On universality of countable and weak products of sigma hereditarily disconnected spaces

by

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Abstract. Suppose a metrizable separable space $Y$ is sigma hereditarily disconnected, i.e., it is a countable union of hereditarily disconnected subspaces. We prove that the countable power $X^\omega$ of any subspace $X \subseteq Y$ is not universal for the class $\mathcal{A}_2$ of absolute $G_{\delta\sigma}$-sets; moreover, if $Y$ is an absolute $F_{\sigma\delta}$-set, then $X^\omega$ contains no closed topological copy of the Nagata space $\mathcal{N} = W(I, P)$; if $Y$ is an absolute $G_{\delta}$-set, then $X^\omega$ contains no closed copy of the Smirnov space $\mathcal{S} = W(I, 0)$.

On the other hand, the countable power $X^\omega$ of any absolute retract of the first Baire category contains a closed topological copy of each $\sigma$-compact space having a strongly countable-dimensional completion.

We also prove that for a Polish space $X$ and a subspace $Y \subseteq X$ admitting an embedding into a $\sigma$-compact sigma hereditarily disconnected space $Z$ the weak product $W(X, Y) = \{(x_i) \in X^\omega : \text{almost all } x_i \in Y\} \subseteq X^\omega$ is not universal for the class $\mathcal{M}_3$ of absolute $G_{\delta\sigma\delta}$-sets; moreover, if the space $Z$ is compact then $W(X, Y)$ is not universal for the class $\mathcal{M}_2$ of absolute $F_{\sigma\delta}$-sets.

A topological space $X$ is called $\mathcal{C}$-universal, where $\mathcal{C}$ is a class of spaces, if for every space $C \in \mathcal{C}$ there is a closed embedding $f : C \to X$. It is well known that the Hilbert cube $Q = [0, 1]^\omega$ is $\mathcal{M}_0$-universal, whereas its pseudointerior $s = (0, 1)^\omega$ is $\mathcal{M}_1$-universal, where $\mathcal{M}_0$ and $\mathcal{M}_1$ are the Borel classes of compact and Polish spaces, respectively (all spaces considered in this paper are metrizable and separable, all maps are continuous). Let us remark that both $Q$ and $s$ are countable products of finite-dimensional spaces. This raises the following question: can the countable power $X^\omega$ of a finite-dimensional space $X$ be $\mathcal{C}$-universal for a higher Borel class $\mathcal{C}$? Taking into account results of [BR] and [Ca$_1$], it was conjectured in [Ba] that the

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countable power $X^\omega$ of any finite-dimensional (resp. strongly countable-dimensional) space $X$ is not $A_1$-universal (resp. $A_2$-universal). Here $A_1$ and $A_2$ are the Borel classes of $\sigma$-compact and absolute $G_{\delta\sigma}$-spaces, respectively.

In this paper we confirm this conjecture. We define a space $X$ to be \textit{sigma hereditarily disconnected} provided $X$ can be written as a countable union $X = \bigcup_{n=1}^{\infty} X_n$ of hereditarily disconnected spaces. Recall that a space $X$ is \textit{hereditarily disconnected} if it contains no connected subset containing more than one point (see [En, 1.4.2]).

For a class $C$ of spaces we denote by $C(c.d.)$ and $C(s.c.d.)$ the subclasses of $C$ consisting of countable-dimensional and strongly countable-dimensional spaces $C \in C$, respectively. Let us remark that each strongly countable-dimensional space is countable-dimensional and each countable-dimensional space is sigma hereditarily disconnected.

**Theorem 1.** (1) If a space $X$ has a sigma hereditarily disconnected completion, then the countable power $X^\omega$ is not $A_1(s.c.d.)$-universal.

(2) If a space $X$ embeds into a sigma hereditarily disconnected absolute $F_{\sigma\delta}$-space, then $X^\omega$ is not $A_2(c.d.)$-universal.

(3) If a space $X$ is sigma hereditarily disconnected, then $X^\omega$ is not $A_2$-universal.

For a class $C$ of spaces let $C(s.c.d.c.)$ denote the subclass of $C$ consisting of spaces with a strongly countable-dimensional completion. The class $A_1(s.c.d.)$ from the first statement of Theorem 1 is the best possible in the following sense.

**Theorem 2.** If $X$ is an absolute retract of the first Baire category, then the countable power $X^\omega$ is $A_1(s.c.d.c.)$-universal.

Clearly, there exist finite-dimensional $\sigma$-compact absolute retracts of the first Baire category, for example the space $X = D \setminus E$, where $D$ is a dendrite with a dense set $E$ of end-points.

Countable powers are partial cases of \textit{weak products}

$$W(X, A) = \{(x_i) \in X^\omega : x_i \in A \text{ for all but finitely many indices } i\},$$

where $A$ is a subset of a space $X$.

The most known and important weak products are the Smirnov space $\sigma = W(I, \{0\})$ and the Nagata space $N = W(I, \mathbb{P})$, where $I = [0, 1]$ and $\mathbb{P}$ is the set of irrational numbers in $I$. Note that both $\sigma$ and $N$ are subsets of the Hilbert cube $Q = I^\omega$. It is well known that the Smirnov space $\sigma$ is $A_1(s.c.d.)$-universal [Mo1] and the Nagata space $N$ is $A_2(c.d.)$-universal [Mo2]. Let us remark that according to Theorem 1 the Smirnov space $\sigma$ admits no sigma hereditarily disconnected completion, while the Nagata space $N$ admits no embedding into a sigma hereditarily disconnected absolute $F_{\sigma\delta}$-space. This answers Question 1.3 of [Mo2]. Recently T. Radul...
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[Ra] (see also [BRZ, §4.1, Ex. 3]) has shown that the weak product $W(Q, \sigma)$ is universal for the additive Borel class $A_3$ of absolute $F_{\sigma\delta\sigma}$-spaces. Can the weak product $W(X, Y)$ be $C$-universal for a higher Borel class, if $Y$ is finite-dimensional or strongly countable-dimensional? In particular, can $W(X, Y)$ be universal for the multiplicative Borel classes $M_2$ and $M_3$ of absolute $F_{\sigma\delta}$- and $G_{\delta\sigma\delta}$-spaces, respectively?

