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Resolving a question of Arkhangel'skiĭ's

by

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Abstract. We construct in ZFC a cosmic space that, despite being the union of countably many metrizable subspaces, has covering dimension equal to 1 and inductive dimensions equal to 2.

1. Introduction. On the class of separable metrizable spaces, the three main topological dimension functions, i.e., covering dimension, dim, small inductive dimension, ind, and large inductive dimension, Ind, coincide and exhibit all the properties that intuition requires of the concept of dimension. A regular space is called *cosmic* if it has a countable network or, equivalently, if it is the continuous image of a separable metrizable space. Cosmic spaces exhibit many desirable properties. Cosmicity is a hereditary and countably productive property, and a cosmic space is Lindelöf and perfectly normal and therefore satisfies the Urysohn inequality, the subset theorem and the countable sum theorem for both dim and Ind, as well as the inequality dim $\leq$ ind and the equality ind $=$ Ind. One would naturally want to know whether the equality dim $=$ ind $=$ Ind extends from metric separable to cosmic spaces. This question was first raised by A. V. Arkhangel'skiĭ [1]. Let us note that the equality dim $=$ Ind is known to hold on the class of μ-spaces, i.e., those spaces that can be embedded in a countable product of paracompact F_σ-metrizable spaces [7, 9]. A paracompact, perfectly normal space that can be expressed as the union of finitely many metric spaces is a μ-space [9], and S. Oka [8] raised the question of the equality dim $=$ Ind for paracompact, perfectly normal spaces that can be expressed as the union of countably many metric spaces. The space of the abstract provides an answer to both questions just mentioned.

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A cosmic space that is the union of countably many metrizable subspaces and has dim $=1$ and Ind >1 was first announced, assuming the continuum hypothesis, by Delistathis and Watson in [3], leaving open the question of the precise value of Ind. The seminal idea is to exploit an example due to Kuratowski [6] of a function on the Cantor set whose graph has positive dimension. Their space is the limit of an inverse sequence of length ω_1 of spaces constructed by the method of resolutions so as to incorporate sufficiently many graphs of Kuratowski functions. However, the lemmas (Lemmas 2.2 and 2.3 of [3]) necessary for their use of resolutions are incorrect, and the need for an appropriate modification of their lengthy constructions has been acknowledged as early as October 2001. It is not, however, clear that there is a straightforward repair that corresponds to what the authors had in mind.

In this paper, the natural numbers, the closed interval $[0,1]$ and the real numbers, all with their usual topology, are denoted by $\mathbb{N}$, $\mathbb{I}$, and $\mathbb{R}$, respectively. A *rational circle* means a circle in $\mathbb{R}^2$ whose radius as well as both coordinates of its centre are rational numbers. $\mathbb{P}$ denotes the union of all rational circles, and $\mathbb{Q}$ the countable set of all points that lie in the intersection of two distinct rational circles. $\mathbb{A}$ will denote a fixed subset of $\mathbb{R}^2$ whose complement in $\mathbb{R}^2$ is a countable dense set disjoint from $\mathbb{P}$. Unless explicitly stated, the rest of our notation agrees with that of Engelking's books [4, 5], where the reader is referred to for all topological facts quoted in this paper without proof.

Delistathis and Watson announce in the introduction of [3] that at each step of their inductive construction they refine the topology of $\mathbb{I}^2$, taking care not to disturb the topology of a fixed countable collection of subspaces whose union is $\mathbb{I}^2$. Theorem 1 below sets up a framework for doing this in a manner that suffices for our purpose. The detailed proof that we give is long but elementary. Let $\{K_\alpha : \alpha < |2^{\mathbb{N}}|\}$ be an enumeration of the proper continua in $\mathbb{R}^2$. For each $\alpha < |2^{\mathbb{N}}|$, Theorem 1 provides a refinement τ_α of the euclidean topology of $\mathbb{R}^2$ such that τ_α on $\mathbb{R}^2 \setminus \mathbb{P}$ and $S \setminus \mathbb{Q}$ is euclidean for each rational circle S and, for $\alpha \neq \beta$, the supremum of τ_α and τ_β restricted to $\mathbb{A} \cap K_\alpha$ contains the graph of a Kuratowski-like function. The supremum of all τ_α on $\mathbb{A}$ will give the space of the abstract. The details are given in Sections 3 and 4.

2. The necessary lemmas. In what follows, $\Gamma(f)$ denotes the graph of a function f and $|A|$ the cardinality of a set A. When considering a product of a number of spaces, we use π_i to denote the projection onto the ith factor.

