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Matrix factorizations and link homology

by

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Abstract. For each positive integer n the HOMFLYPT polynomial of links specializes to a one-variable polynomial that can be recovered from the representation theory of quantum $sl(n)$. For each such n we build a doubly-graded homology theory of links with this polynomial as the Euler characteristic. The core of our construction utilizes the theory of matrix factorizations, which provide a linear algebra description of maximal Cohen–Macaulay modules on isolated hypersurface singularities.

Contents

1. Introduction. The HOMFLYPT polynomial of oriented links in $\mathbb{R}^3$ is uniquely determined by the skein relation in Figure 1 and its value on the unknot (see [HOMFLY], [PT]).

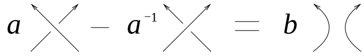

Fig. 1. The HOMFLYPT skein relation

The specialization $a = q^n$ and $b = q - q^{-1}$, for integer n, produces a one-variable link polynomial which can be interpreted via representation theory

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of the quantum group $U_q(sl(n))$ if n is positive (see [RT]), $U_q(sl(-n))$ if n is negative, and $U_q(gl(1|1))$ if $n = 0$ (see [KS]). This polynomial is invariant under changing q to q^{-1} simultaneously with passing to the mirror image of the link; therefore, we do not lose any information by restricting to non-negative n. We denote this one-variable polynomial by $P_n(L)$, where L is an oriented link; the normalization is

$$P_n(\text{unknot}) = [n] := \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{if } n > 0, \quad P_0(\text{unknot}) = 1.$$

For $n = 0, 1, 2, 3$ there exists a doubly-graded homology theory of links whose Euler characteristic is $P_n(L)$. Let us denote this theory by $H'_n(L)$.

- $P_0(L)$ is the Alexander polynomial, and $H'_0(L)$ was constructed by Peter Ozsváth and Zoltán Szabó [OS], and, independently, by Jacob Rasmussen [Ra]. Their theory exists in greater generality and, in particular, encompasses knots in homology spheres.
- $P_1(L) = 1$ and $H'_1(L) \cong \mathbb{Z}$ for any oriented link L, with $\mathbb{Z}$ in bidegree $(0, 0)$. It will be clear subsequently that this is a natural choice for $H'_1(L)$.
- $P_2(L)$ is the Jones polynomial; $H'_2(L)$ was defined in [Kh1], and denoted by $\mathcal{H}(L)$ there. $H'_2(\text{unknot})$ is isomorphic to the integral cohomology ring of the 2-sphere.
- $H'_3(L)$ was constructed in [Kh3]. $H'_3(\text{unknot})$ is isomorphic to the integral cohomology ring of $\mathbb{CP}^2$.

The goal of the present paper is to construct, for each $n > 0$, a doubly-graded homology theory $H_n(L)$ with Euler characteristic $P_n(L)$. The polynomial $P_n(L)$ can be computed by breaking up each crossing into a linear combination of diagrams of flat trivalent graphs, as in Figure 2.

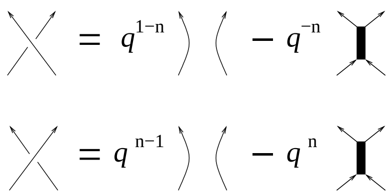

Fig. 2. Reducing to planar graphs

In these planar graphs some edges are oriented so that the neighborhood of each unoriented edge (depicted by a thick line, and referred to from now on as a "wide edge") looks as on the rightmost pictures in Figure 2. Two oriented edges "enter" the wide edge at one vertex and two oriented edges "leave" it at the other. Of course, this arrangement could be used to provide each wide edge with a canonical orientation, but we will not need it. In addition, oriented loops are allowed (an oriented loop is a crossingless plane projection of the oriented unknot).

There is a unique way to assign a Laurent polynomial $P_n(\Gamma) \in \mathbb{Z}[q, q^{-1}]$ to each such graph Γ so as to satisfy all skein relations in Figure 3.

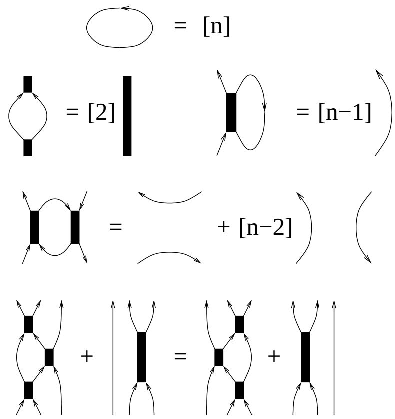

Fig. 3. Graph skein relations, $[i] = \dfrac{q^i - q^{-i}}{q - q^{-1}}$

In the representation theory language, an oriented edge stands for the vector representation V of quantum $sl(n)$, and a wide edge for its (quantum) exterior power $\Lambda^2 V$. The trivalent vertex is the unique (up to scaling) intertwiner between $V^{\otimes 2}$ and $\Lambda^2 V$. The polynomial $P_n(\Gamma)$ has *nonnegative* coefficients (see [MOY]).

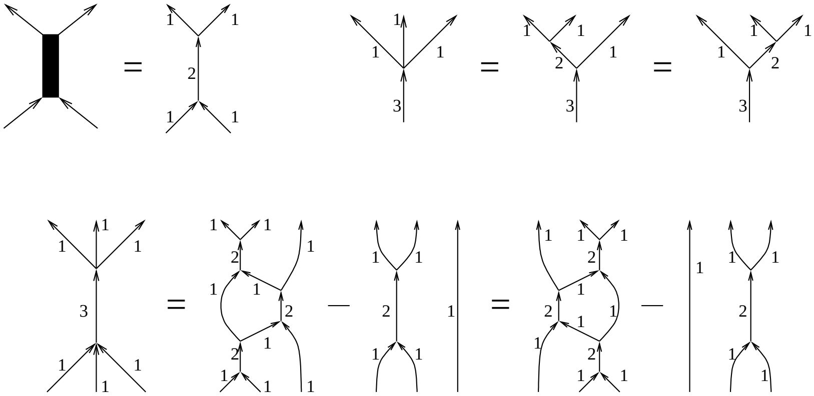

Fig. 4. Murakami–Ohtsuki–Yamada terminology and the appearance of edges labeled by 3

This calculus of planar graphs and its generalization to arbitrary exterior powers of V was developed by Murakami, Ohtsuki and Yamada [MOY]. Oriented edges of graphs in their calculus carry labels from 1 to $n-1$ that denote fundamental weights of $sl(n)$. The part of their calculus that we use deals only with edges labeled by 1 and 2. In our notation we omit these labels; instead, we indicate edges labeled by 2 by wide lines (see Figure 4, top left). Consistency of Figure 3 relations is shown in [MOY].

The last relation in Figure 3 can be rewritten to introduce edges labeled by 3 (see Figure 4), and to say that the difference of certain two endomorphisms of $V^{\otimes 3}$ is a multiple of the projection onto the irreducible summand $\Lambda^3 V$.

The polynomial invariant $P_n(L)$ of an oriented link L can be computed by choosing a plane diagram D of L, resolving each crossing as shown in Figure 2 and summing $P_n(\Gamma)$ weighted by powers of q over all resolutions Γ of D:

$$P_n(L) = P_n(D) := \sum_{\text{resolutions } \Gamma} q^{\alpha(\Gamma)} P_n(\Gamma),$$

with $\alpha(\Gamma)$ determined by Figure 2 rules. Independence from the choice of D follows from the equations in Figure 3. They imply that $P_n(D_1) = P_n(D_2)$ whenever D_1 and D_2 are related by a Reidemeister move.

To construct the homology theory H_n, we first categorify $P_n(\Gamma)$, by defining in a rather roundabout way a graded $\mathbb{Q}$-vector space $H(\Gamma) = \bigoplus_{j\in\mathbb{Z}} H^j(\Gamma)$ such that

$$P_n(\Gamma) = \sum_{j\in\mathbb{Z}} \dim_{\mathbb{Q}} H^j(\Gamma)\, q^j.$$

With Γ we associate a 2-periodic complex $C(\Gamma)$ of graded $\mathbb{Q}$-vector spaces

$$C^0(\Gamma) \to C^1(\Gamma) \to C^0(\Gamma)$$

and define $H(\Gamma)$ as the degree i cohomology of this complex, where i is the parity of the number of components of link L. The construction of $C(\Gamma)$ is based on the notion of a matrix factorization. An (R, w)-*factorization* M over a commutative ring R (where $w \in R$) consists of two free R-modules and two R-module maps

$$M^0 \xrightarrow{d} M^1 \xrightarrow{d} M^0$$

such that $d^2(m) = wm$ for any $m \in M$. The case most commonly considered in the literature is when R is the ring of power series, and w satisfies a certain nondegeneracy assumption (that the quotient $R/(w)$ is an isolated singularity). Such an M is called a *matrix factorization.* When w is homogeneous, one can switch from power series to polynomials.

Matrix factorizations appeared in commutative algebra in early and mid-eighties [E1], [B], [Kn], [S], [BEH] in the study of isolated hypersurface singularities, and much more recently in string theory, as boundary conditions for strings in Landau–Ginzburg models [KL1–3].

The tensor product $M \otimes_R N$ of an (R, w)-factorization M and an $(R, -w)$-factorization N is a 2-periodic complex of R-modules, with well-defined cohomology. More generally, given a finite set $\{w_1, \ldots, w_m\}$ of elements of R that sum to zero, and an (R, w_i)-factorization M_i for each i, the tensor product

$$M_1 \otimes_R \cdots \otimes_R M_m$$

is a 2-periodic complex and its cohomology $H(\bigotimes_i M_i)$ is a $\mathbb{Z}_2$-graded R-module.

Starting with a resolution Γ of a link diagram, we denote by E the set of oriented edges of Γ and by R the ring of polynomials in x_j, $j \in E$. We give each x_j degree 2, making R graded. Assume for simplicity that Γ has no oriented loops and each wide edge of Γ borders exactly four oriented edges (no oriented edge shares both endpoints with the same wide edge). Let T be the set of wide edges. Choosing a $t \in T$, denote the oriented edges at t by $1, 2, 3, 4$ (we think of $1, 2, 3, 4$ as elements of E) so that the four corresponding variables are x_1, x_2, x_3, x_4 (see Figure 5).

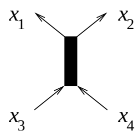

Fig. 5. Near a wide edge

Assign the polynomial

$$w_t = x_1^{n+1} + x_2^{n+1} - x_3^{n+1} - x_4^{n+1}$$

to the edge t. This polynomial lies in the ideal generated by $x_1 + x_2 - x_3 - x_4$ and $x_1x_2 - x_3x_4$ (since $x^{n+1} + y^{n+1}$ is a polynomial $g(x+y, xy)$ in $x+y$ and xy). Therefore, we can write

$$w_t = (x_1 + x_2 - x_3 - x_4)u_1 + (x_1x_2 - x_3x_4)u_2$$

for some polynomials u_1, u_2. The latter are not uniquely defined, but the indeterminacy is easy to describe. We choose

$$u_1 = \frac{x_1^{n+1} + x_2^{n+1} - g(x_3 + x_4, x_1x_2)}{x_1 + x_2 - x_3 - x_4},$$
$$u_2 = \frac{g(x_3 + x_4, x_1x_2) - x_3^{n+1} - x_4^{n+1}}{x_1x_2 - x_3x_4}.$$

Let C_t be the tensor product of the factorizations

$$R \xrightarrow{x_1+x_2-x_3-x_4} R \xrightarrow{u_1} R$$

and

$$R \xrightarrow{x_1x_2-x_3x_4} R \xrightarrow{u_2} R.$$

Then C_t is an (R, w_t)-factorization. Define $C(\Gamma)$ to be the tensor product of the C_t over all wide edges t,

$$C(\Gamma) := \bigotimes_{t\in\Gamma} C_t.$$

The square of the differential in $C(\Gamma)$ is the sum of w_t over all wide edges t,

$$d^2 = \sum_t w_t = 0.$$

The sum is 0, since for each oriented edge i the term x_i^{n+1} appears twice in the sum, once with positive and once with negative sign. We see that $C(\Gamma)$ is a 2-periodic complex of $\mathbb{Q}$-vector spaces.

If, in addition, Γ contains k oriented loops, to define $C(\Gamma)$ we tensor the product of C_t's with k copies of the vector space $\mathrm{H}^*(\mathbb{CP}^{n-1}, \mathbb{Q})$ and, if k is odd, shift the complex. The shift $M\langle 1\rangle$ of a factorization $M^0 \to M^1 \to M^0$ is

$$M^1 \to M^0 \to M^1.$$

If, for some t, some variables (say, x_2 and x_3) belong to the same oriented edge (see Figure 6), we quotient the ring R and the complex by the corresponding relation $(x_2 = x_3)$.

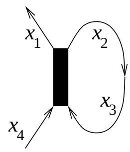

Fig. 6. x_2 and x_3 are on the same edge

We let $H(\Gamma)$ be the cohomology of $C(\Gamma)$. The latter has cohomology only in degree equal to the parity of the number of components of the link L. Additional internal grading in $C(\Gamma)$ coming from the grading in the polynomial ring R induces a $\mathbb{Z}$-grading on $H(\Gamma)$. This, in the nutshell, is how we define $H(\Gamma)$, which serve as building blocks for complexes $C(D)$ assigned to plane diagrams D.

In this paper we work in greater generality, first by allowing graphs that lie inside a disc and have points on its boundary. To such a graph we assign a factorization, rather than a 2-periodic complex. Second, we place one or more marks on each oriented edge, with variables x_i assigned to the marks, rather than just to the edges (see Figure 7). Boundary points also count as marks. Let E be the set of marks and R the ring of polynomials in variables x_i for $i \in E$.

To an arc bounded by marks x_i, x_j and oriented from x_j to x_i we assign the factorization L^i_j:

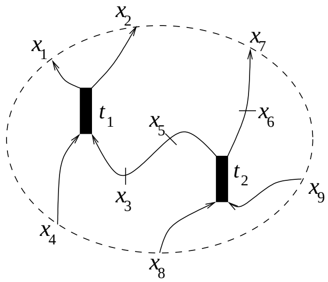

Fig. 7. A graph with boundary points and marks

$$R \xrightarrow{\pi_{ij}} R \xrightarrow{x_i - x_j} R, \quad \text{where} \quad \pi_{ij} = \frac{x_i^{n+1} - x_j^{n+1}}{x_i - x_j}.$$

We then form the tensor product $C(\Gamma)$ of the factorizations C_t over all wide edges t, and factorizations L_j^i over all arcs. Now we ignore the variables x_i assigned to internal marks, and view $C(\Gamma)$ as an (R', w)-factorization, with R' being the ring of polynomials in x_i over all boundary points i, and

$$w = \sum_i \pm x_i^{n+1},$$

with signs determined by the orientation of Γ near boundary points. We prove that, in the homotopy category of factorizations, $C(\Gamma)$ does not depend on how we place internal marks.

For example, if Γ is as in Figure 7, to the wide edges t_1 and t_2 we assign polynomials w_{t_1} and w_{t_2} in variables x_1, x_2, x_3, x_4 and x_5, x_6, x_8, x_9, respectively, and factorizations C_{t_1}, C_{t_2}. Form the tensor product factorization

$$C(\Gamma) = C_{t_1} \otimes_R C_{t_2} \otimes_R L_5^3 \otimes_R L_6^7,$$

where R is the ring of polynomials in $x_1, \dots, x_9$. We then ignore the variables x_3, x_5, x_6 that came from internal marks, and treat $C(\Gamma)$ as an (R', w)-factorization with

$$w = x_1^{n+1} + x_2^{n+1} - x_4^{n+1} + x_7^{n+1} - x_8^{n+1} - x_9^{n+1},$$

and R' being the ring of polynomials in the "boundary" variables x_1, x_2, x_4, x_7, x_8, x_9. As an R'-module, $C(\Gamma)$ is free but has infinite rank. However, by stripping off contractible summands, $C(\Gamma)$ can be reduced to a finite rank factorization.

We prove that the factorizations $C(\Gamma)$ have direct sum decompositions that mimic skein relations in Figure 3, and define homomorphisms χ_0 and χ_1 between factorizations $C(\Gamma^0)$ and $C(\Gamma^1)$ in Figure 8.

Then we consider oriented tangles in a 3-ball B^3 such that all boundary points of a tangle lie on a fixed great circle of the boundary sphere. Let D be a generic projection of a tangle L onto the plane of this great circle.

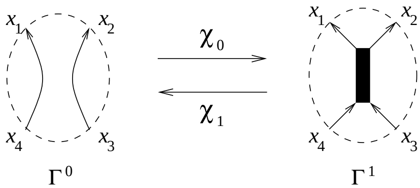

Fig. 8. Graphs Γ^0 and Γ^1

We separate crossings of D into positive and negative, following the rule in Figure 45. To each crossing we assign two planar graphs (resolutions of this crossing); see Figure 9.

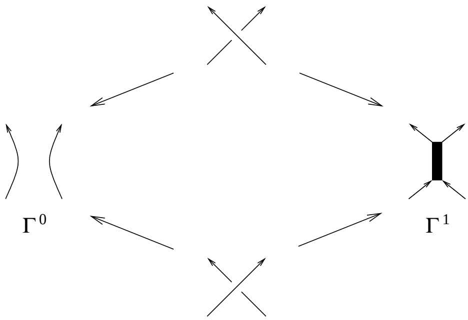

Fig. 9. Resolving crossings

To a diagram of a single crossing we assign the complex of factorizations

$$0 \to C(\Gamma^0) \xrightarrow{\chi_0} C(\Gamma^1) \to 0$$

if the crossing is positive, and the complex

$$0 \to C(\Gamma^1) \xrightarrow{\chi_1} C(\Gamma^0) \to 0$$

if the crossing is negative (shifts in the internal grading should be added to match powers of q that appear in Figure 2 formulas). In both cases we place $C(\Gamma^0)$ in cohomological degree 0.

In general, we place marks on each internal edge of D, form a commutative cube of factorizations $C(\Gamma)$ over all resolutions Γ of D, and take the total complex $C(D)$ of the cube. Each factorization $C(\Gamma)$ has additional $\mathbb{Z}$-grading, induced by the grading of the polynomial ring R, and the differential is grading-preserving. If we ignore the differentials, $C(\Gamma)$ is a $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$-graded R-module.

We prove that if D_1 and D_2 are related by a Reidemeister move, then $C(D_1)$ and $C(D_2)$ are isomorphic as objects in a suitable homotopy category. Therefore, the isomorphism class of $C(D)$ in this category is an invariant of the tangle L.

A cobordism S between tangles T_0 and T_1 is an oriented surface properly embedded in $B^3 \times [0,1]$ with $S \cap (B^3 \times \{i\}) = T_i$, and some additional standard assumptions. S can be presented by a "movie"—a sequence of plane diagrams of its cross-sections with $B^3 \times k$ for various $k \in [0,1]$. The sequence starts with a diagram D_0 of T_0 and ends with a diagram D_1 of T_1. To such a sequence we associate a homomorphism between complexes of factorizations $C(D_0)$ and $C(D_1)$ (with certain grading shifts thrown in), and show that, up to null-homotopies and multiplications by nonzero rationals, the homomorphism does not depend on the movie presentation of S (as long as D_0 and D_1 are the first and last slides). In this way, for each positive n, we obtain an invariant of tangle cobordism (trivial invariant if $n = 1$, and a variation of the one in [Kh4] if $n = 2$).

If L is a link, then $C(\Gamma)$, for any resolution Γ of D, is a 2-periodic complex of graded $\mathbb{Q}$-vector spaces. It has cohomology groups only in degree equal to the parity of the number of components in L. Thus, after removing the contractible summands, $C(\Gamma)$ reduces to a graded vector space, and $C(D)$ to a complex of graded vector spaces. Its cohomology groups

$$H_n(D) = \bigoplus_{i,j\in\mathbb{Z}} H_n^{i,j}(D)$$

do not depend, up to isomorphism, on the choice of the projection of L. The invariant of link cobordism is a homomorphism between these cohomology groups, well-defined up to rescaling by nonzero rational numbers. Due to this, $H_n(D)$ are canonically (up to rescaling) associated to L, rather than just to its diagram D. Thus, notation $H_n(L)$ is justified. The Euler characteristic of $H_n(L)$ is the polynomial $P_n(L)$:

$$P_n(L) = \sum_{i,j\in\mathbb{Z}} (-1)^i q^j \dim_{\mathbb{Q}} H_n^{i,j}(L).$$

$H(\Gamma)$ is the cohomology of the complex $C(\Gamma)$ which has countable dimension as a $\mathbb{Q}$-vector space. However, since $H(\Gamma)$ is finite-dimensional, it is nontrivial in finitely many degrees only, and an obvious upper bound on $|j|$ with $H^j(\Gamma) \neq 0$ is ne where e is the number of edges in Γ (both oriented and unoriented). The complex $C(D)$ is built out of finitely many $H(\Gamma)$, and the differential, being the signed sum of maps χ_0, χ_1, can be determined combinatorially as well. Therefore, there exists a combinatorial algorithm to find the homology groups $H_n(L)$, given L.

If $n = 1$, the factorization C_t is contractible, and any graph with a wide edge has trivial homology. The homology of a circle is $\mathbb{Q}$. This implies that $H_1(L) \cong \mathbb{Q}$ for any link L, with $\mathbb{Q}$ in bidegree $(0,0)$.

When $n = 2$ and Γ is closed (has no boundary points), its cohomology groups $H(\Gamma)$ are isomorphic to $A_2^{\otimes k}$, where A_2 is the cohomology ring of the 2-sphere and k the number of circles in the diagram given by deleting all wide edges of Γ. The equivalence of H_2 with a version of homology theory in [Kh1] follows easily from this observation. Specifically, for each link L there is an isomorphism

$$H_2^{i,j}(L) \cong \mathcal{H}^{i,-j}(L^!) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $L^!$ is the mirror image of L, and $\mathcal{H}$ is the homology theory defined in [Kh1, Section 7] (programs computing $\mathcal{H}(L)$ were implemented by Dror Bar-Natan [BN1] and Alexander Shumakovitch [Sh]).

We conjecture that H_3 is isomorphic to the homology theory constructed in [Kh3], after the latter is tensored with $\mathbb{Q}$. Both theories are doubly-graded, have the same Euler characteristic, and are built out of similar long exact sequences. The language of foams used in [Kh3] and [R] should extend from $n = 3$ to all n and lead to a better understanding of H_n.

Our categorification of the polynomials $P_n(L)$ comes with several obvious caveats:

- The homology groups $H_n(L)$ are finite-dimensional $\mathbb{Q}$-vector spaces rather than finitely generated abelian groups (as in [Kh1], [Kh3] for $n = 2, 3$).
- Defining $H_n(L)$ requires a choice of a plane diagram. A more intrinsic definition would be most welcome.
- Our invariants of link and tangle cobordisms are projective (defined up to overall multiplication by a non-zero rational number).
- Reconstructing the Ozsváth–Szabó–Rasmussen theory (the $n=0$ case) using this approach would require additional ideas, lacking at the moment.

Construction of $C(\Gamma)$ involves tensoring factorizations

$$R \xrightarrow{a} R \xrightarrow{b} R$$

for various a's and b's. We do an elementary study of these tensor products in Section 2. Section 3 contains a review of matrix factorizations and their properties. In Section 4 we explain how to view factorizations as functors and present the identity functor via a factorization. These two sections also introduce general framework for diagrammatical interpretation of matrix factorizations. Section 5 treats graded factorizations and explains how to modify the material of previous sections to cover this case. Section 6 is the computational core of the paper. We define a factorization $C(\Gamma)$ assigned to a planar graph Γ, construct morphisms χ_0, χ_1 between graphs Γ^0, Γ^1 in Figure 8, and prove direct sum decompositions of the factorizations $C(\Gamma)$ that lift the skein relations of Figure 3. In Section 7 we associate a complex of factorizations $C(D)$ to a plane diagram D of a tangle and state Theorem 2. This theorem, claiming the invariance of $C(D)$ under Reidemeister moves in a suitable category of complexes of factorizations, is proved in Section 8. In

Section 9 we use matrix factorizations to construct 2-dimensional topological quantum field theories with corners. The next section deals with functoriality of our tangle invariant, extending it to tangle cobordisms. In the last section we briefly outline an approach to categorification of quantum invariants of links colored by arbitrary fundamental representations of $sl(n)$.

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2. The cyclic Koszul complex. Let R be a Noetherian commutative ring. The *Koszul complex* $R(a_1,\dots,a_m)$ associated to a sequence $a_1,\dots,a_m$, where $a_i \in R$, is the tensor product (over R) of complexes

$$0 \to R \xrightarrow{a_i} R \to 0$$

for $i = 1,\dots,m$. A sequence $(a_1,\dots,a_m)$ is called *R-regular* if a_i is not a zero divisor in the quotient ring $R/(a_1,\dots,a_{i-1})R$ for each $i = 1,\dots,m$, and $R/(a_1,\dots,a_m)R \neq 0$. If the sequence is R-regular, its Koszul complex has cohomology in the rightmost degree only. For the converse to be true it suffices for R to be local (see [E1, Section 17]).

The Koszul complex can be written in a more intrinsic way. Let N be a free R-module of rank m and $a : R \to N$ an R-module homomorphism. The Koszul complex of a is the complex of exterior powers of N:

$$0 \to R \to N \to \Lambda^2 N \to \Lambda^3 N \to \cdots,$$

with the differential being the exterior product with $a(1) \in N$. Choosing a basis of N and writing $a(1)$ in coordinates as $(a_1,\dots,a_m)$, we recover the Koszul complex of this sequence.

Definition 1. Let $w \in R$. An (R,w)-*duplex* consists of two R-modules M^0, M^1 and module maps

$$M^0 \xrightarrow{d} M^1 \xrightarrow{d} M^0$$

such that $d^2 = w$ (as endomorphisms of M^0 and M^1).

In other words, $d^2 m = wm$ for any $m \in M^0 \oplus M^1$. "Duplex" is a term from [FKS], where it was used in a more general situation, with R a (possibly noncommutative) ring and w its central element.

A homomorphism $f : M \to N$ of duplexes is a pair of homomorphisms $f^0 : M^0 \to N^0$ and $f^1 : M^1 \to N^1$ that make the diagram below commute.

$$\begin{array}{ccccc} M^0 & \xrightarrow{d^0} & M^1 & \xrightarrow{d^1} & M^0 \\ {\scriptstyle f^0}\downarrow & & {\scriptstyle f^1}\downarrow & & {\scriptstyle f^0}\downarrow \\ N^0 & \xrightarrow{d^0} & N^1 & \xrightarrow{d^1} & N^0 \end{array}$$

The category of (R, w)-duplexes and duplex homomorphisms is abelian.

A homotopy h between homomorphisms $f, g : M \to N$ of duplexes is a pair of maps $h^i : M^i \to N^{i-1}$ (with indices understood modulo 2) such that

$$f - g = hd_M + d_N h.$$

Null-homotopic morphisms constitute an ideal in the category of duplexes. We call the quotient category by this ideal the category of duplexes up to homotopy, or simply the *homotopy category of duplexes*. This category is triangulated. We denote the shift functor by $\langle 1 \rangle$.

An $(R, 0)$-duplex M is equivalent to a 2-complex (periodic complex with period two) of R-modules. We denote the cohomology of an $(R, 0)$-duplex M by $H(M) \cong H^0(M) \oplus H^1(M)$.

Definition 2. A *factorization* (or *matrix factorization*) is an (R, w)-duplex such that M^0, M^1 are free R-modules.

The homotopy category of (R, w)-factorizations is triangulated.

To a pair of elements $a, b \in R$ we associate the (R, ab)-factorization

$$R \xrightarrow{a} R \xrightarrow{b} R,$$

denoted $\{a, b\}$.

If $\mathbf{a} = (a_1, \dots, a_m)$ and $\mathbf{b} = (b_1, \dots, b_m)$ are two sequences of elements of R, we consider the tensor product factorization $\{\mathbf{a}, \mathbf{b}\} := \otimes_i \{a_i, b_i\}$, where the tensor product is over R. We call $\{\mathbf{a}, \mathbf{b}\}$ the *Koszul factorization* of the pair $(\mathbf{a}, \mathbf{b})$. We say that a pair $(\mathbf{a}, \mathbf{b})$ is *orthogonal* if

$$\mathbf{ab} := \sum_i a_i b_i = 0.$$

The tensor product $\{\mathbf{a}, \mathbf{b}\}$ is a 2-complex iff $(\mathbf{a}, \mathbf{b})$ is orthogonal, in which case we call it the *periodic* (or *cyclic*) *Koszul complex* of $(\mathbf{a}, \mathbf{b})$.

Remark. We allow the case $m = 0$. Then $\mathbf{a}$ and $\mathbf{b}$ are empty sequences, $w = 0$, and the Koszul factorization is $R \to 0 \to R$.

Remark. Factorizations $\{\mathbf{a}, \mathbf{b}\}$ were defined in [BGS] and used there in the classification of finite CM-type singularities.

PROPOSITION 1. *If the entries of* $\mathbf{a}$ *and* $\mathbf{b}$ *generate* R *as an* R*-module, then the factorization* $\{\mathbf{a},\mathbf{b}\}$ *is contractible* (*its identity endomorphism is null-homotopic*).

Proof. Write $1=\sum_i(a'_i a_i + b'_i b_i)$. The pair $(\mathbf{a}',\mathbf{b}')$, for $\mathbf{a}'=(a'_1,\dots,a'_m)$ and $\mathbf{b}=(b'_1,\dots,b'_m)$, defines a homotopy between the identity and the zero endomorphism of $\{\mathbf{a},\mathbf{b}\}$. ∎

COROLLARY 1. *If* $(\mathbf{a},\mathbf{b})$ *is orthogonal and the entries of* $\mathbf{a}$ *and* $\mathbf{b}$ *generate* R *as an* R*-module, then the* 2*-complex* $\{\mathbf{a},\mathbf{b}\}$ *is acyclic.*

Let $I_{\mathbf{a},\mathbf{b}}\subset R$ be the ideal of R generated by the entries of $\mathbf{a}$ and $\mathbf{b}$.

PROPOSITION 2. $H(\{\mathbf{a},\mathbf{b}\})$ *is an* $R/I_{\mathbf{a},\mathbf{b}}$*-module.*

Proof. Multiplications by a_i and b_i are null-homotopic endomorphisms of $\{a_i,b_i\}$, and therefore of $\{\mathbf{a},\mathbf{b}\}$. ∎

Let $\mathbf{a}^i,\mathbf{b}^i$ be the sequences obtained from $\mathbf{a}$ and $\mathbf{b}$ by omitting a_i and b_i, respectively. The factorization $\{\mathbf{a},\mathbf{b}\}$ is the "total factorization" of the bifactorization

$$\{\mathbf{a}^i,\mathbf{b}^i\}\xrightarrow{a_i}\{\mathbf{a}^i,\mathbf{b}^i\}\xrightarrow{b_i}\{\mathbf{a}^i,\mathbf{b}^i\}.$$

If b_i is a nonzerodivisor in R, the second map is injective (as a map between R-modules), and the direct sum of the middle $\{\mathbf{a}^i,\mathbf{b}^i\}$ and its image under the differential is contractible. Furthermore, suppose that $(\mathbf{a},\mathbf{b})$ is orthogonal. Then the quotient of $\{\mathbf{a},\mathbf{b}\}$ by this contractible subcomplex is isomorphic to $\{\mathbf{a}^i,\mathbf{b}^i\}_{R'}$, the periodic Koszul complex of the pair $(\mathbf{a}^i,\mathbf{b}^i)$ in the quotient ring $R':=R/(b_i)$. The quotient map induces an isomorphism on cohomology,

$$H(\{\mathbf{a},\mathbf{b}\})\cong H(\{\mathbf{a}^i,\mathbf{b}^i\}_{R'}),$$

which is an isomorphism of R'-modules.