We recall that a space $X$ is defined to be $\sigma$-complete if $X$ can be written as a countable union $X = \bigcup_{i=1}^{\infty} X_i$, where each $X_i$ is complete-metrizable and closed in $X$.

**Theorem 3.** Let $Y$ be a subspace of a Polish space $X$.

1. If $Y$ has a sigma hereditarily disconnected completion, then the weak product $W(X, Y)$ is not $M_2$-universal;
2. If $Y$ embeds into a $\sigma$-complete sigma hereditarily disconnected space, then $W(X, Y)$ is not $M_3$-universal.

The proofs of our theorems rely on simple homological arguments, so we need to recall some standard notations from homology theory. For every integer $q \geq 0$ let $H_q(X)$ denote the $q$th singular homology group of a space $X$ (reduced in dimension zero so that $H_0(X) = 0$ if and only if $X$ is path-connected) and let $H_*(X) = \bigoplus_{q=0}^{\infty} H_q(X)$. For closed subsets $B \subset A$ of the Hilbert cube $Q$ we denote by $j_B^A$ the homomorphism of $H_*(Q \setminus A)$ into $H_*(Q \setminus B)$ induced by inclusion. A closed subset $A$ of $Q$ is defined to be an irreducible barrier for an element $\alpha \in H_q(Q \setminus A)$ if $\alpha \neq 0$ but $j_B^A(\alpha) = 0$ for any closed proper subset $B \subset A$; and $A$ is an irreducible barrier in $Q$ if either $A = Q$ or $A$ is a closed irreducible barrier for some (non-trivial) element $\alpha \in H_q(X \setminus A)$, $q \geq 0$.

The following lemma plays a crucial role in the proof of Theorems 1, 3 and seems to have an independent value.

**Main Lemma.** For every countable cover $\{X_n\}_{n \in \mathbb{N}}$ of an irreducible barrier $A$ in the Hilbert cube $Q$, one of the sets $X_n$ contains a connected subset $C \subset X_n$ whose closure $\overline{C}$ is an irreducible barrier in $Q$.

**Proof of Main Lemma.** We need the following two homological lemmas proven in [Ca_2].

**Lemma 1.** Suppose $A$ is a closed subset of the Hilbert cube $Q$ such that $H_q(Q \setminus A) \neq 0$ for some $q \geq 0$. Then $A$ contains an irreducible barrier $B$ for some $\alpha \in H_q(Q \setminus B)$.

**Lemma 2.** If $A$ is an irreducible barrier in $Q$ then for every closed subset $B \subset A$ separating $A$ we have $H_*(Q \setminus B) \neq 0$. 
To prove the Main Lemma assume on the contrary that \( \{X_n\}_{n=1}^{\infty} \) is a countable cover of an irreducible barrier \( A \subset Q \) such that no \( X_n \) contains a connected subset \( C \) whose closure is an irreducible barrier in \( Q \). To get a contradiction we will construct a decreasing sequence \( A = A_0 \supset A_1 \supset \ldots \) of irreducible barriers in \( Q \) such that \( A_n \cap X_n = \emptyset \) for every \( n \geq 1 \).

Then by compactness of \( A \) we will find a point \( a \in \bigcap_{n=1}^{\infty} A_n \subseteq A \) that does not belong to \( \bigcup_{n=1}^{\infty} X_n \supseteq A \), a contradiction.

The construction of \( \{A_n\} \) is inductive. Set \( A_0 = A \) and suppose that for an \( n \geq 0 \) irreducible barriers \( A_0 \supset \ldots \supset A_n \) satisfying \( A_k \cap X_k = \emptyset \) for \( 1 \leq k \leq n \) have been constructed. By our hypothesis, \( A_n \cap X_{n+1} \) is either disconnected or not dense in \( A_n \). In both cases, one may easily construct a closed subset \( B \) separating \( A_n \) and missing \( X_{n+1} \).

By Lemma 2, we have \( H_s(Q \setminus B) \neq 0 \), and by Lemma 1, \( B \) contains an irreducible barrier \( A_{n+1} \) in \( Q \). Evidently, \( A_{n+1} \) is as required because \( A_{n+1} \cap X_{n+1} = \emptyset \).

\[ \text{Some auxiliary results.} \]

For any \( n \geq 0 \) irreducible barriers \( A_0 \supset \ldots \supset A_n \) satisfying \( A_k \cap X_k = \emptyset \) for \( 1 \leq k \leq n \) have been constructed. By our hypothesis, \( A_n \cap X_{n+1} \) is either disconnected or not dense in \( A_n \). In both cases, one may easily construct a closed subset \( B \) separating \( A_n \) and missing \( X_{n+1} \). By Lemma 2, we have \( H_s(Q \setminus B) \neq 0 \), and by Lemma 1, \( B \) contains an irreducible barrier \( A_{n+1} \) in \( Q \). Evidently, \( A_{n+1} \) is as required because \( A_{n+1} \cap X_{n+1} = \emptyset \).