LEMMA 1. *Let X, Y be metrizable spaces, A a countable subset of X and $f : X \to Y$ a function which is continuous at every point of $X \setminus A$. Then*

$\Gamma(f)$ is a G_δ-subset of $X \times Y$. Hence $\Gamma(f)$ is Čech-complete whenever X and Y are Čech-complete.

Proof. Consider a point (x, y) of $X \times Y \setminus \Gamma(f)$ with $x \notin A$. Then $y \neq f(x)$ and there are disjoint open sets V_1, V_2 of Y with $f(x) \in V_1$ and $y \in V_2$. By continuity of f at x, there is an open set U of X with $x \in U \subset f^{-1}(V_1)$. Consequently, $(x, y) \in U \times V_2$ and $U \times V_2 \cap \Gamma(f) = \emptyset$. Thus, $\overline{\Gamma(f)} \setminus \Gamma(f) \subset \pi_1^{-1}(A)$, where $\pi_1 : X \times Y \to X$ denotes the canonical projection. Hence

$$\Gamma(f) = \overline{\Gamma(f)} \setminus \bigcup_{a \in A} \{a\} \times (Y \setminus \{f(a)\}).$$

As singletons and other closed subsets of metrizable spaces are G_δ-sets and A is countable, one readily sees that $\Gamma(f)$ is a G_δ-set in $X \times Y$. ∎

Lemma 2. *Let X, Y be topological spaces, K a closed subset of X and A a subset of K. Let $f : X \to Y$ be a function such that $f|_{X \setminus A}$ is continuous and $f|_K$ is continuous at each point of $K \setminus A$. Then f is continuous at each point of $X \setminus A$.*

Proof. Consider a point $x \in X \setminus A$ and an open neighbourhood H of $f(x)$. If $x \notin K$, then $f^{-1}(H) \setminus K$ is an open neighbourhood of x in X inside $f^{-1}(H)$. If $x \in K \setminus A$, then x has open neighbourhoods U, V in X such that $U \setminus A \subset f^{-1}(H)$ and $V \cap K \subset f^{-1}(H)$. Then $U \cap V$ is an open neighbourhood of x in X inside $f^{-1}(H)$. ∎

Lemma 3. *Let X, Y be topological spaces with Y compact. Let $f : X \to Y$ be a function and x_0 a point of X such that $\pi_1^{-1}(x_0) \cap \overline{\Gamma(f)}$ is a singleton. Then f is continuous at x_0.*

Proof. Consider an open neighbourhood V of $f(x_0)$. Then $(\{x_0\} \times (Y \setminus V)) \cap \overline{\Gamma(f)} = \emptyset$ and, by compactness of $Y \setminus V$, there are open neighbourhoods U of x_0 and W of $Y \setminus V$ such that $(U \times W) \cap \Gamma(f) = \emptyset$. This implies $f(U) \subset V$. ∎

Lemma 4. *Let d be a metric inducing a Čech-complete topology on a set X. Let F be a closed subset of $X \times \mathbb{I}$ such that $A = \{a \in X : |\pi_1^{-1}(a) \cap F| > 1\}$ is countable. Let E be a subset of $\pi_1(F)$ that contains A and $f : E \to \mathbb{I}$ a function with $\Gamma(f) \subset F$. Then there is a metric $\varrho \geq d$ inducing a Čech-complete topology on X and such that E with the topology induced by ϱ is homeomorphic to $\Gamma(f)$, and if $S \subset (F \setminus \pi_1^{-1}(A)) \cup \Gamma(f)$ and $|\pi_2(S)| = 1$, then $\varrho = d$ on $\pi_1(S)$. Furthermore, every ϱ-neighbourhood of a point $x \notin A$ is also a d-neighbourhood of x.*

Proof. Evidently, f can be uniquely extended to $\pi_1(F)$ so that $\Gamma(f) \subset F$, and we will assume henceforth that f is defined on the whole of $\pi_1(F)$. For $x \in \pi_1(F) \setminus A$, $\pi_1^{-1}(x) \cap \overline{\Gamma(f)}$ is a singleton and, by Lemma 3, f is continuous at x. As $X \setminus A$ is a normal space and $\pi_1(F)$ is a closed subset of X, by Tietze's extension theorem, there is an extension $g : X \to \mathbb{I}$ of f such that $g|_{X\setminus A}$ is continuous. By Lemma 2, g is continuous at each point of $X \setminus A$ and, by Lemma 1, $\Gamma(g)$ is Čech-complete. To complete the proof, we simply identify a point x of X with the point $(x, g(x))$ of $\Gamma(g)$, thereby identifying $Y \subset X$ with $\Gamma(g|_Y)$, and we let ϱ denote the restriction of the product metric to $\Gamma(g)$.