COROLLARY 2. *If* $(\mathbf{a},\mathbf{b})$ *is orthogonal and* $\mathbf{b}$ *is* R*-regular then*

$$H^0(\{\mathbf{a},\mathbf{b}\})\cong R/(b_1,\dots,b_m)\quad \textit{and}\quad H^1(\{\mathbf{a},\mathbf{b}\})=0.$$

Likewise, if a_i is a nonzerodivisor in R and $(\mathbf{a},\mathbf{b})$ is orthogonal, then the quotient map

$$\{\mathbf{a},\mathbf{b}\}\langle 1\rangle\to\{\mathbf{a}^i,\mathbf{b}^i\}_{R/(a_i)}$$

induces an isomorphism on cohomology.

Motivated in part by Corollary 2, we introduce

DEFINITION 3. An orthogonal pair $(\mathbf{a},\mathbf{b})$ is called *homologically* R*-regular* if $H^0(\{\mathbf{a},\mathbf{b}\})\neq 0$ and $H^1(\{\mathbf{a},\mathbf{b}\})=0$.

If $(\mathbf{a},\mathbf{b})$ is homologically R-regular then $I_{\mathbf{a},\mathbf{b}}$ is a proper ideal in R.

Example. Let $m = 1$. An orthogonal pair (a_1, b_1) is homologically R-regular iff $a_1R = \mathrm{Ann}(b_1)$ and b_1R is a proper subset of $\mathrm{Ann}(a_1)$. Furthermore, when R is a domain, (a_1, b_1) is homologically R-regular iff $a_1 = 0$ and b_1 is not invertible.

Example. Let $\mathbf{0} = (0, \dots, 0)$. The pair $(\mathbf{0}, \mathbf{b})$ is homologically R-regular iff the Koszul complex of $\mathbf{b}$ has cohomology in the rightmost degree only.

Suppose there is a subset $J \subset \{1, \dots, m\}$ with even number of elements such that the sequence $(h_1, \dots, h_m)$ is R-regular, where $h_i = a_i$ if $i \in J$ and $h_i = b_i$ otherwise. Then $(\mathbf{a}, \mathbf{b})$ is homologically R-regular and $H^0(\{\mathbf{a}, \mathbf{b}\}) \cong A/(h_1, \dots, h_m)$.

We can likewise define the notion of homologically M-regular pair $(\mathbf{a}, \mathbf{b})$ for any R-module M, and even for any (R, w)-duplex M. For the latter, we require $\mathbf{ab} = -w$, $H^0(M \otimes_R \{\mathbf{a}, \mathbf{b}\}) \neq 0$ and $H^1(M \otimes_R \{\mathbf{a}, \mathbf{b}\}) = 0$.

Let N be a finitely-generated free R-module and $\alpha : R \to N, \beta : N \to R$ be R-module maps. Consider the factorization $\{\alpha, \beta\}$ given by

$$(1) \qquad \Lambda^{\mathrm{even}} N \xrightarrow{\wedge\alpha + \neg\beta} \Lambda^{\mathrm{odd}} N \xrightarrow{\wedge\alpha + \neg\beta} \Lambda^{\mathrm{even}} N$$

where $\wedge\,\alpha$ is the wedge product with $\alpha(1)$,

$$\wedge\,\alpha : \Lambda^i N \to \Lambda^{i+1} N,$$

$\neg\beta$ is the contraction with β,

$$\neg\beta : \Lambda^i N \to \Lambda^{i-1} N,$$

and

$$\Lambda^{\mathrm{even}} N = \bigoplus_i \Lambda^{2i} N, \qquad \Lambda^{\mathrm{odd}} N = \bigoplus_i \Lambda^{2i+1} N.$$

$\{\alpha, \beta\}$ is a 2-complex iff $\beta\alpha = 0$, in which case we say that α and β are *orthogonal*. Choosing a basis of N, we can write $\alpha = \mathbf{a} = (a_1, \dots, a_m)^T$ and $\beta = \mathbf{b} = (b_1, \dots, b_m)$. Then $\{\alpha, \beta\}$ is isomorphic to $\{\mathbf{a}, \mathbf{b}\}$.

Thus, with each element $\gamma \in N \oplus N^*$ we associate a factorization

$$\{\gamma\} := \{\alpha, \beta\},$$

where $\gamma = \alpha + \beta$ and $\alpha \in N$, $\beta \in N^*$.

Suppose $g : N \to N$ is an automorphism of the R-module N. Write the composition $\beta\alpha$ as $\beta\gamma^{-1}\gamma\alpha$. The factorizations $\{\alpha, \beta\}$ and $\{\gamma\alpha, \beta\gamma^{-1}\}$ are isomorphic, and $GL(N, R)$ acts on the set of pairs (α, β) preserving the isomorphism classes of factorizations $\{\alpha, \beta\}$.

We let the group $H = (\mathbb{Z}_2)^{\times m}$ act on pairs by permuting a_i with a_i in $(a_1, a_1), \dots, (a_m, a_m)$ for $1 \le i \le m$.

For $\sigma \in H$, the factorizations $\{\mathbf{a}, \mathbf{b}\}$ and $\{\sigma(\mathbf{a}, \mathbf{b})\}$ are isomorphic if σ is an even permutation in $H \subset \mathbb{S}_{2m}$. If σ is odd, the factorization $\{\sigma(\mathbf{a}, \mathbf{b})\}$ is isomorphic to $\{\mathbf{a}, \mathbf{b}\}\langle 1\rangle$.

The natural R-linear pairing between N^* and N induces a symmetric bilinear form on $N \oplus N^*$. Let $G' \subset GL(N \oplus N^*, R)$ be the subgroup generated by $GL(N, R)$ and H that act as described above. The action of G' preserves the inner product on $N \oplus N^*$ (with values in R). Let G be the subgroup of G' that consists of all products $g_1\sigma_1 \dots g_r\sigma_r$ of elements from $GL(N, R)$ and H (over all r) such that $\sigma_1 \dots \sigma_r$ is an even permutation.

The following result is clear.

PROPOSITION 3. *$\{\gamma\}$ and $\{g\gamma\}$ are isomorphic factorizations for any $g \in G$ and $\gamma \in N \oplus N^*$.*

The cyclic Koszul complex can be thought of as the square root of the Koszul complex. Namely,

$$\mathrm{Hom}_R(\{\mathbf{a}, \mathbf{b}\}, \{\mathbf{a}, \mathbf{b}\}) \cong \{\mathbf{a}, \mathbf{b}\} \otimes_R \{-\mathbf{b}, \mathbf{a}\} \cong R(\mathbf{a}, \mathbf{b}),$$

where $R(\mathbf{a}, \mathbf{b})$ is the Koszul complex of the length $2m$ sequence given by concatenating $\mathbf{a}$ and $\mathbf{b}$, with the grading collapsed from $\mathbb{Z}$ to $\mathbb{Z}_2$.

3. Potentials, isolated singularities, and matrix factorizations

Potentials and their Jacobian algebras. We start with a finite set $x = \{x_1, \dots, x_k\}$ of variables and let

$$R = \mathbb{Q}[[x]] := \mathbb{Q}[[x_1, \dots, x_k]]$$

be the algebra of power series in $x_1, \dots, x_k$ with rational coefficients. Denote by $\mathfrak{m}$ the unique maximal ideal in R (generated by $x_1, \dots, x_k$).

We say that a polynomial $w = w(x) \in \mathfrak{m}^2$ is a *potential* if the algebra R/I_w is finite-dimensional, where $I_w \subset R$ is the ideal generated by the partial derivatives $\partial_i w := \partial_{x_i} w$. Any polynomial $w \in R$ defines an algebraic map $\mathbb{C}^n \to \mathbb{C}$. This map has an isolated singularity at 0 iff w is a potential.

A polynomial $w \in \mathfrak{m}^2$ is a potential iff $\partial_1 w, \dots, \partial_k w$ is a regular sequence in R, that is, $\partial_i w$ is not a zero divisor in $R/(\partial_1 w, \dots, \partial_{i-1} w)R$ for each $1 \le i \le k$.

The quotient algebra $R_w := R/I_w$ is called the *local algebra of the singularity* and the *Jacobian* (or *Milnor*) *algebra* of w. Its dimension is the *Milnor number* of the singularity.

PROPOSITION 4. *The Jacobian algebra R_w is symmetric.*

A nondegenerate $\mathbb{C}$-linear trace $\mathrm{Tr} : R_w \otimes_{\mathbb{Q}} \mathbb{C} \to \mathbb{C}$ is given by the residue formula

$$\mathrm{Tr}(a) = \frac{1}{(2\pi i)^k} \int\limits_{|\partial_j w| = \varepsilon} \frac{a\, dx_1 \dots dx_k}{\partial_1 w \dots \partial_k w} \tag{2}$$

(proved in [GH, Chapter 5] and [AGV]). Therefore, the $\mathbb{C}$-algebra $R_w \otimes_{\mathbb{Q}} \mathbb{C}$ is symmetric. From [NN, Theorem 5] we deduce that R_w is a symmetric $\mathbb{Q}$-algebra. ∎

Exterior sum of potentials. Let x and y be two disjoint finite sets of variables. Define the exterior sum of potentials $w_1(x)$ and $w_2(y)$ as the potential $w_1(x) + w_2(y)$ in variables x and y. The Jacobian algebra of the exterior sum of potentials is the tensor product of the Jacobian algebras:

$$R_{w_1(x)+w_2(y)} \cong R_{w_1(x)} \otimes_{\mathbb{Q}} R_{w_2(y)}.$$

Matrix factorizations. A *matrix factorization* of a potential w is a collection of two free A-modules M^0, M^1 and two module maps $d^0 : M^0 \to M^1$ and $d^1 : M^1 \to M^0$ such that

$$d^0 d^1 = w \cdot \mathrm{Id}_{M^1}, \qquad d^1 d^0 = w \cdot \mathrm{Id}_{M^0}.$$

We will often write a matrix factorization as

$$M^0 \xrightarrow{d} M^1 \xrightarrow{d} M^0.$$

The modules M^0 and M^1 are not required to have finite rank. Since w is invertible in the field of fractions $\mathbb{Q}((x))$ of R, and the rank of a free R-module N can be defined as the dimension of the $\mathbb{Q}((x))$-vector space $N \otimes_R \mathbb{Q}((x))$, we see that M^0 and M^1 have equal ranks (when the ranks are finite). In general, $\mathrm{rk}(M^0) = \mathrm{rk}(M^1)$, understood as equality of ordinals. We call $\mathrm{rk}(M^0)$ the *rank of* M.

A homomorphism $f : M \to N$ of factorizations is a pair of homomorphisms $f^0 : M^0 \to N^0$ and $f^1 : M^1 \to N^1$ that make the diagram below commute.

$$\begin{array}{ccccc} M^0 & \xrightarrow{d^0} & M^1 & \xrightarrow{d^1} & M^0 \\ \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^0} \\ N^0 & \xrightarrow{d^0} & N^1 & \xrightarrow{d^1} & N^0 \end{array}$$

Denote the set of homomorphisms from M to N by $\mathrm{Hom}_{\mathrm{MF}}(M, N)$. It is an R-module, with the action $a(f^0, f^1) = (af^0, af^1)$ for $a \in R$.

Let $\mathrm{MF}_w^{\mathrm{all}}$ be the category whose objects are matrix factorizations and morphisms are homomorphisms of factorizations. This category is additive and R-linear. Direct sum of matrix factorizations M and N is defined in the obvious way:

$$(M \oplus N)^i = M^i \oplus N^i, \qquad d^i_{M \oplus N} = d^i_M + d^i_N.$$

The shifted factorization $M\langle 1\rangle$ is given by

$$M\langle 1\rangle^i = M^{i+1}, \qquad d^i_{M\langle 1\rangle} = -d^{i+1}_M, \qquad i = 0, 1 \bmod 2.$$

$\langle 1\rangle$ is a functor in $\mathrm{MF}_w^{\mathrm{all}}$ whose square is the identity, and $\langle 2\rangle \cong \mathrm{Id}$.

Example. If the set of variables x is empty, then $R = \mathbb{Q}$, $f = 0$, and a matrix factorization is a pair of vector spaces and a pair of maps between them such that their composition in any order is 0. We call such data a 2-*complex* (short for 2-*periodic complex*).

Matrix factorizations first appeared in the work of David Eisenbud [E1], who related them to maximal Cohen–Macaulay modules over isolated hypersurface singularities (see also [Y1, Chapter 7], [S], and references therein). A module N over a commutative Noetherian local ring is called *maximal Cohen–Macaulay* if the depth of N equals the Krull dimension of the ring. Given a matrix factorization M, the $R/(f)$-module $\mathrm{Coker}(d^1)$ is maximal Cohen–Macaulay, and any maximal Cohen–Macaulay module over $R/(f)$ can be presented in this way (see the above references for details). For a reader-friendly treatment of the background concepts leading to maximal Cohen–Macaulay modules we suggest the book [E2].

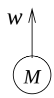

Fig. 10. A matrix factorization

Graphical notation. We denote a matrix factorization M with potential w as in Figure 10. We allow reversing orientation of the arc attached to M simultaneously with changing potential w to $-w$ (see Figure 11).

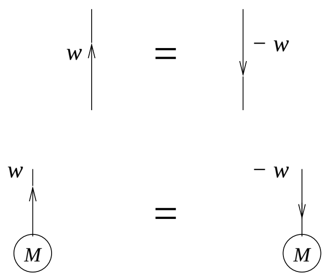

Fig. 11. Orientation reversal

If w is the exterior sum of potentials, we may introduce several arcs at M, one for each summand (see Figure 12).

Factorizations of finite rank. Let MF'_w be the category of finite rank matrix factorizations. It is a full subcategory in $\mathrm{MF}^{\mathrm{all}}_w$. Choose bases in free R-modules M^0 and M^1. The maps d^0, d^1 can be written as $m \times m$ matrices D_0, D_1 with coefficients in R. These matrices satisfy the equations

$$D_0 D_1 = w \cdot \mathrm{Id}, \qquad D_1 D_0 = w \cdot \mathrm{Id} \tag{3}$$

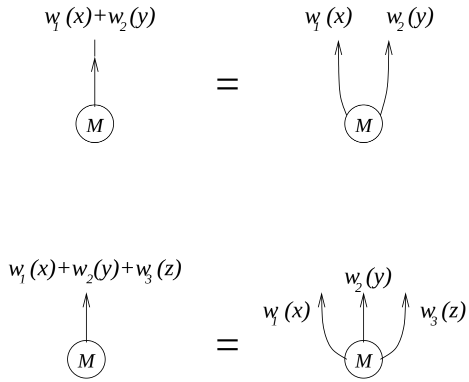

Fig. 12. Diagrams for a factorization over exterior sum of potentials

(of course, any one of these equations implies the other). Alternatively, we can describe this factorization by a $2m\times 2m$ matrix with off-diagonal blocks D_0 and D_1:

$$D=\begin{pmatrix} 0 & D_1 \\ D_0 & 0 \end{pmatrix}, \qquad D^2 = w\cdot \mathrm{Id}\,.$$

Matrix description of objects in MF'_w extends to infinite rank factorizations. The matrices D_0 and D_1 then have infinite rank, but each of their columns has only finitely many nonzero entries.

If factorizations M and N are written in matrix form, $M=(D_0,D_1)$, and $N=(D'_0,D'_1)$, a homomorphism $f:M\to N$ is a pair of matrices (F_0,F_1) such that $F_1D_0=D'_0F_0$ and $F_0D_1=D'_1F_1$ (note, though, that the two equations are equivalent).

Homotopies of factorizations. A homotopy h between maps $f,g:M\to N$ of factorizations is a pair of maps $h^i:M^i\to N^{i-1}$ such that

$$f-g=hd_M+d_Nh.$$

Null-homotopic morphisms constitute an ideal in the category $\mathrm{MF}^{\mathrm{all}}_w$. Let $\mathrm{HMF}^{\mathrm{all}}_w$ be the quotient category by this ideal. It has the same objects as $\mathrm{MF}^{\mathrm{all}}_w$, but fewer morphisms:

$$\mathrm{Hom}_{\mathrm{HMF}}(M,N):=\mathrm{Hom}_{\mathrm{MF}}(M,N)/\{\text{null-homotopic morphisms}\}.$$

Choose bases in M^0 and M^1 and write d as a matrix D. Differentiating the equation $D^2=w$ with respect to x_i we get

$$D(\partial_iD)+(\partial_iD)D=\partial_iw.$$

Therefore, the multiplication by ∂_iw endomorphism of M is homotopic to 0, and we obtain

Proposition 5. *The action of* R *on* $\mathrm{Hom}_{\mathrm{HMF}}(M,N)$ *factors through the action of the Jacobian algebra* R_w.

Let HMF'_w be the category with objects finite rank matrix factorizations and morphisms homomorphisms of factorizations modulo those homotopic to 0. This is a full subcategory in $\mathrm{HMF}^{\mathrm{all}}_w$.

The free R-module $\mathrm{Hom}_R(M,N)$ is a 2-complex

$$\mathrm{Hom}^0_R(M,N) \xrightarrow{d} \mathrm{Hom}^1_R(M,N) \xrightarrow{d} \mathrm{Hom}^0(M,N),$$

where

$$\begin{aligned}\mathrm{Hom}^0_R(M,N) &= \mathrm{Hom}_R(M^0,N^0) \oplus \mathrm{Hom}_R(M^1,N^1),\\ \mathrm{Hom}^1_R(M,N) &= \mathrm{Hom}_R(M^0,N^1) \oplus \mathrm{Hom}_R(M^1,N^0),\end{aligned}$$

and

$$(df)m = d_N(f(m)) + (-1)^i f(d_M(m)) \quad \text{for } f \in \mathrm{Hom}^i_R(M,N).$$

Denote the cohomology of this 2-complex by

$$\mathrm{Ext}(M,N) = \mathrm{Ext}^0(M,N) \oplus \mathrm{Ext}^1(M,N).$$

It is clear from the definitions that

$$\mathrm{Ext}^0(M,N) \cong \mathrm{Hom}_{\mathrm{HMF}}(M,N), \tag{4}$$

$$\mathrm{Ext}^1(M,N) \cong \mathrm{Hom}_{\mathrm{HMF}}(M,N\langle 1\rangle). \tag{5}$$

Proposition 6. $\mathrm{Ext}(M,N)$ *is a finite-dimensional* R_w*-module if* M *and* N *are finite rank factorizations.*

Proof. We need to show that $\mathrm{Hom}_{\mathrm{HMF}}(M,N)$ is finite-dimensional. The latter is an R-module quotient of $\mathrm{Hom}_{\mathrm{MF}}(M,N)$, which is a submodule of the free finite rank R-module $\mathrm{Hom}_R(M,N)$. Since R is Noetherian, subquotients of finitely generated R-modules are finitely generated. The action of R on $\mathrm{Hom}_{\mathrm{HMF}}(M,N)$ factors through the action of the Jacobian ring R_w, by Proposition 5. Therefore, $\mathrm{Hom}_{\mathrm{HMF}}(M,N)$ is finite-dimensional, being finitely generated over a finite-dimensional algebra. ∎

Remark. This proposition and many that follow fail to hold if $w \in R$ is not a potential. However, (R,w)-factorizations for such degenerate w will make important intermediate appearances, as our main examples come from tensor products of factorizations with degenerate w (when $R/(w)$ does not have an isolated singularity at 0).

Definition 4. A polynomial $p \in \mathfrak{m}^2$ is called a *degenerate potential* if the Jacobian algebra R_w is infinite-dimensional.

Any polynomial in $\mathfrak{m}^2$ is either a potential or a degenerate potential (but never both). Unless specified otherwise, w denotes a potential.

The Tyurina algebra. The *Tyurina algebra* R_w^T is defined as the quotient of the Jacobian algebra R_w by the ideal generated by w. Multiplication by w is homotopic to 0 in $\mathrm{Hom}_{\mathrm{MF}}(M, M)$, for any factorization M, and therefore $\mathrm{Hom}_{\mathrm{HMF}}(M, N)$ and $\mathrm{Ext}(M, N)$ are modules over the Tyurina algebra (this is a slight extension of Proposition 5).

Cohomology of factorizations. The quotient $M/\mathfrak{m}M$ is a 2-complex of vector spaces (the square of the differential is 0 since $w \in \mathfrak{m}$):

$$M^0/\mathfrak{m}M^0 \xrightarrow{d} M^1/\mathfrak{m}M^1 \xrightarrow{d} M^0/\mathfrak{m}M^0.$$

Denote the cohomology of this 2-complex by $H(M)$ and call it the *cohomology of* M. Note that $H(M)$ is $\mathbb{Z}_2$-graded, $H(M) = H^0(M) \oplus H^1(M)$. Cohomology of factorizations is a functor from $\mathrm{MF}_w^{\mathrm{all}}$ and $\mathrm{HMF}_w^{\mathrm{all}}$ to the category of $\mathbb{Z}_2$-graded $\mathbb{Q}$-vector spaces and grading-preserving linear maps.

PROPOSITION 7. *The following conditions on* $M \in \mathrm{MF}_w^{\mathrm{all}}$ *are equivalent*:

(i) $H(M) = 0$.
(ii) $H^0(M) = 0$.
(iii) $H^1(M) = 0$.
(iv) *M is isomorphic to the zero factorization in the category* $\mathrm{HMF}_w^{\mathrm{all}}$.
(v) *M is isomorphic in* $\mathrm{MF}_w^{\mathrm{all}}$ *to the (possibly infinite) direct sum of*

$$R \xrightarrow{1} R \xrightarrow{w} R \tag{6}$$

and

$$R \xrightarrow{w} R \xrightarrow{1} R. \tag{7}$$

Proof. The implications (v)⇒(iv)⇒(i)⇒(ii) and (i)⇒(iii) are obvious. It suffices to establish (ii)⇒(v). Choose R-module bases in M^0 and M^1 and write factorization maps in matrix form. If one of the entries does not lie in the maximal ideal $\mathfrak{m} \subset R$, this entry is invertible, and a change of bases makes the matrices block-diagonal with two blocks, one of which is either $(1, w)$ or $(w, 1)$. Therefore, M is isomorphic to the direct sum of some factorization N and either (6) or (7). Applying Zorn's lemma, we conclude that any factorization M decomposes into a direct sum $M \cong M_{es} \oplus M_c$ where M_c is a direct sum of factorizations (6) or (7) and M_{es} does not have any invertible elements in its matrix presentation (equivalently, M_{es} does not contain any summands isomorphic to (6) or (7)). Since $M_{es}^0 \cong V \otimes_{\mathbb{Q}} R$ as R-modules, for some vector space V, we have

$$H^0(M) \cong H^0(M_{es}) \cong V.$$

Therefore, if M satisfies (ii), then $V = 0$ and $M_{es} = 0$. The proposition follows. ∎

A factorization is called *contractible* if it satisfies one of the five equivalent conditions in Proposition 7.

Define the dimension $\dim(M)$ of the factorization M as the dimension of the $\mathbb{Q}$-vector space $H(M)$. We have $\dim(M) \leq 2\,\mathrm{rk}(M)$. If M is a finite rank factorization, equality holds iff M does not contain contractible summands.

Example. A factorization of rank 1 has the form

$$R \xrightarrow{a_0} R \xrightarrow{a_1} R$$

for $a_0, a_1 \in R$ such that $w = a_0 a_1$. Denote this factorization by N. Then

$$\mathrm{Hom}_{\mathrm{MF}}(N,N) \cong R \quad \text{and} \quad \mathrm{Hom}_{\mathrm{HMF}}(N,N) \cong R/(a_0, a_1),$$

the quotient of R by the ideal generated by a_0 and a_1. The factorization N is contractible iff either a_0 or a_1 is invertible (does not lie in $\mathfrak{m}$). It is nontrivial when both a_0 and a_1 belong to the maximal ideal (then $H(M) \cong \mathbb{Q} \oplus \mathbb{Q}$). This is only possible if k, the number of variables in the set x, is at most 2 (since we need $w = a_0 a_1$ to be nondegenerate).

Proposition 8. *The following properties of a morphism $f : M \to N$ of factorizations are equivalent*:

(i) *f is an isomorphism in the homotopy category of factorizations.*
(ii) *f induces an isomorphism between the cohomologies of M and N.*

Proof. The implication (i)⇒(ii) is obvious. To prove (ii)⇒(i), choose decompositions $M \cong M_{es} \oplus M_c$ and $N \cong N_{es} \oplus N_c$. The composition

$$f_{es} : M_{es} \xrightarrow{i} M \xrightarrow{f} N \xrightarrow{p} N_{es},$$

where i and p are the obvious inclusion and projection, induces an isomorphism $H(f_{es}) : H(M_{es}) \cong H(N_{es})$. Given two free R-modules L_1 and L_2, an R-module map $L_1 \to L_2$ which induces an isomorphism of quotients $L_1/\mathfrak{m}L_1 \cong L_2/\mathfrak{m}L_2$ is an isomorphism of R-modules. Therefore, $f_{es} : M_{es} \to N_{es}$ is an isomorphism of R-modules, and f is an isomorphism in the homotopy category of factorizations. ∎

Remark. We see from this proposition that, although $\mathrm{HMF}^{\mathrm{all}}_w$ was defined as a "homotopy" category, it also has the flavor of a "derived" category: in the latter, a morphism which induces an isomorphism in cohomology is an isomorphism. For an explanation and generalizations of this phenomenon see Buchweitz [B] and Orlov [O].

The proof of Proposition 8 implies

Corollary 3. *A decomposition $M \cong M_{es} \oplus M_c$ of M into a factorization without contractible summands and a contractible factorization is unique up to isomorphism.*

We call M_{es} the *essential summand* of M.

Corollary 4. *Any factorization M with finite-dimensional cohomology is a direct sum of a finite rank factorization and a contractible factorization.*

Let MF_w be the category whose objects are factorizations with finite-dimensional cohomology and morphisms are homomorphisms of factorizations. Let HMF_w be the quotient of MF_w by the ideal of null-homotopic morphisms. Any finite rank factorization has finite-dimensional cohomology, therefore, we have (full and faithful) inclusions of categories $\mathrm{MF}'_w \subset \mathrm{MF}_w$ and $\mathrm{HMF}'_w \subset \mathrm{HMF}_w$. Corollaries 3 and 4 imply

Corollary 5. *The inclusion $\mathrm{HMF}'_w \subset \mathrm{HMF}_w$ is an equivalence of categories.*

Although we will be doing most of our work in the categories HMF_w for various w, the other five categories provide a useful supporting framework. To reduce the confusion of six different categories, we arranged them into a table.

	all factorizations	finite rank	factorizations with finite-dimensional cohomology
all homomorphisms	$\mathrm{MF}_w^{\mathrm{all}}$	MF'_w	MF_w
modulo homotopic to 0	$\mathrm{HMF}_w^{\mathrm{all}}$	HMF'_w	$\cong$ HMF_w

The categories in the bottom row are triangulated, and the two rightmost categories in this row are equivalent. Even though most of the time our constructions start with finite rank factorizations, our functors take them to infinite rank factorizations, but with finite-dimensional cohomology. The latter factorizations lie in HMF_w (they can be reduced to finite rank factorizations, but this operation, sometimes necessary on concrete factorizations, is awkward from the categorical viewpoint). At the same time, the collection of categories HMF_w over various w is closed under all functors that are essential in our work, and is a natural and thrifty choice.

Dualities. The free R-module $M^* = \mathrm{Hom}_R(M, R)$ admits a factorization

$$(M^0)^* \xrightarrow{(d^1)^*} (M^1)^* \xrightarrow{(d^0)^*} (M^0)^*.$$

An inclusion of factorizations $M \subset M^{**}$ is an isomorphism if M has finite rank, and an isomorphism in the homotopy category if M has finite-dimensional cohomology. Thus, $M \to M^*$ is a contravariant equivalence in the categories $\mathrm{MF}'_w, \mathrm{HMF}'_w$, and HMF_w.

Assume that M has finite rank, and choose R-module bases in M^0, M^1. The factorization is described by two matrices (D_0, D_1). The dual factorization M^* is described by the transposed matrices (D_1^t, D_0^t).

For $M \in \mathrm{MF}_w^{\mathrm{all}}$ denote by M_- the factorization

$$M^0 \xrightarrow{-d^0} M^1 \xrightarrow{d^1} M^0.$$

This assignment extends to an equivalence of categories of factorizations with potentials w and $-w$.

Let $M_\bullet = (M^*)_-$. This assignment is a contravariant functor between categories of factorizations with potentials w and $-w$. Observe that

$$\{\mathbf{a}, \mathbf{b}\}_\bullet \cong \{-\mathbf{b}, \mathbf{a}\}.$$

Excluding a variable. Let $R = \mathbb{Q}[[x_1, \dots, x_m]]$ and $R' = \mathbb{Q}[[x_2, \dots, x_m]] \subset R$. Suppose that $w \in R'$ is a potential, and $w = \mathbf{ab}$ for a pair $\{\mathbf{a}, \mathbf{b}\}$ as in Section 2, where $a_i, b_i \in R$. Suppose $b_i - x_1 \in R'$ for some i. Let $c = b_i - x_1$, and $\mathbf{a}^i, \mathbf{b}^i$ be the sequences obtained from $\mathbf{a}$ and $\mathbf{b}$ by omitting a_i and b_i. Let $\psi : R \to R'$ be the homomorphism $\psi(x_j) = x_j$ for $j \neq 1$, and $\psi(x_1) = -c$ (so that $\psi(b_i) = 0$). Note that $\psi(w) = w$.

Let $\psi(\mathbf{a}^i), \psi(\mathbf{b}^i)$ be the sequences obtained by applying ψ to every entry of $\mathbf{a}^i$ and $\mathbf{b}^i$. Then $\{\psi(\mathbf{a}^i), \psi(\mathbf{b}^i)\}$ is an (R', w)-factorization. By treating R as an R'-module, we can view $\{\mathbf{a}, \mathbf{b}\}$ as an (R', w)-factorization (of infinite rank). Let

$$f : \{\mathbf{a}, \mathbf{b}\} \to \{\psi(\mathbf{a}^i), \psi(\mathbf{b}^i)\}$$

be the following homomorphism of (R', w)-factorizations:

$$\begin{array}{ccccc} \{\mathbf{a}^i, \mathbf{b}^i\} & \xrightarrow{a_i} & \{\mathbf{a}^i, \mathbf{b}^i\} & \xrightarrow{b_i} & \{\mathbf{a}^i, \mathbf{b}^i\} \\ \downarrow\psi & & \downarrow & & \downarrow\psi \\ \{\psi(\mathbf{a}^i), \psi(\mathbf{b}^i)\} & \longrightarrow & 0 & \longrightarrow & \{\psi(\mathbf{a}^i), \psi(\mathbf{b}^i)\} \end{array}$$

The top line is the factorization $\{\mathbf{a}, \mathbf{b}\}$ written as the total factorization of a bifactorization.