\[ \text{Lemma 3. If } A \subset Q \text{ is an irreducible barrier for some } \alpha \in H_q(Q \setminus A) \text{, then for any subcube } P \text{ of } Q \text{ whose interior meets } A \text{ we have } H_q(P \setminus A) \neq 0. \]

\[ \text{Lemma 4. If } A \text{ is an irreducible barrier in } Q \times Q \text{ and } Y \text{ is an } \infty \text{-dense subset in } Q, \text{ then there is a point } y \in Y \text{ such that } A \cap (\{y\} \times Q) \text{ contains an irreducible barrier } B \text{ in } \{y\} \times Q. \]

For any \( q \geq 0 \) let \( N_q = \{(t_i)_{i \in \omega} \in Q : \text{at most } q \text{ coordinates } t_i \text{ are rational}\} \) denote the analog of the Nöbeling space in the Hilbert cube. It is easily seen that \( N_q \) is a \( G_\delta \)-set in \( Q \) and \( \mathcal{N} = \bigcup_{q=0}^{\infty} N_q \).

\[ \text{Lemma 5. For every } q \geq 0 \text{ the sets } \sigma, s, Q \setminus s, N, \text{ and } N_q \text{ are } q \text{-dense in } Q. \]

\[ \text{Proof. The } q \text{-density of } \sigma, s, Q \setminus s \text{ in } Q \text{ is easily seen and well known.} \]

The \( q \)-density of \( N_q \) in \( Q \) can be proven by analogy with the proof of the universality of the Nöbeling space (see [En, 1.11.5]). Finally, the \( q \)-density of \( N \) in \( Q \) follows from the \( q \)-density of \( N_q \) in \( Q \) and the inclusion \( N_q \subset N \).

\[ \text{Lemma 6. If } A \subset Q \text{ is an irreducible barrier for some } \alpha \in H_q(Q \setminus A) \text{ then } A \cap X \text{ is dense in } A \text{ for every } (q+1) \text{-dense subset } X \subset Q. \]
Thus of the sets $\beta$ barrier in $G$ the intersection the coordinate projection $pr$ such that contradiction. $h$ and $(h)$ then in $X$ each $j$ such that $(h)$ is irreducible barrier $A$ such that $(h)$ has a sigma hereditarily disconnected completion we may a closed embedding this embedding extends to an embedding $A$ that $A$ has a $(q+1)$-dimensional polyhedron $K$, a function $f : K \to Q \setminus A$, and an element $\beta \in H_q(K)$ with $f_\ast(\beta) = \alpha$. Since $j_B^A(\alpha) = 0$, there exists a $(q+1)$-dimensional polyhedron $L$ containing $K$ and a function $g : L \to Q \setminus B$ such that $g|K = f$ and $i_\ast(\beta) = 0$, where $i$ is the embedding of $K$ into $L$ (see [Ma, p. 293]). If $h : L \to X$ is sufficiently near to $g$, then $h(L) \subset Q \setminus B$ and $h|K$ is homotopic to $f$ in $Q \setminus A$. This yields $f_\ast(\beta) = (h|K)_\ast(\beta)$ and from $h(L) \subset Q \setminus B$, we get $h(L) \cap A \subset (A \setminus B) \cap X = U \cap A \cap X = \emptyset$. Then in $H_q(Q \setminus A)$ we have $\alpha = f_\ast(\beta) = (h|K)_\ast(\beta) = h_\ast \circ i_\ast(\beta) = 0$, a contradiction. 

In what follows we will need the following modification of the Main Lemma.

**Lemma 7.** Suppose $X$ is an $\infty$-dense $G_\delta$-set in $Q$ and $A$ is an irreducible barrier in $Q$. If $\{X_n\}_{n \in \mathbb{N}}$ is a countable cover of the set $A \cap X$, then one of the sets $X_n$ contains a connected subset $C \subset X_n$ whose closure $\overline{C}$ is an irreducible barrier in $Q$.

**Proof.** Since $X$ is a $G_\delta$-set in $Q$, we may write $A \setminus X = \bigcup_{n \in \mathbb{N}} A_n$, where each $A_n$ is compact. Then we have a countable cover $\{A_n, X_n\}_{n \in \mathbb{N}}$ of the irreducible barrier $A$. By the Main Lemma, there is a connected set $C \subset Q$ such that $\overline{C}$ is an irreducible barrier in $Q$ and either $C \subset A_n$ or $C \subset X_n$. The case $C \subset A_n$ is impossible. Indeed, by the compactness of $A_n$, $\overline{C} \subset A_n$. Thus $C \cap X = \emptyset$, a contradiction with Lemma 6. 

Finally, we need the following particular case of [BRZ, 3.1.1]:

**Lemma 8.** Let $X$ be a Polish space and $Y \subset X$.

(1) If $Y$ is $A_2$-universal, then there is an embedding $\varphi : Q^\omega \to X$ such that $\varphi^{-1}(Y) = W(Q, s)$.

(2) If $Y$ is $M_2$-universal, then there is an embedding $\varphi : Q^\omega \to X$ such that $\varphi^{-1}(Y) = Q^\omega \setminus W(Q, s)$.

(3) If $Y$ is $M_3$-universal, then there is an embedding $\varphi : Q^\omega \to X$ such that $\varphi^{-1}(Y) = Q^\omega \setminus W(Q, \sigma)$.

**Proof of Theorem 1.** (1) Suppose $X^\omega$ is $A_1(s.c.d.)$-universal and $X$ has a sigma hereditarily disconnected completion $Y$. Since $\sigma \in A_1(s.c.d.)$, we may fix a closed embedding $\varphi : \sigma \to X^\omega$. By Lavrent’ev’s Theorem, this embedding extends to an embedding $\overline{\varphi} : G \to Y^\omega$ of some $G_\delta$-set $G \subset Q$ containing $\sigma$. Since $\varphi(\sigma)$ is closed in $X^\omega$ and dense in $\overline{\varphi}(G)$, we have $\overline{\varphi}^{-1}(X^\omega) = \sigma$. For $m \geq 0$ denote by $\varphi_m : G \to Y$ the composition of $\varphi$ with the coordinate projection $pr_m : Y^\omega \to Y$. 

