \frac{1}{b_{k_{2i-1}}^{\alpha_{m(i)}}} F_i(b_{k_{2i}}^{\alpha_{m(i)}+\beta} x) \bmod 1.$$

Then $f_i \circ (b_{k_{2i}}^{\alpha_{m(i)}+\beta}) = (b_{k_{2i-1}}^{\alpha_{m(i)}}) \circ f_i'$. So $\deg(f_i) b_{k_{2i}}^{\alpha_{m(i)}+\beta} = b_{k_{2i-1}}^{\alpha_{m(i)}} \deg(f_i')$. Thus

$$\deg(f_i') = \frac{\deg(f_i) b_{\alpha_{m(i)}}^{\alpha_{m(i)}+\beta}}{b_{k_{2i-1}}^{k_{2i}}} = \frac{a}{b} b'. \ \blacksquare$$

6. Continuum-wise expansive homeomorphisms. In this section we examine properties of continuum-wise expansive homeomorphisms on solenoids. In particular, we show the relationship between h being positively continuum-wise expansive and $M(h)$. Then it is concluded that if Σ is a solenoid defined by a type 4 sequence, then h is not an expansive homeomorphism. First several technical propositions are needed.

A continuum X is *decomposable* if there exist proper subcontinua H, K such that $X = H \cup K$. A continuum is *indecomposable* if it is not decomposable. It is well known that all solenoids not homeomorphic to a simple closed curve are indecomposable.

The following theorem follows from Corollary (2.7) in [4]:

THEOREM 24. *If $h : X \to X$ is a continuum-wise expansive homeomorphism on indecomposable continuum X, then either h or h^{-1} is positively continuum-wise expansive.*

The following theorem follows from Corollary (2.4) in [4]:

THEOREM 25. *If $h : X \to X$ is a positively continuum-wise expansive homeomorphism, then there exists $\delta > 0$ such that for every $\gamma > 0$ there exists $N_\gamma \in \mathbb{N}$ such that if A is a subcontinuum with* $\operatorname{diam}(A) > \gamma$ *then* $\operatorname{diam}(h^n(A)) > \delta$ *for every $n \geq N_\gamma$.*

For the rest of this section suppose the following:

(1) $\{n_i\}_{i=1}^{\infty}$ is an increasing sequence of positive integers such that $n_{i+1} - n_i > \beta > 0$,
(2) $\{b_i\}_{i=1}^{\infty}$ is a sequence of positive integers such that $b_{n_i}^{n_i+\beta} = b$ for all i,
(3) $\Sigma = \varprojlim (S_i, (b_i))_{i=1}^{\infty}$,
(4) $f_i : S_{n_i+\beta} \to S_{n_i}$ is a map such that $\deg(f_i) = a$ for each i,
(5) $h : \Sigma \to \Sigma$ is a homeomorphism such that

$$h(\langle x_i \rangle_{i=1}^{\infty}) = \bigcap_{j=1}^{\infty} \overline{\bigcup_{i=j}^{\infty} \pi_{n_i}^{-1}(f_i(x_{n_i+\beta}))},$$

(6) $\{\epsilon_i\}_{i=1}^{\infty}$ is a decreasing sequence of positive numbers with $\epsilon_i \to 0$,
(7) $\mathrm{d}_{\mathbb{R}}(F_i(b_{n_i+\beta}^{n_j+\beta}x), b_{n_i}^{n_j}F_j(x)) < \epsilon_k$ for all $j > i > k$ where F_i and F_j are the respective lifts of f_i and f_j,
(8) $\widetilde{F}_i^j(x) = (1/b_{n_i}^{n_j})F_i(b_{n_i+\beta}^{n_j+\beta}x)$.

Proposition 26. $b_{n_i}^{n_j} = b_{n_i+\beta}^{n_j+\beta}$.

Proof. This follows from the fact that

$$b_{n_i+\beta}^{n_j+\beta} = \frac{b_{n_i}^{n_j+\beta}}{b_{n_i}^{n_i+\beta}} = \frac{b_{n_j}^{n_j+\beta}b_{n_i}^{n_j}}{b_{n_i}^{n_i+\beta}} = \frac{bb_{n_i}^{n_j}}{b} = b_{n_i}^{n_j}. \blacksquare$$

Proposition 27. *If* $\mathrm{d}_{\mathbb{R}}(F_i \circ b_{n_i+\beta}^{n_j+\beta}, b_{n_i}^{n_j} \circ F_j) < \epsilon$, *then* $\mathrm{d}_{\mathbb{R}}(\widetilde{F}_i^j, F_j) < \epsilon/b_{n_i}^{n_j}$.

Proof. Since $b_{n_i}^{n_j}\widetilde{F}_i^j(x) = F_i(b_{n_i+\beta}^{n_j+\beta}x)$, it follows that

$$b_{n_i}^{n_j}\mathrm{d}_{\mathbb{R}}(\widetilde{F}_i^j, F_j) = \mathrm{d}_{\mathbb{R}}(b_{n_i}^{n_j}\widetilde{F}_i^j, b_{n_i}^{n_j}F_j) < \epsilon. \blacksquare$$

Proposition 28. *For every* $\epsilon > 0$ *and* $i \in \mathbb{N}$ *there exists* N_ϵ^i *such that if* $j > N_\epsilon^i$, *then* $\mathrm{d}_{\mathbb{R}}(\widetilde{F}_i^j, a) < \epsilon$.

Proof. Fix i. Since $b_{n_i}^{n_j} \to \infty$ as $j \to \infty$, there exists N_ϵ^i such that if $j > N_\epsilon^i$ then $1/b_{n_i}^{n_j} < \epsilon/\mathrm{d}_{\mathbb{R}}(F_i, a)$. Also, since

$$\mathrm{d}_{\mathbb{R}}(F_i, a) = \sup_{x\in\mathbb{R}} \mathrm{d}_{\mathbb{R}}(F_i(b_{n_i+\beta}^{n_j+\beta}x), a(b_{n_i+\beta}^{n_j+\beta}x)),$$

it follows that for each $j > N_\epsilon^i$,

$$\mathrm{d}_{\mathbb{R}}(\widetilde{F}_i^j, a) = \mathrm{d}_{\mathbb{R}}\left(\frac{1}{b_{n_i}^{n_j}}F_i \circ b_{n_i+\beta}^{n_j+\beta}, \frac{1}{b_{n_i}^{n_j}}ab_{n_i+\beta}^{n_j+\beta}\right) \le \frac{1}{b_{n_i}^{n_j}}\mathrm{d}_{\mathbb{R}}(F_i, a) < \epsilon. \blacksquare$$

Proposition 29. *For every* $\epsilon > 0$ *there exists an integer* $N_\epsilon^a > 0$ *such that if* $j > N_\epsilon^a$ *then* $\mathrm{d}_{\mathbb{R}}(F_j, a) < \epsilon$.

Proof. Let N_1 be such that $\mathrm{d}_{\mathbb{R}}(F_i \circ b_{n_i+\beta}^{n_j+\beta}, b_{n_i}^{n_j} \circ F_j) < \epsilon/2$ for all $i, j > N_1$ where $i < j$. Fix $i > N_1$. Let $N_{\epsilon/2}^i > N_1$ be as in Proposition 28 and choose $j > N_{\epsilon/2}^i$. Then by Proposition 27,

$$\mathrm{d}(\widetilde{F}_i^j, F_j) < \frac{\epsilon}{2b_{n_i}^{n_j}} < \epsilon/2,$$

and by Proposition 28, $\mathrm{d}(\widetilde{F}_i^j, a) < \epsilon/2$. Hence, by the triangle inequality, $\mathrm{d}(F_j, a) < \epsilon$ for all $j > N_{\epsilon/2}^i$. Let $N_\epsilon^a = N_{\epsilon/2}^i$ and the proposition follows. $\blacksquare$

Proposition 30. *For every* $\epsilon > 0$ *there exists an integer* $N_\epsilon > 0$ *such that if* $N_\epsilon < j < k$, *then* $\mathrm{d}_{\mathbb{R}}(F_k, a) < \epsilon/b_{b_j}^{n_k}$.