Proposition 9. *f is an isomorphism in the homotopy category of (R', w)-factorizations.*

Proof. It would suffice to show that f induces an isomorphism on cohomology. Multiplication by b_i is an injective endomorphism of the R'-module $\{\mathbf{a}^i, \mathbf{b}^i\}$. Decompose

$$\{\mathbf{a}^i, \mathbf{b}^i\} \cong b_i\{\mathbf{a}^i, \mathbf{b}^i\} \oplus M,$$

where the decomposition is that of R'-modules, and M consists of vectors with all coordinates in R' in the standard R-module basis of $\{\mathbf{a}^i, \mathbf{b}^i\}$. The R'-subfactorization

$$b_i\{\mathbf{a}^i, \mathbf{b}^i\} \xrightarrow{a_i} \{\mathbf{a}^i, \mathbf{b}^i\} \xrightarrow{b_i} b_i\{\mathbf{a}^i, \mathbf{b}^i\}$$

of $\{\mathbf{a}, \mathbf{b}\}$ is contractible, while $f(M) = \{\psi(\mathbf{a}^i), \psi(\mathbf{b}^i)\}$ as R'-modules. It follows immediately that f induces an isomorphism on cohomology. ■

Let $R = \mathbb{Q}[[x_1, \ldots, x_{k+m}]]$ and $R' = \mathbb{Q}[[x_{k+1}, \ldots, x_{k+m}]] \subset R$. Suppose that $w \in R'$ is a potential, and $w = \mathbf{ab}$ for a pair $\{\mathbf{a}, \mathbf{b}\}$ with $a_j, b_j \in R$. Assume that $b_1 - x_1$ is a polynomial in $x_2, \ldots, x_{k+m}$, and, more generally, $b_j - x_j$, for each j between 1 and k, is a polynomial in $x_{j+1}, \ldots, x_{k+m}$, modulo the ideal generated by $b_1, b_2, \ldots, b_{j-1}$. Let ψ be the homomorphism $R \to R'$ uniquely determined by the conditions $\psi(x_j) = x_j$ for $j > k$, and $\psi(b_j) = 0$ for $j \leq k$. Let $\mathbf{a}_k, \mathbf{b}_k$ be the sequences obtained from $\mathbf{a}, \mathbf{b}$ by omitting a_j, b_j for all $j \leq k$. Let $\psi(\mathbf{a}_k), \psi(\mathbf{b}_k)$ be the sequences obtained by applying ψ to every entry of $\mathbf{a}_k$ and $\mathbf{b}_k$. Then $\{\psi(\mathbf{a}_k), \psi(\mathbf{b}_k)\}$ is an (R', w)-factorization. Treating R as an R'-module, we view $\{\mathbf{a}, \mathbf{b}\}$ as an (R', w)-factorization (of infinite rank).

Proposition 10. $\{\mathbf{a}, \mathbf{b}\}$ *and* $\{\psi(\mathbf{a}_k), \psi(\mathbf{b}_k)\}$ *are isomorphic in the homotopy category of* (R', w)*-factorizations.*

Proof. The proof of the previous proposition generalizes to this setup without difficulty. ■

Frobenius structure. The following duality theorem was proved by Ragnar-Olaf Buchweitz [B] in much greater generality.

Theorem 1. *Suppose that* w *is a potential in an even number of variables,* $w = w(x_1, \ldots, x_{2k})$*. There exists a collection of trace maps*

$$\mathrm{Tr}_M : \mathrm{Hom}_{\mathrm{HMF}}(M, M) \to \mathbb{Q},$$

indexed by objects of HMF_w*, such that for any* $M, N \in \mathrm{Ob}(\mathrm{HMF}_w)$ *the composition*

$$\mathrm{Hom}_{\mathrm{HMF}}(M, N) \otimes \mathrm{Hom}_{\mathrm{HMF}}(N, M) \to \mathrm{Hom}_{\mathrm{HMF}}(M, M) \xrightarrow{\mathrm{Tr}_M} \mathbb{Q}$$

is a nondegenerate bilinear pairing.

Proof. See Theorem 7.7.5, Proposition 10.1.5, Example 10.1.6, and Corollary 10.3.3 in [B]. ■

Although Theorem 1 is not used explicitly in this paper, the Frobenius structure of the categories HMF_w is implicit in several of our constructions and proofs. We expect that Theorem 1 will become indispensable in further investigations of the interplay between matrix factorizations and link homology.

4. Factorizations as functors

Internal tensor product. Let $M \in \mathrm{MF}_w^{\mathrm{all}}$ and $N \in \mathrm{MF}_{-w}^{\mathrm{all}}$. The tensor product $M \otimes_R N$ is a 2-complex

$$(M \otimes_R N)^0 \xrightarrow{d} (M \otimes_R N)^1 \xrightarrow{d} (M \otimes_R N)^0,$$

where

$$(M \otimes_R N)^j = \bigoplus_{i \in \{0,1\}} M^i \otimes_R N^{j-i}$$

and

$$d(m \otimes n) = d_M(m) \otimes n + (-1)^i m \otimes d_N(n), \qquad m \in M^i.$$

PROPOSITION 11. *$M \otimes_R N$ has finite-dimensional cohomology if M and N are factorizations with finite-dimensional cohomology.*

Proof. This is a special case of Proposition 13 below. ∎

PROPOSITION 12. *If M has finite rank, there is a natural isomorphism of 2-complexes*

$$\mathrm{Hom}_R(M, N) \cong N \otimes_R M_\bullet. \tag{8}$$

Proof. The usual isomorphism of R-modules on the left and right hand sides of (8) intertwines the differentials in these 2-complexes. ∎

COROLLARY 6. *If M has finite-dimensional cohomology, there is a natural isomorphism of cohomology groups*

$$\mathrm{Ext}(M, N) \cong \mathrm{H}(N \otimes_R M_\bullet), \tag{9}$$

and (in the homotopy category) of 2-complexes

$$\mathrm{Hom}_R(M, N) \cong N \otimes_R M_\bullet. \tag{10}$$

The internal tensor product $M \otimes N$ will be depicted by gluing the diagram's arcs (see Figure 13 right).

Fig. 13. External and internal tensor products

Tensor product for sums of potentials. To add dynamics to the world of matrix factorizations we need a large supply of functors between categories of factorizations over various isolated singularities w. The tensor product (over $\mathbb{Q}[[y]]$) with a matrix factorization $M \in \mathrm{MF}_{w_1(x)-w_2(y)}$ is a functor from $\mathrm{MF}^{\mathrm{all}}_{w_2(y)}$ to $\mathrm{MF}^{\mathrm{all}}_{w_1(x)}$. A slightly more general construction requires three potentials w_1, w_2, w_3 and two factorizations $M \in \mathrm{MF}^{\mathrm{all}}_{w_1(x)-w_2(y)}$, $N \in \mathrm{MF}^{\mathrm{all}}_{w_2(y)-w_3(z)}$. We define their tensor product $M \otimes_y N$ by

$$(M \otimes_y N)^i = \bigoplus_{j \in \{0,1\}} (M^j \otimes_y N^{i-j}) \tag{11}$$

and

$$d(m \otimes n) = d_M(m) \otimes n + (-1)^i m \otimes d_N(n) \quad \text{if } m \in M^i.$$

Here $M^j \otimes_y N^{i-j}$ denotes the completed tensor product $M^j \widehat{\otimes}_{Q[[y]]} N^{i-j}$ (in the sense that the ring $\mathbb{Q}[[x,y,z]]$ of power series in the variables x, y, z is a completion of $\mathbb{Q}[[x,y]] \otimes_{\mathbb{Q}[[y]]} \mathbb{Q}[[y,z]]$).

If M or N is contractible, so is their tensor product. The tensor product can be viewed as a bifunctor

$$\mathrm{MF}^{\mathrm{all}}_{w_1(x)-w_2(y)} \times \mathrm{MF}^{\mathrm{all}}_{w_2(y)-w_3(z)} \to \mathrm{MF}^{\mathrm{all}}_{w_1(x)-w_3(z)},$$
$$\mathrm{HMF}^{\mathrm{all}}_{w_1(x)-w_2(y)} \times \mathrm{HMF}^{\mathrm{all}}_{w_2(y)-w_3(z)} \to \mathrm{HMF}^{\mathrm{all}}_{w_1(x)-w_3(z)}.$$

The tensor product does not preserve the finite rank property (since $\mathbb{Q}[[x,y,z]]$ has infinite rank as a $\mathbb{Q}[[x,z]]$-module if the set of variables y is nonempty). However, we have

PROPOSITION 13. *If M and N have finite-dimensional cohomology, so does their tensor product.*

This proposition can be restated by saying that tensor product restricts to bifunctors

$$\mathrm{MF}_{w_1(x)-w_2(y)} \times \mathrm{MF}_{w_2(y)-w_3(z)} \to \mathrm{MF}_{w_1(x)-w_3(z)}, \tag{12}$$
$$\mathrm{HMF}_{w_1(x)-w_2(y)} \times \mathrm{HMF}_{w_2(y)-w_3(z)} \to \mathrm{HMF}_{w_1(x)-w_3(z)}. \tag{13}$$

Proof. It suffices to show that $T = M \otimes_y N$ has finite-dimensional cohomology if M and N have finite rank. This cohomology $H(T)$ is a module over the Jacobian algebra $R_{f_2(y)}$, where $R = \mathbb{Q}[[y]]$, and a subquotient of the finitely generated free $\mathbb{Q}[[y]]$-module $T/(x,z)T$. Thus, $H(T)$ is a finitely generated $R_{f_2(y)}$-module, and necessarily has finite dimension. ∎

The tensor product of factorizations M and N will be depicted by joining the matching ends of their diagrams and placing a mark at the joint, as Figure 14 illustrates. Sometimes we will write M as M^x_y and N as N^y_z, and their tensor product as $M^x_y N^y_z$.

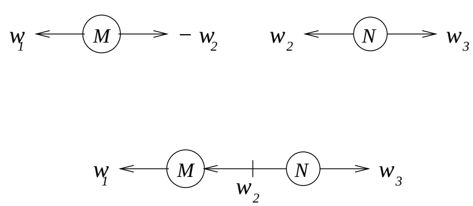

Fig. 14. Graphical presentation of the tensor product over $\mathbb{Q}[[y]]$

If the summands $w_1(x)$ and $w_3(z)$ of the potentials for M and N are themselves sums of potentials, and we want to emphasize these decompositions, we will denote the tensor product $M \otimes_y N$ as in Figure 15.

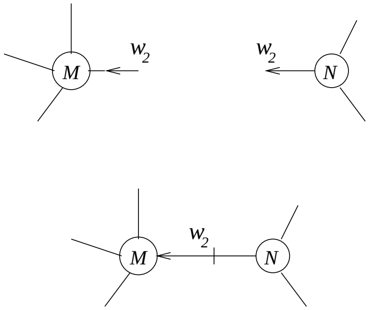

Fig. 15. Tensor product over $\mathbb{Q}[[y]]$ when w_1 and w_3 are sums

REMARK. Matrix factorizations over exterior sums of potentials are studied in [EP], [HP], [P], and [Y2].

If a factorization M has potential $w(x) - w(y) + w_1(z)$, then the quotient $M/(x-y)M$ is a module over the ring $\mathbb{Q}[[x, y, z]]/(x-y)$. We treat $M/(x-y)M$ as a factorization, necessarily of infinite rank, with potential $w_1(z)$ over the ring $\mathbb{Q}[[z]]$ (were we to consider $M/(x-y)M$ over the larger ring $\mathbb{Q}[[x, z]]$, the potential would have been degenerate).

PROPOSITION 14. *If a factorization M with potential $w(x)-w(y)+w_1(z)$ has finite-dimensional cohomology, so does $M/(x-y)M$.*

Proof. Similar to the one of Proposition 13. ∎

We depict the quotient factorization by joining the x and y legs of M, and placing a mark where the legs were joined (see Figure 16).

Fig. 16. M and its quotient $M/(x-y)M$

A closed diagram of factorizations (as in Figure 17) gives rise to a tensor product, the latter a two-periodic complex of modules over a suitable power series ring. Each oriented edge has a finite set of variables and a potential assigned to it, $d^2 = 0$ since the potentials cancel. If each factorization in the diagram has finite-dimensional cohomology, the complex will have finite-dimensional cohomology as well. The network does not even have to be planar, and does not need to be embedded anywhere. In our paper, however, all such diagrams are going to be planar.

External tensor product. When the intermediate set of variables is empty, the (completed) tensor product is over $\mathbb{Q}$, and we call it the *external tensor product* $M \otimes_{\mathbb{Q}} N$. This operation was investigated by Yoshino [Y2].

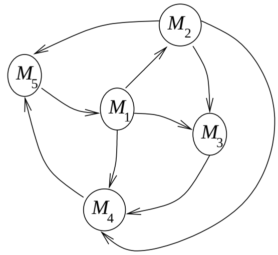

Fig. 17. A closed diagram of factorizations; $d^2 = 0$ in the tensor product, marks on the edges are omitted.

The disjoint union of two diagrams denotes the external tensor product of corresponding factorizations (see Figure 13 left).

Hiding minus signs in tensor products. Suppose I is a finite set and M_a, $a \in I$, is a collection of factorizations (possibly with degenerate potentials). To define the differential in the tensor product $\bigotimes_{a\in I} M_a$ in a manifestly intrinsic way, consider the Clifford ring $Cl(I)$ of the set I. It has generators $a \in I$ and relations

$$a^2 = 1, \qquad ab + ba = 0, \qquad a \neq b.$$

As an abelian group, $Cl(I)$ has rank $2^{|I|}$, where $|I|$ is the cardinality of I, and breaks down into a direct sum

$$Cl(I) = \bigoplus_{J\subset I} \mathbb{Z}_J.$$

$\mathbb{Z}_J$ has as generators all ways to order elements of the set J, and relations

$$a \dots bc \dots e + a \dots cb \dots e = 0$$

for all orderings $a \dots bc \dots e$ of J. The group $\mathbb{Z}_J$ is isomorphic to $\mathbb{Z}$, but there is no canonical isomorphism.

$r_a^2 = 1$ where r_a is the right multiplication by a endomorphism of $Cl(I)$. For each $J \subset I$ which does not contain a we have a 2-periodic sequence

$$\mathbb{Z}_J \xrightarrow{r_a} \mathbb{Z}_{J\sqcup\{a\}} \xrightarrow{r_a} \mathbb{Z}_J.$$

Now define the tensor product factorization of M_a's as the sum, over all $J \subset I$, of

$$(\bigotimes_{a\in J} M_a^1) \otimes (\bigotimes_{b\in I\setminus J} M_b^0) \otimes_{\mathbb{Z}} \mathbb{Z}_J,$$

with the differential

$$d = \sum_{a\in I} d_a \otimes r_a$$

where d_a is the differential in M_a. We denote this tensor product by $\bigotimes_{a\in I} M_a$.

When forming tensor products of factorizations, we will use this trick, and would need to assign labels to each term in a tensor product. If a label

a is assigned to a factorization M, we write M as

$$M^0(\emptyset) \to M^1(a) \to M^0(\emptyset).$$

Factorization of the identity functor. Start with the one-variable potential $w(x) = x^{n+1}$ and consider the potential $w(x) - w(y)$ in two variables. Let

$$\pi_{xy} = \frac{w(x) - w(y)}{x - y} = x^n + x^{n-1}y + \cdots + y^n.$$

Denote by L^x_y the factorization

$$R \xrightarrow{\pi_{xy}} R \xrightarrow{x-y} R,$$

where $R = \mathbb{Q}[[x, y]]$. We have

$$\mathrm{Hom}_{\mathrm{HMF}}(L^x_y, L^x_y) \cong \mathbb{Q}[x]/(x^n), \tag{14}$$

$$\mathrm{Hom}_{\mathrm{HMF}}(L^x_y, L^x_y\langle 1\rangle) \cong 0. \tag{15}$$

Indeed, $\mathrm{Hom}_R(L^x_y, L^x_y)$ is isomorphic to the Koszul complex of the sequence $(x - y, \pi_{xy})$, with the grading collapsed from $\mathbb{Z}$ to $\mathbb{Z}_2$. Regularity of this sequence implies the formulas (of course, it is easy to check the above two formulas directly; for instance, the second formula follows since $x - y$ and π_{xy} are relatively prime). If we assign label (a) to L^x_y, we can write this factorization as

$$R(\emptyset) \xrightarrow{\pi_{xy}} R(a) \xrightarrow{x-y} R(\emptyset).$$

Here $R(\emptyset), R(a)$ are free R-modules of rank 1 with basis vectors $1(\emptyset), 1(a)$. The differential takes $1(\emptyset)$ to $\pi_{xy} \cdot 1(a)$, and $1(a)$ to $(x - y) \cdot 1(\emptyset)$.

We depict L^x_y by an arc oriented from y to x (see Figure 18). If the potential assigned to an endpoint of a diagram has the form x^{n+1} for some variable x, we just write the variable at the endpoint, rather than the potential.

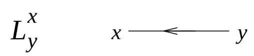

Fig. 18. Arc with endpoints x and y

Proposition 15. *There is a natural isomorphism*

$$L^x_y M^y_z \cong M^x_z,$$

where $w_1(z)$ is any potential in variables z, and M^y_z any factorization over $y^{n+1} - w_1(z)$.

(Notation $L^x_y M^y_z$ was explained several pages earlier.)

Proof. Assign labels a, b, c to factorizations L^x_y, M^y_z, and M^x_z, respectively. The map of $w(x) - w_1(z)$ factorizations

$$\tau_1 : L^x_y \otimes_y M^y_z \to M^x_z$$

defined by taking $R(a) \otimes_y M_z^y$ to 0 and $R(\emptyset) \otimes_y M_z^y \cong \mathbb{Q}[[x, y]] \otimes_y M_z^y$ onto M_z^x by adding the relation $x = y$ induces an isomorphism on cohomologies of the two factorizations. ∎

This isomorphism is functorial in M and implies

Corollary 7. *Tensor product with L_y^x is an invertible functor from the homotopy category of matrix factorizations with potential $y^{n+1} - w_1(z)$ to the homotopy category of matrix factorizations with potential $x^{n+1} - w_1(z)$, for any potential $w_1(z)$, and is isomorphic to the substitution functor that relabels y as x.* ∎

We could say informally that the tensor product with L_y^x is the identity functor. We depict $L_y^x M_z^y$ by gluing their y-endpoints and placing a mark at the gluing point (see Figure 19 top). The proposition can be interpreted graphically as an isomorphism in Figure 19.

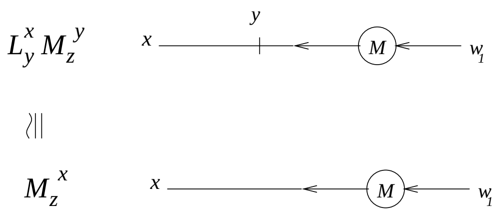

Fig. 19. Removing a mark

Proposition 16. *The diagram in Figure 20 is commutative.*

Proof. In other words, for two marks on different arcs, the order in which they are removed does not matter. The proof is a simple computation with the maps τ_1. ∎

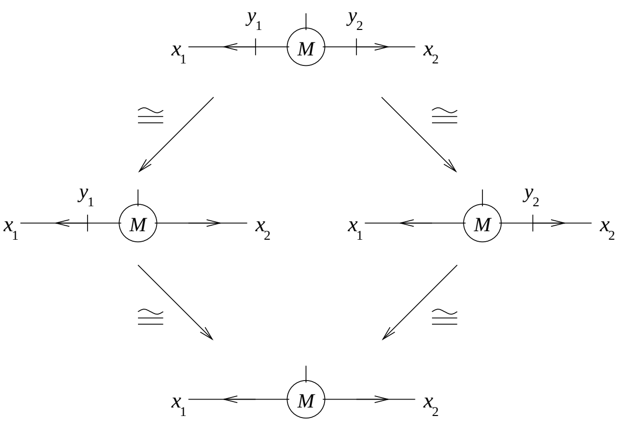

Fig. 20. Commutative diagram of mark removals

PROPOSITION 17. *There is a natural isomorphism*

$$M^z_x L^x_y \cong M^z_y,$$

where $w_1(z)$ is any potential in the variables z, and M^x_z any factorization over $w_1(z) - x^{n+1}$.

Proof. Assign labels a, b, c to factorizations M^z_x, L^x_y, and M^y_z, respectively. The map of $w_1(z) - y^{n+1}$ factorizations

$$\tau_2 : M^z_x \otimes_x L^x_y \to M^z_y$$

defined by taking $M^z_x \otimes_y R(b)$ to 0 and $M^z_x \otimes_x R(\emptyset) \cong M^z_x \otimes_x \mathbb{Q}[[x, y]]$ onto M^z_y by adding the relation $x = y$ induces an isomorphism on cohomologies of the two factorizations. ∎

The proposition can be interpreted graphically as an isomorphism in Figure 21.

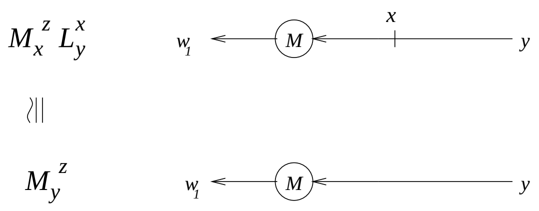

Fig. 21. Removing a mark

The isomorphism is functorial in M and implies

COROLLARY 8. *Tensor product with L^x_y is an invertible functor from the homotopy category of matrix factorizations with potential $w_1(z) - x^{n+1}$ to the homotopy category of matrix factorizations with potential $w_1(z) - y^{n+1}$, for any potential $w_1(z)$, and is isomorphic to the substitution functor that relabels x as y.* ∎

Commutative diagram 20 admits three other versions, with reversed orientation in one or two of the x_1, x_2 legs of M and mark removal using the morphism τ_2 instead of τ_1. Each of these three diagrams is commutative.

In Figure 22 we have two marks on an arc connecting the factorizations N and M. We could remove the mark labeled y using τ_1, or we could remove the other mark (via τ_2), and then relabel y as x.

PROPOSITION 18. *The two resulting morphisms from the top to the bottom left factorizations in Figure 22 are equal.*

Proof. Direct computation. ∎

There are other versions of Figure 22, with the orientation of the marked arc reversed, and with N and M being just one factorization (so that the

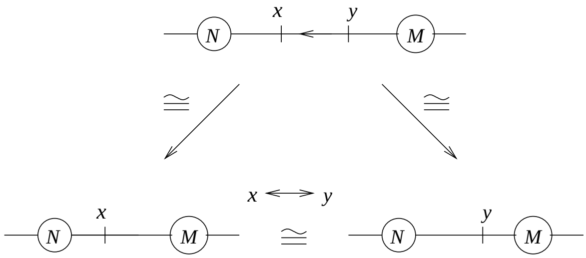

Fig. 22. Yet another commutative diagram

arc starts and ends in the same factorization). Commutativity holds in all of these cases.

We now specialize to the case when M is itself an arc. The tensor product $L^x_y L^y_z$ is the factorization

$$(16)\qquad \begin{pmatrix} R(\emptyset) \\ R(ab) \end{pmatrix} \xrightarrow{\begin{pmatrix} \pi_{xy} & y-z \\ \pi_{yz} & y-x \end{pmatrix}} \begin{pmatrix} R(a) \\ R(b) \end{pmatrix} \xrightarrow{\begin{pmatrix} x-y & y-z \\ \pi_{yz} & -\pi_{xy} \end{pmatrix}} \begin{pmatrix} R(\emptyset) \\ R(ab) \end{pmatrix}$$

where $R = \mathbb{Q}[[x, y, z]]$, and we assigned labels a, b to L^x_y, L^y_z, respectively. The minus sign in front of π_{xy} in the right 2×2 matrix comes from the relation $ba = -ab$.

L^x_z is given by

$$R'(\emptyset) \xrightarrow{\pi_{xz}} R'(c) \xrightarrow{x-z} R'(\emptyset)$$

where $R' = \mathbb{Q}[[x, z]]$ (and notice label c).

We can specialize maps τ_1, τ_2, introduced earlier, to this case.

The map $\tau_1 : L^x_y L^y_z \to L^x_z$ is given by the pair of matrices

$$((\phi_{y\to x}, 0), (0, \phi_{y\to x})),$$

where $\phi_{y\to x} : R \to R'$ is the algebra homomorphism that takes x to x, z to z, and y to x.

The map $\tau_2 : L^x_y L^y_z \to L^x_z$ is given by the pair of matrices

$$((\phi_{y\to z}, 0), (0, \phi_{y\to z})),$$

where $\phi_{y\to z} : R \to R'$ is the algebra homomorphism that takes x to x, and y, z to z.

Proposition 19. *The maps $\tau_1, \tau_2 : L^x_y L^y_z \to L^x_z$ are homotopic.*

Proof. Straightforward. ■

From now on, isomorphism means isomorphism in the homotopy category of factorizations, unless specified otherwise. Since τ_1, τ_2 are homotopic, they describe the same morphism in the homotopy category, denoted τ_y.

Let $\tau'_y : L^x_z \to L^x_y L^y_z$ be given by the pair of matrices

$$\tau'_y = \left(\begin{pmatrix} 1 \\ -e_{xyz} \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right), \quad \text{where} \quad e_{xyz} = \sum_{i+j+k=n-1} x^i y^j z^k.$$

PROPOSITION 20. *τ_y and τ'_y are mutually inverse isomorphisms in the homotopy category of factorizations with potential $w(x) - w(z)$.*

Proof. It is clear that τ'_y is a homomorphism of factorizations, and $\tau_1 \tau'_y$ is the identity endomorphism of $L^x_z(c)$. The proposition follows, since $\tau_y = \tau_1$ is an isomorphism of factorizations. ∎

PROPOSITION 21. *τ is associative: there is an equality*

$$\tau_z(\tau_y \otimes \mathrm{Id}) = \tau_y(\mathrm{Id} \otimes \tau_z)$$

of maps $L^x_y L^y_z L^z_w \to L^x_w$.

Proof. Direct computation. ∎

COROLLARY 9. *τ' is associative.*

COROLLARY 10. *For any m and k there is a canonical isomorphism of factorizations*

$$L^x_{z_1} L^{z_1}_{z_2} \dots L^{z_{m-1}}_{z_m} L^{z_m}_y \cong L^x_{v_1} L^{v_1}_{v_2} \dots L^{v_{k-1}}_{v_k} L^{v_k}_y.$$

These isomorphisms are consistent.

$L^x_y L^y_z$ is depicted by two arcs glued together along matching endpoints, with a mark at the gluing point. The morphism τ_y corresponds to removing a mark, and τ'_y to adding a mark (see Figure 23).

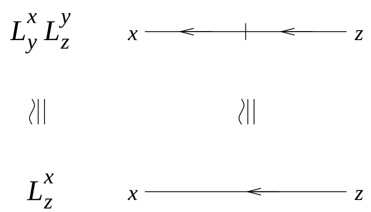

Fig. 23. Adding or removing a mark

Proposition 20 says that removing a marked point on an arc does not change the isomorphism class of a factorization, Proposition 21 says that arc removal is associative (Figure 24), while Corollary 10 asserts that two arcs, each with an arbitrary number of marks, are canonically isomorphic (Figure 25).

Denote by L^x_x the quotient of L^x_y by the relation $y = x$. Then L^x_x is a 2-complex of $\mathbb{Q}[[x]]$-modules

$$\mathbb{Q}[[x]] \xrightarrow{\pi_{xx}} \mathbb{Q}[[x]] \xrightarrow{0} \mathbb{Q}[[x]],$$

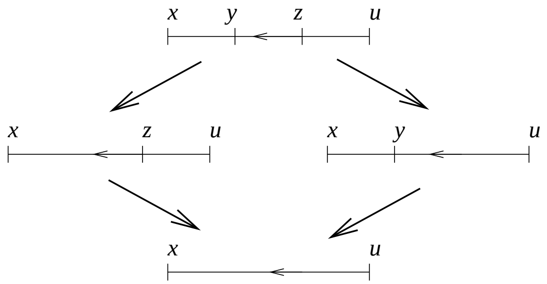

Fig. 24. Associativity of mark removal

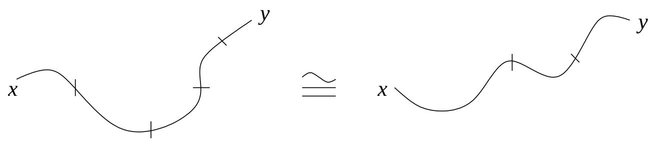

Fig. 25. Canonical isomorphism of marked arcs

and has cohomology only in degree 1. We depict L_x^x by an oriented circle with one mark x.

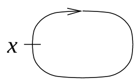

Fig. 26. 2-complex L_x^x

Let

$$A := \mathrm{H}^*(\mathbb{CP}^{n-1}, \mathbb{Q}) = \mathbb{Q}[X]/(X^n),$$

and $\iota : \mathbb{Q} \to A$ be the unit map, $\iota(1) = 1$. Choose a nonzero rational number ζ, and define $\varepsilon : A \to \mathbb{Q}$ as the trace map

$$\varepsilon(X^{n-1}) = \zeta, \quad \varepsilon(X^i) = 0 \quad \text{if } i \neq n-1.$$

We identify the Milnor ring $R_{w(x)} \cong \mathbb{Q}[x]/(x^n)$ with A by taking $x^i \in R_w$ to $X^i \in A$. To an oriented circle without marks we associate the 2-periodic complex $0 \to A \to 0$, denoted $A\langle 1\rangle$.

We fix an isomorphism $\nu_x : A\langle 1\rangle \cong L_x^x$ of 2-periodic complexes of vector spaces, up to homotopies, by taking $X^i \in A$ to $x^i \in \mathbb{Q}[[x]] \cong (L_x^x)^1$ for $0 \leq i \leq n-1$. Graphically, this isomorphism means adding a mark to a circle without marks (see Figure 27).

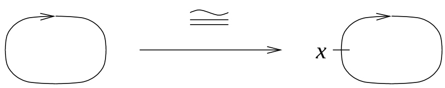

Fig. 27. Adding a mark to a circle

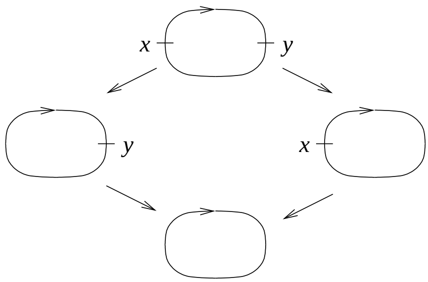

Fig. 28. Commutativity of mark removal

It is easy to see that the maps

$$L_y^x L_x^y \xrightarrow{\tau_y} L_x^x \xrightarrow{\nu_x^{-1}} A\langle 1\rangle \quad \text{and} \quad L_y^x L_x^y \xrightarrow{\tau_x} L_y^y \xrightarrow{\nu_y^{-1}} A\langle 1\rangle$$

are homotopic. This implies consistency between two ways to remove two marks from a circle (Figure 28).