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Using the fact that $Y \setminus X$ and $\sigma$ are sigma hereditarily disconnected, write $Y \setminus X = \bigcup_{n=1}^{\infty} Y_n$ and $\sigma = \bigcup_{n=1}^{\infty} Z_n$, where $Y_n$ and $Z_n$ are hereditarily disconnected. Write also $Q \setminus G = \bigcup_{n=1}^{\infty} G_n$, where each $G_n$ is compact. Since $\sigma \subset G$, we have $\sigma \cap G_n = \emptyset$ for $n \geq 1$.

Thus, the Hilbert cube $Q = \sigma \cup (Q \setminus G) \cup (G \setminus \sigma)$ has the countable cover $\{Z_n, G_n, \varphi^{-1}_m(Y_n)\}_{n,m \in \mathbb{N}}$. By the Main Lemma, there is a connected set $C \subset Q$ such that $C$ is an irreducible barrier in $Q$ and either $C \subset Z_n$, $C \subset G_n$, or $C \subset \varphi^{-1}_m(Y_n)$ for some $n, m \in \mathbb{N}$.

Since all $Z_n$’s are hereditarily disconnected, no $Z_n$ can contain the (connected) set $C$. Next, assuming that $C \subset G_n$ for some $n$, we derive from the compactness of $G_n$ that $\overline{C} \subset G_n$ and thus $\overline{C} \cap \sigma = \emptyset$, a contradiction with Lemmas 6 and 5.

Thus $C \subset \varphi^{-1}_m(Y_n)$ for some $n, m \in \mathbb{N}$. Then $\varphi_m(C)$, being a connected subset of a hereditarily disconnected space, is a single point $y \in Y_n \subset Y \setminus X$. Since $\varphi^{-1}_m(y)$ is a closed subset in $G$ missing $\sigma$, it follows that $\overline{C}$ is an irreducible barrier in $Q$ missing $\sigma$, contrary to Lemmas 6 and 5 again.

(2) Suppose $X^\omega$ is $\mathcal{A}_2$-(c.d.)-universal and $X$ embeds into a sigma hereditarily disconnected $F_{\sigma\delta}$-space $Y$. Since $\mathcal{N} \in \mathcal{A}_2$-(c.d.), we may fix a closed embedding $\varphi : \mathcal{N} \rightarrow X^\omega$. It follows easily from the Lavrent’ev Theorem that this embedding extends to an embedding $\overline{\varphi} : G \rightarrow Y^\omega$ of some $F_{\sigma\delta}$-set $G \subset Q$ containing the Nagata space $\mathcal{N}$. As in the preceding case, observe that $\overline{\varphi}^{-1}(X^\omega) = \mathcal{N}$. For $m \geq 0$ let $\varphi_m = \text{pr}_m \circ \varphi : G \rightarrow Y$.

Using the fact that $Y \setminus X$ and $\mathcal{N}$ are sigma hereditarily disconnected, write $Y \setminus X = \bigcup_{n=1}^{\infty} Y_n$ and $\mathcal{N} = \bigcup_{n=1}^{\infty} Z_n$, where $Y_n$ and $Z_n$ are hereditarily disconnected. The complement $Q \setminus G$, being a $G_{\delta\sigma}$-subset of $Q^\omega$, can be written as $Q \setminus G = \bigcup_{n=1}^{\infty} G_n$, where each $G_n$ is a $G_{\delta}$-set in $Q^\omega$. Observe that $\mathcal{N} \cap G_n = \emptyset$ for $n \geq 1$.

Thus, $Q$ has the countable cover $\{Z_n, G_n, \varphi^{-1}_m(Y_n)\}_{n,m \in \mathbb{N}}$. By the Main Lemma, there is a connected set $C \subset Q$ such that $\overline{C}$ is an irreducible barrier for some non-trivial $\alpha \in H_q(Q \setminus \overline{C})$ and either $C \subset G_n$, $C \subset Z_n$, or $C \subset \varphi^{-1}_m(Y_n)$ for some $n, m \in \mathbb{N}$.

As in the preceding case we can show that the last two inclusions are impossible. Thus, $C \subset G_n$ for some $n \geq 1$. Since $C$ is dense in $\overline{C}$, we find that $\overline{C} \cap G_n$ is a dense $G_{\delta}$-set in $\overline{C}$. By Lemma 6, $\overline{C} \cap \mathcal{N}_{q+1}$ is a dense $G_{\delta}$-set in $\overline{C}$ as well. Then by the Baire Theorem, $\overline{C} \cap \mathcal{N}_{q+1} \cap G_n$ is dense in $\overline{C}$. But $G_n \cap \mathcal{N}_{q+1} = \emptyset$ by construction, a contradiction.

(3) Suppose $X$ is sigma hereditarily disconnected and $X^\omega$ is $\mathcal{A}_2$-universal. Let $Y$ be any completion of $X$. By Lemma 8, there is a map $\varphi : Q^\omega \rightarrow Y^\omega$ such that $\varphi^{-1}(X^\omega) = W(Q, s)$.

For $g_0, \ldots, q_n \in Q$ let $Q(q_0, \ldots, q_n) = \{(q_0, \ldots, q_n)\} \times \prod_{i>n} Q \subset Q^\omega$ and $S(q_0, \ldots, q_n) = \{(q_0, \ldots, q_n)\} \times \prod_{i>n} s \subset Q^\omega$. For $n \geq 0$ let $\varphi_n = \text{pr}_n \circ \varphi : Q^\omega \rightarrow Y$. 