Proof. Let N_1 be such that $d_{\mathbb{R}}(F_j \circ b_{n_j+\beta}^{n_k+\beta}, b_{n_j}^{n_k} \circ F_k) < \epsilon/2$ for all $k > j > N_1$ and let $N_{\epsilon/2}^a > N_1$ be as in Proposition 29. Then by Proposition 26,

$$d_{\mathbb{R}}(F_j \circ b_{n_j+\beta}^{n_k+\beta}, ab_{n_j}^{n_k}) = d_{\mathbb{R}}(F_j \circ b_{n_j+\beta}^{n_k+\beta}, ab_{n_j+\beta}^{n_k+\beta}) = d_{\mathbb{R}}(F_j, a) < \epsilon/2$$

for each $N_{\epsilon/2}^a < j < k$. Thus, by the triangle inequality

$$b_{n_j}^{n_k} d_{\mathbb{R}}(F_k, a) = d_{\mathbb{R}}(b_{n_j}^{n_k} \circ F_k, ab_{n_j}^{n_k}) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Consequently, the proposition follows by letting $N_\epsilon = N_{\epsilon/2}^a$. ∎

PROPOSITION 31. *For every $\epsilon > 0$ there exists an integer $N_\epsilon > 0$ such that if $N_\epsilon < j < k$, then $d_S(f_k, (a)) < \epsilon/b_{n_j}^{n_k}$.*

Proof. This follows from Proposition 30 and the fact that

$$d_S(f_k, (a)) = \min\{d_{\mathbb{R}}(F_k, a) \bmod 1, 1 - (d_{\mathbb{R}}(F_k, a) \bmod 1)\}. \ ∎$$

LEMMA 32. *For every $\epsilon > 0$ there exists N_ϵ such that*

$$d_S(\pi_{n_k} \circ h, (a) \circ \pi_{n_k+\beta}) < \epsilon/b_{n_j}^{n_k} \quad \textit{for all } k > j > N_\epsilon.$$

Proof. By Proposition 31 there exists $N_{\epsilon/2}$ such that

$$d_S(f_k \circ \pi_{n_k+\beta}, (a) \circ \pi_{n_k+\beta}) < \epsilon/(2b_{n_j}^{n_k})$$

for all $k > j > N_{\epsilon/2}$. Also, by the definition of h, there exists $N_{\epsilon/2}^1$ such that

$$d_S(\pi_{n_k} \circ h, f_k \circ \pi_{n_k+\beta}) < \epsilon/(2b_{n_j}^{n_k})$$

for all $k > j > N_{\epsilon/2}^1$. Thus by letting $N_\epsilon = \max\{N_{\epsilon/2}, N_{\epsilon/2}^1\}$, the lemma follows from the triangle inequality. ∎

THEOREM 33. *Let $\Sigma = \varprojlim(S, (b_i))_{i=1}^\infty$ and $h : \Sigma \to \Sigma$ be a homeomorphism such that $M(h) \le 1$. Then h is not positively continuum-wise expansive.*

Proof. Let δ be as in Theorem 25 and $p \in \mathbb{N}$ be such that $p \ge 3$ and $1/p < \delta$. Notice that $M(h^{-1}) \ge 1$. Let $M(h^{-1}) = a/b$ where $a \ge b$. Next let $\{A_k^j\}_{j=1}^{b_1^{n_k}}$ be subcontinua of Σ such that

$$\pi_{n_k+\beta}(A_k^j) = \left[\frac{j-1}{pb_1^{n_k+\beta}}, \frac{j}{pb_1^{n_k+\beta}}\right] \quad \text{for } j \in \{1, \ldots, b_1^{n_k}\}$$

and let $A_k = \bigcup_{j=1}^{b_1^{n_k}} A_k^j$. Then $\operatorname{diam}(A_k^j) \le 1/p$, $\pi_{n_k+\beta}(A_k) = [0, 1/(pb)]$ and $\pi_{n_k}(A_k) = [0, 1/p]$. Let

$$r(m) = \sum_{i=0}^{m-1} \left(\frac{a}{b}\right)^i.$$

By Lemma 32 there exists a sequence $\{i(m)\}_{m=1}^\infty$ such that

$$d_S(\pi_{n_{i(m)}} \circ h^{-1}, (a) \circ \pi_{n_{i(m)}+\beta}) < \frac{1}{p^2 r(m)}.$$

Then $\left[\frac{1}{p^2r(m)}, \frac{a}{pb} - \frac{1}{p^2r(m)}\right] \subset \pi_{n_{i(m)}} \circ h^{-1}(A_{i(m)})$ and it follows inductively that

$$\begin{aligned}[1/p^2, 2/p^2] &\subset \left[\frac{1}{p^2r(m)}\sum_{i=0}^{m-1}\left(\frac{a}{b}\right)^i, \frac{a^m}{pb^m} - \frac{1}{p^2r(m)}\sum_{i=0}^{m-1}\left(\frac{a}{b}\right)^i\right] \\ &\subset \pi_{n_{i(m)}} \circ h^{-m}(A_{i(m)}).\end{aligned}$$

Thus, by the Pigeon-Hole Principle, there exists $A^m = A^{j_m}_{i(m)}$ such that

$$\operatorname{diam}(\pi_{n_{i(m)}} \circ h^{-m}(A^m)) \geq \frac{1}{p^2 b_1^{n_{i(m)}}}.$$

Let $B_m = h^{-m}(A^m)$. Then $\operatorname{diam}(B_m) \geq 1/(2p^2)$ but $\operatorname{diam}(h^m(B_m)) \leq 1/p < \delta$ for each m. Hence h is not positively continuum-wise expansive by Theorem 25. ∎

The following results can be found in Clark [1]. However, they follow quickly from the results in this section, so for completeness, their proofs are given:

COROLLARY 34. *If $h : \Sigma \to \Sigma$ is a homeomorphism such that $M(h) = 1$ then h is not continuum-wise expansive and hence not expansive.*

Proof. It follows from Theorem 33 that h is not positively continuum-wise expansive. Since $M(h^{-1}) = 1$, h^{-1} is also not positively continuum-wise expansive. It now follows from Theorem 24 that since Σ is indecomposable, h cannot be continuum-wise expansive and hence not expansive. ∎

COROLLARY 35. *If $\Sigma = \varprojlim (S_i, (b_i))_{i=1}^{\infty}$ where $\{b_i\}_{i=1}^{\infty}$ is a type 4 sequence, then Σ does not admit an expansive homeomorphism.*

Proof. Let $h : \Sigma \to \Sigma$ be a homeomorphism.

CLAIM. $M(h) = 1$.

Suppose on the contrary that $M(h) = a/b \neq 1$. Then there exists a prime number p that divides ab. Thus, by Corollary 20, there exists an increasing sequence $\{m_i\}_{i=1}^{\infty}$ such that $b_{m_i} = p$. However, this contradicts the fact that $\{b_i\}_{i=1}^{\infty}$ is a type 4 sequence.

Since $M(h) = 1$, it follows from Corollary 34 that h is not expansive. ∎

7. Growth of small intervals in continuum-wise expansive homeomorphisms. In this technical section we show that arbitrarily small arcs must grow at a certain rate under f_i in a fully continuum-wise expansive homeomorphism. In this section, we have the same assumptions as in the previous section with the additional assumption that $f_i(0) = 0$ for all i.

PROPOSITION 36. *Suppose* $r_n > 0$, $\lim_{n\to\infty} b_1^n c_n = \alpha$ *and* $\lim_{n\to\infty} r_n = 1$. *Then*

$$\mathrm{d}_H(\pi_n^{-1}([0, r_n c_n]), \pi_n^{-1}([0, \alpha/b_1^n])) \to 0 \quad \text{as } n \to \infty.$$

Proof. Let $\epsilon > 0$. Then by the multiplication rule for limits there exists a positive integer N such that $|b_1^n r_n c_n - \alpha| < \epsilon/2$ and $\sum_{i=n}^{\infty} 1/2^i < \epsilon/2$ for all $n \geq N$.