Note that for $x \notin A$ and any open rectangle $G \times H$ containing $(x, g(x))$, as g is continuous at x, there is a d-neighbourhood U of x inside $G \cap g^{-1}(H)$. Then $\Gamma(g|_U) \subset G \times H$, and this establishes the last assertion of the lemma. ∎

3. Topologies on $\mathbb{R}^2$. With regard to the sets $\mathbb{P}, \mathbb{Q}$ and $\mathbb{A}$ defined in the introduction, note that $\mathbb{P} \subset \mathbb{A}$, $\dim \mathbb{A} = 1$ and $\dim(\mathbb{R}^2 \setminus \mathbb{P}) = 0$. A *rational arc* will mean a non-empty open arc of a rational circle. Observe that $\mathbb{Q} \cap S$ is dense in any rational arc S. Also, a point x of $\mathbb{P} \setminus \mathbb{Q}$ belongs to a unique rational circle, which we denote by S_x. For $x \in \mathbb{Q}$, we set $S_x = \{x\}$.

When considering subsets of $\mathbb{R}^2$, $\overline{A}$ denotes the Euclidean closure of A, while $\overline{A}^{\tau}$ denotes closure with respect to some topology τ on $\mathbb{R}^2$. Also, A_τ denotes A with the relative topology induced by τ and, when no subscript appears, it will be understood that A carries the euclidean topology. Other similar conventions will be clear from the context.

We investigate the class **GT** of all topologies τ on $\mathbb{R}^2$ that satisfy the following conditions.

(1) τ is finer than the euclidean topology on $\mathbb{R}^2$.
(2) τ is metrizable and Čech-complete.
(3) Apart from a countable number of *exceptional* points belonging to $\mathbb{P}$, the neighbourhoods of all other points are euclidean.
(4) For $x \in \mathbb{P}$, a τ-neighbourhood of x in S_x is also a euclidean neighbourhood.

Every member of **GT** is clearly a separable topology.

PROPOSITION 1. *Let* $\tau_n \in \mathbf{GT}$ *for each* $n \in \mathbb{N}$. *Then* $\tau = \sup_{n\in\mathbb{N}} \tau_n \in \mathbf{GT}$.

Proof. $\mathbb{R}^2_\tau$ is metrizable and Čech-complete because it is homeomorphic to the diagonal of the product $\prod_{n\in\mathbb{N}} \mathbb{R}^2_{\tau_n}$. The rest follows from the observation that the sets of the form $G_1 \cap \cdots \cap G_n$, where $G_i \in \tau_i$, constitute a base for τ. ∎

PROPOSITION 2. *Let τ be the supremum of a set of members of* **GT**. *Then $\mathbb{R}^2_\tau$ is a cosmic space that is the union of countably many subspaces of $\mathbb{R}^2$. Furthermore*, $\operatorname{Ind} \mathbb{A}_\tau \le 2$.

Proof. Let $\{S_n : n \in \mathbb{N}\}$ be an enumeration of all rational circles. Property (4) ensures that each $S_n \setminus \mathbb{Q}$ has the euclidean topology with respect to any member of **GT** and, therefore, with respect to τ. Observe that the dimension in every sense of these sets is zero. Property (3) shows that $(\mathbb{R}^2 \setminus \mathbb{P})_\tau$ has the euclidean topology and hence $\operatorname{Ind}(\mathbb{R}^2 \setminus \mathbb{P})_\tau = 0$. It is now clear that $\mathbb{R}^2_\tau$ is the union of countably many euclidean subspaces of $\mathbb{R}^2$ and thus it is a cosmic space. Recall that a countable set is zero-dimensional in every respect and apply the Urysohn inequality to obtain $\operatorname{Ind}(S_n)_\tau \le 1$ for each $n \in \mathbb{N}$. By the countable sum theorem, $\operatorname{Ind} \mathbb{P}_\tau \le 1$. By the Urysohn inequality $\operatorname{Ind} \mathbb{R}^2_\tau \le 2$. Finally, the subset theorem gives $\operatorname{Ind} \mathbb{A}_\tau \le 2$. ∎

REMARK 1. Let K be a proper continuum and consider a point $x \in K$ and an annulus A with centre x that leaves a point of K outside it. Then $K \cap Z \neq \emptyset$ for any circle Z inside A. Hence $K \cap \mathbb{P}$ is dense in K.

Suppose additionally that $x \in \mathbb{P}$ and K contains no rational arc. Then $A \cap S_x$ is either empty or consists of one or two rational arcs, and there is an annulus B with centre x inside A such that $B \cap S_x \subset S_x \setminus K$. Hence some rational circle Z inside B contains a point of $A \cap K \cap \mathbb{P} \setminus S_x$.