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By induction, for every \( n \geq 0 \) we will construct points \( x_n \in X \), \( q_n \in Q \) and a closed subset \( A_n \subset Q(q_0, \ldots, q_n) \) such that

1. \( q_n \notin s \);
2. \( A_n \supset A_{n+1} \);
3. \( A_n \) is an irreducible barrier in \( Q(q_0, \ldots, q_n) \);
4. \( \varphi_n(A_n) = \{x_n\} \subset X \).

To get a contradiction, observe that the point \( q = (q_n)_{n \geq 0} \in Q^\omega \), being the intersection of \( A_n \)'s, belongs to \( \varphi^{-1}(X^\omega) = W(Q,s) \) by (4). On the other hand, (1) implies \( q \notin W(Q,s) \).

**Inductive step.** Let \( A_{-1} = Q^\omega \). Suppose that for some \( n \geq -1 \) the points \( q_0, \ldots, q_n \in Q \) and the irreducible barrier \( A_n \subset Q(q_0, \ldots, q_n) \) have been constructed. Write \( X = \bigcup_{i=1}^\infty X_i \), where \( X_i \) are hereditarily disconnected. Observe that \( s(q_0, \ldots, q_n) \subset W(Q,s) \) is a \( \omega \)-dense \( G_\delta \) set in \( Q(q_0, \ldots, q_n) \).

Since the collection \( \{A_n \cap \varphi_{n+1}^{-1}(X_i)\}_{i \in \mathbb{N}} \) covers \( A_n \cap s(q_0, \ldots, q_n) \), we may apply Lemma 7 to find an \( i \in \mathbb{N} \) and a connected set \( C \subset A_n \cap \varphi_{n+1}^{-1}(X_i) \) such that \( C \) is an irreducible barrier in \( Q(q_0, \ldots, q_n) \). Since \( \varphi_{n+1}(C) \) is a connected subset of the hereditarily disconnected space \( X_i \), we have \( \varphi_{n+1}(C) = \{x_{n+1}\} \) for some \( x_{n+1} \in X_i \subset X \). Then \( \varphi_{n+1}(C) = \{x_{n+1}\} \) as well. By Lemma 4, there is a \( q_{n+1} \in Q \setminus s \) such that \( C \cap Q(q_0, \ldots, q_{n+1}) \) contains an irreducible barrier \( A_{n+1} \) in \( Q(q_0, \ldots, q_{n+1}) \). Evidently, the points \( x_{n+1}, q_{n+1} \), and the set \( A_{n+1} \) satisfy the conditions (1)–(4). 

**Proof of Theorem 3.** Let \( Y \) be a subspace of a Polish space \( X \).

(1) Suppose \( Y \) has a sigma hereditarily disconnected completion \( \hat{Y} \) and the weak product \( W(X,Y) \) is \( \mathcal{M}_2 \)-universal. By Lemma 8, there is an embedding \( \varphi : Q^\omega \to X^\omega \) such that \( \varphi^{-1}(W(X,Y)) = Q^\omega \setminus W(Q,s) \). For \( n \geq 0 \) let \( \varphi_n : Q^\omega \to X \) be the composition of \( \varphi \) and the coordinate projection \( \text{pr}_n : X^\omega \to X \).

By induction, for every \( n \geq 0 \) we will construct a point \( q_n \in Q \) and a closed subset \( A_n \subset Q(q_0, \ldots, q_n) \) such that

1. \( A_n \supset A_{n+1} \);
2. \( A_n \) is an irreducible barrier in \( Q(q_0, \ldots, q_n) \);
3. either \( \varphi_n(A_n) \subset Y \) or \( \varphi_n(A_n) \subset X \setminus Y \);
4. \( q_n \notin s \) if and only if \( \varphi_n(A_n) \subset Y \).

To get a contradiction, observe that the point \( q = (q_n)_{n \geq 0} \in Q^\omega \) is the intersection of the sets \( A_n \). Let \( x = (x_n)_{n \geq 0} = \varphi(q) \in X^\omega \). By (3) and (4), \( x_n \in Y \) if and only if \( q_n \in s \). This yields \( \varphi(q) = (x_n) \in W(X,Y) \) if and only if \( q = (q_n) \in W(Q,s) \), contrary to \( \varphi^{-1}(W(X,Y)) = Q^\omega \setminus W(Q,s) \).

**Inductive step.** Let \( A_{-1} = Q^\omega \). Suppose that for some \( n \geq -1 \) the points \( q_0, \ldots, q_n \in Q \) and the irreducible barrier \( A_n \subset Q(q_0, \ldots, q_n) \) have
been constructed. According to the Lavrent’ev Theorem, we may assume \( \widehat{Y} \) to be a subspace of \( X \). Write \( \widehat{Y} = \bigcup_{i=1}^{\infty} Y_i \) and \( X \setminus \widehat{Y} = \bigcup_{i=1}^{\infty} F_i \), where for every \( i \geq 1 \), \( Y_i \) is a hereditarily disconnected set and \( F_i \) is closed in \( X \). Because the countable collection \( \{ \varphi_{n+1}^{-1}(Y_i), \varphi_{n+1}^{-1}(F_i) : i \in \mathbb{N} \} \) covers the irreducible barrier \( A_n \), we may apply the Main Lemma to find a connected set \( C \subset A_n \) such that \( \overline{C} \) is an irreducible barrier in \( Q(q_0, \ldots, q_n) \) and either \( C \subset \varphi_{n+1}^{-1}(F_i) \) or \( C \subset \varphi_{n+1}^{-1}(Y_i) \) for some \( i \).