CASE 1: $r_n c_n > \alpha/b_1^n$ for some $n \geq N$. Then $\pi_n^{-1}([0, \alpha/b_1^n])$ is a proper subset of $\pi_n^{-1}([0, r_n c_n])$. So there exists $\mathbf{x} \in \pi_n^{-1}([0, r_n c_n]) - \pi_n^{-1}([0, \alpha/b_1^n])$. Thus $\pi_n(\mathbf{x}) = x_n \in (\alpha/b_1^n, r_n c_n]$. Therefore $|x_n - \alpha/b_1^n| < \epsilon/(2b_1^n)$. So $\mathrm{d}(\mathbf{x}, \pi_n^{-1}([0, \alpha/b_1^n])) < \epsilon$ and hence

$$\mathrm{d}_H(\pi_n^{-1}([0, r_n c_n]), \pi_n^{-1}([0, \alpha/b_1^n])) < \epsilon.$$

CASE 2: $r_n c_n < \alpha/b_1^n$ for some $n \geq N$. The proof is similar to Case 1.

CASE 3: $r_n c_n = \alpha/b_1^n$ for some $n \geq N$. Then clearly

$$\mathrm{d}_H(\pi_n^{-1}([0, r_n c_n]), \pi_n^{-1}([0, \alpha/b_1^n])) = 0 < \epsilon. \ \blacksquare$$

PROPOSITION 37. *If* $\{H_i\}_{i=1}^{\infty}$ *and* $\{K_i\}_{i=1}^{\infty}$ *are collections of subsets of* X *such that* $\lim_{i\to\infty} \mathrm{d}_H(H_i, K_i) = 0$ *then*

$$\bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} K_i} = \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} H_i}.$$

Proof. Let $\epsilon > 0$ and $x \in \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} K_i}$. Then there exists an increasing sequence $\{i_k\}_{k=1}^{\infty}$ such that $\mathrm{d}(x, K_{i_k}) < \epsilon/2$ for each k. Also, there exists $N > 0$ such that $\mathrm{d}_H(H_{i_k}, K_{i_k}) < \epsilon/2$ for each $k > N$. Thus, by the triangle inequality, $\mathrm{d}(x, H_{i_k}) < \epsilon$ for each $k > N$. Since ϵ is arbitrary, $x \in \overline{\bigcup_{i=n}^{\infty} H_i}$ for each n. Thus, $x \in \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} H_i}$. The proof of the other inclusion is similar. $\blacksquare$

PROPOSITION 38. *Suppose that* $\{c_i\}_{i=1}^{\infty}$ *is sequence of positive numbers such that* $\lim_{i\to\infty} b_1^i c_i = \alpha$. *Then*

$$\lim_{i\to\infty} \mathrm{d}_H(\pi_{n_i}^{-1}(f_i([0, c_{n_i+\beta}])), \pi_{n_i}^{-1}(f_i([0, \alpha/b_1^{n_i+\beta}]))) = 0.$$

Proof. By Proposition 36,

$$\lim_{i\to\infty} \mathrm{d}_H(h(\pi_{n_i}^{-1}([0, c_{n_i+\beta}])), h(\pi_{n_i}^{-1}([0, \alpha/b_1^{n_i+\beta}]))) = 0.$$

Then since

$$\lim_{i\to\infty} \mathrm{d}_H(\pi_{n_i}^{-1}(f_i([0, c_{n_i+\beta}])), h(\pi_{n_i}^{-1}([0, c_{n_i+\beta}]))) = 0$$

and

$$\lim_{i\to\infty} \mathrm{d}_H(h(\pi_{n_i}^{-1}([0, \alpha/b_1^{n_i+\beta}])), \pi_{n_i}^{-1}(f_i([0, \alpha/b_1^{n_i+\beta}]))) = 0,$$

the proposition follows from the triangle inequality. $\blacksquare$

The following "calculus" notation will be used. If $\{r_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that $r_n > c$ for all but a finite number of n and $\lim_{n\to\infty} r_n = c$, then we will write $\lim_{n\to\infty} r_n = c^+$.

PROPOSITION 39. *Suppose* $\lim_{i\to\infty} b_1^{n_i} c_{n_i} = \alpha$, $I_\alpha = \bigcap_{i=1}^{\infty} \pi_{n_i}^{-1}([0,\alpha/b_1^{n_i}])$ *and* $f_i([0, c_{n_i+\beta}]) \subset [0, br_{n_i+\beta}c_{n_i+\beta}]$ *where* $\lim_{i\to\infty} r_{n_i} = 1^+$. *Then* $h(I_\alpha) \subset I_\alpha$.

Proof. By Propositions 37 and 38,

$$h(I_\alpha) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} \pi_{n_i}^{-1}(f_i([0,\alpha/b_1^{n_i+\beta}]))} = \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} \pi_{n_i}^{-1}(f_i([0,c_{n_i+\beta}]))}.$$

Also, since $f_i([0, c_{n_i+\beta}]) \subset [0, br_{n_i+\beta}c_{n_i+\beta}]$ we have

$$\bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} \pi_{n_i}^{-1}(f_i([0,c_{n_i+\beta}]))} \subset \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} \pi_{n_i}^{-1}([0,br_{n_i+\beta}c_{n_i+\beta}])}.$$

Next by Propositions 36 and 37 it follows that

$$\begin{aligned}\bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} \pi_{n_i}^{-1}([0,br_{n_i+\beta}c_{n_i+\beta}])} &= \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} \pi_{n_i}^{-1}([0,\alpha/b_1^{n_i}])} \\ &= \bigcap_{i=1}^{\infty} \pi_{n_i}^{-1}([0,\alpha/b_1^{n_i}]) = I_\alpha. \ \blacksquare\end{aligned}$$

THEOREM 40. *Suppose* $h : \Sigma \to \Sigma$ *is a positive continuum-wise expansive homeomorphism and* $f_i([0, c_{n_i+\beta}]) \subset [0, br_{n_i+\beta}c_{n_i+\beta}]$ *where* $\lim_{i\to\infty} r_{n_i} = 1^+$. *Then* $\lim_{i\to\infty} b_1^{n_i} c_{n_i+\beta} = 0$.

Proof. To prove this theorem we must first prove two claims:

CLAIM. *There exists* k *such that* $\limsup_{i\to\infty} b_{n_k}^{n_i} c_{n_i+\beta} < 1$.

By Theorem 33 we may assume that $a > b$. Let N_r be such that $r_{n_j} < (a+b)/(2b)$ for each $j > N_r$. Let $\epsilon < (a-b)/4$ and N_ϵ be as in Proposition 31. Let $k > \max\{N_r, N_\epsilon\}$. Then for every $i \ge k$,

$$\mathrm{d}_S(f_i,(a)) < \frac{\epsilon}{b_{n_k}^{n_i}} < \frac{a-b}{4b_{n_k}^{n_i}}.$$

Since $f_i([0, c_{n_i+\beta}]) \subset [0, br_{n_i+\beta}c_{n_i+\beta}]$, we have

$$f_i(c_{n_i+\beta}) \le br_{n_i+\beta}c_{n_i+\beta} < \frac{a+b}{2}c_{n_i+\beta} < ac_{n_i+\beta}.$$

Hence

$$ac_{n_i+\beta} - \frac{a+b}{2}c_{n_i+\beta} \le \mathrm{d}_S(f_i(c_{n_i+\beta}),(a)(c_{n_i+\beta})) < \frac{a-b}{4b_{n_k}^{n_i}}.$$

It follows that $c_{n_i+\beta} < 1/(2b_{n_k}^{n_i})$ for each $i > k$, which yields the Claim.

CLAIM. *Suppose that* $\limsup_{i\to\infty} b_{n_k}^{n_i} c_{n_i+\beta} = \alpha_k$ *where* $\alpha_k < 1$ *for some* k. *Then* $\alpha_k = 0$.

Suppose on the contrary that there is an increasing sequence $\{i(j)\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} b_k^{n_{i(j)}} c_{n_{i(j)}+\beta} = \alpha_k > 0$. Let $I_{\alpha_k} = \bigcap_{j=1}^{\infty} \pi_{i(j)}^{-1}([0, \alpha_k / b_k^{n_{i(j)}}])$, which is a subarc of Σ. Then by Proposition 39, $h(I_{\alpha_k}) \subset I_{\alpha_k}$. Since arcs do not admit continuum-expansive homeomorphisms (see [4, Theorem 2.6]), this contradicts the fact that h is a positive continuum-wise expansive homeomorphism.