Combining with the associativity property for mark removal and addition, we conclude that the factorizations assigned to two circles with arbitrary number of marks are canonically isomorphic in the homotopy category of 2-complexes of $\mathbb{Q}$-vector spaces (Figure 29).

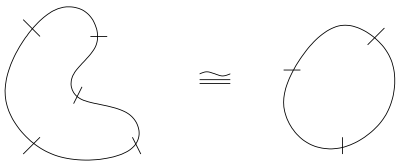

Fig. 29. Canonical isomorphism of 2-periodic complexes assigned to marked circles

Remark. This canonical isomorphism allows us to view ι and ε as maps between $\mathbb{Q}$ and the 2-complex assigned to a circle with an arbitrary number of marks.

Suppose we have a collection of factorizations M_i, $i \in I$, each with potential w_i which is a signed sum of x_j^{n+1} for j in a subset of I. Let us tensor M_i's and a number of arc factorizations L together in some way to produce a network of factorizations (as in Figure 30 top left). Each arc in the network has potential x^{n+1}. We divide arcs into internal and external arcs. External arcs are those with at least one loose endpoint. The resulting tensor product $Z_1 = \bigotimes_i M_i$ is a factorization with potential which is a signed sum of x_j^{n+1}, over all loose endpoints j. The sign is determined by the orientation of the network near j.

Suppose now we tensor M_i's and several arc factorizations together so as to produce the same network but, possibly, with different marks (as in Figure 30 top right). Denote this tensor product by Z_2. Our results imply

PROPOSITION 22. *The factorizations Z_1 and Z_2 are canonically isomorphic (in the homotopy category). The isomorphism is natural in M_i's, in the following sense. If $f : M_r \to N_r$ is a homomorphism of factorizations, for some r, and Z'_1 (respectively, Z'_2) is obtained from Z_1 (respectively, Z_2) by substituting N_r for M_r in the tensor product, then the diagram*

$$\begin{array}{ccc} Z_1 & \xrightarrow{\cong} & Z_2 \\ {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow \\ Z'_1 & \xrightarrow{\cong} & Z'_2 \end{array}$$

commutes (see Figure 30).

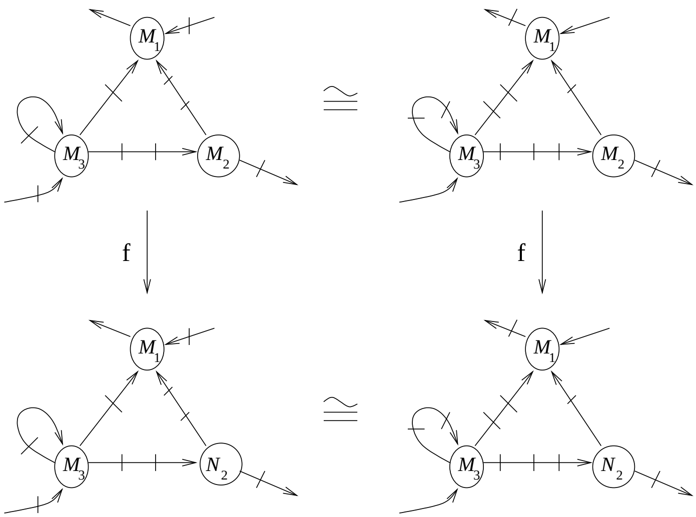

Fig. 30. Canonical isomorphism of networks; a map $f : M_2 \to N_2$ induces a homomorphism of factorizations assigned to whole networks; the diagram is commutative.

Realization of the identity functor (multi-variable case). Given a multi-variable potential $w = w(x_1, \ldots, x_k)$, we can write

$$w(x) - w(y) = \sum_{i=1}^{k} w_i(x_i - y_i)$$

where

$$w_i = \frac{w(y_1, \ldots, y_{i-1}, x_i, \ldots, x_k) - w(y_1, \ldots, y_i, x_{i+1}, \ldots, x_k)}{x_i - y_i}$$

are polynomials in x's and y's. Let L be the tensor product (over i from 1 to k) of factorizations

$$R \xrightarrow{w_i} R \xrightarrow{x_i - y_i} R$$

where $R = \mathbb{Q}[[x, y]]$.

PROPOSITION 23. *For any potential $w_1(z)$ and any factorization M with potential $w(y) - w_1(z)$ the tensor product $L \otimes_y M$ is isomorphic in the homotopy category of factorizations to the factorization M with y's relabeled as x's. The isomorphism is functorial in M.*

Proof. Apply Proposition 9 repeatedly to exclude $y_1, \ldots, y_k$. ∎

Proposition 23 says, essentially, that tensoring with L is the identity functor. Also, we have

$$\mathrm{Hom}_{\mathrm{HMF}}(L, L) \cong \mathbb{Q}[[x]]_{w(x)}, \qquad \mathrm{Ext}^1_{\mathrm{HMF}}(L, L) \cong 0,$$

i.e., the endomorphism ring of L is the Jacobian algebra of w. Note that Theorem 1 applied to L implies Proposition 4.

The results of the previous subsection and of Section 9 can be generalized from x^{n+1} to arbitrary multi-variable potentials, but we postpone this analysis to a later paper.

5. Homogeneous potentials and graded factorizations

Homogeneous potentials and graded factorizations. Suppose that each variable x_i is given a positive integer degree p_i and the potential w is homogeneous of degree p. Then w belongs to the Jacobian ideal,

$$w = \frac{1}{p} \sum_i p_i x_i \frac{\partial f}{\partial x_i}$$

(Euler's formula), so that the Milnor and Tyurina algebras are isomorphic.

The ring of power series with coefficients in a field is local. After switching from rings to graded rings, the role of the power series rings is played by polynomial rings whose generators are in positive degrees only, since these rings have only one maximal homogeneous ideal. From now on in this paper we work with homogeneous potentials and switch from the power series ring to the ring of polynomials. From here on R is a polynomial ring.

A graded (or homogeneous) factorization with a homogeneous potential w of degree $2(n+1)$ consists of free graded R-modules M^0, M^1 and degree $n+1$ homomorphisms d^0, d^1 such that $d^1 d^0 = w$ (this implies $d^0 d^1 = w$),

$$M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^0.$$

A homomorphism of graded factorizations is required to have degree 0, while a homotopy should have degree $-n-1$. Each of the six categories of factorizations described in Section 3 has a graded version, denoted in lowercase letters. For instance, hmf_w is the homotopy category of graded factorizations of w with finite-dimensional cohomology. This category is triangulated.

We denote by $\{m\}$ the grading shift up by m. The factorization $M\{m\}$ has the form

$$M^0\{m\} \xrightarrow{d^0} M^1\{m\} \xrightarrow{d^1} M^0\{m\}.$$

The cohomological shift functor $\langle 1\rangle$ does not change the grading of M^0, M^1, and commutes with the grading shift functor $\{m\}$.

All results and constructions of Sections 2, 3 and 4 easily extend to graded factorizations. There is no need to complete tensor products of factorizations in the graded case. The factorization $\{a, b\}$ is defined for homogeneous a, b with $\deg(a) + \deg(b) = 2(n + 1)$, and has the form

$$R \xrightarrow{a} R\{n + 1 - \deg(a)\} \xrightarrow{b} R.$$

We shifted the degree of the middle R so that the differentials would have degree $n + 1$. Then

$$\{a, b\}\langle 1\rangle = \{b, a\}\{\deg(b) - n - 1\}.$$

Notice two different uses of curly brackets: to denote Koszul factorizations and shifts.

When extending the subsection 'Realization of the identity functor (one-variable case)' of Section 4 to the graded case, we give the variables x, y, z degree 2. The factorization L^x_y has the form

$$R \xrightarrow{\pi_{xy}} R\{1 - n\} \xrightarrow{x-y} R.$$

The isomorphism in Proposition 15 has degree 0, assuming M is a graded factorization.

The cohomology $H(M)$ of a homogeneous factorization is a $\mathbb{Z}_2\oplus\mathbb{Z}$-graded $\mathbb{Q}$-vector space,

$$H(M) = \bigoplus_{i\in\{0,1\},\, j\in\mathbb{Z}} H^{i,j}(M).$$

We define the graded dimension of M as

$$\mathrm{gdim}(M) := \sum_{j\in\mathbb{Z},\, i\in\{0,1\}} \dim H^{i,j}(M)\, q^j s^i.$$

Then, for external tensor product $M \otimes N$,

$$\mathrm{gdim}(M \otimes N) = \mathrm{gdim}(M)\,\mathrm{gdim}(N),$$

assuming the relation $s^2 = 1$. The dimension of M is the $q = s = 1$ specialization of the graded dimension.

The algebra A (cohomology of $\mathbb{CP}^{n-1}$), defined in the previous section, is naturally graded. We shift the grading of A down by $n-1$ so that $\deg(1) = 1 - n$, and, more generally, $\deg(X^i) = 2i + 1 - n$. The unit map $\iota : \mathbb{Q} \to A$ and the trace map $\varepsilon : A \to \mathbb{Q}$ have degree $1 - n$, while the multiplication in A has degree $n-1$. To a circle without marks we assign the 2-complex $A\langle 1\rangle$ of graded vector spaces.

Let M be a homogeneous factorization of finite rank. If the endomorphism ring $\mathrm{Hom}_{\mathrm{mf}}(M,M)$ contains a degree 0 idempotent e, we can decompose M into a direct sum of factorizations, $M = eM \oplus (1-e)M$. Suppose instead that the quotient ring $\mathrm{Hom}_{\mathrm{hmf}}(M,M)$ contains an idempotent e. Let I be the kernel of the quotient map

$$\mathrm{Hom}_{\mathrm{mf}}(M,M) \xrightarrow{f} \mathrm{Hom}_{\mathrm{hmf}}(M,M),$$

and J the ideal in $\mathrm{Hom}_{\mathrm{mf}}(M,M)$ of endomorphisms that induce the zero map on cohomology. Note that $I \subset J$, and J is a nilpotent ideal in $\mathrm{Hom}_{\mathrm{mf}}(M,M)$, since a degree 0 endomorphism cannot have coefficients of arbitrary large degrees. Thus, $J^N = 0$ for some N, and therefore, $I^N = 0$. Nilpotent ideals have the lifting idempotents property (exercise, or see [Be, Theorem 1.7.3]). There exists an idempotent $\widetilde{e} \in \mathrm{Hom}_{\mathrm{mf}}(M,M)$ that lifts e, that is, $f(\widetilde{e}) = e$. It allows us to decompose $M = \widetilde{e}M \oplus (1-\widetilde{e})M$. We state this as

PROPOSITION 24. *The category* hmf_w *has the splitting idempotents property.*

REMARK. The category HMF_w has this property as well, which can be seen by a slight modification of the above argument, using the fact that $\bigcap_N J^N = 0$ and lifting e recursively.

An additive category is called *Krull–Schmidt* if any object has the unique decomposition property. In other words, if $M \cong \bigoplus_{i\in I} M_i$ and $M \cong \bigoplus_{j\in J} N_j$, for some sets I, J and indecomposables M_i, N_j, then there is a bijection $z : I \to J$ such that $M_i \cong N_{z(i)}$.

PROPOSITION 25. *The category* hmf_w *is Krull–Schmidt.*

Proof. Any object of hmf_w is isomorphic to a finite rank factorization. This and Proposition 24 imply that the endomorphism ring of any indecomposable in hmf_w is local. Proposition 25 follows. ∎

For graded factorizations M, N with finite-dimensional cohomology there is an isomorphism of graded R-modules

$$\mathrm{Hom}_{\mathrm{HMF}}(M,N) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{hmf}}(M\{i\}, N).$$

Given a graded cyclic Koszul factorization $\{\mathbf{a},\mathbf{b}\}$, its graded dual is

$$\{\mathbf{a},\mathbf{b}\}_\bullet \cong \{-\mathbf{b},\mathbf{a}\} \cong \{\mathbf{a},-\mathbf{b}\}\langle m\rangle\Big\{\sum_i \deg(a_i) - m(n+1)\Big\}, \tag{17}$$

where the sequences $\mathbf{a}$ and $\mathbf{b}$ have length m.

Quasi-homogeneous potentials. A potential that is homogeneous with respect to some basis in R is called *quasi-homogeneous.* A quasi-homogeneous potential lies in its Jacobian ideal. From the work of K. Saito [Sa] we know that the converse is true: only quasi-homogeneous potentials lie in their Jacobian ideal (so that the Milnor and Tyurina algebras are isomorphic). To a mathematician, quasi-homogeneous singularities are distinguished by the existence of the associated Frobenius manifold [BV], [D], and other special properties [AGV], [Di], [Ku]. From a string theorist's viewpoint, each quasi-homogeneous singularity gives rise to the rich structure of a (super) conformal 2D field theory [VW], [M], while an arbitrary isolated singularity only produces a 2D topological field theory, the latter equivalent to a commutative Frobenius algebra (a 0-dimensional Gorenstein ring).

In the rest of the paper we are dealing exclusively with homogeneous potentials.

6. Planar graphs and factorizations

Factorization from a planar graph. We consider graphs of a particular kind embedded in a disc (for an example see Figure 7). A graph can have both unoriented and oriented edges, and oriented loops. Unoriented edges are called "wide" and depicted correspondingly. Any unoriented edge has two oriented edges entering it at one vertex, and two oriented edges leaving it at the other. Oriented edges might end on the boundary of the disc. Inside the disc we allow only trivalent vertices where a wide edge and a pair of oriented edges meet. An oriented edge is *internal* if none of its endpoints is on the boundary of the disc. Otherwise, the edge is called *external.* Any internal edge has one or more marks placed on it. We also treat boundary points as marks; this ensures that each external edge has a mark. Additional marks on external edges are allowed. An oriented loop may or may not have marks.

If Γ is such a graph, we denote by $m(\Gamma)$ the set of its marks and by $\partial\Gamma$ the set of boundary points, the latter a subset of $m(\Gamma)$. If $i \in \partial\Gamma$, the sign of i, denoted $s(i)$, is 1 if the edge at i is oriented outward, and -1 if the edge is oriented inward. For instance, boundary points marked $1, 2, 7$ in Figure 7 have sign 1, and points $4, 8, 9$ have sign -1.

Let $R = \mathbb{Q}[x_i : i \in m(\Gamma)]$ be the ring of polynomials in variables x_i, over all marks i, and R' be its subring $\mathbb{Q}[x_i : i \in \partial\Gamma]$. We introduce a grading on R and R' by giving each x_i degree 2.

Assign to Γ the potential

$$w(\Gamma) = \sum_{i \in \partial\Gamma} s(i) x_i^{n+1}.$$

To Γ we now associate a graded factorization $C(\Gamma)$ over the ring R' with potential $w(\Gamma)$. First, to a wide edge t bounded by marks $1,2,3,4$ as in Figure 5 we assign a factorization with potential

$$w_t = x_1^{n+1} + x_2^{n+1} - x_3^{n+1} - x_4^{n+1}.$$

Starting with formal variables x, y, we can write $x^{n+1}+y^{n+1}$ as a polynomial in $x+y$ and xy. Let g be this polynomial,

$$g(x+y, xy) = x^{n+1} + y^{n+1}.$$

Explicitly,

$$g(s_1, s_2) = s_1^{n+1} + (n+1) \sum_{1 \le i \le (n+1)/2} \frac{(-1)^i}{i} \binom{n-i}{i-1} s_2^i s_1^{n+1-2i}.$$

w_t can be written as

$$\begin{aligned} w_t &= g(x_1+x_2, x_1x_2) - g(x_3+x_4, x_3x_4) \\ &= g(x_1+x_2, x_1x_2) - g(x_3+x_4, x_1x_2) \\ &\quad + g(x_3+x_4, x_1x_2) - g(x_3+x_4, x_3x_4) \\ &= \frac{g(x_1+x_2, x_1x_2) - g(x_3+x_4, x_1x_2)}{x_1+x_2-x_3-x_4}(x_1+x_2-x_3-x_4) \\ &\quad + \frac{g(x_3+x_4, x_1x_2) - g(x_3+x_4, x_3x_4)}{x_1x_2-x_3x_4}(x_1x_2-x_3x_4). \end{aligned}$$

Let

$$u_1 = u_1(x_1,x_2,x_3,x_4) = \frac{g(x_1+x_2, x_1x_2) - g(x_3+x_4, x_1x_2)}{x_1+x_2-x_3-x_4}, \tag{18}$$

$$u_2 = u_2(x_1,x_2,x_3,x_4) = \frac{g(x_3+x_4, x_1x_2) - g(x_3+x_4, x_3x_4)}{x_1x_2-x_3x_4}. \tag{19}$$

Note that u_1 and u_2 are polynomials, and

$$w_t = u_1(x_1+x_2-x_3-x_4) + u_2(x_1x_2-x_3x_4).$$

To t we assign the graded factorization

$$C_t := \{(u_1, u_2), (x_1+x_2-x_3-x_4, x_1x_2-x_3x_4)\}\{-1\}.$$

In other words, C_t is the tensor product of graded factorizations

$$R_t \xrightarrow{u_1} R_t\{1-n\} \xrightarrow{x_1+x_2-x_3-x_4} R_t$$

and

$$R_t \xrightarrow{u_2} R_t\{3-n\} \xrightarrow{x_1x_2-x_3x_4} R_t,$$

with the grading shifted down by 1.

In general, a wide edge t will be bounded by marks i, j, k, l. Then C_t is defined as above, with $1,2,3,4$ converted into i, j, k, l.

If α is an arc in an oriented edge (or in an oriented loop) bounded by marks i and j and oriented from j to i, we denote by L^i_j the factorization

$$R_\alpha \xrightarrow{\pi_{ij}} R_\alpha \xrightarrow{x_i - x_j} R_\alpha$$

where $R_\alpha = \mathbb{Q}[x_i, x_j]$ and

$$\pi_{ij} = \frac{x_i^{n+1} - x_j^{n+1}}{x_i - x_j}.$$

This factorization was introduced earlier, in Section 4, as the factorization assigned to an arc.

To an oriented loop without marks we assign the 2-complex $A\langle 1\rangle$ (see Section 4).

Finally, we define $C(\Gamma)$ as the tensor product of C_t over all wide edges t, of L^i_j over all α, and of $A\langle 1\rangle$ over all markless loops in Γ. The tensor product is formed over suitable intermediate rings so that $C(\Gamma)$ is a free module of finite rank over R. For instance, to form $C(\Gamma)$ for Γ in Figure 7, we first tensor C_{t_1} with L^3_5 over the ring $\mathbb{Q}[x_3]$, and then tensor the result with C_{t_2} over $\mathbb{Q}[x_5]$. In conclusion, we tensor $C_{t_1} \otimes L^3_5 \otimes C_{t_2}$ with L^7_6 over $\mathbb{Q}[x_6]$.

Properties. Clearly, $C(\Gamma)$ is a factorization with potential $w(\Gamma)$. We treat it as a graded factorization over the ring R' of polynomials in boundary variables. $w(\Gamma)$ is a nondegenerate potential in this ring. If a graph has at least one internal mark, $C(\Gamma)$ has infinite rank as an R'-module.

PROPOSITION 26. *For any graph Γ, the factorization $C(\Gamma)$ lies in* $\mathrm{hmf}_{w(\Gamma)}$.

Proof. In other words, $C(\Gamma)$ has finite-dimensional cohomology. This follows at once from results of Section 3. ∎

Suppose that Γ' is obtained from Γ by placing a different collection of internal marks on oriented edges and loops of Γ. Then the two graphs have the same potential $w(\Gamma') = w(\Gamma)$ assigned to them, and factorization $C(\Gamma')$ belongs to the category $\mathrm{hmf}_{w(\Gamma)}$ since the graphs share the same set of boundary points.

PROPOSITION 27. *There is a canonical isomorphism in* $\mathrm{hmf}_{w(\Gamma)}$

$$C(\Gamma') \cong C(\Gamma).$$

Proof. This is a special case of Proposition 22. ∎

Maps χ_0 and χ_1. Consider the graphs Γ^0, Γ^1 depicted in Figure 8. The factorization $C(\Gamma^0)$ is the tensor product of the factorizations L^1_4 and L^2_3, and is given by

$$\begin{pmatrix} R(\emptyset) \\ R(ab)\{2-2n\} \end{pmatrix} \xrightarrow{P_0} \begin{pmatrix} R(a)\{1-n\} \\ R(b)\{1-n\} \end{pmatrix} \xrightarrow{P_1} \begin{pmatrix} R(\emptyset) \\ R(ab)\{2-2n\} \end{pmatrix}$$

where

$$P_0 = \begin{pmatrix} \pi_{14} & x_2 - x_3 \\ \pi_{23} & x_4 - x_1 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} x_1 - x_4 & x_2 - x_3 \\ \pi_{23} & -\pi_{14} \end{pmatrix},$$

$$\pi_{ij} = \sum_{k=0}^{n} x_i^k x_j^{n-k},$$

and we assigned labels a, b to L_4^1 and L_3^2. The factorization $C(\Gamma^1)$ is

$$\begin{pmatrix} R(\emptyset)\{-1\} \\ R(a'b')\{3-2n\} \end{pmatrix} \xrightarrow{Q_1} \begin{pmatrix} R(a')\{-n\} \\ R(b')\{2-n\} \end{pmatrix} \xrightarrow{Q_2} \begin{pmatrix} R(\emptyset)\{-1\} \\ R(a'b')\{3-2n\} \end{pmatrix},$$

with

$$Q_1 = \begin{pmatrix} u_1 & x_1x_2 - x_3x_4 \\ u_2 & x_3 + x_4 - x_1 - x_2 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} x_1 + x_2 - x_3 - x_4 & x_1x_2 - x_3x_4 \\ u_2 & -u_1 \end{pmatrix}.$$

We assigned labels a', b' to the two factorizations (with degenerate potentials) whose tensor product is $C(\Gamma^1)$.

A map between $C(\Gamma^0)$ and $C(\Gamma^1)$ can be described by a pair of 2×2 matrices (U_0, U_1). Let $\chi_0 : C(\Gamma^0) \to C(\Gamma^1)$ be given by the pair

$$U_0 = \begin{pmatrix} x_4 - x_2 + \mu(x_1 + x_2 - x_3 - x_4) & 0 \\ a_1 & 1 \end{pmatrix},$$

$$U_1 = \begin{pmatrix} x_4 + \mu(x_1 - x_4) & \mu(x_2 - x_3) - x_2 \\ -1 & 1 \end{pmatrix},$$

where

$$a_1 = (\mu - 1)u_2 + \frac{u_1 + x_1u_2 - \pi_{23}}{x_1 - x_4}$$

and $\mu \in \mathbb{Z}$. Different choices of μ give homotopic maps. The map χ_0 has degree 1.

Let $\chi_1 : C(\Gamma^1) \to C(\Gamma^0)$ be given by the pair of matrices (V_0, V_1):

$$V_0 = \begin{pmatrix} 1 & 0 \\ a_2 & a_3 \end{pmatrix}, \qquad V_1 = \begin{pmatrix} 1 & x_3 + \lambda(x_2 - x_3) \\ 1 & x_1 + \lambda(x_4 - x_1) \end{pmatrix},$$

where

$$a_2 = \lambda u_2 + \frac{u_1 + x_1u_2 - \pi_{23}}{x_4 - x_1}, \qquad a_3 = \lambda(x_3 + x_4 - x_1 - x_2) + x_1 - x_3$$

and $\lambda \in \mathbb{Z}$. Different choices of λ give homotopic maps. The map χ_1 has degree 1.

The composition $\chi_1\chi_0$ is described by the pair (V_0U_0, V_1U_1). Computing the products and specializing to $\mu = 1 - \lambda$ we get

$$V_0U_0 = V_1U_1 = (x_1 - x_3 + \lambda(x_3 + x_4 - x_1 - x_2))\,\mathrm{I}, \tag{20}$$

where I is the identity 2×2 matrix. Therefore, the composition $\chi_1\chi_0$ is homotopic to the multiplication by the $x_1 - x_3$ endomorphism of $C(\Gamma^0)$,

$$\chi_1\chi_0 = m(x_1 - x_3).$$

This is obtained by setting $\lambda = 0$ in (20). Choosing instead $\lambda = 1$, we see that $\chi_1\chi_0$ is homotopic to multiplication by $x_4 - x_2$. There is no contradiction here, since the endomorphism of multiplication by $x_1 + x_2 - x_3 - x_4$ is null-homotopic.

Likewise, the composition $\chi_0\chi_1$ is homotopic to the multiplication by $x_1 - x_3$ (and to the multiplication by $x_4 - x_2$) endomorphism of $C(\Gamma^1)$.

Disjoint union. Given two graphs Γ_1, Γ_2 with potentials w_1, w_2, the potential of their disjoint union $\Gamma_1 \sqcup \Gamma_2$ is the exterior sum $w_1 + w_2$. An example of a disjoint union is depicted in Figure 31. More often than not, this operation is not uniquely determined by Γ_1 and Γ_2, since we can place Γ_2 between any pair of adjacent exterior legs of Γ_1. When Γ_2 has no exterior legs, we can place it inside any region of Γ_1, including those not adjacent to the border of the disc.

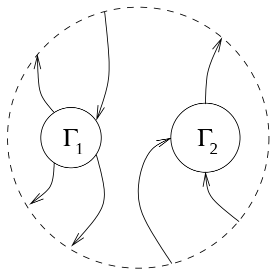

Fig. 31. A disjoint union of Γ_1 and Γ_2

Proposition 28. *There is a canonical isomorphism in* $\mathrm{hmf}_{w_1+w_2}$

$$C(\Gamma_1 \sqcup \Gamma_2) \cong C(\Gamma_1) \otimes_{\mathbb{Q}} C(\Gamma_2).$$

Proof. This is obvious from the definition of $C(\Gamma)$. ∎

Corollary 11. *If Γ_2 is a disjoint union of Γ_1 and a loop, then*

$$C(\Gamma_2) \cong C(\Gamma_1)\langle 1\rangle \otimes_{\mathbb{Q}} A.$$

Direct sum decomposition I. For the graphs Γ_1, Γ depicted in Figure 32, $C(\Gamma_1)$, $C(\Gamma)$ are (R', w)-factorizations, where $R' = \mathbb{Q}[x_2, x_3]$ and $w = x_2^{n+1} - x_3^{n+1}$. Notice that $C(\Gamma)$ has infinite rank as a (R', w)-factorization.

Fig. 32. Graphs Γ_1 and Γ

It can also be viewed as a factorization over the larger ring $\mathbb{Q}[x_1, x_2, x_3]$ but with a degenerate potential.

PROPOSITION 29. *There is an isomorphism in* hmf_w

$$C(\Gamma)\langle 1\rangle \cong \bigoplus_{i=0}^{n-2} C(\Gamma_1)\{2-n+2i\}.$$

Proof. Let Γ_2 be the disjoint union of Γ_1 and a circle with one mark (see Figure 33). Define grading-preserving maps

$$\alpha : C(\Gamma_1)\langle 1\rangle \to C(\Gamma)\{n-2\}, \qquad \beta : C(\Gamma)\{2-n\} \to C(\Gamma_1)\langle 1\rangle$$

as follows. α is the composition (Figure 33)

$$C(\Gamma_1)\langle 1\rangle \xrightarrow{\iota'} C(\Gamma_2)\{n-1\} \xrightarrow{\chi_0} C(\Gamma)\{n-2\},$$

where ι' is the tensor product of the identity of $C(\Gamma_1)$ with the "unit" map ι.

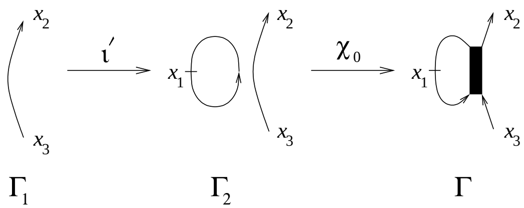

Fig. 33. Map α

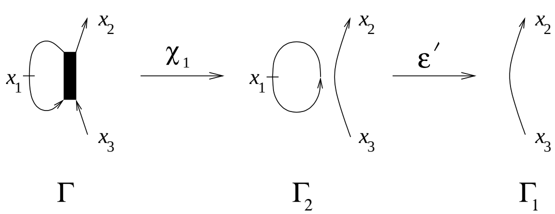

Fig. 34. Map β

β is the composition (Figure 34)

$$C(\Gamma)\{2-n\} \xrightarrow{\chi_1} C(\Gamma_2)\{1-n\} \xrightarrow{\varepsilon'} C(\Gamma_1)\langle 1\rangle,$$

where ε' is the product of the identity of $C(\Gamma_1)$ with the trace map ε.

Define grading-preserving maps α_i, β_i for $0 \le i \le n-2$ by

$$\alpha_i : C(\Gamma_1)\langle 1\rangle \to C(\Gamma)\{n-2-2i\}, \qquad \alpha_i = \sum_{j=0}^{i} m(x_1^j x_2^{i-j})\alpha,$$
$$\beta_i : C(\Gamma)\{n-2-2i\} \to C(\Gamma_1)\langle 1\rangle, \qquad \beta_i = \beta m(x_1^{n-i-2}).$$

Here $m(x_1^j x_2^{i-j})$ denotes the endomorphism of $C(\Gamma)$ which is the multiplication by $x_1^j x_2^{i-j}$.

Lemma 1. $\beta_j\alpha_i = \delta_{i,j}\zeta\,\mathrm{Id}$.

Proof. We have

$$\begin{aligned}\beta_j\alpha_i &= \sum_{k=0}^{i} \beta m(x_1^{n-j-2}) m(x_1^k x_2^{i-k})\alpha = \sum_{k=0}^{i} \varepsilon'\chi_1\chi_0 m(x_1^{n-j-2}) m(x_1^k x_2^{i-k})\iota' \\ &= \sum_{k=0}^{i} \varepsilon' m(x_1 - x_3) m(x_1^{n+k-j-2} x_2^{i-k})\iota' \\ &= \sum_{k=0}^{i} \varepsilon' m(x_1 - x_2) m(x_1^{n+k-j-2} x_2^{i-k})\iota' \\ &= \sum_{k=0}^{i} \varepsilon' m(x_1 - x_2) m(x_1^{n+k-j-2} x_2^{i-k})\iota' \\ &= \varepsilon' m(x_1^{n+i-j-1} - x_1^{n-j-2} x_2^{i+1})\iota' = \varepsilon' m(x_1^{n+i-j-1})\iota' \\ &= \varepsilon(X^{n+i-j-1})\,\mathrm{Id} = \delta_{i,j}\zeta\,\mathrm{Id}\,.\end{aligned}$$

In the third equality we used the fact that $\chi_1\chi_0 = m(x_1 - x_3)$, and in the fourth that $m(x_2) = m(x_3)$ as endomorphisms of $C(\Gamma_2)$. Recall that $\zeta = \varepsilon(X^{n-1})$ is a nonzero rational number. ∎

The lemma implies that the map

$$\alpha' = \sum_{i=0}^{n-2} \alpha_i : \bigoplus_{i=0}^{n-2} C(\Gamma_1)\langle 1\rangle\{2+2i-n\} \to C(\Gamma) \tag{21}$$

is an inclusion of a direct summand, since $\beta'\alpha' = \mathrm{Id}$, where

$$\beta' = \sum_{i=0}^{n-2} \beta_i : C(\Gamma) \to \bigoplus_{i=0}^{n-2} C(\Gamma_1)\langle 1\rangle\{2+2i-n\}.$$

In particular, α' induces an injective map $H(\alpha')$ on cohomology of these factorizations. The factorization $C(\Gamma_1)$ has dimension 2, and the factorization on the left hand side of (21) has dimension $2(n-1)$. To finish the proof of Proposition 29 it suffices to show that $H(\alpha')$ is bijective, which, in turn, follows from the following lemma.