We claim that either \( \varphi_{n+1}(\overline{C}) \subset X \setminus Y \) or \( \varphi_{n+1}(\overline{C}) \subset Y \). Indeed, if \( C \subset \varphi_{n+1}^{-1}(F_i) \), then \( \varphi_{n+1}(\overline{C}) \subset F_i \subset X \setminus Y \) (because \( F_i \) is closed in \( X \)). If \( C \subset \varphi_{n+1}^{-1}(Y_i) \), then because \( C \) is connected and \( Y_i \) is hereditarily disconnected, we deduce that \( \varphi_{n+1}(C) \) consists of a unique point \( y \in Y_i \). Then \( \varphi_{n+1}(\overline{C}) = \{y\} \) and hence \( \varphi_{n+1}(\overline{C}) \subset Y \) if \( y \in Y \) and \( \varphi_{n+1}(\overline{C}) \subset X \setminus Y \) otherwise.

By Lemma 4, there is a point \( q_{n+1} \in Q \) such that \( \overline{C} \cap Q(q_0, \ldots, q_{n+1}) \) contains an irreducible barrier \( A_{n+1} \) in \( Q(q_0, \ldots, q_{n+1}) \). Moreover, since \( s \) and \( Q \setminus s \) are \( \infty \)-dense in \( Q \) the point \( q_{n+1} \) can be chosen so that \( q_{n+1} \in s \) if and only if \( \varphi_{n+1}(\overline{C}) \subset Y \). Evidently, the point \( q_{n+1} \) and the set \( A_{n+1} \) satisfy the conditions (1)–(4).

(2) Suppose \( Y \) embeds into a \( \sigma \)-complete sigma hereditarily disconnected space \( \widehat{Y} \) and the weak product \( W(X, Y) \) is \( \mathcal{M}_3 \)-universal. By Lemma 8, there is an embedding \( \varphi : Q^\omega \to X^\omega \) such that \( \varphi^{-1}(W(X, Y)) = Q^\omega \setminus W(Q, \sigma) \). For \( n \geq 0 \) let \( \varphi_n = \text{pr}_n \circ \varphi : Q^\omega \to X \). Let also \( \pi_n : Q^\omega \to Q \) be the projection onto the \( n \)th coordinate.

According to the Lavrent’ev Theorem, we may assume \( \widehat{Y} \) to be a subspace of \( X \). Write \( \widehat{Y} = \bigcup_{k=1}^{\infty} Y_k \), where each \( Y_k \) is an absolute \( G_\delta \)-set closed in \( \widehat{Y} \). Denote by \( \overline{Y_k} \) the closure of \( Y_k \) in \( X \). Write also \( \sigma = \bigcup_{k=1}^{\infty} I_k \), where \( I_k \) are compact subsets of \( Q \). By induction for every \( k \geq 0 \) we will construct a partition of \( \{0, \ldots, k\} \) into three subsets \( H_i(k) \), \( i = 1, 2, 3 \), so that

(1) for \( i = 1, 2, H_i(k) \subset H_i(k') \) if \( k \leq k' \).

For every \( r \in \bigcup_{k \geq 0} H_1(k) \cup H_2(k) \) we will construct a point \( q_r \in Q \) and for every \( k \geq 0 \) we let

\[
P_k = \bigcap_{r \in H_1(k) \cup H_2(k)} \pi_r^{-1}(q_r) \subset Q^\omega
\]

and \( P_{-1} = Q^\omega \). By (1) we have \( P_k \supset P_{k+1} \) for every \( k \). We shall also construct a subcube \( R_k \) of \( P_k \) and an irreducible barrier \( A_k \) in \( R_k \) such that the following conditions are satisfied for every \( k \):

(2) \( A_k \supset A_{k+1} \);
(3) if \( r \in H_1(k) \), then \( q_r \in \sigma \) and \( \varphi_r(A_k) \subset Y \);
(4) if \( r \in H_2(k) \), then \( q_r \in Q \setminus \sigma \) and \( \varphi_r(A_k) \subset X \setminus Y \);
(5) if \( r \in H_3(k) \), then \( R_k \cap \pi_r^{-1}(I_k) = \emptyset \) and \( \varphi_r(A_k) \cap \overline{Y_k} = \emptyset \).
To get a contradiction, observe that by (2) there exists a point $z = (z_r) \in \bigcap_{k \geq 0} A_k$. By (3)–(5), $z_r \in \sigma$ if and only if $\varphi_r(z) \in Y$. Thus $z \in W(Q, \sigma)$ if and only if $\varphi(z) \in W(X, Y)$, contrary to $\varphi^{-1}(W(X, Y)) = Q^\omega \setminus W(Q, \sigma)$.

**Inductive construction.** Let $R_{-1} = A_{-1} = Q^\omega$. Suppose $k = 0$ or $k \geq 1$ and our objects are constructed up to order $k - 1$. Let $k = r_0, r_1, \ldots, r_l$ be the elements of the set $\{k\} \cup H_3(k - 1)$. We shall construct two finite decreasing sequences

$$R_{k-1} = U_{-1} \supset U_0 \supset \ldots \supset U_l, \quad A_{k-1} = B_{-1} \supset B_0 \supset \ldots \supset B_l,$$

where $U_j$ is a subcube in $R_{k-1}$ and $B_j$ is an irreducible barrier in $U_j$ for $j \leq l$. From the construction of these sets we will see to which of the sets $H_i(k)$ an element $r_j$ should be assigned (elements of $\{0, \ldots, k\} \setminus \{r_0, \ldots, r_l\}$ belong to $H_1(k)$ or $H_2(k)$ according to (1)).