By the two claims there exists a k such that $\lim_{i\to\infty} b_{n_k}^{n_i} c_{n_i+\beta} = 0$. Thus

$$\lim_{i\to\infty} b_1^{n_i} c_{n_i+\beta} = b_1^{n_k} \lim_{i\to\infty} b_{n_k}^{n_i} c_{n_i+\beta} = 0. \ \blacksquare$$

The proofs of the next two theorems are similar to that of Theorem 40:

THEOREM 41. *Suppose that* $h : \Sigma \to \Sigma$ *is a positive continuum-wise expansive homeomorphism and* $f_i([1 - c_{n_i+\beta}, 0]) \subset [1 - br_{n_i+\beta} c_{n_i+\beta}, 0]$ *where* $\lim_{i\to\infty} r_{n_i} = 1^+$. *Then* $\lim_{i\to\infty} b_1^{n_i} c_{n_i+\beta} = 0$.

If $x, y \in S$ where $x < y$, then define $[y, x]_0$ to be $[y, 0] \cup [0, x]$.

THEOREM 42. *Suppose that* $h : \Sigma \to \Sigma$ *is a positive continuum-wise expansive homeomorphism and*

$$f_i([1 - c_{n_i+\beta}, c_{n_i+\beta}]_0) \subset [1 - br_{n_i+\beta} c_{n_i+\beta}, br_{n_i+\beta} c_{n_i+\beta}]_0$$

where $\lim_{i\to\infty} r_{n_i} = 1^+$. *Then* $\lim_{i\to\infty} b_1^{n_i} c_{n_i+\beta} = 0$.

COROLLARY 43. *Let* f_i *be as in Theorem* 40, 41 *or* 42. *Then for every* $\epsilon \in (0, 1/2)$ *there exist* $r_\epsilon \in (1, 2)$ *and* $N_\epsilon \in \mathbb{Z}^+$ *such that*

$$br_\epsilon \operatorname{diam}\left(\left[1 - \frac{\epsilon}{b_1^{\alpha_i}}, \frac{\epsilon}{b_1^{\alpha_i}}\right]_0\right) \le \operatorname{diam}\left(f_i\left(\left[1 - \frac{\epsilon}{b_1^{\alpha_i}}, \frac{\epsilon}{b_1^{\alpha_i}}\right]_0\right)\right) \quad \text{for all } i > N_\epsilon.$$

Proof. Suppose on the contrary that there exists an increasing sequence $\{i_k\}_{k=1}^{\infty}$ such that

$$br_k \operatorname{diam}\left(\left[1 - \frac{\epsilon}{b_1^{\alpha_{i_k}}}, \frac{\epsilon}{b_1^{\alpha_{i_k}}}\right]_0\right) > \operatorname{diam}\left(f_{i_k}\left(\left[1 - \frac{\epsilon}{b_1^{\alpha_{i_k}}}, \frac{\epsilon}{b_1^{\alpha_{i_k}}}\right]_0\right)\right)$$

where $\lim_{k\to\infty} r_k = 1^+$. Thus, since the lift of f_i is increasing (see Theorem 13), one of the following must be true:

$$f_i\left(\left[0, \frac{\epsilon}{b_1^{\alpha_{i_k}}}\right]\right) \subset \left[0, br_k \frac{\epsilon}{b_1^{\alpha_{i_k}}}\right],$$

$$f_i\left(\left[1 - \frac{\epsilon}{b_1^{\alpha_{i_k}}}, 0\right]\right) \subset \left[1 - br_k \frac{\epsilon}{b_1^{\alpha_{i_k}}}, 0\right]$$

or

$$f_i\left(\left[1-\frac{\epsilon}{b_1^{\alpha_{i_k}}},\frac{\epsilon}{b_1^{\alpha_{i_k}}}\right]\right)\subset\left[1-br_k\frac{\epsilon}{b_1^{\alpha_{i_k}}},br_k\frac{\epsilon}{b_1^{\alpha_{i_k}}}\right].$$

However, if $c_{\alpha_{i_k}+\beta}=\epsilon/b_1^{\alpha_{i_k}}$ then

$$\lim_{k\to\infty}b_1^{\alpha_{i_k}}c_{\alpha_{i_k}+\beta}=\lim_{k\to\infty}b_1^{\alpha_{i_k}}\frac{\epsilon}{b_1^{\alpha_{i_k}}}=\epsilon,$$

which contradicts Theorem 40, 41 or 42. ∎

Corollary 44. *Let f_i be as in Theorem 40, 41 or 42. Define $h_i=(1/k)F_i(kx) \bmod 1$. Then for every $\epsilon\in(0,1/(2k))$ there exist $r_\epsilon\in(1,2)$ and $N_\epsilon\in\mathbb{Z}^+$ such that*

$$br_\epsilon\operatorname{diam}\left(\left[\frac{j}{k}-\frac{\epsilon}{b_1^{\alpha_i}},\frac{j}{k}+\frac{\epsilon}{b_1^{\alpha_i}}\right]\right)\le\operatorname{diam}\left(h_i\left(\left[\frac{j}{k}-\frac{\epsilon}{b_1^{\alpha_i}},\frac{j}{k}+\frac{\epsilon}{b_1^{\alpha_i}}\right]\right)\right)$$

for all $i>N_\epsilon$ and $j\in\{1,\dots,k-1\}$.

Proof. Let $r_{\epsilon/k}$ be defined from f_i in Corollary 43. If on the contrary

$$br_{\epsilon/k}\operatorname{diam}\left(\left[\frac{j}{k}-\frac{\epsilon}{b_1^{\alpha_i}},\frac{j}{k}+\frac{\epsilon}{b_1^{\alpha_i}}\right]\right)>\operatorname{diam}\left(h_i\left(\left[\frac{j}{k}-\frac{\epsilon}{b_1^{\alpha_i}},\frac{j}{k}+\frac{\epsilon}{b_1^{\alpha_i}}\right]\right)\right),$$

then

$$br_{\epsilon/k}\operatorname{diam}\left((k)\left(\left[\frac{j}{k}-\frac{\epsilon}{b_1^{\alpha_i}},\frac{j}{k}+\frac{\epsilon}{b_1^{\alpha_i}}\right]\right)\right)>\operatorname{diam}\left((k)\circ h_i\left(\left[\frac{j}{k}-\frac{\epsilon}{b_1^{\alpha_i}},\frac{j}{k}+\frac{\epsilon}{b_1^{\alpha_i}}\right]\right)\right).$$

Thus,

$$\begin{aligned}br_{\epsilon/k}\operatorname{diam}\left(\left[1-\frac{k\epsilon}{b_1^{\alpha_i}},\frac{k\epsilon}{b_1^{\alpha_i}}\right]_0\right)&>\operatorname{diam}\left(f_i\circ(k)\left(\left[\frac{j}{k}-\frac{\epsilon}{b_1^{\alpha_i}},\frac{j}{k}+\frac{\epsilon}{b_1^{\alpha_i}}\right]\right)\right)\\&\ge\operatorname{diam}\left(f_i\left(\left[1-\frac{k\epsilon}{b_1^{\alpha_i}},\frac{k\epsilon}{b_1^{\alpha_i}}\right]_0\right)\right),\end{aligned}$$

which contradicts Corollary 43. ∎

8. Periodic sets. In this section we show that there exist periodic sets of h that are similar to the periodic points in Section 4. First we show that if $h:\Sigma\to\Sigma$ is an expansive homeomorphism, then h must have a fixed point.