LEMMA 2. *The factorization $C(\Gamma)$ has dimension $2(n-1)$.*

Proof. The quotient of $C(\Gamma)$ by the ideal (x_2, x_3) is a 2-complex, and a free module of rank 4 over the ring $R_1 = \mathbb{Q}[x_1]$. It is the cyclic Koszul complex of the pair $\{((n+1)x_1^n, -(n+1)x_1^{n-1}), (0,0)\}$, since

$$u_1|_{x_1=x_4,\, x_2=0,\, x_3=0} = (n+1)x_1^n, \qquad u_2|_{x_1=x_4,\, x_2=0,\, x_3=0} = -(n+1)x_1^{n-1}.$$

By changing coordinates using a suitable automorphism γ of $N = R_1 \oplus R_1$, as explained at the end of Section 2, we see that this 2-complex is isomorphic to the cyclic Koszul complex of the pair $\{(x_1^{n_1}, 0), (0,0)\}$. The complex

$$\{x_1^{n-1}, 0\} : R_1 \xrightarrow{x_1^{n-1}} R_1 \xrightarrow{0} R_1$$

has cohomology of dimension $n-1$. Tensoring it with $\{0,0\}$ doubles the dimension. Thus,

$$\dim H(C(\Gamma)) = \dim H(\{(x_1^{n-1}, 0), (0,0)\}) = 2(n-1). \ \blacksquare$$

Proposition 29 follows. ■

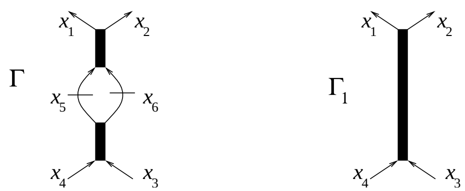

Fig. 35

Direct sum decomposition II. Consider factorizations $C(\Gamma), C(\Gamma_1)$, where Γ, Γ_1 are now the graphs depicted in Figure 35. Both $C(\Gamma)$, $C(\Gamma_1)$ are (R', w)-factorizations, where $R' = \mathbb{Q}[x_1, x_2, x_3, x_4]$ and

$$w = x_1^{n+1} + x_2^{n+1} - x_3^{n+1} - x_4^{n+1}.$$

The factorization $C(\Gamma)$ has infinite rank.

PROPOSITION 30. *There is an isomorphism in* hmf_w

$$C(\Gamma) \cong C(\Gamma_1)\{1\} \oplus C(\Gamma_1)\{-1\}.$$

Proof. Let $R = \mathbb{Q}[x_1, \ldots, x_6]$, $s_1 = x_5 + x_6$, $s_2 = x_5x_6$ and $R_0 = \mathbb{Q}[x_1, x_2, x_3, x_4, s_1, s_2]$. The ring R is a free module of rank 2 over its subring R_0.

We have $C(\Gamma_1) = \{\mathbf{a}, \mathbf{b}\}\{-1\}$, where $\mathbf{a} = (u_1, u_2)$, $\mathbf{b} = (x_1 + x_2 - x_3 - x_4, x_1x_2 - x_3x_4)$, and $u_1 = u_1(x_1, x_2, x_3, x_4)$, $u_2 = u_2(x_1, x_2, x_3, x_4)$.

Hence $C(\mathit{\Gamma}) = \{\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}\}\{-2\}$ where

$$(\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}) = \begin{pmatrix} u_1' & x_1 + x_2 - x_5 - x_6 \\ u_2' & x_1x_2 - x_5x_6 \\ u_1'' & x_5 + x_6 - x_3 - x_4 \\ u_2'' & x_5x_6 - x_3x_4 \end{pmatrix} \tag{22}$$

and

$$\begin{aligned} u_1' &= u_1(x_1, x_2, x_5, x_6), \qquad u_2' = u_2(x_1, x_2, x_5, x_6), \\ u_1'' &= u_1(x_5, x_6, x_3, x_4), \qquad u_2'' = u_2(x_5, x_6, x_3, x_4). \end{aligned}$$

The first column of (22) lists entries of $\widetilde{\mathbf{a}}$, the second lists entries of $\widetilde{\mathbf{b}}$. All coefficients of $\widetilde{\mathbf{a}}$ and $\widetilde{\mathbf{b}}$ lie in R_0. Let $\{\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}\}_0$ be the restriction of $\{\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}\}$ to R_0, i.e., the tensor product of the factorizations

$$R_0 \xrightarrow{a_i} R_0 \xrightarrow{b_i} R_0$$

where a_i, b_i are the entries of $\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}$. We can decompose

$$\{\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}\} \cong \{\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}\}_0 \oplus \{\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}\}_0\{2\}$$

as an (R_0, w)-factorization, since $R \cong R_0 \oplus R_0\{2\}$ as graded R_0-modules. The potential w does not depend on the variables s_1, s_2 in R_0; the third entry of $\widetilde{\mathbf{b}}$ is $s_1 - x_3 - x_4$ and the fourth $s_2 - x_3x_4$. Applying Proposition 10 to exclude s_1 and s_2, we conclude that $\{\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}\}_0$ is isomorphic in the category hmf_w to $\{\mathbf{a}, \mathbf{b}\}$. Proposition 30 follows. ∎

Direct sum decomposition III. Consider the graphs $\mathit{\Gamma}, \mathit{\Gamma}_1, \mathit{\Gamma}_2$ of Figure 36.

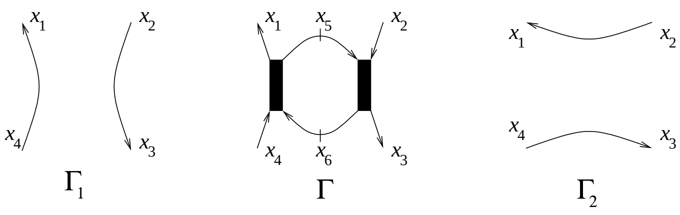

Fig. 36. Graphs $\mathit{\Gamma}_1, \mathit{\Gamma}, \mathit{\Gamma}_2$

$C(\mathit{\Gamma}), C(\mathit{\Gamma}_1), C(\mathit{\Gamma}_2)$ are homogeneous factorizations with the potential

$$w = x_1^{n+1} - x_2^{n+1} + x_3^{n+1} - x_4^{n+1}$$

over the ring $R' = \mathbb{Q}[x_1, x_2, x_3, x_4]$.

Proposition 31. *There is a direct sum decomposition in* hmf_w

$$C(\mathit{\Gamma}) \cong C(\mathit{\Gamma}_2) \oplus \Big(\bigoplus_{i=0}^{n-3} C(\mathit{\Gamma}_1)\langle 1\rangle\{3 - n + 2i\}\Big).$$

Proof. Define grading-preserving maps

$$\alpha : C(\Gamma_1)\langle 1\rangle \to C(\Gamma)\{n-3\}, \qquad \beta : C(\Gamma)\{n-3\} \to C(\Gamma_1)\langle 1\rangle$$

as follows. α is the composition (see Figure 37)

$$C(\Gamma_1)\langle 1\rangle \xrightarrow{\iota'} C(\Gamma_3)\{n-1\} \xrightarrow{\chi_0'} C(\Gamma)\{n-3\},$$

where ι is the tensor product of the identity of $C(\Gamma_1)$ with the "unit" map ι from $\mathbb{Q}\langle 1\rangle$ to the factorization assigned to a circle with two marks. χ_0' is the composition of two χ_0 maps.

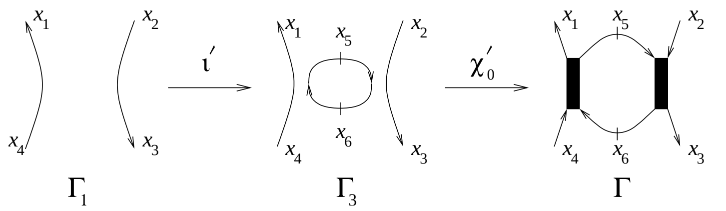

Fig. 37. Map α

β is the dual composition (see Figure 38)

$$C(\Gamma) \xrightarrow{\chi_1'} C(\Gamma_3)\{-2\} \xrightarrow{\varepsilon'} C(\Gamma_1)\langle 1\rangle\{n-3\},$$

where ε' is the tensor product of the identity of $C(\Gamma_1)$ with the trace map, and χ_1' is the composition of two χ_1 maps.

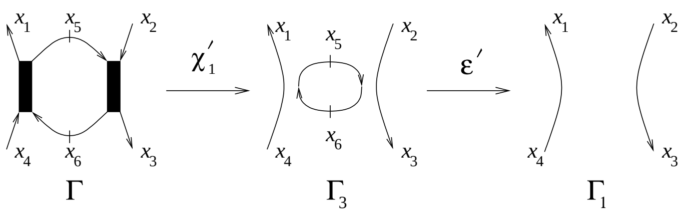

Fig. 38. Map β

Define grading-preserving maps α_i, β_i for $0 \le i \le n-3$ by

$$\alpha_i : C(\Gamma_1)\langle 1\rangle\{3-n+2i\} \to C(\Gamma), \qquad \alpha_i = m(x_5^i)\alpha,$$
$$\beta_i : C(\Gamma) \to C(\Gamma_1)\langle 1\rangle\{3-n+2i\}, \qquad \beta_i = \beta \sum_{a+b+c=n-3-i} m(x_1^a x_3^b x_5^c).$$

LEMMA 3. $\beta_j \alpha_i = \delta_{i,j}\zeta\,\mathrm{Id}$.

Proof. A computation similar to the one in the proof of Lemma 1. We leave the details to the reader. ∎

Let $\alpha' = \sum_{i=0}^{n-3} \alpha_i$ and $\beta' = \zeta^{-1} \sum_{i=0}^{n-3} \beta_i$, considered as grading-preserving maps between

$$N = \bigoplus_{i=0}^{n-3} C(\Gamma_1)\langle 1\rangle\{3-n+2i\}$$

and $C(\Gamma)$. Then $\beta'\alpha' = \mathrm{Id}_N$, so that $\alpha'\beta'$ is an idempotent endomorphism of $C(\Gamma)$ in the category hmf_w. The splitting idempotents property in hmf_w (Proposition 24) implies that $C(\Gamma) \cong N \oplus M$ for some graded factorization M.

Lemma 4. $\operatorname{gdim} C(\Gamma) = sq^{-1}[n-1](1+sq^{1-n})(1+sq^{3-n})$.

Proof. $\operatorname{gdim} C(\Gamma)$ is the graded dimension of the cohomology of the 2-complex $C(\Gamma)/\mathfrak{m}C(\Gamma)$, where $\mathfrak{m}$ is the maximal ideal (x_1,x_2,x_3,x_4) of $R' = \mathbb{Q}[x_1,x_2,x_3,x_4]$. This quotient is a free module of rank 16 over the ring $R_1 = \mathbb{Q}[x_5,x_6]$, and, after the shift by $\{2\}$, the cyclic Koszul complex (over R_1) of the pair

$$(\mathbf{a},\mathbf{b}) = \begin{pmatrix} u_1' & x_5-x_6 \\ u_2' & 0 \\ u_1'' & x_6-x_5 \\ u_2'' & 0 \end{pmatrix},$$

where

$$u_1' = u_1(x_1,x_5,x_6,x_4)|_{x_1=x_4=0}, \quad u_1'' = u_1(x_3,x_6,x_5,x_2)|_{x_2=x_3=0},$$
$$u_2' = u_2(x_1,x_5,x_6,x_4)|_{x_1=x_4=0}, \quad u_2'' = u_2(x_3,x_6,x_5,x_2)|_{x_2=x_3=0}.$$

We can modify this pair using transformations described at the end of Section 2 without changing its graded dimension. For instance, we can add the first entry of the second column to the third, simultaneously with subtracting the third entry of the first column from the first entry of the same column. This corresponds to a suitable change of basis in the free R_1-module $R_1^{\oplus 4}$. The resulting pair is

$$\begin{pmatrix} u_1'-u_1'' & x_5-x_6 \\ u_2' & 0 \\ u_1'' & 0 \\ u_2'' & 0 \end{pmatrix}.$$

Moreover, $u_1' - u_1'' = 0$, since $\{\mathbf{a},\mathbf{b}\}$ is a 2-complex, and $x_5 - x_6$ is not a zero divisor. Using Proposition 9, we can cross out the first row of the above matrix and quotient the rest of the entries by the relation $x_5 = x_6$ without changing the cohomology and graded dimension of this 2-complex. Thus, it suffices to find the graded dimension of the pair

$$(\mathbf{a}_1, \mathbf{b}_1) = \begin{pmatrix} u_2'|_{x_6=x_5} & 0 \\ u_1''|_{x_6=x_5} & 0 \\ u_2''|_{x_6=x_5} & 0 \end{pmatrix}$$

over the ring $\mathbb{Q}[x_5]$. A direct computation tells us that

$$\begin{aligned} u_2'|_{x_6=x_5} &= (n+1)x_5^n, \\ u_1''|_{x_6=x_5} &= -(n+1)x_5^{n-1}, \\ u_2''|_{x_6=x_5} &= -(n+1)x_5^{n-1}. \end{aligned}$$

Adding suitable multiples of the last entry of the first column to other entries of this column and dividing by $n+1$, we simplify the 2-complex to

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ x_5^{n-1} & 0 \end{pmatrix}.$$

The graded dimension of this 2-complex is the product of the graded dimensions of its rows (everything now is over the ring $\mathbb{Q}[x_5]$), since all rows but one consist of zeros. The entries $(n+1)x_5^n$ and $-(n+1)x_5^{n-1}$ have degrees $2n$ and $2n-2$, respectively. Therefore, the graded dimension of the first row is $1+sq^{1-n}$, while the graded dimension of the second row is $1+sq^{3-n}$. The graded dimension of $\{x_5^{n-1}, 0\}$ is $s(q^{n-1}+q^{n-3}+\cdots+q^{3-n}) = sq[n-1]$. The graded dimension of $C(\Gamma)$ is q^{-2} times the product of these three graded dimensions. The lemma follows. ∎

The graded dimension of $C(\Gamma_1)$ is the product of the graded dimensions of the factorizations assigned to its two arcs. Thus,

$$\operatorname{gdim} C(\Gamma_1) = (1+sq^{1-n})^2.$$

Likewise,

$$\operatorname{gdim} C(\Gamma_2) = (1+sq^{1-n})^2.$$

Recall that earlier we decomposed $C(\Gamma) \cong N \oplus M$. Therefore,

$$\operatorname{gdim} C(\Gamma) = \operatorname{gdim} N + \operatorname{gdim} M.$$

Clearly,

$$\operatorname{gdim} N = s[n-2] \operatorname{gdim} C(\Gamma_1).$$

Lemma 5. *$C(\Gamma_2)$ and M have the same graded dimension.*

Proof. This follows from Lemma 4 and the above equations. In the computation we use the fact that $s^2 = 1$. ∎

Lemma 6. $\operatorname{gdim} \operatorname{Ext}_{\mathrm{HMF}}(C(\Gamma_2), C(\Gamma)) = q^{2n-2}[2][n][n-1]$.

The Ext groups here are computed in the category HMF_w, and are naturally $\mathbb{Z}\oplus\mathbb{Z}_2$-graded, since the factorizations are homogeneous. The graded dimension gdim is the Poincaré polynomial of a $\mathbb{Z}\oplus\mathbb{Z}_2$-graded vector space.

Proof of Lemma 6. We start with the isomorphism

$$\mathrm{Ext}_{\mathrm{HMF}}(C(\Gamma_2), C(\Gamma)) \cong H(C(\Gamma)\otimes_{R'} C(\Gamma_2)_\bullet)\{2n-2\}$$

of $\mathbb{Z}\oplus\mathbb{Z}_2$-graded vector spaces, implying the equality of the graded dimensions of the two sides. The 2-complex whose cohomology is computed on the right hand side (without the shift) is isomorphic to the 2-complex $C(\Gamma_4)$, where Γ_4, shown in Figure 39, is given by coupling Γ_2 to Γ.

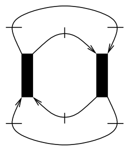

Fig. 39. Graph Γ_4

Applying Propositions 30 and 29, we find that $\mathrm{gdim}\, C(\Gamma_4) = [2][n][n-1]$. Lemma 6 follows. ■

Furthermore,

$$\mathrm{gdim}\,\mathrm{Ext}_{\mathrm{HMF}}(C(\Gamma_2), C(\Gamma_1)) = q^{2n-2}[n]s.$$

Indeed, the left hand side, up to the shift $\{2n-2\}$, is isomorphic to the cohomology of the factorization assigned to the graph given by coupling Γ_1 and Γ_2. This graph is a circle. The variable s appears because Ext^1 is nontrivial.

Since the factorization N is the direct sum of $C(\Gamma_1)$'s with shifts, we can compute the graded dimension of $\mathrm{Ext}_{\mathrm{HMF}}(C(\Gamma_2), N)$, and hence of the groups $\mathrm{Ext}_{\mathrm{HMF}}(C(\Gamma_2), M)$, as the difference of two graded dimensions. The result is

$$\mathrm{gdim}\,\mathrm{Ext}_{\mathrm{HMF}}(C(\Gamma_2), M) = q^{2n-2}[n]^2. \tag{23}$$

Similarly, we compute that

$$\mathrm{gdim}\,\mathrm{Ext}_{\mathrm{HMF}}(M, C(\Gamma_2)) = q^{2n-2}[n]^2. \tag{24}$$

To avoid the last computation, we could invoke Theorem 1 and its graded version (see [B, Example 10.1.6]) to show that the graded dimensions of the two Ext groups are equal.

Since $q^{2n-2}[n]^2 \in 1 + q\mathbb{Z}[q]$, we obtain the following corollary.

Corollary 12. *The vector space of morphisms between $C(\Gamma_2)$ and M in hmf_w is isomorphic to $\mathbb{Q}$.*

In other words, any two nontrivial grading-preserving morphisms from $C(\Gamma_2)$ to M are multiples of each other, and similarly for morphisms in the opposite direction.

After stripping off contractible summands from M we obtain a graded factorization of rank 2 which has the form

$$R' \oplus R'\{2-2n\} \to R'\{1-n\} \oplus R'\{1-n\} \to R' \oplus R'\{2-2n\}.$$

Assume from now on that M has been reduced to this minimal form. Then we strengthen the corollary, by noticing the absence of homotopies of degree $-1-n$ between $C(\Gamma_2)$ and M. We conclude that the space of grading-preserving morphisms between $C(\Gamma_2)$ and M in mf_w is one-dimensional (i.e. even before throwing out null-homotopic morphisms).

Let θ be a nontrivial grading-preserving morphism from $C(\Gamma_2)$ to M, and θ' a nontrivial grading-preserving morphism from M to $C(\Gamma_2)$. Each of these morphisms is unique up to scaling by a nonzero rational number.

Lemma 7. *The composition $\theta'\theta$ is nonzero.*

Proof. Recall that a morphism between two factorizations can be described by a pair of matrices (F_0, F_1) with polynomial coefficients. Define the rank of a morphism as the rank of F_0, treated as a matrix over the quotient field of the polynomial ring. The matrices F_0 and F_1 have the same rank.

The factorization $C(\Gamma_2)$ has the form

$$\begin{pmatrix} R \\ R\{2-2n\} \end{pmatrix} \xrightarrow{P_0} \begin{pmatrix} R\{1-n\} \\ R\{1-n\} \end{pmatrix} \xrightarrow{P_1} \begin{pmatrix} R \\ R\{2-2n\} \end{pmatrix},$$

where

$$P_0 = \begin{pmatrix} \pi_{12} & x_3 - x_4 \\ \pi_{34} & x_2 - x_1 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} x_1 - x_2 & x_3 - x_4 \\ \pi_{34} & -\pi_{12} \end{pmatrix}.$$

The factorization M has the form

$$\begin{pmatrix} R \\ R\{2-2n\} \end{pmatrix} \xrightarrow{V_0} \begin{pmatrix} R\{1-n\} \\ R\{1-n\} \end{pmatrix} \xrightarrow{V_1} \begin{pmatrix} R \\ R\{2-2n\} \end{pmatrix},$$

where V_0, V_1 are matrices with homogeneous entries (of degrees equal to the degrees of the matching entries in P_1, P_2) such that $V_1 V_0 = w \cdot \mathrm{Id}$.

Note that if $\theta'\theta$ is homotopic to 0, then it is actually zero, since there are no homotopies of degree $-n-1$ from $C(\Gamma_2)$ to itself. Assume that $\theta'\theta = 0$. This is only possible if the ranks of both θ' and θ are equal to 1. Let θ be given by a pair of matrices

$$\Theta_0 = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}, \qquad \Theta_1 = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

Since θ has zero degree, we see that all entries of Θ_1 as well as the diagonal entries of Θ_0 are rationals, $f_{12} = 0$, and f_{21} is a polynomial of degree $2n-2$.

Assume that $f_{22} \neq 0$. Then we can rescale a basis vector of M^0 so that $f_{22} = 1$. Since Θ_0 has rank 1, we see that $f_{11} = 0$. By changing the basis in M^1 if necessary, we can assume $g_{21} = 0$. Let

$$V_0 = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}.$$

The equation $\Theta_1 P_0 = V_0 \Theta_0$ implies

$$v_{22} f_{21} = g_{22} \pi_{34}, \qquad v_{22} = g_{22}(x_2 - x_1).$$

This forces $g_{22} = 0$, since π_{34} is not divisible by $x_2 - x_1$. Then $v_{22} = 0$, which implies that $\det(V_0) = -v_{12}v_{21}$. This determinant must divide w^2 (which is the determinant of $V_1 V_0$). This is impossible since v_{12} is a homogeneous linear function of x_1, x_2, x_3, x_4 and

$$w = x_1^{n+1} - x_2^{n+1} + x_3^{n+1} - x_4^{n+1}.$$

Therefore, $f_{22} = 0$. The equation $\Theta_1 P_0 = V_0 \Theta_0$ implies

$$g_{11}(x_3 - x_4) + g_{12}(x_2 - x_1) = 0, \qquad g_{21}(x_3 - x_4) + g_{22}(x_2 - x_1) = 0.$$

Since g_{ij} are rational numbers, they are all zeros. This is a contradiction, since Θ_1 should have rank 1. ∎

The lemma implies that $\theta'\theta$ is a nonzero multiple of the identity morphism of $C(\Gamma_2)$. In addition, we know that $C(\Gamma_2)$ and M have the same graded dimension. Therefore, $C(\Gamma_2)$ and M are isomorphic in hmf_w. Proposition 31 follows. ∎

Direct sum decomposition IV. Consider the graphs $\Gamma_i, 1 \leq i \leq 4$, depicted in Figure 40. The factorizations $C(\Gamma_i)$ have potential

$$w = x_1^{n+1} + x_2^{n+1} + x_3^{n+1} - x_4^{n+1} - x_5^{n+1} - x_6^{n+1}.$$

We view them as factorizations over the ring $R' = \mathbb{Q}[x_1, \ldots, x_6]$.

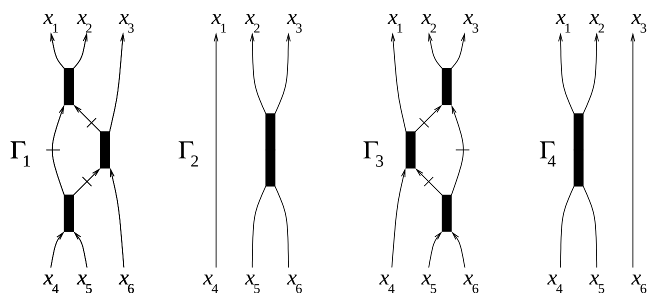

Fig. 40. Graphs $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$

PROPOSITION 32. *There is an isomorphism in* hmf_w

$$C(\Gamma_1) \oplus C(\Gamma_2) \cong C(\Gamma_3) \oplus C(\Gamma_4).$$

Proof. If $n = 1$, then $C(\Gamma_i) = 0$ for all i, and the proposition is trivial. Assuming $n > 1$, we introduce another factorization. Let

$$\begin{aligned} s_1 &= x_1 + x_2 + x_3, & s_4 &= x_4 + x_5 + x_6, \\ s_2 &= x_1x_2 + x_1x_3 + x_2x_3, & s_5 &= x_4x_5 + x_4x_6 + x_5x_6, \\ s_3 &= x_1x_2x_3, & s_6 &= x_4x_5x_6. \end{aligned}$$

Let $R_s \subset R'$ be the subring generated by $s_1, \ldots, s_6$. It is isomorphic to the polynomial ring in $s_1, \ldots, s_6$. As an R_s-module, R' is free of rank 36. Define a 3-variable polynomial h by

$$h(s_1, s_2, s_3) = x_1^{n+1} + x_2^{n+1} + x_3^{n+1}.$$

Then

$$\begin{aligned} w &= h(s_1, s_2, s_3) - h(s_4, s_5, s_6) \\ &= \frac{h(s_1, s_2, s_3) - h(s_4, s_2, s_3)}{s_1 - s_4}(s_1 - s_4) \\ &\quad + \frac{h(s_4, s_2, s_3) - h(s_4, s_5, s_3)}{s_2 - s_5}(s_2 - s_5) \\ &\quad + \frac{h(s_4, s_5, s_3) - h(s_4, s_5, s_6)}{s_3 - s_6}(s_3 - s_6) \\ &= v_1\alpha_1 + v_2\alpha_2 + v_3\alpha_3, \end{aligned}$$

where

$$\begin{aligned} \alpha_1 &= s_1 - s_4 = x_1 + x_2 + x_3 - x_4 - x_5 - x_6, \\ \alpha_2 &= s_2 - s_5 = x_1x_2 + x_1x_3 + x_2x_3 - x_4x_5 - x_4x_6 - x_5x_6, \\ \alpha_3 &= s_3 - s_6 = x_1x_2x_3 - x_4x_5x_6, \\ v_1 &= \frac{h(s_1, s_2, s_3) - h(s_4, s_2, s_3)}{s_1 - s_4}, \\ v_2 &= \frac{h(s_4, s_2, s_3) - h(s_4, s_5, s_3)}{s_2 - s_5}, \\ v_3 &= \frac{h(s_4, s_5, s_3) - h(s_4, s_5, s_6)}{s_3 - s_6}. \end{aligned}$$

v_1, v_2, v_3 are polynomials in $x_1, \ldots, x_6$. The quotient of R_s by the ideal $(\alpha_1, \alpha_2, \alpha_3)$ is isomorphic to $\mathbb{Q}[s_1, s_2, s_3]$. The images of v_1, v_2, v_3 in this quotient ring have the form

$$\frac{\partial h(s_1, s_2, s_3)}{\partial s_1}, \qquad \frac{\partial h(s_1, s_2, s_3)}{\partial s_2}, \qquad \frac{\partial h(s_1, s_2, s_3)}{\partial s_3}.$$

It is well-known that the quotient of $\mathbb{Q}[s_1, s_2, s_3]$ by the ideal generated by these derivatives is isomorphic to the cohomology ring of the Grassmannian of 3-dimensional subspaces of $\mathbb{C}^n$ (see [MC, p. 127], for example). Thus, there is an algebra isomorphism

$$R_s/(\alpha_1, \alpha_2, \alpha_3, v_1, v_2, v_3) \cong \mathrm{H}^*(\mathrm{Gr}(3, n), \mathbb{Q}). \tag{25}$$

In the degenerate case $n = 2$ both sides are zero.

Likewise, the quotient ring $R'/(\alpha_1, \alpha_2, \alpha_3, v_1, v_2, v_3)$ is the cohomology ring of the configuration space

$$\{N_1 \subset N_2 \subset N_3 \supset N'_2 \supset N'_1 \mid \dim N_i = \dim N'_i = i,\ N_i, N'_i \subset \mathbb{C}^n\}.$$

Lemma 8. *The sequence $(\alpha_1, \alpha_2, \alpha_3, v_1, v_2, v_3)$ and each of its permutations are regular sequences in the rings R_s and R'.*

Proof. The length of the sequence equals the number of generators of the polynomial rings R_s and R'. Since the quotient rings by the ideal generated by this sequence are finite-dimensional, the sequence is regular. ∎

Let Υ be the following (R', w)-factorization:

$$\Upsilon := \{(v_1, v_2, v_3), (\alpha_1, \alpha_2, \alpha_3)\}\{-3\}.$$

Υ is the tensor product of the factorizations

$$R' \xrightarrow{v_1} R'\{1-n\} \xrightarrow{\alpha_1} R',$$
$$R' \xrightarrow{v_2} R'\{3-n\} \xrightarrow{\alpha_2} R',$$
$$R' \xrightarrow{v_3} R'\{5-n\} \xrightarrow{\alpha_3} R',$$

shifted by $\{-3\}$. One could think of Υ as the factorization assigned to the diagram in the lower left corner of Figure 4.

Proposition 32 will follow from

Proposition 33. *There is an isomorphism in* hmf_w

$$C(\Gamma_1) \cong \Upsilon \oplus C(\Gamma_4).$$

Indeed, the factorization $C(\Gamma_1)$ turns into $C(\Gamma_3)$, and $C(\Gamma_2)$ into $C(\Gamma_4)$ if we permute x_1 with x_3 and x_4 with x_6. The factorization Υ is invariant under this operation, since α_i's and v_i's are, so that Proposition 33 also implies the isomorphism

$$C(\Gamma_3) \cong \Upsilon \oplus C(\Gamma_2).$$

Then both sides in the equation of Proposition 32 are isomorphic to $\Upsilon \oplus C(\Gamma_2) \oplus C(\Gamma_4)$. Thus, it suffices to establish Proposition 33.

The proof of Proposition 33 occupies the next 10 pages.

Lemma 9. *$C(\Gamma_4)$ is a direct summand of $C(\Gamma_1)$.*

Proof. Let

$$g_0 = \chi_0 r_0, \qquad C(\Gamma_4) \xrightarrow{r_0} C(\Gamma_5) \xrightarrow{\chi_0} C(\Gamma_1)$$

be the composition depicted in Figure 41.