Suppose for $j \geq 0$ the sets $U_{j-1}$ and $B_{j-1}$ are constructed. We distinguish two cases:

(a) $\varphi_{r_j}^{-1}(X \setminus \overline{Y}_k) \cap B_{j-1} \neq \emptyset$. Then we can find a subcube $U_j$ in $U_{j-1}$ whose interior in $U_{j-1}$ meets the barrier $B_{j-1}$ and $U_j \subset \varphi_{r_j}^{-1}(X \setminus \overline{Y}_k)$. By Lemmas 1 and 3, $B_{j-1} \cap U_j$ contains an irreducible barrier $B_j$ in $U_j$. We assign $r_j$ to $H_3(k)$.

(b) $\varphi_{r_j}(B_{j-1}) \subset \overline{Y}_k$. Since $Y_k$ is closed in $\overline{Y}$ we get $\overline{Y}_k \cap \overline{Y} = Y_k$. Recalling that $Y_k$ is a sigma hereditarily disconnected absolute $G_\delta$-set, write $Y_k = \bigcup_{i=1}^\infty D_i$ and $\overline{Y}_k \setminus \overline{Y} = \overline{Y}_k \setminus Y_k = \bigcup_{i=1}^\infty F_i$, where the sets $D_i$ are hereditarily disconnected and $F_i$ are closed in $X$. Then the countable collection $\{\varphi_{r_j}^{-1}(D_i), \varphi_{r_j}^{-1}(F_i) : i \in \mathbb{N}\}$ covers the irreducible barrier $B_{j-1}$. By the Main Lemma, there is a connected subset $C \subset B_{j-1}$ such that $\overline{C}$ is an irreducible barrier in $B_{j-1}$ and either $C \subset \varphi_{r_j}^{-1}(F_i)$ or $C \subset \varphi_{r_j}^{-1}(D_i)$ for some $i$. As in the preceding proof, we have either $\varphi_{r_j}(\overline{C}) \subset Y$ or $\varphi_{r_j}(\overline{C}) \subset X \setminus Y$. Let $U_j = U_{j-1}$, $B_j = \overline{C}$, and assign $z_j$ to $H_1(k)$ if $\varphi_{r_j}(B_j) \subset Y$ and to $H_2(k)$ if $\varphi_{r_j}(B_j) \subset X \setminus Y$.

Thus we constructed the sets $H_i(k)$, $i = 1, 2, 3$. Since the complement of the closed set $\bigcup_{r \in H_3(k)} \pi_r^{-1}(I_k)$ is $\infty$-dense in $U_l$, we may find a subcube $K \subset U_l$ whose interior relative to $U_l$ meets the barrier $B_l$ and such that $K \cap \bigcup_{r \in H_3(k)} \pi_r^{-1}(I_k) = \emptyset$ (see Lemma 6). By Lemma 1, the set $B_l \cap K$ contains an irreducible barrier $B$ in $K$.

Applying Lemma 4 find for every $r \in H_1(k) \setminus H_1(k-1)$ a point $q_r \in \sigma$ and for every $r \in H_1(k) \setminus H_2(k-1)$ a point $q_r \in Q \setminus s$ such that $B \cap P_{q_r}$ contains an irreducible barrier $A_k$ in the subcube $R_k = P_k \cap K$ of $P_k$. Clearly, the constructed objects satisfy the conditions (1)–(5).

**Proof of Theorem 2.** First we recall some definitions. Let $0 \leq n \leq \infty$. A subset $A$ of a space $X$ is called a $Z_n$-set in $X$ if $A$ is closed in $X$ and every
map \( f : I^n \to X \) of the \( n \)-dimensional cube can be uniformly approximated by maps into \( X \setminus A \). A space \( X \) is called a \( \sigma Z_n \)-space if \( X \) can be written as a countable union \( X = \bigcup_{i=1}^{\infty} X_i \) of \( Z_n \)-sets \( X_i \) in \( X \). Note that each \( \sigma Z_n \)-space is a \( \sigma Z_m \)-space for every \( m \leq n \). Observe also that a space \( X \) is of the first Baire category if and only if \( X \) is a \( \sigma Z_0 \)-space.

The following fact is proven in [BT].

**Lemma 9.** If an absolute retract \( X \) is a \( \sigma Z_0 \)-space, then for every \( n \in \mathbb{N} \) its \( n \)-th power \( X^n \) is a \( \sigma Z_{n-1} \)-space.

In Lemma 5.4 of [DMM] T. Dobrowolski, W. Marciszewski, and J. Mogilski have proven that if an absolute retract \( X \) is a \( \sigma Z_\infty \)-space, then for every \( \sigma \)-compact space \( A \) there is a proper map \( f : A \to X \). Modifying their arguments and using results of [To] one may prove

**Lemma 10.** If for some \( n \geq 0 \) an absolute retract \( X \) is a \( \sigma Z_n \)-space, then for every \( n \)-dimensional \( \sigma \)-compact space \( A \) there exists a proper map \( f : A \to X \).

For a class \( C \) of spaces and \( n \geq 0 \) let \( C[n] = \{ C \in C : \dim(C) \leq n \} \). Let us recall that a map \( f : A \to X \) is **proper** provided the preimage \( f^{-1}(K) \) of any compact subset \( K \subset X \) is compact.

**Lemma 11.** If \( X \) is an absolute retract of the first Baire category, then for every \( n \in \mathbb{N} \) its power \( X^{3n+2} \) is \( A_1[n] \)-universal.