Proposition 45. *Let $f:S\to S$ be a map such that $\deg(f)=a$ and let $b\in\mathbb{Z}^+$. Then $\mathrm{d}_{\mathbb{R}}\big(\frac{1}{b}F(bx),ax\big)=\frac{1}{b}\mathrm{d}_{\mathbb{R}}(F(bx),abx)$ where F is the lift of f.*

Proof. This follows from the fact that

$$\mathrm{d}_{\mathbb{R}}\left(\frac{1}{b}F(bx),ax\right)=\mathrm{d}_{\mathbb{R}}\left(\frac{1}{b}F(bx),\frac{1}{b}bax\right)=\frac{1}{b}\mathrm{d}_{\mathbb{R}}(F(bx),abx).\ ∎$$

Proposition 46. *Let $f : S \to S$ be a map such that* $\deg(f) = a$ *and* $\widetilde{f} : S \to S$ *be such that* $\widetilde{f}(x) = (1/k)F(kx) \bmod 1$ *for some $k \in \mathbb{N}$ where F is the lift of f. If m divides k then $\widetilde{f}(i/m) = (a)(i/m)$ for each $i \in \{0, \ldots, m\}$.*

Proof. Since m divides k, $k(i/m)$ is an integer. So $(1/k)F(k(i/m)) = (1/k)ak(i/m) = a(i/m)$. Thus $\widetilde{f}(i/m) = a(i/m) \bmod 1 = (a)(i/m)$. ∎

The following is the well-known coincidence theorem:

Theorem 47. *Suppose that $F, G : \mathbb{R} \to \mathbb{R}$ are maps such that $G(0) \geq F(0)$ and $G(n) \leq F(n)$. Then there exists $c \in [0, n]$ such that $F(c) = G(c)$.*

Proof. Let $D(x) = F(x) - G(x)$. Then the theorem follows from the Intermediate Value Theorem. ∎

Lemma 48. *Suppose $f, g : S \to S$ are such that* $\deg(g) \neq \deg(f)$. *Then there exists $x_c \in S$ such that $f(x_c) = g(x_c)$.*

Proof. Without loss of generality, suppose that $\deg(g) < \deg(f)$. Let F and G be the respective lifts of f and g. There are three cases:

Case 1: $F(0) < G(0)$. Let n be a positive integer such that

$$n > \frac{G(0) - F(0)}{\deg(f) - \deg(g)}.$$

Then

$$F(n) = n \deg(f) + F(0) > n \deg(g) + G(0) = G(n).$$

It follows from Theorem 47 that there exists $c \in [0, n]$ such that $F(c) = G(c)$. Let $x_c = c \bmod 1$. Then $f(x_c) = g(x_c)$.

Case 2: $F(0) > G(0)$. Then the proof is similar to Case 1.

Case 3: $F(0) = G(0)$. Then $f(0) = g(0)$. ∎

Theorem 49. *Suppose that $h : \Sigma \to \Sigma$ is a homeomorphism such that $M(h) \neq 1$. Then h has a fixed point.*

Proof. By Theorem 5, Corollary 23 and Theorem 33 there exists a non-increasing sequence $\{n_i\}_{i=1}^{\infty}$ and maps $h_i : S_{n_{2i}} \to S_{n_{2i-1}}$ such that

- $\deg(h_i)/b_{n_{2i-1}}^{n_{2i}} = M(h) \neq 1$ for each i,
- $\mathrm{d}_H(h(\langle x_i \rangle_{i=1}^{\infty}), \pi_{n_{2i-1}}^{-1}(h_i(x_{n_{2i}}))) \to 0$ as $i \to \infty$.

By Lemma 48, since $\deg(h_i) \neq b_{n_{2i-1}}^{n_{2i}}$, there exists x^{c_i} such that $h_i(x^{c_i}) = (b_{n_{2i-1}}^{n_{2i}})x^{c_i}$ for each i. Let $\{\mathbf{x}^i\}_{i=1}^{\infty} = \{\langle x_j^i \rangle_{j=1}^{\infty}\}_{i=1}^{\infty}$ be a sequence of points such that $\pi_{n_{2i}}(\mathbf{x}^i) = x^{c_i}$ and $\mathbf{y}$ be a limit point of $\{\mathbf{x}^i\}_{i=1}^{\infty}$.

Claim. $\mathbf{y}$ *is a fixed point of h.*

For every $\epsilon > 0$ there exists N_ϵ such that if $i > N_\epsilon$ then

$$\mathrm{d}(h(\mathbf{x}^i), \mathbf{x}^i) < \mathrm{d}_H(h(\mathbf{x}^i), \pi^{-1}_{n_{2i-1}}(x^i_{n_{2i-1}})) = \mathrm{d}_H(h(\mathbf{x}^i), \pi^{-1}_{n_{2i-1}}((b^{n_{2i}}_{n_{2i-1}})x^{c_i}))$$
$$= \mathrm{d}_H(h(\mathbf{x}^i), \pi_{n_{2i-1}}(h_i(x^i_{n_{2i}}))) < \epsilon/3.$$

Furthermore, since $\mathbf{y}$ is a limit point of $\{\mathbf{x}^i\}_{i=1}^\infty$, there exists $i' > N_\epsilon$ such that $\mathrm{d}(\mathbf{x}^{i'}, \mathbf{y}) < \epsilon/3$ and $\mathrm{d}(h(\mathbf{x}^{i'}), h(\mathbf{y})) < \epsilon/3$. Thus $\mathrm{d}(\mathbf{y}, h(\mathbf{y})) < \epsilon$ by the triangle inequality. Since ϵ is arbitrary, we conclude that $h(\mathbf{y}) = \mathbf{y}$. ■

It will be easier to prove the main theorem if we take the fixed point of h to be $\mathbf{0}$. The next theorem shows that we can do this.

THEOREM 50. *If $\widehat{h} : \Sigma \to \Sigma$ is an expansive homeomorphism, then there exists an expansive homeomorphism $h : \Sigma \to \Sigma$ such that $h(\mathbf{0}) = \mathbf{0}$.*

Proof. By Corollary 34, if $\widehat{h}$ is expansive, then $M(\widehat{h}) \neq 1$. Thus by Theorem 49, there exists a fixed point $\mathbf{y}$ of $\widehat{h}$. Also, since Σ is homogeneous [10], there exists a homeomorphism $\phi : \Sigma \to \Sigma$ such that $\phi(\mathbf{y}) = \mathbf{0}$. Then by Theorem 4, $h = \phi \circ \widehat{h} \circ \phi^{-1}$ is an expansive homeomorphism such that

$$h(\mathbf{0}) = \phi \circ \widehat{h} \circ \phi^{-1}(\mathbf{0}) = \phi \circ \widehat{h}(\mathbf{y}) = \phi(\mathbf{y}) = \mathbf{0}. \ ■$$

Next we find periodic sets of h. Define $\chi^\rho_{a,b}(i) = j$ if $ai = bj \bmod \rho$, $(\chi^\rho_{a,b}(i))^2 = \chi^\rho_{a,b}(\chi^\rho_{a,b}(i))$ and inductively define $(\chi^\rho_{a,b}(i))^n = \chi^\rho_{a,b}((\chi^\rho_{a,b}(i))^{n-1})$. Also, if X is a continuum, then define the *hyperspace* of X by

$$C(X) = \{Y \mid Y \text{ is a subcontinuum of } X\}.$$

THEOREM 51. *Suppose*

(1) $\Sigma = \varprojlim (S_i, (b_i))_{i=1}^\infty$ *where* $\{b_i\}_{i=1}^\infty$ *is a sequence of primes,*
(2) $b = b_\alpha^{\alpha+\beta}$ *where* $\alpha, \beta \in \mathbb{N}$,
(3) ρ *is relatively prime to positive integers* a *and* b,
(4) $a/b > 1$,
(5) $h : \Sigma \to \Sigma$ *is a homeomorphism,*
(6) $\mathrm{d}_H((a)x, \pi_\alpha \circ h(\pi^{-1}_{\alpha+\beta}(x))) < 1/(3\rho(a+1))$.

Then there exist disjoint closed subsets $\{Y_i\}_{i=1}^{\rho-1}$ *such that*

- $\mathrm{d}_H(\pi_{\alpha+\beta}(Y_i), i/\rho) < 1/(3\rho(a+1))$,
- $h(Y_i) = Y_{\chi^\rho_{a,b}(i)}$.