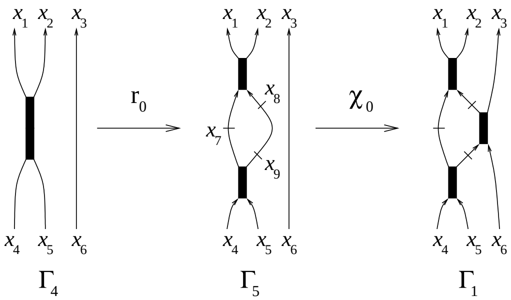

Fig. 41. Map g_0

Proposition 30 implies that $C(\Gamma_5) \cong C(\Gamma_4)\{-1\} \oplus C(\Gamma_4)\{1\}$, and r_0 is defined to be the inclusion of $C(\Gamma_4)$ into $C(\Gamma_5)$ as the direct summand $C(\Gamma_4)\{-1\}$. The maps r_0 and χ_0 have degrees -1 and 1, respectively, so that g_0 is grading-preserving.

Let

$$g_1 = r_1 \chi_1, \qquad C(\Gamma_1) \xrightarrow{\chi_1} C(\Gamma_5) \xrightarrow{r_1} C(\Gamma_4)$$

be the composition depicted in Figure 42.

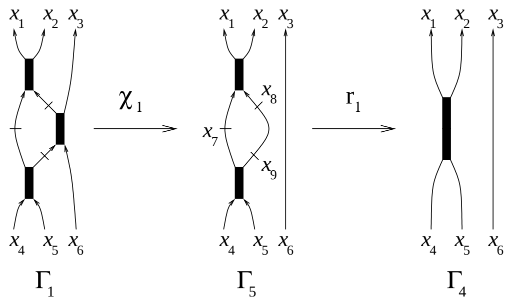

Fig. 42. Map g_1

r_1 is the projection of $C(\Gamma_5) \cong C(\Gamma_4)\{-1\} \oplus C(\Gamma_4)\{1\}$ onto the second direct summand (the direct sum decomposition is the one obtained in the proof of Proposition 30).

We compute

$$g_1 g_0 = r_1 \chi_1 \chi_0 r_0 = r_1 m(x_8 - x_6) r_0 = r_1 m(x_8) r_0,$$

since $r_1 m(x_6) r_0 = 0$. The proof of Proposition 30 implies that $r_1 m(x_8) r_0$ is a nonzero multiple of the identity morphism. Therefore, g_0 is split injective, and $C(\Gamma_4)$ is a direct summand of $C(\Gamma_1)$. ∎

Let M be the complement of $C(\mathit{\Gamma}_4)$ in $C(\mathit{\Gamma}_1)$,

$$C(\mathit{\Gamma}_1) \cong M \oplus C(\mathit{\Gamma}_4).$$

LEMMA 10. $\operatorname{gdim} C(\mathit{\Gamma}_1) = q^{-3}(1+q^2)(1+sq^{1-n})(1+sq^{3-n})^2$ *if* $n > 2$.

Proof. Similar to that of Lemma 4. Up to a shift, $C(\mathit{\Gamma}_1)$ is the Koszul factorization $\{\mathbf{a}, \mathbf{b}\}$ for some length 6 sequences whose entries are polynomials in $x_1, \ldots, x_6$ and in three other variables (say y_1, y_2, y_3) assigned to the three internal marks of $\mathit{\Gamma}_1$. We can exclude two of these variables using Proposition 9, and then specialize to $x_1 = \cdots = x_6 = 0$, since we are computing cohomology. The details are left to the reader. ∎

In particular, if $n > 2$, the dimension of $C(\mathit{\Gamma}_1)$ is 16. If $n = 2$, the factorization $C(\mathit{\Gamma}_1)$ has dimension 8, which is also the dimension of $C(\mathit{\Gamma}_4)$. Hence, $M = 0$ if $n = 2$, and there is an isomorphism $C(\mathit{\Gamma}_1) \cong C(\mathit{\Gamma}_4)$. The factorization Υ is contractible ($v_3 = 3$ if $n = 2$), and Propositions 33, 32 follow. From now on we assume $n > 2$.

It is easy to see that

$$\operatorname{gdim} C(\mathit{\Gamma}_4) = q^{-1}(1+sq^{1-n})^2(1+sq^{3-n}),$$
$$\operatorname{gdim} \Upsilon = q^{-3}(1+sq^{1-n})(1+sq^{3-n})(1+sq^{5-n}).$$

Therefore,

$$\operatorname{gdim} M = \operatorname{gdim} C(\mathit{\Gamma}_1) - \operatorname{gdim} C(\mathit{\Gamma}_4) = \operatorname{gdim} \Upsilon.$$

Thus, M and Υ have the same graded dimension.

LEMMA 11.

$$\operatorname{gdim} \operatorname{Ext}(\Upsilon, C(\mathit{\Gamma}_4)) = q^{3n-3}[2][n][n-1][n-2].$$

Proof. The Ext groups in the lemma are isomorphic to the cohomology of the 2-complex

$$C(\mathit{\Gamma}_4) \otimes \Upsilon_\bullet \cong \{\mathbf{a}, \mathbf{b}\}\langle 1\rangle\{3n-9\},$$

where

$$(\mathbf{a}, \mathbf{b}) = \begin{pmatrix} \pi_{36} & x_3 - x_6 \\ u_1(x_1, x_2, x_4, x_5) & x_1 + x_2 - x_4 - x_5 \\ u_2(x_1, x_2, x_4, x_5) & x_1 x_2 - x_4 x_5 \\ \alpha_1 & v_1 \\ \alpha_2 & v_2 \\ \alpha_3 & v_3 \end{pmatrix}$$

Pass to the subring R'' of $R' = \mathbb{Q}[x_1, \ldots, x_6]$ generated by

$$x_1' = x_1 + x_2, \quad x_2' = x_1 x_2, \quad x_3, \quad x_4' = x_4 + x_5, \quad x_5' = x_4 x_5, \quad x_6,$$

and denote by $\{\mathbf{a},\mathbf{b}\}_0$ the cyclic Koszul complex of the pair $(\mathbf{a},\mathbf{b})$ viewed as a pair in R'' (this is possible since all coefficients of $\mathbf{a}$ and $\mathbf{b}$ lie in R''). Then

$$\operatorname{gdim} H(\{\mathbf{a},\mathbf{b}\}) = q^2[2]^2 \operatorname{gdim} H(\{\mathbf{a},\mathbf{b}\}_0).$$

We use the first three lines of $(\mathbf{a},\mathbf{b})_0$ to exclude x_6, x_4', and x_5', and denote by $(\mathbf{a}^1,\mathbf{b}^1)$ the pair obtained by crossing out the first three lines of $(\mathbf{a},\mathbf{b})_0$ and passing to the quotient ring of R'' by the relations $x_6 = x_4' = x_5' = 0$. Then $\mathbf{a}^1 = (0,0,0)$, $\mathbf{b}^1 = (-v_1,-v_2,-v_3)$,

$$H(\{\mathbf{a},\mathbf{b}\}_0) \cong H(\{\mathbf{a}^1,\mathbf{b}^1\}) \cong H((0,0,0),(v_1,v_2,v_3)),$$

and the lemma follows easily, since (v_1,v_2,v_3) is a regular sequence in the quotient ring $R''' = R''/(x_6,x_4',x_5')$, and $R'''/(v_1,v_2,v_3)$ is isomorphic to the cohomology ring of the partial flag variety

$$\{N_2 \subset N_3 \mid \dim(N_i) = i,\ N_3 \subset \mathbb{C}^n\}. \ \blacksquare$$

LEMMA 12.

$$\operatorname{gdim} \operatorname{Ext}(\Upsilon, C(\Gamma_1)) = q^{3n-3}[2]^3[n][n-1][n-2].$$

Proof. The Ext groups in the lemma are isomorphic to the cohomology of the 2-complex $C(\Gamma_1)\otimes \Upsilon_\bullet$. This is a complex of free R^1-modules, where R^1 is the ring of polynomials in $x_1,\dots,x_6$ and three "internal" variables corresponding to the three marks in Γ_1. Passing to a suitable subring (over which R^1 is a free rank 4 module) and excluding several variables, one can show that $C(\Gamma_1)\otimes \Upsilon_\bullet$ is isomorphic (up to contractible complexes) to the direct sum of four copies of $C(\Gamma_1)\otimes \Upsilon_\bullet$ with shifts, implying

$$\operatorname{gdim} \operatorname{Ext}(\Upsilon, C(\Gamma_1)) = [2]^2 \operatorname{gdim} \operatorname{Ext}(\Upsilon, C(\Gamma_4)). \ \blacksquare$$

The lemmas imply

$$\begin{aligned}\operatorname{gdim} \operatorname{Ext}(\Upsilon, M) &= \operatorname{gdim} \operatorname{Ext}(\Upsilon, C(\Gamma_1)) - \operatorname{gdim} \operatorname{Ext}(\Upsilon, C(\Gamma_4))\\ &= q^{3n-3}[2][3][n][n-1][n-2].\end{aligned}$$

Since $q^{3n-3}[2][3][n][n-1][n-2] \in 1+q\mathbb{Z}[q]$, the space of degree 0 morphisms from Υ to M is one-dimensional. Let θ be a nontrivial (not null-homotopic) degree 0 morphism from Υ to M.

LEMMA 13. *θ is an isomorphism of factorizations.*

Proof. The factorization Υ has the form

$$\begin{pmatrix} R'\{-3\} \\ R'\{5-2n\} \\ R'\{3-2n\} \\ R'\{1-2n\} \end{pmatrix} \xrightarrow{\Lambda_0} \begin{pmatrix} R'\{2-n\} \\ R'\{-n\} \\ R'\{-2-n\} \\ R'\{6-3n\} \end{pmatrix} \xrightarrow{\Lambda_1} \begin{pmatrix} R'\{-3\} \\ R'\{5-2n\} \\ R'\{3-2n\} \\ R'\{1-2n\} \end{pmatrix},$$

where

$$\Lambda_0 = \begin{pmatrix} v_3 & -\alpha_2 & -\alpha_1 & 0 \\ v_2 & \alpha_3 & 0 & -\alpha_1 \\ v_1 & 0 & \alpha_3 & \alpha_2 \\ 0 & v_1 & -v_2 & v_3 \end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix} \alpha_3 & \alpha_2 & \alpha_1 & 0 \\ -v_2 & v_3 & 0 & \alpha_1 \\ -v_1 & 0 & v_3 & -\alpha_2 \\ 0 & -v_1 & v_2 & \alpha_3 \end{pmatrix}.$$

We can assume that the factorization $M = M^0 \oplus M^1$ has no contractible summands. Since it has the same graded dimension as Υ, we can write M as

$$(26) \qquad \begin{pmatrix} R'\{-3\} \\ R'\{5-2n\} \\ R'\{3-2n\} \\ R'\{1-2n\} \end{pmatrix} \xrightarrow{\Pi_0} \begin{pmatrix} R'\{2-n\} \\ R'\{-n\} \\ R'\{-2-n\} \\ R'\{6-3n\} \end{pmatrix} \xrightarrow{\Pi_1} \begin{pmatrix} R'\{-3\} \\ R'\{5-2n\} \\ R'\{3-2n\} \\ R'\{1-2n\} \end{pmatrix}$$

for some 4×4 matrices Π_0, Π_1 with homogeneous entries whose degrees match the degrees of the entries of Λ_0, Λ_1. Let $\Pi_0 = (a_{ij})_{i,j=1}^4$ and $\Pi_1 = (b_{ij})_{i,j=1}^4$. The entry a_{14} is a degree $n+1$ homomorphism from $R'\{1-2n\}$ to $R'\{2-n\}$, and necessarily a rational number. If $a_{14} \neq 0$, then M contains a contractible summand, which is a contradiction. Therefore, $a_{14} = 0$. Likewise, $b_{14} = 0$ if $n > 4$.

We can write the morphism θ via a pair of matrices

$$\Theta_0 = (c_{ij})_{i,j=1}^4, \qquad \Theta_1 = (f_{ij})_{i,j=1}^4.$$

Since θ is a morphism of factorizations, the diagram

$$\begin{array}{ccccc} \Upsilon^0 & \xrightarrow{\Lambda_0} & \Upsilon^1 & \xrightarrow{\Lambda_1} & \Upsilon^0 \\ {\scriptstyle\Theta_0}\downarrow & & {\scriptstyle\Theta_1}\downarrow & & {\scriptstyle\Theta_0}\downarrow \\ M^0 & \xrightarrow{\Pi_0} & M^1 & \xrightarrow{\Pi_1} & M^0 \end{array}$$

is commutative, and the following relations hold:

$$(27) \qquad \Pi_0 \Theta_0 = \Theta_1 \Lambda_0,$$

$$(28) \qquad \Pi_1 \Theta_1 = \Theta_0 \Lambda_1.$$

θ is grading-preserving. If $n > 4$, then the sequences of degree shifts $(-3, 5-2n, 3-2n, 1-2n)$ and $(2-n, -n, -2-n, 6-3n)$ are strictly decreasing, and the matrices Θ_0, Θ_1 are lower-triangular, $c_{ij} = f_{ij} = 0$ for $i < j$, since there are no grading-preserving homomorphisms from $R'\{k_1\}$ to $R'\{k_2\}$ if $k_2 > k_1$. If $n = 4$, the sequences are not strictly decreasing, but changing bases in M^0 and M^1 if necessary, we can assume that Θ_0 and Θ_1 are lower-triangular in this case too. When $n = 3$, we cannot assume right away that Θ_0 and Θ_1 are lower-triangular. We postpone considering the cases $n = 3$ and $n = 4$, and from now on until two paragraphs after the proof of Lemma 18 restrict to the case $n > 4$.

θ is an isomorphism iff both Θ_0 and Θ_1 are invertible over the ring R'. Since they are lower-triangular with rational diagonal entries, they are invertible iff all diagonal entries are nonzero. Moreover, Θ_1 is invertible iff Θ_0 is invertible. This is due to the equation $\Pi_0\Theta_0\Lambda_0^{-1} = \Theta_1$ and the fact that the determinants of Π_0 and Λ_0 are nonzero multiples of w^2 (to verify this property, use the fact that w is an irreducible polynomial, $\Lambda_0\Lambda_1 = \Pi_0\Pi_1 = w\mathrm{I}$, and Π_0 has coefficients in the same degrees as Λ_0).

Thus, θ is an isomorphism iff Θ_0 is invertible.

The matrix equation (27) can be rewritten as 16 equations for the entries of the matrices $\Pi_0, \Theta_0, \Theta_1$. Three of these equations have the form

$$\begin{aligned}
a_{24}c_{44} &= -\alpha_1 f_{22},\\
a_{34}c_{44} &= -\alpha_1 f_{32} + \alpha_2 f_{33},\\
a_{44}c_{44} &= -\alpha_1 f_{42} + \alpha_2 f_{43} + v_3 f_{44}.
\end{aligned}$$

Lemma 14. $c_{44} \neq 0$.

Proof. Assume $c_{44} = 0$. Then the above equations reduce to

$$\begin{aligned}
0 &= -\alpha_1 f_{22},\\
0 &= -\alpha_1 f_{32} + \alpha_2 f_{33},\\
0 &= -\alpha_1 f_{42} + \alpha_2 f_{43} + v_3 f_{44}.
\end{aligned}$$

Since $f_{22}, f_{33}, f_{44} \in \mathbb{Q}$, and the sequence $(\alpha_1, \alpha_2, v_3)$ is regular, the equations imply

$$f_{22} = f_{32} = f_{33} = f_{44} = 0.$$

The last equation of the three above simplifies to

$$f_{42}\alpha_1 = f_{43}\alpha_2.$$

Since α_1 and α_2 are relatively prime,

$$f_{42} = \alpha_2 z_0, \qquad f_{43} = \alpha_1 z_0,$$

for some polynomial z_0.

Three of the equations in (28) are

$$\begin{aligned}
b_{24}f_{44} &= \alpha_1 c_{22},\\
b_{34}f_{44} &= \alpha_1 c_{32} - \alpha_2 c_{33},\\
b_{44}f_{44} &= \alpha_1 c_{42} - \alpha_2 c_{43} + \alpha_3 c_{44}.
\end{aligned}$$

Since $f_{44} = 0$, and $(\alpha_1, \alpha_2, \alpha_3)$ is a regular sequence, we derive

$$c_{22} = c_{32} = c_{33} = c_{44} = 0,$$

and

$$c_{42} = \alpha_2 z_1, \qquad c_{43} = \alpha_1 z_1,$$

for some polynomial z_1.

The remaining terms of the matrix equations (27), (28) imply

$$f_{21} = -a_{24}z_1, \quad f_{31} = -a_{34}z_1, \quad f_{41} = \alpha_3 z_0 - a_{44}z_1,$$
$$c_{21} = b_{24}z_0, \quad c_{31} = b_{34}z_0, \quad c_{41} = b_{44}z_0 - v_3 z_1.$$

Let Z_0, Z_1 be 4×4 matrices with the only nonzero entry z_0 (respectively, z_1) at the intersection of the fourth row and the first column. Then

$$\Theta_0 = \Pi_1 Z_0 - Z_1 \Lambda_0, \quad \Theta_1 = Z_0 \Lambda_1 - \Pi_0 Z_1.$$

We see that θ is homotopic to 0 through the homotopy $(Z_0, -Z_1)$. This contradicts our assumption on θ. Therefore, $c_{44} \neq 0$, and Lemma 14 is proved. ∎

Lemma 15. $f_{44} \neq 0$.

Proof. An argument in the proof of the previous lemma shows that $f_{44} = 0$ implies $c_{44} = 0$. ∎

Since $f_{44} \neq 0$ and $c_{44} \neq 0$, by rescaling the last basis vector in M^0 and in M^1, we can assume

$$f_{44} = c_{44} = 1.$$

To finish the proof of Lemma 13 in the case $n > 3$, we assume to the contrary that θ is not an isomorphism.

Lemma 16. $f_{11} = 0$ *if* θ *is not an isomorphism.*

Proof. Suppose $f_{11} \neq 0$. By changing the first basis vector in M^1 if necessary, we can reduce to the case

$$f_{11} = 1, \quad f_{21} = f_{31} = f_{41} = 0.$$

Entry $(1, 3)$ of equation (27) is

$$a_{13}c_{33} = -\alpha_1 f_{11} = -\alpha_1.$$

Therefore $c_{33} \neq 0$, and we assume, without loss of generality (by changing the third basis vector in M^0), that

$$c_{33} = 1, \quad c_{43} = 0.$$

Then $a_{13} = -\alpha_1$, and equation (27), entry $(1, 2)$, simplifies to

$$-\alpha_2 = a_{12}c_{22} - \alpha_1 c_{32}.$$

Since α_2 is not divisible by α_1, we know that $c_{22} \neq 0$. Changing the second basis vector in M^0, we assume

$$c_{22} = 1, \quad c_{32} = 0, \quad c_{42} = 0, \quad a_{12} = -\alpha_2.$$

Since $c_{22} = c_{33} = c_{44} = 1$ and Θ_0 is not invertible (as we assumed that θ is not an isomorphism), necessarily $c_{11} = 0$.

Entry $(1,1)$ of (27) can be written as

$$v_3 = -\alpha_2 c_{21} - \alpha_1 c_{31}.$$

Contradiction, since v_3 is not in the ideal (α_1, α_2). Lemma 16 follows. ∎

LEMMA 17. *$c_{11} = 0$ if θ is not an isomorphism.*

Proof. Suppose otherwise ($c_{11} \neq 0$) and change the first basis vector of M^0 so that

$$c_{11} = 1, \quad c_{21} = c_{31} = c_{41} = 0.$$

From the previous lemma we know that $f_{11} = 0$. Entry $(1,3)$ of (28) reads $\alpha_1 = b_{13} f_{33}$. We see that $f_{33} \neq 0$, and after modifying the third basis vector of M^1 we assume

$$f_{33} = 1, \quad f_{43} = 0, \quad b_{13} = \alpha_1.$$

Entry $(1,2)$ of the equation (28) reads

$$\alpha_2 = b_{12} f_{22} + \alpha_1 f_{32}.$$

Since α_2 does not factor, $f_{22} \neq 0$, and we can reduce to the case

$$f_{22} = 1, \quad f_{32} = f_{42} = 0, \quad b_{12} = \alpha_2.$$

Entry $(1,1)$ of (28) simplifies to

$$\alpha_3 = \alpha_2 f_{21} + \alpha_1 f_{31}.$$

Contradiction, since α_3 is not in the ideal generated by α_1 and α_2. Lemma 17 follows. ∎

By now, we have reduced our considerations to the case

$$f_{44} = c_{44} = 1, \quad f_{11} = c_{11} = 0.$$

LEMMA 18. *$f_{33} = 0$ if θ is not an isomorphism.*

Proof. If $f_{33} \neq 0$, we can assume $f_{33} = 1$, $f_{43} = 0$. Entry $(1,3)$ of equation (28) becomes $b_{13} = 0$. Note that $b_{12} \neq 0$; otherwise b_{11} (which is a polynomial of degree 3 in x_i's) would be the only nonzero entry in the first row of Π_1, and a divisor of $\det \Pi_1$, which is proportional to w^2. Since w is irreducible, this is impossible, so that $b_{12} \neq 0$.

Entry $(1,2)$ of (28) simplifies to $b_{12} f_{22} = 0$. Thus, $f_{22} = 0$. Entry $(1,1)$ of the same equation reduces to $b_{12} f_{21} = 0$. Thus, $f_{21} = 0$.

Next, entries $(2,3)$ and $(2,4)$ of (28) reduce to

$$b_{24} = \alpha_1 c_{22}, \quad b_{23} = \alpha_1 c_{21}.$$

Entry $(2,2)$ simplifies to

$$b_{23} f_{32} + b_{24} f_{42} = \alpha_2 c_{21} + v_3 c_{22}.$$

Since v_3 is not in the ideal generated by α_1, α_2, we know that $c_{22} = 0$. Then $b_{24} = \alpha_1 c_{22} = 0$ (entry $(2,4)$ of (28)). Entry $(2,3)$ tells us that $b_{23} = \alpha_1 c_{21}$, while entry $(2,2)$ reduces to $\alpha_2 c_{21} = \alpha_1 c_{21} f_{32}$. Since α_2 is not divisible by α_1, we have $c_{21} = 0$ and $b_{23} = 0$.

Switching to equation (27), and looking at entry $(4,4)$, we get $a_{44} = v_3 - \alpha_1 f_{42}$, while entry $(4,3)$ simplifies to

$$-v_2 - \alpha_1 f_{41} = a_{43} c_{33} + (v_3 - \alpha_1 f_{42}) c_{43}.$$

Since v_2 does not lie in the ideal (α_1, v_3), we have $c_{33} \neq 0$. Then we can assume $c_{33} = 1, c_{43} = 0$. Entries $(1,3)$ and $(2,3)$ tell us that $a_{13} = a_{23} = 0$.

To summarize, we have shown that

$$a_{13} = a_{23} = a_{24} = 0.$$

Also, $a_{14} = 0$. Therefore, the matrix Π_0 is block lower-diagonal. Its determinant is a nonzero multiple of w^2. The determinant of Π_0 is divisible by the determinant of its lower-diagonal 2×2 block

$$\begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} \alpha_3 - \alpha_1 f_{31} & \alpha_2 - \alpha_1 f_{32} \\ -v_2 - \alpha_1 f_{41} & v_3 - \alpha_1 f_{42} \end{pmatrix}.$$

The only possibility is for this determinant to be a nonzero multiple of w, leading to the equation

$$(\alpha_3 - \alpha_1 f_{31})(v_3 - \alpha_1 f_{42}) + (\alpha_2 - \alpha_1 f_{32})(v_2 + \alpha_1 f_{41}) = \lambda w$$

for some $\lambda \in \mathbb{Q}$. Using $w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, we reduce the equation to

$$\alpha_1(-v_1 - \alpha_3 f_{42} - v_3 f_{31} + \alpha_1 f_{31} f_{42} + \alpha_2 f_{41} - v_2 f_{32} - \alpha_1 f_{32} f_{41}) = (\lambda - 1) w.$$

Since w is not divisible by α_1, we have $\lambda = 1$, and the equation becomes

$$-v_1 - \alpha_3 f_{42} - v_3 f_{31} + \alpha_1 f_{31} f_{42} + \alpha_2 f_{41} - v_2 f_{32} - \alpha_1 f_{32} f_{41} = 0.$$

Contradiction, since v_1 does not belong to the ideal generated by α_1, α_2, α_3, v_2, v_3. Lemma 18 follows. ■

Entries $(4,3)$ and $(4,4)$ of equation (28) now become

$$b_{44} f_{43} = \alpha_1 c_{41} + v_3 c_{43} + v_2,$$
$$b_{44} = \alpha_1 c_{42} - \alpha_2 c_{43} + \alpha_3,$$

implying

$$v_2 = f_{43}(\alpha_1 c_{42} - \alpha_2 c_{43} + \alpha_3) - \alpha_1 c_{41} - v_3 c_{43}.$$

This is impossible, since v_2 does not lie in the ideal $(\alpha_1, \alpha_2, \alpha_3, v_3)$.

Therefore, θ is invertible and Υ is isomorphic to M if $n > 4$. Lemma 13 and Propositions 33, 32 follow in the case $n > 4$.

If $n = 4$, the sequences of degree shifts in M^0, M^1 (see formula (26)) are $(-3, -3, -5, -7)$ and $(-2, -4, -6, -6)$. These sequences are decreasing, and the matrices Θ_0, Θ_1 are block lower-triangular. By changing bases in M^0 and M^1 if necessary, we can assume that Θ_0 and Θ_1 are lower-triangular. The entry b_{14} of Π_1 is a homogeneous linear polynomial in the x's. If $b_{14} = 0$, Lemmas 14–18 hold for $n = 4$ as well (with the simplification that $z_0 = 0$ in the proof of Lemma 14), and we are done. Assume now that $b_{14} \neq 0$. Then

- $f_{44} = 0$ (from equation (28), entry $(1, 4)$),
- $c_{22} = 0$ (equation (28), entry $(2, 4)$),
- $c_{33} = 0$ (equation (28), entry $(3, 4)$),
- $c_{32} = 0$ (equation (28), entry $(3, 4)$),
- $c_{44} = 0$ (equation (28), entry $(4, 4)$),
- $f_{22} = 0$ (equation (27), entry $(2, 4)$),
- $f_{11} = 0$ (equation (27), entry $(1, 3)$),
- $f_{33} = 0$ (equation (27), entry $(3, 4)$),
- $f_{32} = 0$ (equation (27), entry $(3, 4)$).

Furthermore, $b_{14}f_{43} = \alpha_1 c_{11}$ (equation (28), entry $(1, 3)$), and $b_{14}f_{42} = \alpha_2 c_{11}$ (equation (28), entry $(1, 2)$). Since α_1 and α_2 are relatively prime, this is only possible if $c_{11} = 0$, which, in turn, implies $f_{43} = 0$, $f_{42} = 0$. Next, $c_{21} = 0$ (equation (28), entry $(2, 2)$), and $c_{31} = 0$ (equation (28), entry $(3, 2)$). From the remaining equations we derive

$$c_{41} = zv_3, \qquad c_{42} = -z\alpha_2, \qquad c_{43} = -z\alpha_1,$$
$$f_{21} = za_{24}, \qquad f_{31} = za_{34}, \qquad f_{41} = za_{44},$$

for some rational number z. Therefore, θ is null-homotopic, through the homotopy $(0, Z)$, where Z is the 4×4 matrix with the only nonzero entry z in the lower left corner. Contradiction. Lemma 13 and Propositions 33, 32 follow in the case $n = 4$.

If $n = 3$, the degree conditions force the coefficients

$$a_{14}, b_{23}, c_{21}, c_{14}, c_{23}, c_{24}, c_{34}, f_{12}, f_{13}, f_{14}, f_{23}, f_{43}$$

to be zero. By changing bases in M^0, M^1, if necessary, we can assume that $c_{13} = f_{24} = 0$.

Lemma 19. *$c_{44} \neq 0$ if $n = 3$.*

Proof. Suppose to the contrary that $c_{44} = 0$. Three of the equations (27) reduce to

$$0 = -\alpha_1 f_{22},$$
$$0 = -\alpha_1 f_{42} + v_3 f_{44},$$
$$0 = -\alpha_1 f_{32} + \alpha_2 f_{33} + v_3 f_{34},$$

implying that

$$f_{22} = f_{44} = f_{42} = f_{33} = 0.$$

Equation (28), entry $(2, 4)$, implies $c_{22} = 0$, while entry $(3, 3)$ implies $c_{33} = c_{31} = 0$. From entry $(1, 3)$ we derive $c_{11} = 0$, and from the entry $(1, 3)$ of (27) that $f_{11} = 0$. The remaining entries of the two matrix equations force the other entries of Θ_0, Θ_1 to have the form

$$\begin{array}{lll} c_{12} = z_0 b_{13}, & c_{32} = z_0 b_{33}, & \\ c_{41} = -v_3 z_1, & c_{42} = b_{43} z_0 + \alpha_2 z_1, & c_{43} = z_1 \alpha_1, \\ f_{21} = -z_1 a_{24}, & f_{31} = -z_1 a_{34} - x v_2, & f_{41} = -z_1 a_{44}, \\ f_{32} = z_0 v_3, & f_{34} = z_0 \alpha_1. & \end{array}$$

Therefore, we can write

$$\Theta_0 = Z_1 \Lambda_0 + \Pi_1 Z_0, \qquad \Theta_1 = \Pi_0 Z_1 + Z_0 \Lambda_1,$$

for matrices Z_0, Z_1 which have zero entries save for z_0 in $(3,2)$ in Z_0 and $-z_1$ in $(4,1)$ in Z_1, with $z_0, z_1 \in \mathbb{Q}$. Thus, θ is homotopic to 0 and the lemma follows. ∎

The proof of the lemma mimicked very closely the proof of Lemma 14. The proofs of the following lemmas are omitted, since they are parallel to those of Lemmas 15–18.

Lemma 20. *$f_{33} \neq 0$ if $n = 3$.*

Lemma 21. *$f_{11} = 0$ if $n = 3$ and θ is not an isomorphism.*

Lemma 22. *$c_{22} = 0$ if $n = 3$ and θ is not an isomorphism.*

Lemma 23. *$f_{44} = 0$ if $n = 3$ and θ is not an isomorphism.*

We can assume that $f_{33} = c_{44} = 1$. Entries $(4,3)$ and $(4,4)$ of equation (28) now become

$$b_{43} = \alpha_1 c_{41} + v_3 c_{43} + v_2,$$
$$b_{43} f_{34} = \alpha_1 c_{42} - \alpha_2 c_{43} + \alpha_3,$$

leading to a contradiction. Lemma 13 and Propositions 33, 32 follow in the remaining case $n = 3$. ∎

Shifts. Direct sum decompositions in the above propositions often contain terms shifted by $\langle 1\rangle$. In general, if there is a direct sum decomposition that contains $C(\Gamma_1)$ and $C(\Gamma_2)$ for a pair of graphs Γ_1, Γ_2, to determine whether $C(\Gamma_2)$ will be shifted by $\langle 1\rangle$ relative to $C(\Gamma_1)$ modify Γ_1, Γ_2 by substituting a pair of arcs for each wide edge of these graphs (right to left transformation in Figure 8). The graphs will turn into collections of arcs and circles inside a disc. The two collections share the set of boundary points. Glue them together along their boundaries to get a collection of circles on a 2-sphere. If the number of circles plus half the number of boundary points is odd, there is a shift by $\langle 1\rangle$. Otherwise, $C(\Gamma_1)$ and $C(\Gamma_2)$ are unshifted relative to each other.