Let $A(z_0; r, R)$ denote the annulus $\{z \in \mathbb{R}^2 : r < |z - z_0| < R\}$, where $0 < r < R < \infty$. A *system of annuli* for x will mean a sequence $\{A_{x,n}\}_{n\in\mathbb{N}}$ of annuli, where $A_{x,n} = A(x; r_n, R_n)$, $R_{n+1} \le r_n < R_n$ and $\lim_{n\to\infty} r_n = 0$.

REMARK 2. Let K be a proper continuum that contains no rational arc (resp. some rational arc S). Let $\{A_{x,n}\}_{n\in\mathbb{N}}$ be a system of annuli for a point x of $K \cap \mathbb{P}$ (resp. $S \cap \mathbb{Q}$). Let D_x be an open disc with centre x. For some $m = m(x) \in \mathbb{N}$, $A_{x,m}$ lies inside D_x but does not contain the whole of K (resp. S). By Remark 1, for each $n \in \mathbb{N}$, we can pick a point x_n in $K \cap \mathbb{P} \cap A_{x,m+n} \setminus S_x$ (resp. in $S \cap \mathbb{Q} \cap A_{x,m+n} \setminus S_x$). Pick an open disc D_{x_n} with centre x_n and $\overline{D}_{x_n} \subset A_{x,m+n} \setminus S_x$. Note that any sequence $\{z_n\}$ with $z_n \in D_{x_n}$ converges to x, $\operatorname{diam} D_{x_n} < \frac{1}{2} \operatorname{diam} D_x$ and, moreover, we have the freedom of choosing $\operatorname{diam} D_{x_n}$ as small as we wish.

If $t = (t_1, \ldots, t_n)$ is an n-tuple, $t^\wedge s$ denotes the $(n+1)$-tuple $(t_1, \ldots, t_n, s)$. It is convenient to treat $\emptyset$ as a 0-tuple and to identify $\emptyset^\wedge n$ with n.

THEOREM 1. *Let* $\mathbf{A}$ *consist of a system of annuli* $\{A_{x,n}\}_{n\in\mathbb{N}}$ *for each point* x *of a proper continuum* K *in* $\mathbb{R}^2$. *Then there is a countable subset* $D = D(K, \mathbf{A})$ *of* $K \cap \mathbb{P}$ *and a topology* $\sigma = \sigma(K, \mathbf{A}) \in \mathbf{GT}$ *such that*

(i) D *is the set of exceptional points of* σ.
(ii) *For each* $x \in D$, *there is a fixed sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* D *and some* $m = m(x) \in \mathbb{N}$ *with* $x_n \in A_{x,m+n}$.
(iii) $E = \overline{D}$ *is a subset* $K \cap \mathbb{A}$ *homeomorphic to the Cantor set.*
(iv) *For each* $x \in D$, *convergence to* x *in* $\mathbb{R}^2 \setminus \bigcup_{n\in\mathbb{N}} A_{x,n}$ *is equivalent to* σ*-convergence to* x.
(v) *Whenever* τ *is a perfectly normal topology on* $\mathbb{R}^2$ *refining its usual topology and such that* (α) *for each* $x \in D$, *the fixed sequence* $\{x_n\}$

converges to x with respect to τ and (β) *every countable closed subset of τ has an isolated point*, *then* $\operatorname{Ind} E_{\sigma\vee\tau} > 0$.

Proof. Start with a point $d = d_\emptyset \in K \cap \mathbb{P}$ and an open disc D_d with centre d and radius < 1. If K contains some rational arc, fix one such arc S and pick d in $S \cap \mathbb{Q}$. Remark 2 supplies for each i-tuple t of natural numbers a point $d_t \in K \cap \mathbb{P}$ and an open disc $D_t = D_{d_t}$ with centre d_t through the formula $d_{s^\wedge n} = (d_s)_n$. We let $D = \{d_t : t \text{ an } i\text{-tuple of natural numbers}\}$.

According to Remark 2, for each $x \in D$ and some integer $m = m(x)$, $\overline{D}_{x_n} \subset A_{x,m+n} \setminus S_x$, and any sequence $\{z_n\}$ with $z_n \in D_{x_n}$ converges to x. It can be seen that $\overline{D}_{t^\wedge n} \subset D_t$, $\operatorname{diam} D_t < 2^{-|t|}$, $D_{t_1} \subset D_{t_2}$ iff t_1 is an extension of t_2, and if neither of t_1, t_2 is an extension of the other, then $\overline{D}_{t_1} \cap \overline{D}_{t_2} = \emptyset$. We can further assume that the first $|t|$ points of $\mathbb{R}^2 \setminus \mathbb{A}$ do not lie in $\overline{D}_t$.