Proof. Since a, b and ρ are fixed, let $\chi(i) = \chi^\rho_{a,b}(i)$. Also let

$$I_i = \left[\frac{i}{\rho} - \frac{1}{3(a+1)\rho}, \frac{i}{\rho} + \frac{1}{3(a+1)\rho}\right] \text{ and } \mathcal{T}^0_i = \{Y^0_i \in C(\Sigma) \mid \pi_{\alpha+\beta}(Y^0_i) = I_i\}.$$

CLAIM 1. *$\pi_\alpha \circ h(Y_i^0) \subset \left[\frac{ai}{\rho} - \frac{a+1}{3(a+1)\rho}, \frac{ai}{\rho} + \frac{a+1}{3(a+1)\rho}\right]$ for all $Y_i^0 \in \mathcal{T}_i^0$.*

Pick $x \in \pi_\alpha(h(Y_i^0))$. Then there exists $\widehat{y} \in Y_i^0$ such that $\pi_\alpha(h(\widehat{y})) = x$. Since $\pi_{\alpha+\beta}(\widehat{y}) \in I_i = [\frac{i}{\rho} - \frac{1}{3(a+1)\rho}, \frac{i}{\rho} + \frac{1}{3(a+1)\rho}]$,

$$\mathrm{d}_S((a) \circ \pi_{\alpha+\beta}(\widehat{y}), x) = \mathrm{d}_S((a) \circ \pi_{\alpha+\beta}(\widehat{y}), \pi_\alpha \circ h(\widehat{y})) < \frac{1}{3\rho(a+1)}$$

by hypothesis (6). Also, since

$$(a) \circ \pi_{\alpha+\beta}(\widehat{y}) \in \left[\frac{ai}{\rho} - \frac{a}{3(a+1)\rho}, \frac{ai}{\rho} + \frac{a}{3(a+1)\rho}\right],$$

it follows that

$$x \in \left[\frac{ai}{\rho} - \frac{a+1}{3(a+1)\rho}, \frac{ai}{\rho} + \frac{a+1}{3(a+1)\rho}\right].$$

CLAIM 2. *$\pi_\alpha \circ h\big(\pi^{-1}_{\alpha+\beta}\big(\frac{i}{\rho} - \frac{1}{3\rho(a+1)}\big)\big) \in \left[\frac{ai}{\rho} - \frac{a+1}{3(a+1)\rho}, \frac{ai}{\rho} - \frac{a-1}{3(a+1)\rho}\right]$.*

This follows from the fact that

$$\mathrm{d}_H\left((a)\left(\frac{i}{\rho} - \frac{1}{3\rho(a+1)}\right), \pi_\alpha \circ h\left(\pi^{-1}_{\alpha+\beta}\left(\frac{i}{\rho} - \frac{1}{3\rho(a+1)}\right)\right)\right) < \frac{1}{3\rho(a+1)}.$$

CLAIM 3. *$\pi_\alpha \circ h\big(\pi^{-1}_{\alpha+\beta}\big(\frac{i}{\rho} + \frac{1}{3\rho(a+1)}\big)\big) \in \left[\frac{ai}{\rho} + \frac{a-1}{3(a+1)\rho}, \frac{ai}{\rho} + \frac{a+1}{3(a+1)\rho}\right]$.*

The proof of this claim is similar to the proof of Claim 2.

CLAIM 4. *$\left[\frac{ai}{\rho} - \frac{a-1}{3(a+1)\rho}, \frac{ai}{\rho} + \frac{a-1}{3(a+1)\rho}\right] \subset \pi_\alpha(h(Y_i^0))$.*

Since $h(Y_i^0)$ is a continuum, by Claim 1, $\pi_\alpha(h(Y_i^0))$ is an interval contained in $\left[\frac{ai}{\rho} - \frac{a+1}{3(a+1)\rho}, \frac{ai}{\rho} + \frac{a+1}{3(a+1)\rho}\right]$. By Claims 2 and 3,

$$\pi_\alpha \circ h(Y_i^0) \cap \left[\frac{ai}{\rho} - \frac{a+1}{3(a+1)\rho}, \frac{ai}{\rho} - \frac{a-1}{3(a+1)\rho}\right] \neq \emptyset$$

and

$$\pi_\alpha \circ h(Y_i^0) \cap \left[\frac{ai}{\rho} + \frac{a-1}{3(a+1)\rho}, \frac{ai}{\rho} + \frac{a+1}{3(a+1)\rho}\right] \neq \emptyset.$$

Thus the claim follows.

On to the proof of the theorem: from Claim 4, $bj = ai$ and $b \le a-1$ we have

$$\begin{aligned}(b)(I_j) &= \left[\frac{bj}{\rho} - \frac{b}{3(a+1)\rho}, \frac{bj}{\rho} + \frac{b}{3(a+1)\rho}\right] \\ &= \left[\frac{ai}{\rho} - \frac{b}{3(a+1)\rho}, \frac{ai}{\rho} + \frac{b}{3(a+1)\rho}\right] \\ &\subset \left[\frac{ai}{\rho} - \frac{a-1}{3(a+1)\rho}, \frac{ai}{\rho} + \frac{a-1}{3(a+1)\rho}\right] \subset \pi_\alpha(h(Y_i^0)).\end{aligned}$$

Therefore $I_{\chi(i)} \subset \pi_{\alpha+\beta}(h(Y_i^0))$. Let

$$\mathcal{T}_i^1 = \{Y_i^1 \in C(\Sigma) \mid \text{there exists } Y_i^0 \in \mathcal{T}_i^0 \text{ such that } Y_i^1 \subset Y_i^0 \\ \text{and } \pi_{\alpha+\beta}(h(Y_i^1)) = I_{\chi(i)}\}.$$

Notice that if $Y_i^1 \in \mathcal{T}_i^1$, then $h(Y_i^1) \in \mathcal{T}_{\chi(i)}^0$. Continuing inductively, let

$$\mathcal{T}_i^n = \{Y_i^n \in C(\Sigma) \mid \text{there exists } Y_i^{n-1} \in \mathcal{T}_i^{n-1} \text{ such that } Y_i^n \subset Y_i^{n-1} \\ \text{and } h(Y_i^n) \in \mathcal{T}_{\chi(i)}^{n-1}\}.$$

We now have the following facts:

- for every $Y_i^n \in \mathcal{T}_i^n$ there exists $Y_i^{n-1} \in \mathcal{T}_i^{n-1}$ such that $Y_i^n \subset Y_i^{n-1}$,
- $h(\mathcal{T}_i^n) \subset \mathcal{T}_{\chi(i)}^{n-1}$,
- $\bigcup_{Y_i^{n+1} \in \mathcal{T}_i^{n+1}} Y_i^{n+1} \subset \bigcup_{Y_i^n \in \mathcal{T}_i^n} Y_i^n$,
- $\bigcup_{Y_i^n \in \mathcal{T}_i^n} Y_i^n \neq \emptyset$ for each i and n.

Let

$$\widehat{Y}_i = \bigcap_{n=0}^{\infty} \overline{\bigcup_{Y_i^n \in \mathcal{T}_i^n} Y_i^n}.$$

Then

$$h(\widehat{Y}_i) = h\Big(\bigcap_{n=0}^{\infty} \overline{\bigcup_{Y_i^n \in \mathcal{T}_i^n} Y_i^n}\Big) = \bigcap_{n=0}^{\infty} \overline{\bigcup_{Y_i^n \in \mathcal{T}_i^n} h(Y_i^n)} \\ \subset \bigcap_{n=1}^{\infty} \overline{\bigcup_{Y_{\chi(i)}^{n-1} \in \mathcal{T}_{\chi(i)}^{n-1}} Y_{\chi(i)}^{n-1}} = \widehat{Y}_{\chi(i)}.$$

Let k be a positive integer such that $i = (\chi(i))^k$. Then

$$h^k(\widehat{Y}_i) \subset h^{k-1}(\widehat{Y}_{(\chi(i))^{k-1}}) \subset \cdots \subset \widehat{Y}_i.$$

Next let $Y_i = \bigcap_{n=1}^{\infty} h^{nk}(\widehat{Y}_i)$, which is nonempty since $\widehat{Y}_i$ is nonempty and closed. Therefore, $h^k(Y_i) = Y_i$ and it follows that $h(Y_i) = Y_{\chi(i)}$. ∎

Let $f : S \to S$ and $g : S \to S$ be maps. Suppose $Y_i \subset S$ for $i \in \{0, \ldots, n-1\}$. Then $\{Y_i\}_{i=0}^{n-1}$ is an *ordered collection of n-periodic sets under* (f, g) if $f(Y_i) = g(Y_{i+1 \bmod n})$ for each i. If each $Y_i = \{y_i\}$, then $\{y_i\}_{i=0}^{n-1}$ is an *ordered collection of n-periodic points under* (f, g). If $g = \mathrm{id}_S$, then $\{Y_i\}_{i=0}^{n-1}$ is an *ordered collection of n-periodic sets under f*. Likewise for $\delta > 0$, $\{Y_i\}_{i=0}^{n-1}$ is an *ordered collection of (n, δ)-periodic sets under* (f, g) if $\mathrm{d}_H(f(Y_i), g(Y_{i+1 \bmod n})) < \delta$ for each i.