Closed graphs. We say that Γ is *closed* if it has no boundary points. $C(\Gamma)$ is a 2-complex ($d^2 = 0$) iff Γ is closed. In this subsection we assume that Γ is closed. Define the *parity* of Γ (denoted $p(\Gamma)$) as the number of circles in the modification of Γ described above (erase a neighborhood of each wide edge, and add two oriented arcs in place of this edge). Let $H(\Gamma)$ be the cohomology groups of $C(\Gamma)$. Since $C(\Gamma)$ has cohomology only in degree $p(\Gamma)$, we have

$$H(\Gamma) = H^{p(\Gamma)}(C(\Gamma)).$$

Internal grading of $C(\Gamma)$ makes $H(\Gamma)$ into a $\mathbb{Z}$-graded $\mathbb{Q}$-vector space.

PROPOSITION 34. *The graded dimension of $H(\Gamma)$ is the invariant $P_n(\Gamma)$,*

$$\sum_{j\in\mathbb{Z}} \dim H^j(\Gamma)\, q^j = P_n(\Gamma).$$

This follows from the direct sum decompositions obtained in this section and the well-known fact that skein relations in Figure 3 suffice to evaluate $P_n(\Gamma)$ for any graph Γ.

Examples of homologically regular pairs. We continue to assume that Γ is closed, and hence $C(\Gamma)$ is a 2-complex. It is cohomology $H(\Gamma)$ concentrated in one degree only. $C(\Gamma)$ is the cyclic Koszul complex $\{\mathbf{a},\mathbf{b}\}$ for a suitable pair $(\mathbf{a},\mathbf{b})$ with $\mathbf{ab} = 0$. If $H^0(\Gamma) \neq 0$, then $H^1(\Gamma) = 0$, and the pair $(\mathbf{a},\mathbf{b})$ is homologically regular, in the terminology of Section 2. If $H^1(\Gamma) \neq 0$, we can permute a_i with b_i in the sequences $\mathbf{a},\mathbf{b}$, for some i, to produce a homologically regular pair. Thus, any closed graph gives rise to a homologically regular pair.

The simplest example of a homologically regular pair is $(\mathbf{0},\mathbf{b})$, where $\mathbf{b}$ is a regular sequence.

For any pair, $H(\{\mathbf{a},\mathbf{b}\})$ is a module over the ring R. Note that if $\mathbf{b}$ is regular, the cohomology $H^0(\{\mathbf{0},\mathbf{b}\})$ is a cyclic module over R, isomorphic to $R/(\mathbf{b})$. Modifications of $(\mathbf{0},\mathbf{b})$ using symmetries from the group G in Section 2 do not change the cohomology and its module structure. All examples of homologically regular pairs given in Section 2 have the property that H^0 is a cyclic R-module.

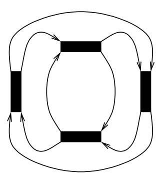

Fig. 43. Graph Γ

Curiously, homologically regular pairs $(\mathbf{a},\mathbf{b})$ assigned to certain graphs Γ have the property that H^0 is not a cyclic R-module. The simplest example of such Γ is shown in Figure 43. We assume that $n > 2$, and each oriented edge has one mark (not shown on the diagram). The graded dimension of $H^0(C(\Gamma)\langle 1\rangle)$ is

$$[n][n-1](2[n-1]+[n-3])$$

and has the form $q^{5-3n}(2+x)$, where $x \in q\mathbb{Z}[q]$. Since the ring R of polynomials in edge variables is nonnegatively graded, with $\mathbb{Q}$ in degree 0, the cohomology of $C(\Gamma)\langle 1\rangle$ is not a cyclic module over R (it is $\mathbb{Q}\oplus\mathbb{Q}$ in the lowest degree). Hence, the pair $(\mathbf{a},\mathbf{b})$ assigned to this graph cannot be obtained from any pair of the form $(\mathbf{0},\mathbf{b}')$ using symmetries from the group G (see the end of Section 2). Thus, the pair assigned to Γ is homologically regular and "intrinsically cyclic", unlike pairs $\{\mathbf{0},\mathbf{b}'\}$ whose cyclic Koszul complexes are just the Koszul complexes of $\mathbf{b}'$ with collapsed grading.

7. Tangle diagrams and complexes of factorizations

Complexes of factorizations. To any additive category $\mathcal{C}$ associate the category $K(\mathcal{C})$ with objects being bounded complexes of objects of $\mathcal{C}$ and morphisms being morphisms of complexes up to homotopy. The category $K(\mathcal{C})$ is triangulated.

Recall that hmf_w is the category of graded factorizations with potential w and finite-dimensional cohomology, up to homotopy. Let $K_w = K(\mathrm{hmf}_w)$, the homotopy category of hmf_w. An object of K_w is $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_2$-graded.

We distinguish the three shift functors in K_w, the shifts functors $\langle 1\rangle, \{1\}$ coming from hmf_w and the shift $[1]$ in the category of complexes. These three functors pairwise commute.

Example. If w is a potential in the empty set of variables, hmf_w is the homotopy category of 2-periodic complexes of graded $\mathbb{Q}$-vector spaces with finite-dimensional cohomology. Any object of hmf_w is then isomorphic to its cohomology, which is a $\mathbb{Z}\oplus\mathbb{Z}_2$-graded vector space, and hmf_w is equivalent to the category of $\mathbb{Z}\oplus\mathbb{Z}_2$-graded finite-dimensional vector spaces. K_w is equivalent to the homotopy category of this category, and therefore to the category of $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_2$-graded finite-dimensional vector spaces.

Tangles in a ball. By a *tangle* L we mean a proper embedding of an oriented compact 1-manifold into a ball B^3. We fix a great circle on the boundary 2-sphere of B^3 and require that the boundary points of the embedded 1-manifold lie on this great circle. A *diagram* D of L is a generic projection of L onto the plane of the great circle. An isotopy of a tangle should not move its boundary points.

A *marked diagram* (also denoted D) is a diagram with several marks placed on D so that any segment bounded by crossings has at least one mark (see an example in Figure 44). Boundary points also count as marks.

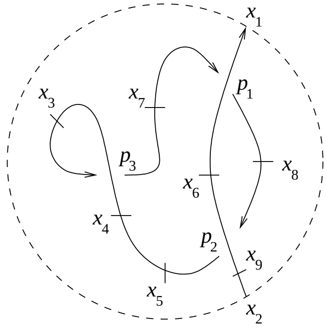

Fig. 44. A marked diagram of a tangle

Let $m(D)$ be the set of marks of D, and ∂D the set of boundary points (it is a subset of $m(D)$). Let R be the ring of polynomials in x_i, $i \in m(D)$, and R' the ring of polynomials in x_i, $i \in \partial D$.

We separate crossings of D into positive and negative as explained in Figure 45.

Fig. 45. Positive and negative crossings

Given a crossing p, let Γ^0, Γ^1 be its two resolutions (see Figure 9). To a positive crossing assign the complex of factorizations (also see Figure 46)

$$0 \to C(\Gamma^0)\{1-n\} \xrightarrow{\chi_0} C(\Gamma^1)\{-n\} \to 0.$$

To a negative crossing assign the complex

$$0 \to C(\Gamma^1)\{n\} \xrightarrow{\chi_1} C(\Gamma^0)\{n-1\} \to 0.$$

In both cases we place $C(\Gamma^0)$ in cohomological degree 0. Denote this complex by C^p.

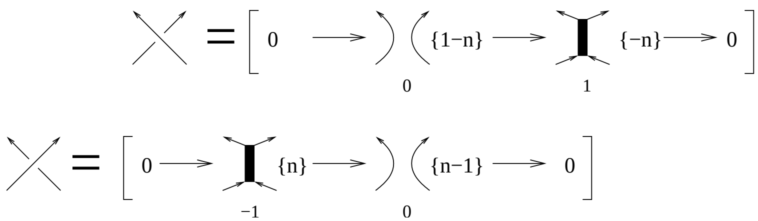

Fig. 46. Complex assigned to a crossing

To D associate the complex of factorizations $C(D)$ which is the tensor product of C^p over all crossings p of L^i_j over all arcs $j \to i$, and of $A\langle 1\rangle$ over all crossingless markless circles of D (if such exist). The tensoring is done over appropriate polynomial rings so that $C(D)$, as an R-module, is free of finite rank.

For instance, to produce $C(D)$ for D in Figure 44, we tensor C^{p_1} with C^{p_2} over $\mathbb{Q}[x_6, x_8]$, and tensor the result with C^{p_3} over $\mathbb{Q}[x_7]$.

We proceed by tensoring $C^{p_1} \otimes C^{p_2} \otimes C^{p_3}$ with L_5^4 over $\mathbb{Q}[x_4, x_5]$, and with L_2^9 over $\mathbb{Q}[x_9]$, so that

$$C(D) = C^{p_1} \otimes C^{p_2} \otimes C^{p_3} \otimes L_5^4 \otimes L_2^9.$$

$C(D)$, for any diagram D, is a complex of graded (R', w)-factorizations, where

$$w = \sum_{i \in \partial D} \pm x_i^{n+1},$$

with signs determined by orientations of D near boundary points. Thus, $C(D)$ is an object of the category K_w.

PROPOSITION 35. *The complexes $C(D)$ and $C(D')$ are canonically isomorphic if D' differs from D only by marks.*

Proof. This follows at once from Proposition 22. ■

THEOREM 2. *The complexes $C(D)$ and $C(D')$ are isomorphic in K_w if D and D' are two diagrams of the same tangle L.*

Proof. It suffices to check the assertion when D and D' are related by a single Reidemeister move. This is done in the next section. ■

COROLLARY 13. *The isomorphism class of the object $C(D)$ in the category K_w is an invariant of the tangle L.*

Link homology. When L is a link, the ring R' equals $\mathbb{Q}$, and hmf_w is isomorphic to the category of finite-dimensional $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$-graded $\mathbb{Q}$-vector spaces. Cohomology groups are nontrivial only in the cyclic degree which is the number of components of L modulo 2. This reduces the grading of cohomology of $C(D)$ to $\mathbb{Z} \oplus \mathbb{Z}$.

Denote the resulting cohomology groups by

$$H_n(D) = \bigoplus_{i,j \in \mathbb{Z}} H_n^{i,j}(D).$$

It is clear from the construction that the Euler characteristic of $H_n(D)$ is the polynomial $P_n(L)$,

$$P_n(L) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim_{\mathbb{Q}} H_n^{i,j}(D).$$

The isomorphism classes of the vector spaces $H_n^{i,j}(D)$ depend only on L. Denote by $h_n^{i,j}(L)$ the dimension of $H_n^{i,j}(D)$. This is an invariant of the link L. For each n, we can put them together into a 2-variable polynomial invariant of L,

$$h_n(L) = \sum_{i,j} t^i q^j h_n^{i,j}(L).$$

Reduced link homology. Choose a component of L, and a mark i on D that belongs to this component. For each resolution Γ of L, the vector space $H(\Gamma)$ is a free module over the ring $A \cong \mathbb{Q}[x_i]/(x_i^n)$. Let $\widetilde{\mathbb{Q}}$ be the one-dimensional graded A-module, placed in degree 0, and

$$\widetilde{C}(D) := C(D) \otimes_A \widetilde{\mathbb{Q}}.$$

The complexes $\widetilde{C}(D)$ and $\widetilde{C}(D')$ are quasi-isomorphic if D and D' represent the same link with the same preferred component. Denote the cohomology groups of $\widetilde{C}(D)$ by $\widetilde{H}_n^{i,j}(D)$, and their dimensions by $\widetilde{h}_n^{i,j}(L)$. Then

$$\widetilde{h}_n(L) := \sum_{i,j} t^i q^j \widetilde{h}_n^{i,j}(L)$$

is a two-variable polynomial invariant of L with a preferred component. Its specialization to $t = -1$ is the one-variable polynomial $P_n(L)/[n]$, which is another common normalization for this one-variable specialization of HOMFLYPT.

8. Invariance under Reidemeister moves

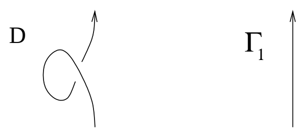

Fig. 47. Type I move

Type I move. Consider the Figure 47 case of the type I move. We follow the notations from the proof of Proposition 29. In particular, Γ, Γ_1, and Γ_2 are as in Figure 33. The complex $C(D)$ has the form

$$0 \to C(\Gamma_2)\langle 1\rangle\{1-n\} \xrightarrow{\chi_0} C(\Gamma)\{-n\} \to 0.$$

Let $\widetilde{\alpha}_i$, for $0 \le i \le n-1$, be the map

$$\widetilde{\alpha}_i : C(\Gamma_1)\{2i+2-2n\} \to C(\Gamma_2)\langle 1\rangle\{1-n\}, \qquad \widetilde{\alpha}_i = \sum_{j=0}^{i} m(x_1^j x_2^{i-j})\iota'.$$

Let $Y_1 \subset C(\Gamma_2)\langle 1\rangle\{1-n\}$ be the image of

$$\bigoplus_{0\le i\le n-2} C(\Gamma_1)\{2i+2-2n\}$$

under the map

$$\widetilde{\alpha} = \sum_{i=0}^{n-2} \widetilde{\alpha}_i,$$

and $Y_2 \subset C(\Gamma_2)\langle 1\rangle\{1-n\}$ the image of $C(\Gamma_1)$ under the map $\widetilde{\alpha}_{n-1}$.

There is a direct sum decomposition in hmf_w

$$C(\Gamma_2)\langle 1\rangle\{1-n\} \cong Y_1 \oplus Y_2.$$

Furthermore, $\chi_0(Y_2) = 0$, and the restriction of χ_0 to Y_1 is an isomorphism from Y_1 to $C(\Gamma)\{-n\}$. Therefore, in the category K_w, the complex $C(D)$ is isomorphic to the direct sum of

$$0 \to Y_1 \xrightarrow{\cong} C(\Gamma)\{-n\} \to 0 \quad \text{and} \quad 0 \to Y_2 \to 0,$$

with Y_2 in cohomological degree 0. Since the first summand is contractible, $C(D)$ is isomorphic to $Y_2 \cong C(\Gamma_1)$ in the category K_w.

The invariance under other cases of the type I move can be verified similarly.

Fig. 48. Type IIa move

Type IIa move. Consider the diagrams D and Γ^0 of Figure 48. The complex $C(D)$,

$$0 \to C^{-1}(D) \xrightarrow{\partial^{-1}} C^0(D) \xrightarrow{\partial^0} C^1(D) \to 0,$$

has the form (see Figure 49)

$$0 \to C(\Gamma_{00})\{1\} \xrightarrow{(f_1,f_3)^t} \begin{matrix} C(\Gamma_{01}) \\ \oplus \\ C(\Gamma_{10}) \end{matrix} \xrightarrow{(f_2,-f_4)} C(\Gamma_{11})\{-1\} \to 0,$$

where f_1, f_4 are given by the map χ_1 (corresponding to the topology change in the lower halves of the diagrams Γ_{00}, Γ_{10}), and f_2, f_3 are given by χ_0 (topology change in the upper halves of Γ_{00}, Γ_{01}).

We know that

$$C(\Gamma_{10}) \cong C(\Gamma^1)\{1\} \oplus C(\Gamma^1)\{-1\}, \tag{29}$$

$$C(\Gamma_{01}) \cong C(\Gamma^0), \tag{30}$$

$$C(\Gamma_{00}) \cong C(\Gamma_{11}) \cong C(\Gamma^1). \tag{31}$$

where Γ^0, Γ^1 are the diagrams of Figure 8, and here and further on the isomorphisms are in the category hmf_w, for the potential $w = x_1^{n+1} + x_2^{n+1} - x_3^{n+1} - x_4^{n+1}$.

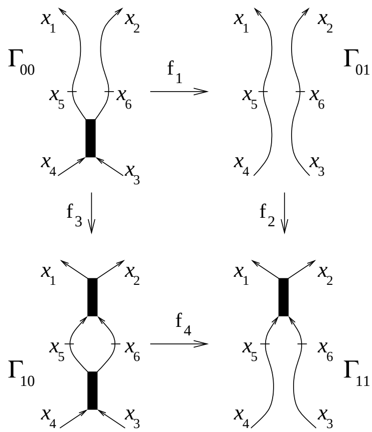

Fig. 49. Four resolutions of D in the type IIa move

Fix isomorphisms (29), (30), (31). Under these isomorphisms, f_3, f_4 become two-component maps

$$f_3 = (f_{03}, f_{13})^t, \qquad f_4 = (f_{04}, f_{14}),$$

where for instance f_{03} (respectively, f_{13}) is a degree 0 (respectively, degree 2) endomorphism of $C(\Gamma^1)$.

LEMMA 24.

(i) *$C(\Gamma^1)$ has no negative degree endomorphisms in* hmf_w. *The only degree* 0 *endomorphisms are multiples of the identity.*
(ii) *The space of degree* 2 *endomorphisms of $C(\Gamma^1)$ is* 3*-dimensional if $n > 2$. Multiplications by x_1, x_2, x_3, x_4 span this space, with the only relation $m(x_1 + x_2 - x_3 - x_4) = 0$.*
(iii) *The space of degree* 2 *endomorphisms of $C(\Gamma^1)$ is* 2*-dimensional if $n = 2$. Multiplications by x_1, x_2, x_3, x_4 span this space, with relations $m(x_1 + x_2) = 0$ and $m(x_3 + x_4) = 0$.*

Proof. Since $C(\Gamma^1)\{1\}$ is the Koszul factorization for the pair

$$((u_1, u_2), (x_1 + x_2 - x_3 - x_4, x_1x_2 - x_3x_4)),$$

the complex $\mathrm{Hom}_R(C(\Gamma^1), C(\Gamma^1))$ is isomorphic to the Koszul complex of the sequence

$$(x_1 + x_2 - x_3 - x_4, x_1x_2 - x_3x_4, u_1, u_2), \tag{32}$$

with the grading collapsed from $\mathbb{Z}$ to $\mathbb{Z}_2$ (for the definition of u_1, u_2 see Section 6). This sequence is regular, so that the cohomology of this 2-complex is

$$\mathbb{Q}[x_1, x_2, x_3, x_4]/(x_1 + x_2 - x_3 - x_4, x_1x_2 - x_3x_4, u_1, u_2).$$

Part (i) of the lemma follows, since this cohomology is isomorphic, as a graded $\mathbb{Q}$-algebra, to $\mathrm{End}_{\mathrm{HMF}}(C(\varGamma^1))$. If $n > 2$, all terms in (32) but the first are homogeneous polynomials in x's that are at least quadratic. Part (ii) follows. If $n = 2$, the term u_2 is linear, and we get the additional relation $m(x_3 + x_4) = 0$. ∎

REMARK. If $n = 1$, there is nothing to prove, since $C(\varGamma) = 0$ whenever $\varGamma$ has a wide edge, and the whole story becomes trivial.

From the lemma we deduce that f_{03} and f_{14} are rational multiples of the identity, while f_{13}, f_{04} are multiplications by linear combinations of x_1, x_2, x_3, x_4.

$C(D)$ is a complex, so that $\partial^0\partial^{-1} = 0$. We compute

$$0 = \partial^0\partial^{-1} = f_2 f_1 - f_4 f_3 = m(x_1 - x_3) - f_{14} f_{13} - f_{04} f_{03}.$$

Therefore, either $f_{14} \neq 0$, or $f_{03} \neq 0$, since multiplication by $x_1 - x_3$ is a nontrivial endomorphism of $C(\varGamma^1)$.

Assume that $f_{14} \neq 0$. Then f_{14} is a nonzero multiple of the identity. By rescaling, we normalize f_{14} to be the identity.

LEMMA 25. *$f_{03} \neq 0$.*

Proof. Suppose that $f_{03} = 0$. Then $f_{13} = m(x_1 - x_3)$. The composition of f_3 with the map χ_1 in the opposite direction, where the topology change takes place around the upper wide edge of $\varGamma_{10}$, is multiplication by $x_1 - x_6$, which is the same as multiplication by $x_1 - x_2$ (since multiplications by x_2 and x_6 are homotopic endomorphisms of $C(\varGamma_{00})$). The map χ_1, restricted to the summand $C(\varGamma^1)\{-1\}$ of $C(\varGamma_{10})$, is multiplication by some rational number z. Under the assumption $f_{03} = 0$, we have

$$m(x_1 - x_2) = \chi_1 f_3 = \chi_1 f_{13} = z \cdot m(x_1 - x_3).$$

Therefore,

$$m((1 - z)x_1 - x_2 + z x_3) = 0.$$

This is impossible, by Lemma 24. Lemma 25 follows. ∎

Likewise, if $f_{03} \neq 0$, we can show that $f_{14} \neq 0$. Thus, both f_{03} and f_{14} are nonzero multiples of the identity morphism. We normalize so that each of them is the identity map. The differential

$$\partial^{-1} : C^{-1}(D) \to C^0(D)$$

is split injective. Since f_{14} is the identity, the differential

$$\partial^0 : C^0(D) \to C^1(D)$$

is split surjective. We can twist the direct sum decomposition

$$C^0(D) \cong C(\varGamma^0) \oplus C(\varGamma^1)\{1\} \oplus C(\varGamma^1)\{-1\}$$

so that f_{03}, f_{14} become the only nonzero entries in the matrices describing the differentials $\partial^{-1}, \partial^0$, and $C(D)$ breaks into the direct sum of three complexes

$$\begin{aligned} 0 \rightarrow \quad & C(\Gamma^0) \quad \rightarrow 0, \\ 0 \rightarrow C(\Gamma^1)\{1\} \overset{\cong}{\rightarrow} \; & C(\Gamma^1)\{1\} \; \rightarrow 0, \\ & 0 \rightarrow C(\Gamma^1)\{-1\} \overset{\cong}{\rightarrow} C(\Gamma^1)\{-1\} \rightarrow 0. \end{aligned}$$

The last two complexes are contractible. Therefore, $C(D)$ and $C(\Gamma^0)$ are isomorphic in the category K_w.

Fig. 50. Type IIb move

Type IIb move. Consider the diagrams D and Γ_2 of Figure 50. The complex $C(D)$ has the form (see Figure 51)

$$0 \rightarrow C(\Gamma_{00})\{1\} \xrightarrow{(f_1, f_3)^t} \begin{matrix} C(\Gamma_{01}) \\ \oplus \\ C(\Gamma_{10}) \end{matrix} \xrightarrow{(f_2, -f_4)} C(\Gamma_{11})\{-1\} \rightarrow 0.$$

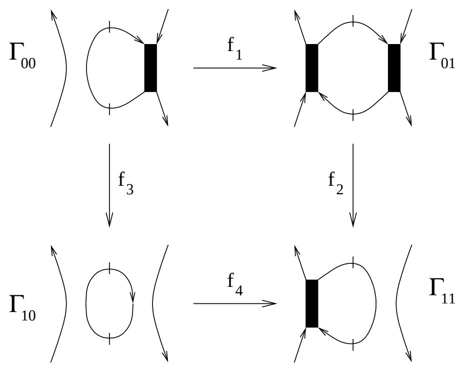

Fig. 51. Commutative square of resolutions of D, type IIb move

There are isomorphisms in hmf_w

$$C(\Gamma_{00})\{1\} \cong \bigoplus_{i=0}^{n-2} C(\Gamma_1)\langle 1\rangle\{3-n+2i\}, \tag{33}$$

$$C(\Gamma_{11})\{-1\} \cong \bigoplus_{i=0}^{n-2} C(\Gamma_1)\langle 1\rangle\{1-n+2i\}, \tag{34}$$

$$C(\Gamma_{10}) \cong \bigoplus_{i=0}^{n-1} C(\Gamma_1)\langle 1\rangle\{1-n+2i\}, \tag{35}$$

$$C(\Gamma_{01}) \cong (\bigoplus_{i=0}^{n-3} C(\Gamma_1)\langle 1\rangle\{3-n+2i\}) \oplus C(\Gamma_2), \tag{36}$$

where Γ_1 and Γ_2 are as in Figure 36.

The proof of the invariance under type I move implies that f_4 is split surjective. Likewise, f_3 is split injective. Since the category hmf_w has splitting idempotents, we can decompose $C^0(D)$ into the direct sum

$$C^0(D) \cong \mathrm{Im}(\partial^{-1}) \oplus Y_1 \oplus Y_2$$

so that ∂^0 restricts to an isomorphism from Y_1 to $C(\Gamma_{11})\{-1\}$ and $\partial^0 Y_2 = 0$. Therefore, $C(D)$ is isomorphic to the direct sum of complexes

$$\begin{aligned} 0 \to \quad Y_2 \quad &\to 0, \\ 0 \to C(\Gamma_{00})\{1\} \xrightarrow{\cong} \mathrm{Im}(\partial^{-1}) &\to 0, \\ 0 \to \quad Y_1 \quad &\xrightarrow{\cong} C(\Gamma_{11})\{-1\} \to 0. \end{aligned}$$

From formulas (33)–(36) we obtain

$$C^0(D) \cong C(\Gamma_{01}) \oplus C(\Gamma_{10}) \cong C(\Gamma_{00})\{1\} \oplus C(\Gamma_{11})\{-1\} \oplus C(\Gamma_2).$$

The category hmf_w is Krull–Schmidt (objects have unique direct sum decompositions into indecomposables). Therefore, $Y_2 \cong C(\Gamma_2)$ and the complexes $C(D)$ and $0 \to C(\Gamma_2) \to 0$ are isomorphic. This concludes our proof of the invariance under the type IIb move.

Type III move. We need to show that $C(D)$ and $C(D')$ are isomorphic for D, D' of Figure 52.

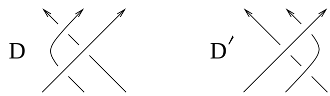

Fig. 52. Type III move

The diagram D has eight resolutions, denoted Γ_{ijk}, for $i,j,k \in \{0,1\}$ (see Figure 53). The complex $C(D)\{-3n\}$ has the form

$$0 \to C(\Gamma_{111}) \xrightarrow{d^{-3}} \begin{pmatrix} C(\Gamma_{011})\{-1\} \\ C(\Gamma_{010})\{-1\} \\ C(\Gamma_{110})\{-1\} \end{pmatrix} \xrightarrow{d^{-2}} \begin{pmatrix} C(\Gamma_{100})\{-2\} \\ C(\Gamma_{010})\{-2\} \\ C(\Gamma_{001})\{-2\} \end{pmatrix} \xrightarrow{d^{-1}} C(\Gamma_{000})\{-3\} \to 0$$

with $C(\Gamma_{000})\{-3\}$ in cohomological degree 0. We depicted this complex in Figure 53.

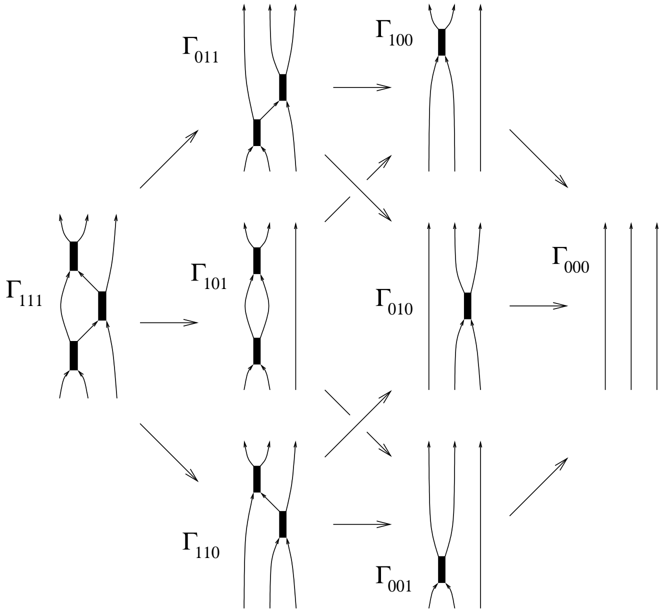

Fig. 53. Resolution cube of D

The diagrams Γ_{100} and Γ_{001} are isotopic, so that $C(\Gamma_{100})$ and $C(\Gamma_{001})$ are isomorphic factorizations. Moreover,

$$C(\Gamma_{101}) \cong C(\Gamma_{100})\{1\} \oplus C(\Gamma_{100})\{-1\}, \tag{37}$$

and from Proposition 33 we know that

$$C(\Gamma_{111}) \cong C(\Gamma_{100}) \oplus \Upsilon.$$

The differential d^{-3} is injective on $C(\Gamma_{100}) \subset C(\Gamma_{111})$. In fact, its middle component (the map to $C(\Gamma_{101})\{-1\}$) is injective, which follows from our construction of the inclusion $C(\Gamma_{100}) \subset C(\Gamma_{111})$ and the proof of the invariance under the type IIa move.

The factorization $d^{-3}(C(\Gamma_{100}))$ is a direct summand of $C^{-2}(D)\{-3n\}$. Thus, $C(D)\{-3n\}$ contains a contractible summand

$$0 \to C(\Gamma_{100}) \xrightarrow{d^{-3}} C(\Gamma_{100}) \to 0. \tag{38}$$

The direct sum decomposition (37) can be selected so that

$$C(\Gamma_{101})\{-1\} \cong p_{101} d^{-3} C(\Gamma_{100}) \oplus C(\Gamma_{100})\{-2\}$$

where p_{101} is the projection onto the middle summand of $C^{-2}(D)\{-3n\}$.

The differential d^{-2} is injective on $C(\Gamma_{100})\{-2\} \subset C(\Gamma_{101})\{-1\}$, since its middle component is a nonzero multiple of the identity. Furthermore, the image of $C(\Gamma_{100})\{-2\} \subset C(\Gamma_{101})\{-1\}$ under d^{-2} is a direct summand of $C^{-1}(D)$. Hence, the complex $C(D)\{-3n\}$ contains a contractible direct summand isomorphic to

$$0 \to C(\varGamma_{100})\{-2\} \xrightarrow{d^{-2}} C(\varGamma_{100})\{-2\} \to 0. \tag{39}$$

After splitting off the contractible direct summands (38) and (39), the complex $C(D)\{-3n\}$ reduces to the complex C of the form

$$0 \to \varUpsilon \xrightarrow{d^{-3}} \begin{pmatrix} C(\varGamma_{011})\{-1\} \\ C(\varGamma_{110})\{-1\} \end{pmatrix} \xrightarrow{d^{-2}} \begin{pmatrix} C(\varGamma_{010})\{-2\} \\ C(\varGamma_{100})\{-2\} \end{pmatrix} \xrightarrow{d^{-1}} C(\varGamma_{000})\{-3\} \to 0$$

(see Figure 54). To the diagram Y on the left of the figure we assign the factorization $\varUpsilon$.