For each $t \in \{0,1\}^i$, $i \in \mathbb{N}$, we define a compact subset E_t of $\mathbb{R}^2$ as follows:

$$E_0 = \{d_\emptyset\} \cup \bigcup_{n=2}^{\infty} \overline{D}_n, \qquad E_1 = \{d_1\} \cup \bigcup_{n=1}^{\infty} \overline{D}_{(1,n)}$$

and if $E_t = \{d_s\} \cup \bigcup_{n=m}^{\infty} \overline{D}_{d_{s^\wedge n}}$, then

$$E_{t^\wedge 0} = \{d_s\} \cup \bigcup_{n=m+1}^{\infty} \overline{D}_{d_{s^\wedge n}} \quad \text{and} \quad E_{t^\wedge 1} = \{d_{s^\wedge m}\} \cup \bigcup_{n=1}^{\infty} \overline{D}_{d_{s^\wedge m^\wedge n}}.$$

Let $E = \bigcap_{k\in\mathbb{N}} \bigcup (E_t : t \in \{0,1\}^k)$. From the properties of the discs D_t, $E_{t_1} \subset E_{t_2}$ iff t_1 is an extension of t_2, and if neither of t_1, t_2 is an extension of the other, then $E_{t_1} \cap E_{t_2} = \emptyset$. Note that for the $s = s(t)$ that occurs in the definition of E_t, $E_t \subset D_{s(t)}$ and $|s(t)|$ is the number of 1's in t. Thus, if t contains n 1's, then $\operatorname{diam} E_t < 2^{-n}$ and E_t misses the first n points of $\mathbb{R}^2 \setminus \mathbb{A}$. One readily sees that for each $(i_1, i_2, \ldots) \in \{0,1\}^{\mathbb{N}}$, $\bigcap_{k\in\mathbb{N}} E_{(i_1,\ldots,i_k)}$ is a singleton $\{x\}$. In fact, $x \in \overline{D} \cap \mathbb{A} \setminus D$ if $(i_1, i_2, \ldots)$ contains infinitely many 1's, otherwise, $x \in D$. Clearly, $E = \overline{D}$ and E is a subspace of $K \cap \mathbb{A}$ homeomorphic to the Cantor set.

Starting with $I_d = I = [0,1]$, we define a closed interval $I_x = [l_x, r_x]$ for each $x \in D$ by

$$I_{x_n} = \begin{cases} \left[l_x, \frac{1}{2}(l_x + r_x)\right] & \text{for } n \text{ odd}, \\ \left[\frac{1}{2}(l_x + r_x), r_x\right] & \text{for } n \text{ even}. \end{cases}$$

We next define T to be the closure in $\mathbb{R}^2 \times \mathbb{I}$ of $\bigcup_{x\in D} \{x\} \times I_x$. Because the first coordinate projection π_1 is closed and continuous and $\pi_1(T)$ contains D, we must have $\pi_1(T) = \overline{D} = E$.

Let $y \in E \setminus D$. Then there is a unique sequence $\{n_k\}$ of natural numbers such that $\{y\} = \bigcap_{k\in\mathbb{N}} \overline{D}_{(n_1,\ldots,n_k)} = \bigcap_{k\in\mathbb{N}} D_{(n_1,\ldots,n_k)}$. Clearly, $\bigcap_{k\in\mathbb{N}} I_{d_{(n_1,\ldots,n_k)}}$ consists of a single point $f(y)$, and $\pi_1^{-1}(y) \cap T = \{(y, f(y))\}$. Letting $f(x)$ be the mid-point of I_x for $x \in D$, we have a function $f : E \to \mathbb{I}$. By Lemma 3, f is continuous at each point of $E \setminus D$.

For x in D, fix an open disc O_x with centre x such that $\overline{O}_x \subset D_x$, and define $F_x = \overline{O}_x \setminus \bigcup_{n\in\mathbb{N}} D_{x_n}$ and

$$F = \bigcup_{x\in D} (F_x \times \{f(x)\}) \cup (\{x\} \times I_x) \cup \Gamma(f) = \bigcup_{x\in D} (F_x \times \{f(x)\}) \cup T.$$

Observe that $\{F_x : x \in D\}$ consists of mutually disjoint closed sets and $F_x \cap E = \{x\}$. Hence $\{z \in \mathbb{R}^2 : |\pi_1^{-1}(z) \cap F| > 1\} = D$. Note also that if $z \in D_x$ and $(z, s) \in F \setminus T$, then either $z \in F_x$ or $z \in D_{x_n}$ for some $n \in \mathbb{N}$, and hence $s \in I_x$.