The next theorem gives a useful condition when $\{Y_i\}_{i=0}^{n-1}$ is not (n, δ)-periodic.

THEOREM 52. *Suppose the following:*

(1) $f,(b) : S \to S$ *where* $b \in \mathbb{Z}^+$,
(2) $\{x_i\}_{i=0}^{n-1}$ *is an ordered collection of* n*-periodic points under* $(f,(b))$,
(3) *there exist* $r > b$ *and* $0 < \theta < \psi/3 < \psi < 1/b$ *such that if* $\theta < \mathrm{d}_S(x,x_i) < \psi$ *then* $\mathrm{d}_S(f(x),f(x_i)) > r\mathrm{d}_S(x,x_i)$,
(4) $\{Y_i\}_{i=0}^{n-1}$ *is a collection of closed subsets of* S *with* $d_H(Y_i,x_i) < \psi/3$,
(5) $\delta < \min\{\frac{r-b}{3}\theta, \frac{\psi}{3}\}$,
(6) *there exist* $i \in \{0,\dots,n-1\}$ *such that* $\theta < d_H(Y_i,x_i)$.

Then $\{Y_i\}_{i=0}^{n-1}$ *is not* (n,δ)*-periodic under* $(f,(b))$.

Proof. First, we need the following claim:

CLAIM 1. *If* $\theta < \mathrm{d}_H(Y_k,x_k)$ *then*

$$\mathrm{d}_H(f(Y_k),f(x_k)) > \mathrm{d}_H((b)(Y_k),(b)(x_k)) + 3\delta.$$

Since Y_k is closed, there exists $y_k \in Y_k$ such that $\mathrm{d}_S(y_k,x_k) = \mathrm{d}_H(Y_k,x_k)$. Thus $\theta < \mathrm{d}_S(y_k,x_k) < \psi/3$. Therefore it follows that

$$\begin{aligned}\mathrm{d}_H(f(Y_k),f(x_k)) &\geq \mathrm{d}_S(f(y_k),f(x_k))\\ &> r\mathrm{d}_S(y_k,x_k) = (r-b)\mathrm{d}_S(y_k,x_k) + b\mathrm{d}_S(y_k,x_k)\\ &> (r-b)\theta + b\mathrm{d}_H(Y_k,x_k) > 3\delta + \mathrm{d}_H((b)(Y_k),(b)(x_k)).\end{aligned}$$

By hypothesis (6) we may assume that $\mathrm{d}_H(x_0,Y_0) > \theta$. Otherwise just reorder $\{x_i\}_{i=0}^{n-1}$ and $\{Y_i\}_{i=0}^{n-1}$. Also, assume that $\mathrm{d}_H(f(Y_{i-1}),(b)(Y_i)) < \delta$ for $i \in \{1,\dots,n-1\}$. We will show that $\mathrm{d}_H(f(Y_{n-1}),(b)(Y_0)) \geq \delta$.

CLAIM 2. $\mathrm{d}_H(f(Y_k),f(x_k)) > \mathrm{d}_H((b)(Y_0),(b)(x_0)) + 2(k+1)\delta$ *for* $k \in \{0,\dots,n-1\}$.

The proof is by induction on k.

Base case: $\mathrm{d}_H(f(Y_0),f(x_0)) > \mathrm{d}_H((b)(Y_0),(b)(x_0)) + 2\delta$. This follows from the assumption that $\mathrm{d}_H(Y_0,x_0) > \theta$ and Claim 1.

Induction step: Suppose $\mathrm{d}_H(f(Y_{k-1}),f(x_{k-1})) > \mathrm{d}_H((b)(Y_0),(b)(x_0)) + 2(k)\delta$ for $k \geq 2$. Since

$$\mathrm{d}_H(f(Y_{k-1}),(b)(Y_k)) + \mathrm{d}_H((b)(Y_k),(b)(x_k)) \geq \mathrm{d}_H(f(Y_{k-1}),(b)(x_k)),$$

it follows that

$$\begin{aligned}b\mathrm{d}_H(Y_k,x_k) &= \mathrm{d}_H((b)(Y_k),(b)(x_k))\\ &\geq \mathrm{d}_H(f(Y_{k-1}),(b)(x_k)) - \mathrm{d}_H(f(Y_{k-1}),(b)(Y_k))\\ &> \mathrm{d}_H(f(Y_{k-1}),f(x_{k-1})) - \delta > \mathrm{d}_H((b)(Y_0),(b)(x_0)) + 2k\delta - \delta\\ &> b\mathrm{d}_H(Y_0,x_0) + (2k-1)\delta > b\theta.\end{aligned}$$

Notice that $\mathrm{d}_H(Y_k,x_k) > \theta$. Thus by Claim 1,

$$\mathrm{d}_H(f(Y_k),f(x_k)) > \mathrm{d}_H((b)(Y_k),(b)(x_k)) + 3\delta.$$

Therefore

$$\begin{aligned}\mathrm{d}_H(f(Y_k), f(x_k)) &> \mathrm{d}_H((b)(Y_0), (b)(x_0)) + (2k-1)\delta + 3\delta \\ &= \mathrm{d}_H((b)(Y_0), (b)(x_0)) + 2(k+1)\delta.\end{aligned}$$

Thus Claim 2 is proved.

Next since

$$\mathrm{d}_H(f(Y_{n-1}), (b)(x_0)) = \mathrm{d}_H(f(Y_{n-1}), f(x_{n-1})) \geq \mathrm{d}_H((b)(Y_0), (b)(x_0)) + 2(n)\delta,$$

it follows from the triangle inequality that

$$\delta < 2(n)\delta \leq \mathrm{d}_H(f(Y_{n-1}), (b)(Y_0))$$

for $n \geq 1$. If $n = 0$, then $\delta < \mathrm{d}_H(f(Y_0), (b)(Y_0))$ follows from Claim 1. ∎

9. Main result. In order to prove the main result, we first prove a technical lemma whose hypothesis we have worked hard to satisfy:

LEMMA 53. *Suppose the following:*

(1) $\{b_i\}_{i=1}^{\infty}$ *is a sequence of primes and* $\Sigma = \varprojlim (S, (b_i))_{i=1}^{\infty}$,
(2) $\{\alpha'_i\}_{i=1}^{\infty}$ *and* $\{s_i\}_{i=1}^{\infty}$ *are increasing sequences such that* $s_i < \alpha'_i$,
(3) *there exists a non-negative integer* β *such that* $b_{\alpha'_i}^{\alpha'_i+\beta} = b$ *for each* i ,
(4) *there is a collection of maps* $h_i : S_{\alpha'_i+\beta} \to S_{\alpha'_i}$ *such that* $\deg(h_i) = a$ *where* $a > b$,
(5) $h : \Sigma \to \Sigma$ *is a homeomorphism such that*

$$h(\langle x_j\rangle_{j=1}^{\infty}) = \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} \overline{\pi_{\alpha'_i}^{-1}(h_i(x_{\alpha'_i+\beta}))} \quad \textit{and} \quad h(\mathbf{0}) = \mathbf{0},$$