**Proof.** Fix \( n \in \mathbb{N} \) and a \( \sigma \)-compact space \( A \) with \( \dim(A) \leq n \). By Lemmas 9 and 10 there exists a proper map \( f : A \to X^{n+1} \). Since \( X \), being an absolute retract, contains a topological copy of the interval \( I \), we can apply the classical Menger–Nöbeling–Lefschetz Theorem [En, 1.11.4] to find an embedding \( g : A \to X^{2n+1} \). Then \( e = (f, g) : A \to X^{n+1} \times X^{2n+1} = X^{3n+2} \) is a closed embedding. \( \blacksquare \)

**Proof of Theorem 2.** By [To, 4.1, 2.4] the space \( X \) embeds into a complete-metrizable absolute retract \( \tilde{X} \) so that \( X \) is homotopy dense in \( \tilde{X} \). The latter means that there is a homotopy \( h : \tilde{X} \times [0,1] \to \tilde{X} \) such that \( h(\tilde{X} \times (0,1)) \subset X \) and \( h(x,0) = x \) for every \( x \in \tilde{X} \).

Let \( A \in A_1(\text{s.c.d.c.}) \), i.e., \( A \) is a \( \sigma \)-compact space having a strongly countable-dimensional completion \( C \). By the Compactification Theorem [En, 5.3.5] the space \( C \) has a strongly countable-dimensional metrizable compactification \( K \). Write \( K = \bigcup_{i=0}^{\infty} K_i \), where each \( K_i \subset K_{i+1} \) is a compact finite-dimensional subspace of \( K \).

By Lemma 11, the countable power \( X^\omega \) is \( A_1[n] \)-universal for all \( n \in \mathbb{N} \). Then Theorem 3.1.1 of [BRZ] implies that for every \( i \) there exists an embedding \( f_i : K_n \to \tilde{X}^\omega \) with \( f_i^{-1}(X^\omega) = K_n \cap A \). Since \( X^\omega \) is homotopy dense in the absolute retract \( \tilde{X}^\omega \), the map \( f_i \) can be extended to a map \( \overline{f}_i : K \to \tilde{X}^\omega \).
such that $\overline{f_i}(K \setminus K_i) \subset X^\omega$. Consider the map $f = (\overline{f_i})_{i=0}^\infty : K \to (\overline{X})^\omega$ and notice that it is an embedding with $f^{-1}((X^\omega)^\omega) = A$. Thus the restriction $f|A : A \to (X^\omega)^\omega$ is a closed embedding, i.e., the space $X^\omega$, being homeomorphic to $(X^\omega)^\omega$, is $A_1(\text{s.c.d.c.})$-universal.

**Some questions and comments.** The exponent $3n + 2$ in Lemma 11 is not optimal. In fact, for every locally path-connected space $X$ of the first Baire category the power $X^{2n+1}$ is $A_1[n]$-universal for every $n \geq 0$. The proof of this statement requires more involved arguments and will be given in another paper.

**Question 1.** For which Borel classes $\mathcal{C}$ is there an absolute retract $A \in \mathcal{C}[1]$ whose power $A^{n+1}$ is $\mathcal{C}[n]$-universal for every $n \in \mathbb{N}$?

**Question 2.** Suppose that $X$, $Y$ are finite-dimensional $\sigma$-compact absolute retracts of the first Baire category. Are their countable powers $X^\omega$ and $Y^\omega$ homeomorphic?

Note that by Theorem 2 each of the spaces $X^\omega$, $Y^\omega$ embeds as a closed subset into the other. By Lemma 9 these spaces are $\sigma Z_n$-spaces for every $n \in \mathbb{N}$. By Theorem 1 and Lemma 5.4 of [DMM], they are not $\sigma Z$-spaces, so that the standard technique of absorbing spaces (see [BRZ]) cannot be applied to answer Question 2.

Let us remark that the second assertion of Theorem 3 generalizes [Ca$_3$], the first assertion of Theorem 1 generalizes a result of [BR], and the third one generalizes [Ca$_1$]. As mentioned in the introduction, the Nagata space $\mathcal{N}$ admits no embedding into a sigma hereditarily disconnected absolute $F_{\sigma\delta}$-space. In this context it would be interesting to know answers to the following questions.

**Question 3.** Suppose $F \supset \mathcal{N}$ is an $F_{\sigma\delta}$-subset in $Q$ containing the Nagata space $\mathcal{N}$.

(a) Does $F$ contain a Hilbert cube (cf. [En, 5.3.6])?

(b) Is $F$ strongly infinite-dimensional?

(c) Does $F \setminus \mathcal{N}$ contain an arc? Note that $F \setminus \mathcal{N}$ is connected, moreover, $A \cap (F \setminus \mathcal{N})$ is connected for every irreducible barrier $A$ in $Q$.

(d) Does $F$ contain a copy $I$ of $[0,1]$ such that $I \cap \mathcal{N}$ is a countable dense subset of $I$? Note that $F$ always contains a copy $K$ of the Cantor set such that $K \cap \mathcal{N}$ is countable and dense in $K$.

**Question 4.** Does there exist a countable-dimensional absolute $F_{\sigma\delta}$-space containing a copy of each countable-dimensional compactum?

According to [En, 5.3.11 and 7.1.33], the Smirnov space $\sigma$ contains a copy of all Smirnov cubes. This shows that there are $\sigma$-compact strongly
countable-dimensional spaces containing compacta of arbitrary high transfinite dimension ind.

The Main Lemma implies that irreducible barriers in $Q$ are not sigma hereditarily disconnected. In fact, every sigma hereditarily disconnected compactum is weakly infinite-dimensional [Kr, §6]. It is not clear if the converse is also true.

**Question 5. Is every weakly infinite-dimensional compactum sigma hereditarily disconnected?**

It was remarked by R. Pol that this question is connected with the known open problem on existence of a weakly infinite-dimensional compactum whose square is strongly infinite-dimensional: such a compactum cannot be sigma hereditarily disconnected. Observe that the example of an uncountable-dimensional weakly infinite-dimensional compactum constructed by R. Pol [Po] is sigma hereditarily disconnected.

**References**


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