To prove that F is a closed subset of $\mathbb{R}^2 \times \mathbb{I}$, consider a sequence $\{(z^n, s^n)\}$ of distinct points of $F \setminus T$ that converges in $\mathbb{R}^2 \times \mathbb{I}$ to a point (z, s). The following three cases may arise.

1. For some $x \in D$, $F_x \times \{f(x)\}$ contains infinitely many terms of the sequence, in which case it also contains (z, s).
2. For some $x \in D$, there are strictly increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $z^{m_k} \in D_{x_{n_k}}$. In this case, $z = x$, $s^{m_k} \in I_{x_{n_k}} \subset I_x$, and hence $s \in I_x$.
3. There are sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $\{m_k\}$ is strictly increasing and $z^{m_k} \in D_{d_{(n_1,\dots,n_k)}}$. In this case, $z = \lim_{k\to\infty} d_{(n_1,\dots,n_k)} \in E \setminus D$. Also, $s^{m_k} \in I_{d_{(n_1,\dots,n_k)}}$ and s is the unique point in the intersection of these segments, i.e., $f(z)$. Thus, $(z, s) \in \Gamma(f)$.

In any case, $(z, s) \in F$, and therefore F is a closed subset of $\mathbb{R}^2 \times \mathbb{I}$.

Now we apply Lemma 4, taking X to be $\mathbb{R}^2$, d its usual metric and $A = D$. The metric ϱ supplied by Lemma 4 induces a topology σ on $\mathbb{R}^2$ that is Čech-complete and finer than the usual one, and every σ-neighbourhood of a point outside D is euclidean. For $x \in D$, observe that $\varrho = d$ on $O_x \setminus \bigcup_{n\in\mathbb{N}} \overline{D}_{x_n}$ and recall that $\overline{D}_{x_n} \subset A_{x,m+n} \setminus S_x$ for some integer m. Therefore in S_x every σ-neighbourhood of x is a euclidean neighbourhood, and convergence to x with respect to σ in $\mathbb{R}^2 \setminus \bigcup_{n\in\mathbb{N}} A_{x,n}$ is equivalent to convergence with respect to d. Thus, $\sigma = \sigma(K, \mathbf{A}) \in \mathbf{GT}$ and items (i)–(iv) of Theorem 1 are satisfied.

Let τ be a topology on $\mathbb{R}^2$ satisfying the conditions of (v). By Lemma 4, E_σ is homeomorphic to $\Gamma(f)$ as a subspace of $\mathbb{R}^2 \times \mathbb{I}$ and, because τ is finer than the euclidean topology on $\mathbb{R}^2$, $E_{\sigma\vee\tau}$ is homeomorphic to $\Gamma(f)$ as a subspace of $\mathbb{R}^2_\tau \times \mathbb{I}$, which henceforth we denote by Γ. Note that, because τ is perfectly normal, $\mathbb{R}^2_\tau \times \mathbb{I}$ is perfectly normal.

Suppose $\operatorname{Ind} \Gamma \le 0$. Then there exist two disjoint open sets U, V of $\mathbb{R}^2_\tau \times \mathbb{I}$ such that $(d, f(d)) \in U$, $(d, 1) \in V$ and $\Gamma \subset U \cup V$. Let $A = T \setminus U \cup V$. Consider the set B consisting of all points $y \in D$ such that one of the points $(y, l_y), (y, r_y)$ belongs to the member of $\{U, V\}$ that does not contain $(y, f(y))$. Evidently, $d \in B \subset \pi_1(A) \subset D$, and $\pi_1(A)$ is a countable closed subset of $\mathbb{R}^2_\tau$. Property (β) of τ implies that $\overline{B}^\tau$ and hence B contains an isolated point b.

Let G be a τ-neighbourhood of b. We may suppose without loss of generality that $(b, f(b)) \in U$ and $(b, r_b) \in V$. Observe that $I_{b_{2n}} = [l_{b_{2n}}, r_{b_{2n}}] = [f(b), r_b]$ and, by property (α), the sequence $\{b_{2n}\}$ τ-converges to b. Hence $\{(b_{2n}, l_{b_{2n}})\}$ and $\{(b_{2n}, r_{b_{2n}})\}$ converge respectively to $(b, f(b))$ and (b, r_b) in $\mathbb{R}^2_\tau \times \mathbb{I}$. Hence, for some $m \in \mathbb{N}$, $(b_{2m}, l_{b_{2m}}) \in U \cap \pi_1^{-1}(G)$ and $(b_{2m}, r_{b_{2m}}) \in V \cap \pi_1^{-1}(G)$. As the point $(b_{2m}, f(b_{2m}))$ of Γ belongs to one of U, V, it is clear that $b_{2m} \in G \cap B$, contradicting the fact that b is an isolated point of B. It follows that $\operatorname{Ind} \Gamma > 0$, which concludes the proof of Theorem 1. ∎

The proof of $\dim \Gamma > 0$ is essentially due to Kuratowski (cf. [6], Problem 1.2.E of [5] and Proposition 2.1 of [3]).