(6) $\rho_i = b_{s_i}$ *is relatively prime to both* a *and* b,
(7) *for each* i *there is an integer* $k_i > 3\rho_i(a+1)$ *such that*

$$h_i\left(\frac{j}{k_i\rho_i}\right) = (a)\frac{j}{k_i\rho_i} \quad \textit{for each } i \in \mathbb{Z}^+ \textit{ and } j \in \{0, \ldots, k_i p_i\},$$

(8) $d_H((a)x, \pi_{\alpha'_i} \circ h(\pi_{\alpha'_i}^{-1}(x))) < 1/(3\rho_i(a+1))$,
(9) *for every* $\epsilon \in (0, 1/2)$ *there exist* $N_\epsilon \in \mathbb{N}$ *and* $r_\epsilon \in (1, 2)$ *such that*

$$\begin{aligned} br_\epsilon \operatorname{diam}\left(\left[\frac{j}{\rho_i} - \frac{\epsilon}{b_1^{\alpha'_i+\beta}}, \frac{j}{\rho_i} + \frac{\epsilon}{b_1^{\alpha'_i+\beta}}\right]\right) \\ \leq \operatorname{diam}\left(h_i\left(\left[\frac{j}{\rho_i} - \frac{\epsilon}{b_1^{\alpha'_i+\beta}}, \frac{j}{\rho_i} + \frac{\epsilon}{b_1^{\alpha'_i+\beta}}\right]\right)\right)\end{aligned}$$

for all $i > N_\epsilon$ *and* $j \in \{1, \ldots, \rho_i - 1\}$.

Then h *is not an expansive homeomorphism.*

Proof. Suppose on the contrary that there exists an expansive constant $c > 0$ for h. Since $\{s_i\}_{i=1}^{\infty}$ is increasing, there exists an N_1 such that $\mathrm{diam}(\pi_{s_i}^{-1}(0)) < c/6$ for all $i > N_1$. Then since $(b_{s_i})(j/\rho_i) = (\rho_i)(j/\rho_i) = 0$, it follows that $\mathrm{diam}(\pi_{s_i+1}^{-1}(\{j/\rho_i\}_{j=1}^{\rho_i-1})) < c/6$ for all $i > N_1$. Let

$$\delta_i = \min\left\{ \frac{c}{6b_1^{\alpha_i'+\beta}}, \frac{r_{c/6}-1}{3} b \frac{c}{6b_1^{\alpha_i'+\beta}} \right\}$$

where $r_{c/6}$ is as in Corollary 44.

By hypothesis (5), there exists $N_2 > 0$ such that for every $i \geq N_2$,

$$\mathrm{d}_H(h(\langle x_j\rangle_{j=1}^{\infty}), \pi_{\alpha_i'}^{-1}(h_i(x_{\alpha_i'+\beta}))) < \min\left\{ \frac{c}{6b}, \frac{r_{c/6}-1}{18} c \right\}.$$

So $\mathrm{d}_H(\pi_{\alpha_i'} \circ h(\langle x_j\rangle_{j=1}^{\infty}), h_i(x_{\alpha_i'+\beta})) < \delta_i$ for each $i > N_2$.

By Theorem 51, there exists a collection $\{Y_j^i\}_{j=1}^{\rho_i-1}$ of closed sets such that $h(Y_j^i) = Y^i_{\chi_{a,b}^{\rho_i}(j)}$ and $\mathrm{d}_H(\pi_{\alpha_i'+\beta}(Y_j^i), j/\rho_i) < 1/(3\rho_i(a+1))$. Thus,

$$\begin{aligned}
\mathrm{d}_H((b) \circ \pi_{\alpha_i'+\beta}(Y^i_{\chi_{a,b}^{\rho_i}(j)}), h_i \circ \pi_{\alpha_i'+\beta}(Y_j^i)) &= \mathrm{d}_H(\pi_{\alpha_i'}(Y^i_{\chi_{a,b}^{\rho_i}(j)}), h_i \circ \pi_{\alpha_i'+\beta}(Y_j^i)) \\
&= \mathrm{d}_H(\pi_{\alpha_i'} \circ h(Y_j^i), h_i \circ \pi_{\alpha_i'+\beta}(Y_j^i)) \\
&< \delta_i
\end{aligned}$$

for all $i > N_2$. Let $\widetilde{Y}_1^i = \pi_{\alpha_i'+\beta}(Y_1^i)$, $\widetilde{Y}_j^i = \pi_{\alpha_i'+\beta}(Y^i_{(\chi_{a,b}^{\rho_i}(1))^j})$, $x_1^i = 1/\rho_i$ and $x_j^i = (\chi_{a,b}^{\rho_i}(1))^j/\rho_i$. Then $\{\widetilde{Y}_j^i\}_{j=1}^{\rho_i-1}$ is an ordered collection of (ρ_i-1, δ_i)-periodic sets and $\{x_j^i\}_{j=1}^{\rho_i-1}$ is an ordered collection of (ρ_i-1)-periodic points under $(h_i, (b))$ such that $\mathrm{d}_H(x_j^i, \widetilde{Y}_j^i) < 1/(3\rho_i(a+1))$ for each i. Let $\theta_i = c/(3b_1^{\alpha_i'+\beta})$. Then $\delta_i < (r_{c/6}-1)b\theta_i/3$. So by hypothesis (9), we can apply Theorem 52 to conclude

$$\mathrm{d}_H(x_j^i, \widetilde{Y}_j^i) \leq \theta_i = \frac{c}{3b_1^{\alpha_i'+\beta}}$$

for each $i > \max\{N_1, N_2\}$ and $j \in \{1, \dots, \rho_i - 1\}$. However, it then follows that

$$\mathrm{d}_H\left(\pi_{\alpha_i'+\beta}^{-1}(\{x_j^i\}_{j=1}^{\rho_i-1}), \bigcup_{j=1}^{\rho_i-1} Y_j^i\right) \leq \frac{c}{3}.$$

Since $\mathrm{diam}(\pi_{\alpha_i'+\beta}^{-1}(\{x_j^i\}_{j=1}^{\rho_i-1})) < c/6$, we deduce that

$$\begin{aligned}
\mathrm{diam}\left(\bigcup_{j=1}^{\rho_i-1} Y_j^i\right) &\leq \mathrm{d}_H\left(\pi_{\alpha_i'+\beta}^{-1}(\{x_j^i\}_{j=1}^{\rho_i-1}), \bigcup_{j=1}^{\rho_i-1} Y_j^i\right) + \mathrm{diam}(\pi_{\alpha_i'+\beta}^{-1}(\{x_j^i\}_{j=1}^{\rho_i-1})) \\
&< \frac{c}{2}.
\end{aligned}$$

Therefore, since $h(\bigcup_{j=1}^{\rho_i-1} Y_j^i) = \bigcup_{j=1}^{\rho_i-1} Y_j^i$ and $|\bigcup_{j=1}^{\rho_i-1} Y_j^i)| \geq 2$, we conclude that c is not an expansive constant for h. ∎

In order to prove the main theorem, we must show that all of the hypotheses in Lemma 53 hold:

THEOREM 54. *If $h : \Sigma \to \Sigma$ is an expansive homeomorphism of a solenoid Σ, then Σ must be homeomorphic to Σ_b for some $n \geq b$.*

Proof. Suppose that Σ is a solenoid not homeomorphic to Σ_n for $n \geq 2$. We may assume that Σ is not the simple closed curve by [8]. By Corollary 6, we may assume hypothesis (1). If $\{b_i\}_{i=1}^{\infty}$ is a type 5A sequence, then Σ is homeomorphic to a regular solenoid. If $\{b_i\}_{i=1}^{\infty}$ is a type 4 sequence, then Σ does not admit an expansive homeomorphism by Corollary 35. Thus, hypotheses (2), (3) and (4) follow from Theorem 21 and Corollary 23. Hypothesis (5) follows from Theorems 5 and 50. Hypothesis (6) follows from Theorems 19 and 22. Hypothesis (7) follows from Proposition 46. Hypothesis (8) follows from Lemma 32 and hypothesis (7). Hypothesis (9) follows from Corollary 44. Thus the theorem follows from Lemma 53. ∎

By taking Theorem 54 along with Theorem 14 we get the main result Theorem 2.

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