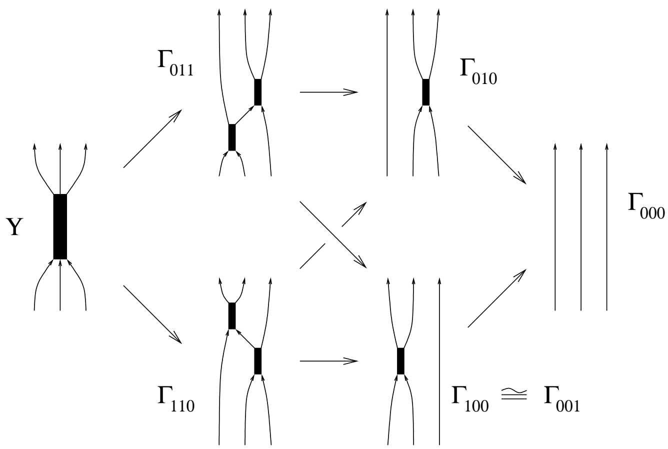

Fig. 54. Complex C

In the rest of the section we assume that $n > 2$ (proofs for $n = 2$ are easier since then $\varUpsilon = 0$).

Lemma 26. *For each arrow in Figure* 54 *with some diagram* Z_1 *as the source and* Z_2 *as the target*, *the space of grading-preserving morphisms*

$$\mathrm{Hom}_{\mathrm{hmf}}(C(Z_1), C(Z_2)\{-1\})$$

is one-dimensional.

Proof. Straightforward, similar to the proof of Corollary 12. The details are omitted. ∎

Denote by $c_1, \dots, c_8$ the nontrivial morphisms from the above lemma. Each morphism is determined up to multiplication by a nonzero rational number. Figure 55 shows how the indices i of c_i's match the eight arrows of Figure 54. We choose c_i's so that each square in Figure 55 commutes.

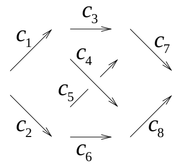

Fig. 55. Indices and arrows

The differential in C is grading-preserving. Therefore, it has the form

$$d^{-3} = \begin{pmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \end{pmatrix}, \quad d^{-2} = \begin{pmatrix} \lambda_3 c_3 & \lambda_5 c_5 \\ \lambda_4 c_4 & \lambda_6 c_6 \end{pmatrix}, \quad d^{-1} = \begin{pmatrix} \lambda_7 c_7 \\ \lambda_8 c_8 \end{pmatrix}$$

for some rational numbers $\lambda_1, \ldots, \lambda_8$.

Lemma 27. *For any two composable arrows $Z_1 \to Z_2 \to Z_3$ in Figure* 54, *the corresponding homomorphism of factorizations*

$$c_j c_i : C(Z_1) \to C(Z_3)\{-2\}$$

is nonzero in hmf_w.

Proof. For instance, to show that $c_7 c_3 \neq 0$, we note that this composition is the product of two χ_1 maps (up to rescaling by a nonzero rational number), one for each wide edge of Γ_{011}. Compose it with the "dual" product of two χ_0 maps, as in Figure 56. Assign variables $x_1, \ldots, x_6$ to the endpoints of our diagrams in the same way as in Figure 42. Then the Figure 56 map is $m(x_1 - x_5)m(x_2 - x_6)$. This is a nontrivial endomorphism of $C(\Gamma_{000})$, so that the composition $c_7 c_3$ is nonzero.

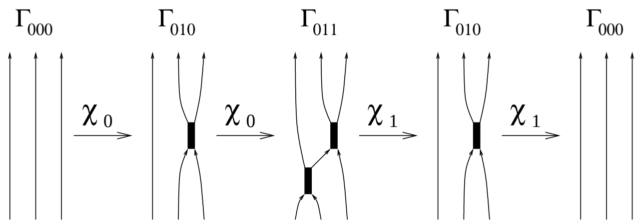

Fig. 56. Composing $c_7 c_3$ with its dual

Identical arguments take care of the other three compositions that end in Γ_{000}. Similar but longer computations can be used to check nontriviality of each of the four compositions $c_j c_i$ that start at Y. ∎

The category K_w has an automorphism (tensoring with the complex assigned to the inverse braid of D) that takes $C(D)$ to $C(D'')$ where D'' is the diagram made up of three disjoint oriented arcs. Since $C(D'')$ is indecomposable in K_w, the same is true of $C(D)$ and $C \cong C(D)$. Indecomposability of C together with the above lemma implies that none of the coefficients

$\lambda_1, \dots, \lambda_8$ is 0. By rescaling, we can set

$$\lambda_1 = \lambda_2 = \lambda_4 = \lambda_5 = \lambda_7 = \lambda_8 = 1, \qquad \lambda_3 = \lambda_6 = -1.$$

The complex C is therefore invariant under the "flip" which transposes x_1 with x_3 and x_4 with x_6. This flip takes $C(D)$ to $C(D')$, thus, C is isomorphic in K_w to $C(D')$. We have $C(D) \cong C \cong C(D')$. The invariance under the Reidemeister move III of Figure 52 follows.

9. Factorizations and 2-dimensional TQFTs with corners. Let I be a finite set and $s : I \to \{1, -1\}$ a map, which we call the *orientation map*. We say that s is *balanced* if the sets $s^{-1}(1)$ and $s^{-1}(-1)$ have the same cardinality (i.e., s takes as many elements of I to 1 as it does to -1). From now on we assume that s is balanced.

To (I, s) we assign the category $\mathrm{Cob}_{I,s}$ whose objects are oriented one-dimensional manifolds N with boundary I such that the orientation of N induces the orientation s on $I = \partial N$. Morphisms from N_0 to N_1 are oriented 2-dimensional surfaces S with boundary $N_0 \cup -N_1 \cup I \times [0,1]$ and corners $I \times \{0\} \sqcup I \times \{1\}$.

Let R_I be the ring of polynomials in x_i, $i \in I$, and set

$$w(I, s) = \sum_{i \in I} s(i) x_i^{n+1} \in R_I.$$

To an object $N \in \mathrm{Cob}_{I,s}$ we can assign a factorization $C(N)$ with potential $w(I, s)$. The factorization is the tensor product (over $\mathbb{Q}$) of L^i_j over all arcs $j \to i$ in N, and of $A\langle 1 \rangle$, one for each circle in N. We would like to extend this assignment to a functor from $\mathrm{Cob}_{I,s}$ to the category of factorizations. To do this, pick a morphism S in $\mathrm{Cob}_{I,s}$ and write it as a composition of simple cobordisms (cobordisms with only one critical point). To the cobordisms of creation and annihilation of a circle we assign the maps ι, ε, defined in Section 4. To the saddle point cobordism between 1-manifolds N_0, N_1 of Figure 57 we want to associate a map

$$\eta : C(N_0) \to C(N_1)\langle 1 \rangle \{1 - n\}.$$

To define η, assign labels a_1, a_2, b_1, b_2 to the components of N_0, N_1 as indicated in Figure 57.

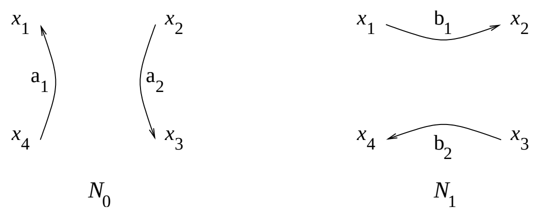

Fig. 57

The factorization $C(N_0)$ has the form

$$\begin{pmatrix} R(\emptyset) \\ R(a_1a_2)\{2-2n\} \end{pmatrix} \xrightarrow{P_0} \begin{pmatrix} R(a_1)\{1-n\} \\ R(a_2)\{1-n\} \end{pmatrix} \xrightarrow{P_0'} \begin{pmatrix} R(\emptyset) \\ R(a_1a_2)\{2-2n\} \end{pmatrix},$$

where

$$P_0 = \begin{pmatrix} \pi_{14} & x_3 - x_2 \\ \pi_{32} & x_4 - x_1 \end{pmatrix}, \qquad P_0' = \begin{pmatrix} x_1 - x_4 & x_3 - x_2 \\ \pi_{32} & -\pi_{14} \end{pmatrix},$$

and $R = \mathbb{Q}[x_1, x_2, x_3, x_4]$.

The factorization $C(N_1)\langle 1\rangle$ is given by

$$\begin{pmatrix} R(b_1)\{1-n\} \\ R(b_2)\{1-n\} \end{pmatrix} \xrightarrow{P_1} \begin{pmatrix} R(\emptyset) \\ R(b_1b_2)\{2-2n\} \end{pmatrix} \xrightarrow{P_1'} \begin{pmatrix} R(b_1)\{1-n\} \\ R(b_2)\{1-n\} \end{pmatrix},$$

where

$$P_1 = \begin{pmatrix} x_2 - x_1 & x_4 - x_3 \\ -\pi_{34} & \pi_{12} \end{pmatrix}, \qquad P_1' = \begin{pmatrix} -t_{12} & x_4 - x_3 \\ -\pi_{34} & x_1 - x_2 \end{pmatrix},$$

and $R = \mathbb{Q}[x_1, x_2, x_3, x_4]$.

η is given by the pair of matrices

$$\begin{pmatrix} e_{123} + e_{124} + (x_4 - x_3)r & 1 \\ -e_{134} - e_{234} + (x_1 - x_2)r & 1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 1 \\ -e_{123} - e_{234} + (x_1 - x_4)r & -e_{134} - e_{124} + (x_3 - x_2)r \end{pmatrix},$$

with

$$e_{ijk} = \sum_{a+b+c=n-1} x_i^a x_j^b x_k^c$$

and r being an arbitrary polynomial of degree $n-2$ in $x_1, \dots, x_4$. Up to chain homotopy, η does not depend on r.

η generates $\mathrm{Hom}_{\mathrm{HMF}}(C(N_0), C(N_1)\langle 1\rangle)$ as an R-module, and we should assign η to the saddle point cobordism. There is a problem, though. The diagrams in Figure 57 are invariant under rotation by 180°, but η acquires a sign after the rotation. Namely, if we transpose x_1 with x_3, x_2 with x_4, a_1 with a_2, and b_1 with b_2 in the formula for η, the resulting map is $-\eta$. Thus, η can be canonically defined only up to sign.

If the saddle point cobordism takes place not between arcs, but between components of N_0, N_1, some of which are circles, we add marks to each component, select a suitable pair of arcs bounded by marks (and, possibly, by boundary points), apply η to that pair, and finally erase marks. It is easy to see that, up to sign, the resulting map does not depend on the intermediate choices.

Let $\mathrm{H}'_{I,s}$ be the category with the same objects as $\mathrm{hmf}_{w(I,s)}$ but morphisms being the Ext groups $\mathrm{Ext}_{\mathrm{HMF}}(M_0, M_1)$, with f and $-f$ identified for all $f \in \mathrm{Ext}_{\mathrm{HMF}}(M_0, M_1)$. We use Ext groups rather than just Hom's since the shift $\langle 1 \rangle$ is built into ι, ε, and η.

Given a surface S which is an object of $\mathrm{Cob}_{I,s}$, write it as a product of cobordisms with only one critical point and define $C(S)$ as the corresponding product of ι, ε, and η's. Recall that our definition of ε contained a parameter $\zeta \in \mathbb{Q}^*$. To make ε compatible with η we must set ζ to either $1/(n+1)$ or $-1/(n+1)$ (so that $\varepsilon\eta = \pm\mathrm{Id}$ if the saddle point cobordism goes from an arc to the union of a circle and an arc).

PROPOSITION 36. *$\pm C(S)$ does not depend on the presentation of S as a product of elementary cobordisms.*

The proof is left to the reader.

Thus, we obtain a functor from the cobordism category $\mathrm{Cob}_{I,s}$ to the category $\mathrm{H}'_{I,s}$. The shifts $\upsilon_1(S), \upsilon_2(S)$ in

$$\pm C(S) : C(N_0) \to C(N_1)\langle \upsilon_1(S)\rangle\{\upsilon_2(S)\}$$

are as follows. Glue N_0 and N_1 along the common boundary I, and count the number υ of components in the closed 1-manifold that results. $\upsilon_1(S)$ is the parity of $\upsilon + |I|/2$, while

$$\upsilon_2(S) = (n-1)\left(\chi(S) - \frac{|I|}{2}\right)$$

where $\chi(S)$ is the Euler characteristic of S.

2-functor. In the above construction we fixed the boundary of one-manifolds. If we put together all categories $\mathrm{Cob}_{I,s}$ over various I and s (and consider decompositions of I into pairs of disjoint sets, to view N with $\partial N = I$ as a morphism) we get a 2-category of oriented surfaces with corners. The functors

$$C : \mathrm{Cob}_{I,s} \to \mathrm{H}'_{I,s}$$

extend to a 2-functor from this 2-category of cobordisms to the 2-category of factorizations with potentials as objects, factorizations as morphisms, and elements of Ext groups between factorizations (with f and $-f$ identified) as 2-morphisms. We leave the details to the reader.

10. Projective invariance for cobordisms of tangles. As before, we consider oriented tangles in a ball B^3. Fix a great circle on the ball's boundary, choose a finite subset I of this circle and a balanced "orientation" function $s : I \to \{1, -1\}$. Let $\mathrm{TC}_{I,s}$ be the category of tangle cobordisms with objects oriented tangles L in B^3 with oriented boundary (I, s) and morphisms from L_0 to L_1 oriented surfaces S embedded in $B^3 \times [0,1]$ with boundary

$$\partial S = L_0 \times \{0\} \cup L_1 \times \{1\} \cup I \times [0,1]$$

and corners $I \times \{0\} \cup I \times \{1\}$, up to isotopy that fixes the boundary.

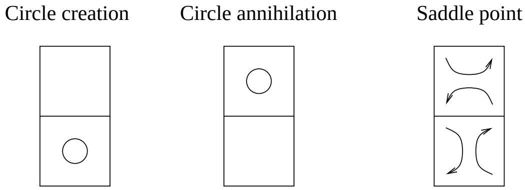

Fig. 58. Movie moves of simple cobordisms

A cobordism S admits a combinatorial description via a sequence of plane diagrams of its cross-sections with $B^3 \times k$, for various $k \in [0,1]$. Each consecutive pair of diagrams differ either by a Reidemeister move, or a Morse move, the latter describing a simple cobordism with one critical point (Figure 58). Such sequences are referred to as *movies*. Two sequences describe the same cobordism if they can be connected through a finite sequence of *movie moves*, shown in Figures 59, 60. We assume that the reader is familiar with this theory, and refer to [CS1], [CS2] and references therein for details.

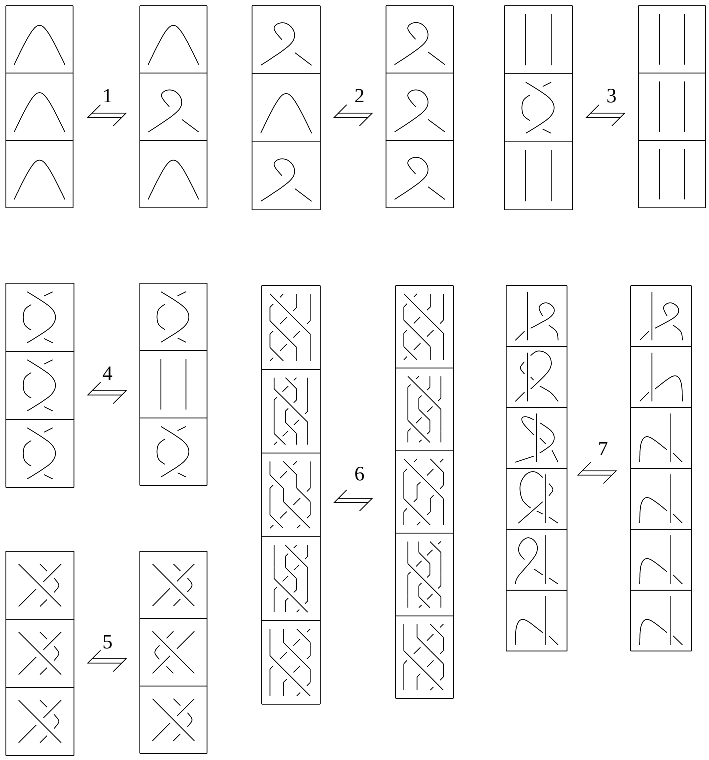

Fig. 59. Movie moves 1–7

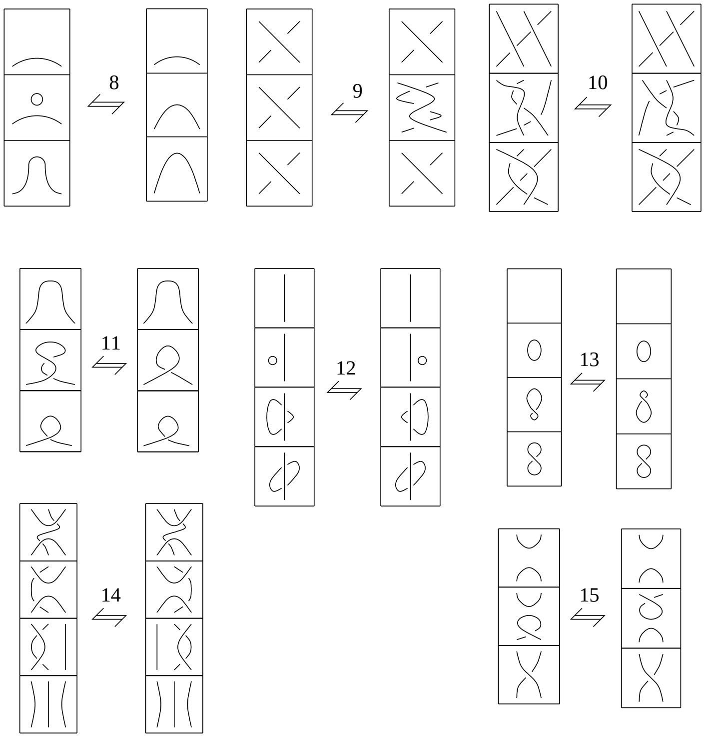

Fig. 60. Movie moves 8–15

Let $w = w(I, s)$ be the potential defined in the preceding section. Earlier, we associated an isomorphism $C(D_1) \cong C(D_2)$ in the category K_w to a Reidemeister move between the tangle diagrams D_1 and D_2. Aping the last section, assign the maps ι, ε, η to the circle creation, circle annihilation, and saddle point moves (see Figure 58), respectively.

Given a movie $z = (z_0, \ldots, z_m)$ representing a surface S, assign to z the morphism in K_w:

$$C(z) : C(z_0) \to C(z_m)\langle \upsilon_1(S)\rangle\{\upsilon_2(S)\}$$

by composing the maps associated to each move $z_i \to z_{i+1}$ in z. The quantities $\upsilon_1(S), \upsilon_2(S)$ were defined at the end of the previous section.

Proposition 37. *Up to overall multiplication by nonzero rational numbers, the map $C(z)$ (viewed as a morphism in the category K_w) does not depend on the movie presentation z of S.*

Proof. For a given movie move of Figures 59, 60, denote the top frame by b_1, the bottom frame by b_2, the left movie by S_l and the right movie by S_r. We need to show that the morphisms $C(S_l)$ and $C(S_r)$ are proportional, $C(S_l) = \lambda C(S_r)$ for some $\lambda \in \mathbb{Q}^*$.

The movies S_l and S_r in movie move 6 are compositions of Reidemeister moves. Therefore,

$$C(S_l), C(S_r) : C(b_1) \to C(b_2)$$

are isomorphisms. Let b be the tangle diagram which consists of four parallel disjoint segments (the crossingless diagram of the trivial braid). There is an automorphism of K_w (tensoring with $C(b_1')$, where b_1' is the "inverse" of the braid diagram b_1) which takes $C(b_1)$ to $C(b)$. Therefore,

$$\mathrm{Hom}_{K_w}(C(b_1), C(b_1)) \cong \mathrm{Hom}_{K_w}(C(b), C(b)).$$

The vector space on the right hand side is isomorphic to $\mathbb{Q}$ (the only degree 0 endomorphisms of $C(b)$ are multiples of the identity). Hence

$$\mathrm{Hom}_{K_w}(C(b_1), C(b_2)) \cong \mathrm{Hom}_{K_w}(C(b_1), C(b_1)) \cong \mathbb{Q}$$

(the isomorphisms are not canonical, though), and $C(S_l), C(S_r)$ are proportional.

This argument applies to moves 1, 2, 3, 4, 5, 7, 9, 10, 11 as well, and to all versions of these moves (various orientations, overcrossing/undercrossing variations, etc.)

Move 8 follows from the compatibility of ι, ε, and η (see Section 9).

In move 12, the maps

$$C(S_l), C(S_r) : C(b_1) \to C(b_2)\langle 1\rangle\{n-1\}$$

are nontrivial and lie in the one-dimensional $\mathbb{Q}$-vector space

$$\mathrm{Hom}_{K_w}(C(b_1), C(b_2)\langle 1\rangle\{n-1\}).$$

Therefore, the maps are proportional. The same argument works for other versions of this move, and for all versions of moves $13, 14, 15$ (for moves $14, 15$ change $\{n-1\}$ to $\{1-n\}$). Proposition 37 follows. ∎

The above proof is based on the observation that the space of homs between $C(b_1)$ and $C(b_2)$ (with a suitable shift) is one-dimensional. The same approach was used in [Kh4] to show functoriality of the homology theory $\mathcal{H}$ from [Kh1] (see Jacobsson [J] for a different proof), and by Dror Bar-Natan [BN2] to prove functoriality of his refinement of $\mathcal{H}$.

We denote by $C(S)$ the set $\{\lambda C(z) \mid \lambda \in \mathbb{Q}^*\}$. This is an invariant of the cobordism S.

In particular, given a diagram D of a tangle L, the object $C(D)$ of K_w is canonically (up to rescaling) associated to L. We denote this object by $C(L)$ (recall that it is a complex of graded matrix factorizations, treated as an object of K_w).

Given a diagram D of an oriented link L, the homology groups $H_n(D)$ are $\mathbb{Q}$-vector spaces canonically assigned to L (up to overall rescaling by nonzero rational numbers). We denote these groups by $H_n(L)$ and their graded summands by $H_n^{i,j}(L)$.

Thus, an oriented link cobordism S between L_0 and L_1 induces a homomorphism

$$C(S) : H_n(L_0) \to H_n(L_1),$$

well-defined up to rescalings by nonzero rationals, and for each i, j restricts to homomorphisms

$$H_n^{i,j}(L_0) \to H_n^{i,j+(1-n)\chi(S)}(L_1),$$

where $\chi(S)$ is the Euler characteristic of S. The collection of maps $C(S)$ over all link cobordisms S is a functor from the category of link cobordisms to the category of bigraded $\mathbb{Q}$-vector spaces (with morphisms being graded linear maps with the equivalence relation $f \sim \lambda f$ for $\lambda \in \mathbb{Q}^*$). The Euler characteristic of $H_n(L)$ is the polynomial

$$P_n(L) = \sum_{i,j} (-1)^i q^j \dim_{\mathbb{Q}} H_n^{i,j}(L).$$

2-functor. So far we considered tangles with a fixed oriented boundary (I, s). By switching from tangles in a ball to tangles in $\mathbb{R}^2 \times [0, 1]$ and varying possible boundaries one can form the 2-category of tangle cobordisms (see [F], [BL], and references therein). Our construction can be extended to a 2-functor from the 2-category of oriented tangle cobordisms to a 2-category with potentials $w(I, s)$ as objects, complexes of matrix factorizations as 1-morphisms, and homomorphisms between (suitably shifted) complexes as 2-morphisms (of course, we will have to quotient by null-homotopic morphisms, and identify morphisms that are multiples of each other, $f \cong \lambda f$ for $\lambda \in \mathbb{Q}^*$). This 2-functor is braided monoidal.

11. A generalization. Each complex simple Lie algebra $\mathfrak{g}$ gives rise to a polynomial invariant of links whose components are decorated by finite-dimensional irreducible representations of $\mathfrak{g}$ (see [RT]). The polynomial P_n results if $\mathfrak{g} = \mathfrak{sl}_n$ and every component is assigned the fundamental n-dimensional representation V of $\mathfrak{sl}_n$. Murakami, Ohtsuki, and Yamada [MOY] develop a calculus of trivalent graphs that helps understand the polynomial invariant of links with components colored by arbitrary exterior powers of V (the ith exterior power $\Lambda^i V$ is often called the ith fundamental representation of $\mathfrak{sl}_n$). Although their construction also extends the invariant to spacial trivalent graphs, here we will only look at planar graphs. Each edge of a graph is oriented and labeled by a number from 1 to $n-1$. Every vertex is trivalent and the sum of the labels at the edges entering the vertex minus the sum of the labels at the edges leaving it is a multiple of n.

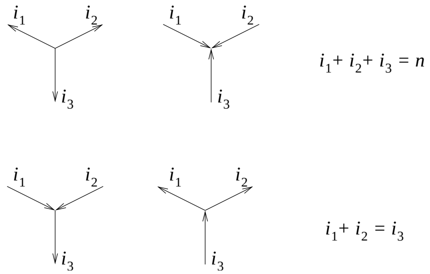

Fig. 61. The four types of vertices

To every such graph Γ an invariant $\langle \Gamma \rangle$ is assigned, taking values in $\mathbb{Z}[q, q^{-1}]$. The invariant is unchanged if the orientation of an edge is reversed simultaneously with changing the label from i to $n-i$. We use this transformation to reduce our considerations to graphs with all labels at most $n/2$. We split the vertices into four types by the number (zero to three) of edges oriented into the vertex (see Figure 61).

The construction of Section 6 generalizes to a homology theory $H(\Gamma)$ for graphs Γ as above. We will now sketch this generalization and its conjectural extension to link homology. For convenience, assume that n is even.

Denote by $i(e)$ the number assigned to an edge e, and select a set $s(e)$ of cardinality $i(e)$ such that the sets assigned to different edges are disjoint. Suppose that a vertex v bounds the edges e_1, e_2, e_3. To v we assign the potential

$$w_v = \sum_{j \in s(e_1) \sqcup s(e_2) \sqcup s(e_3)} \pm x_j^{n+1}$$

where the sign is $+$ if the edge e_k leaves v and $j \in s(e_k)$, and $-$ otherwise (note that the potential of an edge e is $\pm \sum_{j \in s(e)} x_j^{n+1}$).

Let $\mathbb{Q}[s(e)]$ be the ring of polynomials in x_j, $j \in s(e)$, and denote by $S(e)$ its subring of symmetric polynomials. Let

$$R_v := S(e_1) \otimes_{\mathbb{Q}} S(e_2) \otimes_{\mathbb{Q}} S(e_3)$$

be the tensor product of the three rings. Then $w_v \in R_v$.

Consider first the case when the edges e_1, e_2, e_3 are oriented away from v. Then $i(e_1) + i(e_2) + i(e_3) = n$. Let $\sigma_k(v)$ be the kth elementary symmetric function in the x_j's for j in the set $s(e_1) \sqcup s(e_2) \sqcup s(e_3)$. Write

$$w_v = \sum_j x_j^{n+1} = \sum_{k=1}^{n} \sigma_k(v) g_k(v)$$

for some $g_k(v)$'s (which are not uniquely defined, of course).

To v we assign the factorization C_v which is the tensor product of

$$R_v \xrightarrow{g_k(v)} R_v \xrightarrow{\sigma_k(v)} R_v$$

over $1 \le k \le n$.

If the edges e_1, e_2, e_3 are all oriented towards v, then

$$w_v = -\sum_j x_j^{n+1} = -\sum_{k=1}^{n} \sigma_k(v) g_k(v),$$

and to v we assign the factorization C_v which is the tensor product of

$$R_v \xrightarrow{g_k(v)} R_v \xrightarrow{-\sigma_k(v)} R_v$$

over $1 \le k \le n$.

Suppose now that, say, e_1, e_2 are oriented away from v and e_3 towards v. Then $i(e_3) = i(e_1) + i(e_2)$ and

$$w_v = \sum_{j \in s(e_1) \sqcup s(e_2)} x_j^{n+1} - \sum_{j \in s(e_3)} x_j^{n+1}.$$

Let σ'_k be the kth elementary symmetric function in x_j, $j \in s(e_1) \sqcup s(e_2)$, and σ''_k be the kth elementary symmetric function in x_j, $j \in s(e_3)$. Since w_v belongs to the ideal of R_v generated by the differences $\sigma'_k - \sigma''_k$, we can write

$$w_v = \sum_{k=1}^{i(e_3)} (\sigma'_k - \sigma''_k) g_k(v)$$

for some $g_k(v) \in R_v$, $1 \le k \le i(e_3)$. Now assign to the vertex v the factorization C_v which is the tensor product of

$$R_v \xrightarrow{g_k(v)} R_v \xrightarrow{\sigma'_k - \sigma''_k} R_v$$

for $1 \le k \le i(e_3)$.

Likewise, assign to v the tensor product of

$$R_v \xrightarrow{g_k(v)} R_v \xrightarrow{\sigma''_k - \sigma'_k} R_v$$

if two edges e_1, e_2 are oriented inwards, and e_3 outwards.

If Γ does not contain loops, define $H(\Gamma)$ as the homology of the 2-complex $C(\Gamma)$ which is the tensor product of C_v over all vertices v of Γ. The tensor product is taken over intermediate rings $S(e)$ for various edges e, so that $C(\Gamma)$ is a finite-rank module over the ring $\bigotimes_e S(e)$.

When loops are present in Γ, a loop labeled i will "contribute" to the tensor product C_v the cohomology of the Grassmannian $\mathrm{Gr}(i, n)$ of i-dimensional subspaces in $\mathbb{C}^n$.

We conjecture that $H(\Gamma)$ has cohomology in only one out of its two degrees. An additional $\mathbb{Z}$-grading on $H(\Gamma)$ comes from grading on the rings $S(e)$, with $\deg(x_j) = 2$. We conjecture that the graded dimension of $H(\Gamma)$ is the invariant $\langle \Gamma \rangle$ (up to obvious normalizations such as multiplication by a power of q, etc.).

The above construction should work when n is even. If n is odd, add a variable x_v to each vertex with all edges oriented in, and to each vertex with all edges out. Then add x_v^2 to the potential w_v, enlarge the ring R_v by adjoining the variable x_v, and modify the factorization C_v correspondingly. We omit the details.

Given a diagram D of an oriented framed link L colored by numbers from 1 to $n-1$, each crossing can be resolved in a number of ways into planar graphs, and the invariant $P_n(L)$ is a linear combination of $\langle \Gamma \rangle$ for various resolutions Γ of D, with coefficients which are plus or minus powers of q (see Section 5 of [MOY]). We conjecture that, likewise, the $\mathbb{Q}$-vector spaces $H(\Gamma)$ can be naturally strung together into a complex $C(D)$ whose (bigraded) cohomology groups are invariants of L and have graded Euler characteristic $P_n(L)$. For each n, this homology theory of colored oriented framed links in $\mathbb{R}^3$ should be functorial under (oriented, framed, and suitably decorated) link cobordisms; it should also extend to a homology theory of (decorated) spatial trivalent graphs, and be functorial under (carefully defined) graph cobordisms in $\mathbb{R}^3 \times [0,1]$.

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