4. A cosmic space with dim < ind**.** The set $\mathbb{N}$ has a family of almost disjoint infinite subsets of cardinality $|2^{\mathbb{N}}|$. Let $\{N_\alpha : \alpha < |2^{\mathbb{N}}|\}$ be such a family and write $N_\alpha = \{\alpha_1, \alpha_2, \ldots\}$, where $\alpha_i < \alpha_{i+1}$. For each $x \in \mathbb{R}^2, n \in \mathbb{N}$ and $\alpha < |2^{\mathbb{N}}|$, let $A^\alpha_{x,n} = \{z \in \mathbb{R}^2 : 1/(\alpha_n + 1) < |z - x| < 1/\alpha_n\}$. The important property of the annuli just defined is that

(∗) for all $\alpha \neq \beta$, there is some integer $k = k(\alpha, \beta)$ such that $A^\alpha_{x,m} \cap A^\beta_{x,n} = \emptyset$ for $m > k$ and all n.

Let $\{K_\alpha : \alpha < |2^{\mathbb{N}}|\}$ be an enumeration of the proper continua in $\mathbb{R}^2$. For each $\alpha < |2^{\mathbb{N}}|$, let $\mathbf{A}_\alpha$ consist of the system of annuli $\{A^\alpha_{x,n}\}$ for each point x of K_α. Taking $K = K_\alpha$ and $\mathbf{A} = \mathbf{A}_\alpha$ in Theorem 1, we obtain subsets $D^\alpha = D(K_\alpha, \mathbf{A}_\alpha)$ of $K_\alpha \cap \mathbb{P}$ and $E^\alpha = \overline{D^\alpha}$ of $K_\alpha \cap \mathbb{A}$ and a topology $\sigma_\alpha = \sigma(K_\alpha, \mathbf{A}_\alpha) \in \mathbf{GT}$. We define $\tau = \sup\{\sigma_\alpha : \alpha < |2^{\mathbb{N}}|\}$ and $\tau_\alpha = \sup\{\sigma_\beta : \beta \neq \alpha\}$.

By Proposition 2, τ and each τ_α, as well as any restriction of them to a subset of $\mathbb{R}^2$, is the union of a countable number of euclidean subspaces of $\mathbb{R}^2$ and is therefore cosmic. Now τ is Lindelöf and every open set of it is the countable union of members of topologies of the form $\sigma_{\alpha_1} \vee \cdots \vee \sigma_{\alpha_k}$. Hence, given a countable number of open sets G_i and a countable number of closed sets F_i of τ, there are $\alpha_1, \alpha_2, \ldots < |2^{\mathbb{N}}|$ such that each G_i is open and each F_i is closed in $\sup\{\sigma_{\alpha_i}\}$, which, by Proposition 1, as a member of $\mathbf{GT}$ is separable metrizable and Čech-complete. One consequence of this is that $\mathbb{R}^2_\tau$ is a Baire space and, therefore, any countable closed subset of it contains an isolated point. Another consequence is that the euclidean interior, $\operatorname{int} G$, of a τ-open subset G of $\mathbb{R}^2$ is obtained from G by subtracting only a countable number of points. Evidently, each $\mathbb{R}^2_{\tau_\alpha}$ has similar properties.

LEMMA 5. $\operatorname{Ind} E^\alpha_\tau > 0$.

Proof. For $x \in D^\alpha$, Theorem 1 supplies a fixed sequence of points x^α_n of D^α and an integer m such that $x^\alpha_n \in A^\alpha_{x,m+n}$. Now by $(*)$, for $\beta \neq \alpha$, a tail of the sequence $\{x^\alpha_n\}$ lies outside $\bigcup_{n\in\mathbb{N}} A^\beta_{x,n}$. By Theorem 1(iv), this implies that $\{x^\alpha_n\}$ converges to x with respect to σ_β, and hence with respect to $\tau_\alpha = \sup\{\sigma_\beta : \beta \neq \alpha\}$. Now the result follows from Theorem 1(v), since $\tau = \sigma_\alpha \vee \tau_\alpha$. ∎

PROPOSITION 3. $\operatorname{Ind} \mathbb{A}_\tau = 2$.

Proof. Let B_r denote the open disc with centre 0 and radius r. Suppose $\operatorname{Ind} \mathbb{A}_\tau \le 1$. Then there are disjoint τ-open sets U_1, U_2 of $\mathbb{A}$ such that $\mathbb{A} \cap \overline{B}_2 \subset U_1$, $\mathbb{A} \setminus B_3 \subset U_2$ and $\operatorname{Ind} L_\tau \le 0$, where $L = \mathbb{A} \setminus U_1 \cup U_2$. Let $M = \mathbb{R}^2 \setminus \operatorname{int} V_1 \cup \operatorname{int} V_2$, where V_i is the largest τ-open set of $\mathbb{R}^2$ whose trace on the dense set $\mathbb{A}$ is U_i. Then $M \setminus L$ is countable and, by the countable sum theorem, $\operatorname{Ind}(M \cap \mathbb{A})_\tau \le 0$. Also, $B_2 \subset V_1$, $\mathbb{R}^2 \setminus \overline{B}_3 \subset V_2$ and $V_1 \cap V_2 = \emptyset$. Thus, the compact set M is a partition in $\mathbb{R}^2$ between $\overline{B}_1$ and $\mathbb{R}^2 \setminus B_4$. Hence $\dim M > 0$, M contains a proper continuum and, for some $\alpha < |2^{\mathbb{N}}|$, $K_\alpha \subset M$. But then E^α is a τ-closed subset of $K_\alpha \cap \mathbb{A}$ and, in view of Lemma 5, $M \cap \mathbb{A}$ has positive inductive dimension with respect to τ. This shows that $\operatorname{Ind} \mathbb{A}_\tau > 1$. Finally, by Proposition 2, $\operatorname{Ind} \mathbb{A}_\tau = 2$. ∎

LEMMA 6. *Let* $\sigma = \sigma_{\alpha_1} \vee \cdots \vee \sigma_{\alpha_k}$. *Then* $\dim \mathbb{A}_\sigma \le 1$.

Proof. Let $D = D^{\alpha_1} \cup \cdots \cup D^{\alpha_k}$ and $E = E^{\alpha_1} \cup \cdots \cup E^{\alpha_k}$. As D is countable, $\dim D \le 0$. As $(E^{\alpha_i} \setminus D)_\sigma$ is a subspace of the Cantor set E^{α_i}, it has $\dim \le 0$. By the Urysohn inequality, $\dim (E^{\alpha_i})_\sigma \le 1$. Observe that $(\mathbb{A} \setminus E)_\sigma = \mathbb{A} \setminus E$ can be written as a countable union of closed subsets of $\mathbb{A}$ with $\dim \le \dim \mathbb{A} \le 1$. Finally, by the countable sum theorem, $\dim \mathbb{A}_\sigma \le 1$. ∎

PROPOSITION 4. $\dim \mathbb{A}_\tau = 1$.

Proof. The sets of the form $G_1 \cap \cdots \cap G_n$, where $G_i \in \sigma_{\alpha_i}$, constitute a base for τ. It follows that the Lindelöf space $\mathbb{A}_\tau$ is the limit space of the inverse limit system over the set of finite subsets of $2^{\mathbb{N}}$, directed by inclusion, whose bonding maps are the identity functions $\mathbb{A}_{\sigma_{\alpha_1} \vee \cdots \vee \sigma_{\alpha_m} \vee \cdots \vee \sigma_{\alpha_n}} \to \mathbb{A}_{\sigma_{\alpha_1} \vee \cdots \vee \sigma_{\alpha_m}}$. By Lemma 6, each $\mathbb{A}_{\sigma_{\alpha_1} \vee \cdots \vee \sigma_{\alpha_m}}$ has $\dim \le 1$. Now the inverse limit theorem for dim, which holds when the limit space is Lindelöf (see e.g. Proposition 1 in [2]), gives $\dim \mathbb{A}_\tau \le 1$. As $\dim \le 0$ implies $\operatorname{Ind} \le 0$, in view of Proposition 3, we must have $\dim \mathbb{A}_\tau = 1$. ∎

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on 18 February 2006, following our observation that the use of CH, which in the original version was restricted to the construction of annuli with property $(*)$, could be trivially avoided.

The reader should note that, independently, A. Dow and K. P. Hart have described under the assumption of Martin's Axiom a cosmic space X with $\dim X = 1 < 2 \leq \operatorname{Ind} X$; this was unknown to us before submission. Two versions of their preprint *Cosmic dimensions*, dated 5 Sep 2005 and 15 Nov 2005, respectively, appear at http://arxiv.org/abs/math/0509097.

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